2 □ LIMITS AND DERIVATIVES

2.1 The Tangent and Velocity Problems

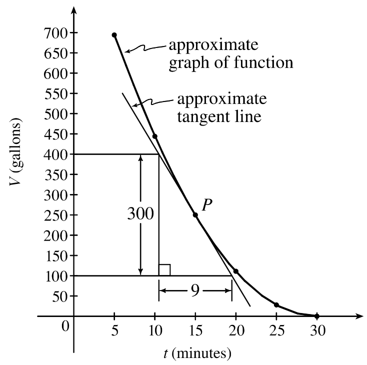
1. (a) Using $P(15, 250)$, we construct the following table:

t	Q	slope $= m_{PQ}$
5	$(5, 694)$	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	$(10, 444)$	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	$(20, 111)$	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	$(25, 28)$	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	$(30, 0)$	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9} = -33.\overline{3}$.



2. (a) Slope $= \frac{2948 - 2530}{42 - 36} = \frac{418}{6} \approx 69.67$

(b) Slope $= \frac{2948 - 2661}{42 - 38} = \frac{287}{4} = 71.75$

(c) Slope $= \frac{2948 - 2806}{42 - 40} = \frac{142}{2} = 71$

(d) Slope $= \frac{3080 - 2948}{44 - 42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats/minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. (a) $y = \dfrac{1}{1-x}$, $P(2, -1)$

	x	$Q(x, 1/(1-x))$	m_{PQ}
(i)	1.5	$(1.5, -2)$	2
(ii)	1.9	$(1.9, -1.111\,111)$	1.111 111
(iii)	1.99	$(1.99, -1.010\,101)$	1.010 101
(iv)	1.999	$(1.999, -1.001\,001)$	1.001 001
(v)	2.5	$(2.5, -0.666\,667)$	0.666 667
(vi)	2.1	$(2.1, -0.909\,091)$	0.909 091
(vii)	2.01	$(2.01, -0.990\,099)$	0.990 099
(viii)	2.001	$(2.001, -0.999\,001)$	0.999 001

(b) The slope appears to be 1.

(c) Using $m = 1$, an equation of the tangent line to the curve at $P(2, -1)$ is $y - (-1) = 1(x - 2)$, or $y = x - 3$.

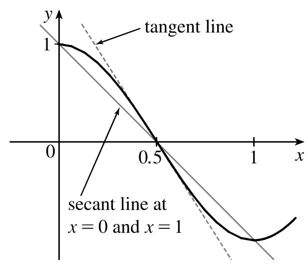
4. (a) $y = \cos \pi x$, $P(0.5, 0)$

	x	Q	m_{PQ}
(i)	0	$(0, 1)$	-2
(ii)	0.4	$(0.4, 0.309017)$	-3.090170
(iii)	0.49	$(0.49, 0.031411)$	-3.141076
(iv)	0.499	$(0.499, 0.003142)$	-3.141587
(v)	1	$(1, -1)$	-2
(vi)	0.6	$(0.6, -0.309017)$	-3.090170
(vii)	0.51	$(0.51, -0.031411)$	-3.141076
(viii)	0.501	$(0.501, -0.003142)$	-3.141587

(b) The slope appears to be $-\pi$.

(c) $y - 0 = -\pi(x - 0.5)$ or $y = -\pi x + \frac{1}{2}\pi$.

(d)



5. (a) $y = y(t) = 40t - 16t^2$. At $t = 2$, $y = 40(2) - 16(2)^2 = 16$. The average velocity between times 2 and $2 + h$ is

$$v_{\text{ave}} = \frac{y(2+h) - y(2)}{(2+h) - 2} = \frac{\left[40(2+h) - 16(2+h)^2\right] - 16}{h} = \frac{-24h - 16h^2}{h} = -24 - 16h, \text{ if } h \neq 0.$$

(i) $[2, 2.5]$: $h = 0.5$, $v_{\text{ave}} = -32$ ft/s

(ii) $[2, 2.1]$: $h = 0.1$, $v_{\text{ave}} = -25.6$ ft/s

(iii) $[2, 2.05]$: $h = 0.05$, $v_{\text{ave}} = -24.8$ ft/s

(iv) $[2, 2.01]$: $h = 0.01$, $v_{\text{ave}} = -24.16$ ft/s

(b) The instantaneous velocity when $t = 2$ (h approaches 0) is -24 ft/s.

6. (a) $y = y(t) = 10t - 1.86t^2$. At $t = 1$, $y = 10(1) - 1.86(1)^2 = 8.14$. The average velocity between times 1 and $1 + h$ is

$$v_{\text{ave}} = \frac{y(1+h) - y(1)}{(1+h) - 1} = \frac{\left[10(1+h) - 1.86(1+h)^2\right] - 8.14}{h} = \frac{6.28h - 1.86h^2}{h} = 6.28 - 1.86h, \text{ if } h \neq 0.$$

(i) $[1, 2]$: $h = 1$, $v_{\text{ave}} = 4.42$ m/s

(ii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = 5.35$ m/s

(iii) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = 6.094$ m/s

(iv) $[1, 1.01]$: $h = 0.01$, $v_{\text{ave}} = 6.2614$ m/s

(v) $[1, 1.001]$: $h = 0.001$, $v_{\text{ave}} = 6.27814$ m/s

(b) The instantaneous velocity when $t = 1$ (h approaches 0) is 6.28 m/s.

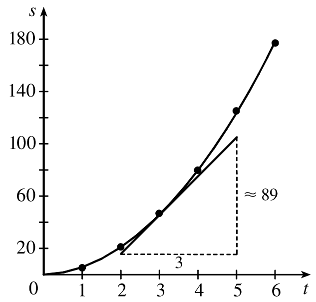
7. (a) (i) On the interval $[2, 4]$, $v_{\text{ave}} = \dfrac{s(4) - s(2)}{4 - 2} = \dfrac{79.2 - 20.6}{2} = 29.3$ ft/s.

(ii) On the interval $[3, 4]$, $v_{\text{ave}} = \dfrac{s(4) - s(3)}{4 - 3} = \dfrac{79.2 - 46.5}{1} = 32.7$ ft/s.

(iii) On the interval $[4, 5]$, $v_{\text{ave}} = \dfrac{s(5) - s(4)}{5 - 4} = \dfrac{124.8 - 79.2}{1} = 45.6$ ft/s.

(iv) On the interval $[4, 6]$, $v_{\text{ave}} = \dfrac{s(6) - s(4)}{6 - 4} = \dfrac{176.7 - 79.2}{2} = 48.75$ ft/s.

(b) Using the points $(2, 16)$ and $(5, 105)$ from the approximate tangent line, the instantaneous velocity at $t = 3$ is about $\frac{105 - 16}{5 - 2} = \frac{89}{3} \approx 29.7$ ft/s.



8. (a) (i) $s = s(t) = 2\sin\pi t + 3\cos\pi t$. On the interval $[1, 2]$, $v_{\text{ave}} = \frac{s(2) - s(1)}{2 - 1} = \frac{3 - (-3)}{1} = 6$ cm/s.

(ii) On the interval $[1, 1.1]$, $v_{\text{ave}} = \frac{s(1.1) - s(1)}{1.1 - 1} \approx \frac{-3.471 - (-3)}{0.1} = -4.71$ cm/s.

(iii) On the interval $[1, 1.01]$, $v_{\text{ave}} = \frac{s(1.01) - s(1)}{1.01 - 1} \approx \frac{-3.0613 - (-3)}{0.01} = -6.13$ cm/s.

(iv) On the interval $[1, 1.001]$, $v_{\text{ave}} = \frac{s(1.001) - s(1)}{1.001 - 1} \approx \frac{-3.00627 - (-3)}{0.001} = -6.27$ cm/s.

(b) The instantaneous velocity of the particle when $t = 1$ appears to be about -6.3 cm/s.

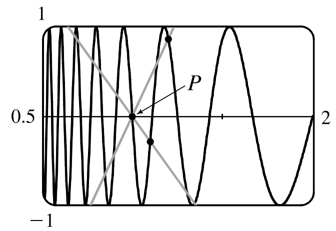
9. (a) For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

x	Q	m_{PQ}
2	$(2, 0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	-1.0847
1.3	$(1.3, -0.8230)$	-2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	-2.8173

x	Q	m_{PQ}
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	-2.1651
0.7	$(0.7, 0.7818)$	-2.6061
0.8	$(0.8, 1)$	-5
0.9	$(0.9, -0.3420)$	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108. So we estimate that the slope of the tangent line at P is about -31.4.

2.2 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim\limits_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).

(b) $\lim\limits_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

4. (a) As x approaches 2 from the left, the values of $f(x)$ approach 3, so $\lim\limits_{x \to 2^-} f(x) = 3$.

(b) As x approaches 2 from the right, the values of $f(x)$ approach 1, so $\lim\limits_{x \to 2^+} f(x) = 1$.

(c) $\lim\limits_{x \to 2} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

(d) When $x = 2$, $y = 3$, so $f(2) = 3$.

(e) As x approaches 4, the values of $f(x)$ approach 4, so $\lim\limits_{x \to 4} f(x) = 4$.

(f) There is no value of $f(x)$ when $x = 4$, so $f(4)$ does not exist.

5. (a) As x approaches 1, the values of $f(x)$ approach 2, so $\lim\limits_{x \to 1} f(x) = 2$.

(b) As x approaches 3 from the left, the values of $f(x)$ approach 1, so $\lim\limits_{x \to 3^-} f(x) = 1$.

(c) As x approaches 3 from the right, the values of $f(x)$ approach 4, so $\lim\limits_{x \to 3^+} f(x) = 4$.

(d) $\lim\limits_{x \to 3} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

(e) When $x = 3$, $y = 3$, so $f(3) = 3$.

6. (a) $h(x)$ approaches 4 as x approaches -3 from the left, so $\lim\limits_{x \to -3^-} h(x) = 4$.

(b) $h(x)$ approaches 4 as x approaches -3 from the right, so $\lim\limits_{x \to -3^+} h(x) = 4$.

(c) $\lim\limits_{x \to -3} h(x) = 4$ because the limits in part (a) and part (b) are equal.

(d) $h(-3)$ is not defined, so it doesn't exist.

(e) $h(x)$ approaches 1 as x approaches 0 from the left, so $\lim\limits_{x \to 0^-} h(x) = 1$.

(f) $h(x)$ approaches -1 as x approaches 0 from the right, so $\lim\limits_{x \to 0^+} h(x) = -1$.

(g) $\lim\limits_{x \to 0} h(x)$ does not exist because the limits in part (e) and part (f) are not equal.

(h) $h(0) = 1$ since the point $(0, 1)$ is on the graph of h.

(i) Since $\lim\limits_{x \to 2^-} h(x) = 2$ and $\lim\limits_{x \to 2^+} h(x) = 2$, we have $\lim\limits_{x \to 2} h(x) = 2$.

(j) $h(2)$ is not defined, so it doesn't exist.

(k) $h(x)$ approaches 3 as x approaches 5 from the right, so $\lim_{x\to 5^+} h(x) = 3$.

(l) $h(x)$ does not approach any one number as x approaches 5 from the left, so $\lim_{x\to 5^-} h(x)$ does not exist.

7. (a) $\lim_{t\to 0^-} g(t) = -1$ (b) $\lim_{t\to 0^+} g(t) = -2$

(c) $\lim_{t\to 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim_{t\to 2^-} g(t) = 2$ (e) $\lim_{t\to 2^+} g(t) = 0$

(f) $\lim_{t\to 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.

(g) $g(2) = 1$ (h) $\lim_{t\to 4} g(t) = 3$

8. (a) $\lim_{x\to -3} A(x) = \infty$ (b) $\lim_{x\to 2} A(x)$ does not exist. (c) $\lim_{x\to 2^-} A(x) = -\infty$

(d) $\lim_{x\to 2^+} A(x) = \infty$ (e) $\lim_{x\to -1} A(x) = -\infty$

(f) The equations of the vertical asymptotes are $x = -3$, $x = -1$ and $x = 2$.

9. (a) $\lim_{x\to -7} f(x) = -\infty$ (b) $\lim_{x\to -3} f(x) = \infty$ (c) $\lim_{x\to 0} f(x) = \infty$

(d) $\lim_{x\to 6^-} f(x) = -\infty$ (e) $\lim_{x\to 6^+} f(x) = \infty$

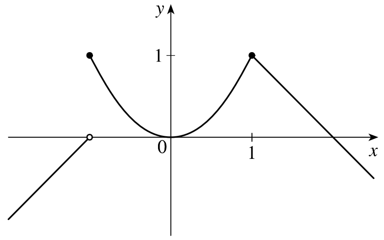
(f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

10. $\lim_{t\to 12^-} f(t) = 150$ mg and $\lim_{t\to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

11. From the graph of

$$f(x) = \begin{cases} 1 + x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 1, \\ 2 - x & \text{if } x \ge 1 \end{cases}$$

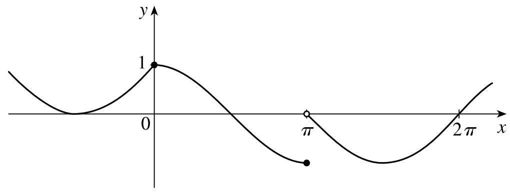
we see that $\lim_{x\to a} f(x)$ exists for all a except $a = -1$. Notice that the right and left limits are different at $a = -1$.



12. From the graph of

$$f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \le x \le \pi, \\ \sin x & \text{if } x > \pi \end{cases}$$

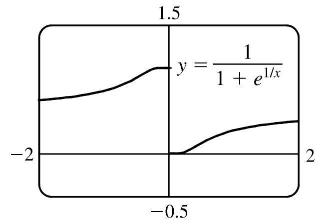
we see that $\lim_{x\to a} f(x)$ exists for all a except $a = \pi$. Notice that the right and left limits are different at $a = \pi$.



13. (a) $\lim\limits_{x\to 0^-} f(x) = 1$

(b) $\lim\limits_{x\to 0^+} f(x) = 0$

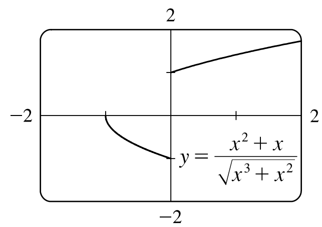
(c) $\lim\limits_{x\to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.



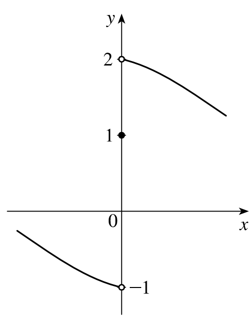
14. (a) $\lim\limits_{x\to 0^-} f(x) = -1$

(b) $\lim\limits_{x\to 0^+} f(x) = 1$

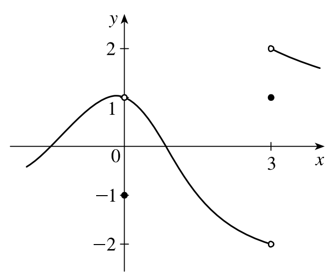
(c) $\lim\limits_{x\to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.



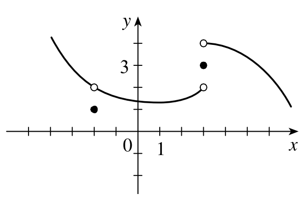
15. $\lim\limits_{x\to 0^-} f(x) = -1, \quad \lim\limits_{x\to 0^+} f(x) = 2, \quad f(0) = 1$



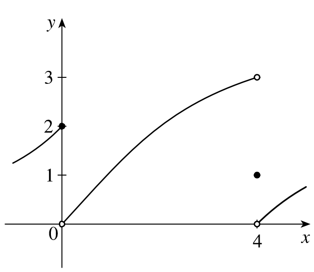
16. $\lim\limits_{x\to 0} f(x) = 1$, $\lim\limits_{x\to 3^-} f(x) = -2$, $\lim\limits_{x\to 3^+} f(x) = 2$, $f(0) = -1$, $f(3) = 1$



17. $\lim\limits_{x\to 3^+} f(x) = 4, \quad \lim\limits_{x\to 3^-} f(x) = 2$, $\lim\limits_{x\to -2} f(x) = 2$, $f(3) = 3, \quad f(-2) = 1$



18. $\lim\limits_{x\to 0^-} f(x) = 2$, $\lim\limits_{x\to 0^+} f(x) = 0$, $\lim\limits_{x\to 4^-} f(x) = 3$, $\lim\limits_{x\to 4^+} f(x) = 0$, $f(0) = 2$, $f(4) = 1$



19. For $f(x) = \dfrac{x^2-3x}{x^2-9}$:

x	$f(x)$
3.1	0.508 197
3.05	0.504 132
3.01	0.500 832
3.001	0.500 083
3.0001	0.500 008

x	$f(x)$
2.9	0.491 525
2.95	0.495 798
2.99	0.499 165
2.999	0.499 917
2.9999	0.499 992

It appears that $\lim\limits_{x\to 3} \dfrac{x^2-3x}{x^2-9} = \dfrac{1}{2}$.

20. For $f(x) = \dfrac{x^2 - 3x}{x^2 - 9}$:

x	$f(x)$
-2.5	-5
-2.9	-29
-2.95	-59
-2.99	-299
-2.999	-2999
-2.9999	$-29{,}999$

x	$f(x)$
-3.5	7
-3.1	31
-3.05	61
-3.01	301
-3.001	3001
-3.0001	$30{,}001$

It appears that $\lim\limits_{x \to -3^+} f(x) = -\infty$ and that $\lim\limits_{x \to -3^-} f(x) = \infty$, so $\lim\limits_{x \to -3} \dfrac{x^2 - 3x}{x^2 - 9}$ does not exist.

21. For $f(t) = \dfrac{e^{5t} - 1}{t}$:

t	$f(t)$
0.5	22.364 988
0.1	6.487 213
0.01	5.127 110
0.001	5.012 521
0.0001	5.001 250

t	$f(t)$
-0.5	1.835 830
-0.1	3.934 693
-0.01	4.877 058
-0.001	4.987 521
-0.0001	4.998 750

It appears that $\lim\limits_{t \to 0} \dfrac{e^{5t} - 1}{t} = 5$.

22. For $f(h) = \dfrac{(2+h)^5 - 32}{h}$:

h	$f(h)$
0.5	131.312 500
0.1	88.410 100
0.01	80.804 010
0.001	80.080 040
0.0001	80.008 000

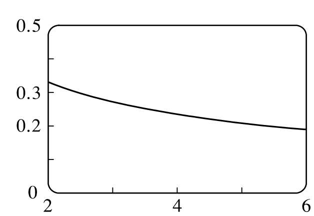
h	$f(h)$
-0.5	48.812 500
-0.1	72.390 100
-0.01	79.203 990
-0.001	79.920 040
-0.0001	79.992 000

It appears that $\lim\limits_{h \to 0} \dfrac{(2+h)^5 - 32}{h} = 80$.

23. For $f(x) = \dfrac{\ln x - \ln 4}{x - 4}$:

x	$f(x)$
3.9	0.253 178
3.99	0.250 313
3.999	0.250 031
3.9999	0.250 003

x	$f(x)$
4.1	0.246 926
4.01	0.249 688
4.001	0.249 969
4.0001	0.249 997

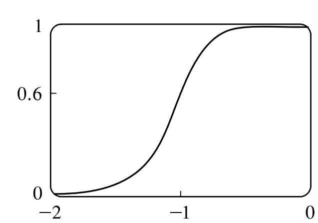


It appears that $\lim\limits_{x \to 4} f(x) = 0.25$. The graph confirms that result.

24. For $f(p) = \dfrac{1 + p^9}{1 + p^{15}}$:

p	$f(p)$
-1.1	0.427 397
-1.01	0.582 008
-1.001	0.598 200
-1.0001	0.599 820

p	$f(p)$
-0.9	0.771 405
-0.99	0.617 992
-0.999	0.601 800
-0.9999	0.600 180



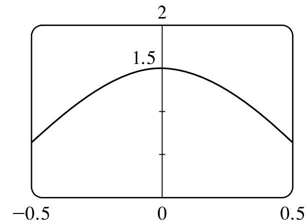
It appears that $\lim\limits_{p \to -1} f(p) = 0.6$. The graph confirms that result.

25. For $f(\theta) = \dfrac{\sin 3\theta}{\tan 2\theta}$:

θ	$f(\theta)$
± 0.1	$1.457\,847$
± 0.01	$1.499\,575$
± 0.001	$1.499\,996$
± 0.0001	$1.500\,000$

It appears that $\lim\limits_{\theta \to 0} \dfrac{\sin 3\theta}{\tan 2\theta} = 1.5$.

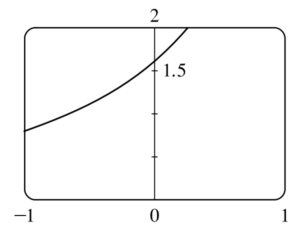
The graph confirms that result.



26. For $f(t) = \dfrac{5^t - 1}{t}$:

t	$f(t)$
0.1	$1.746\,189$
0.01	$1.622\,459$
0.001	$1.610\,734$
0.0001	$1.609\,567$

t	$f(t)$
-0.1	$1.486\,601$
-0.01	$1.596\,556$
-0.001	$1.608\,143$
-0.0001	$1.609\,308$



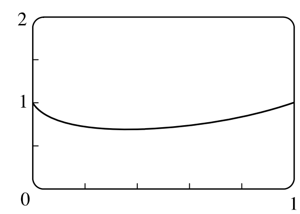
It appears that $\lim\limits_{t \to 0} f(t) \approx 1.6094$. The graph confirms that result.

27. For $f(x) = x^x$:

x	$f(x)$
0.1	$0.794\,328$
0.01	$0.954\,993$
0.001	$0.993\,116$
0.0001	$0.999\,079$

It appears that $\lim\limits_{x \to 0^+} f(x) = 1$.

The graph confirms that result.

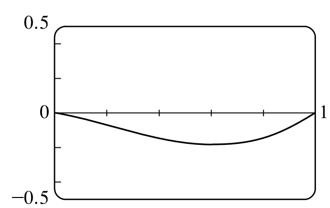


28. For $f(x) = x^2 \ln x$:

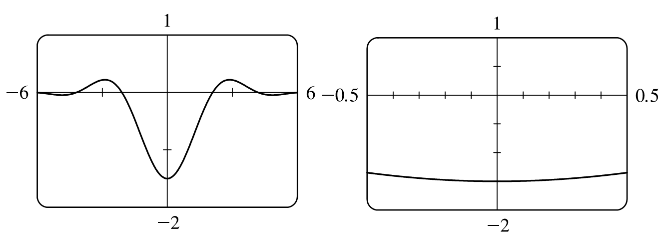
x	$f(x)$
0.1	$-0.023\,026$
0.01	$-0.000\,461$
0.001	$-0.000\,007$
0.0001	$-0.000\,000$

It appears that $\lim\limits_{x \to 0^+} f(x) = 0$.

The graph confirms that result.



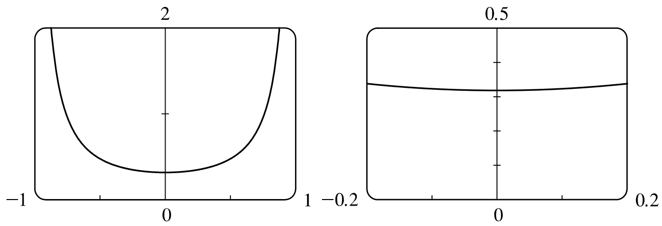
29. (a) From the graphs, it seems that $\lim\limits_{x \to 0} \dfrac{\cos 2x - \cos x}{x^2} = -1.5$.



(b)

x	$f(x)$
± 0.1	$-1.493\,759$
± 0.01	$-1.499\,938$
± 0.001	$-1.499\,999$
± 0.0001	$-1.500\,000$

30. (a) From the graphs, it seems that $\lim\limits_{x\to 0}\dfrac{\sin x}{\sin \pi x} = 0.32$.



(b)

x	$f(x)$
± 0.1	0.323 068
± 0.01	0.318 357
± 0.001	0.318 310
± 0.0001	0.318 310

Later we will be able to show that the exact value is $\dfrac{1}{\pi}$.

31. $\lim\limits_{x\to 5^+}\dfrac{x+1}{x-5} = \infty$ since the numerator is positive and the denominator approaches 0 from the positive side as $x \to 5^+$.

32. $\lim\limits_{x\to 5^-}\dfrac{x+1}{x-5} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $x \to 5^-$.

33. $\lim\limits_{x\to 1}\dfrac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

34. $\lim\limits_{x\to 3^-}\dfrac{\sqrt{x}}{(x-3)^5} = -\infty$ since the numerator is positive and the denominator approaches 0 from the negative side as $x \to 3^-$.

35. Let $t = x^2 - 9$. Then as $x \to 3^+$, $t \to 0^+$, and $\lim\limits_{x\to 3^+}\ln(x^2-9) = \lim\limits_{t\to 0^+}\ln t = -\infty$ by (5).

36. $\lim\limits_{x\to 0^+}\ln(\sin x) = -\infty$ since $\sin x \to 0^+$ as $x \to 0^+$.

37. $\lim\limits_{x\to(\pi/2)^+}\dfrac{1}{x}\sec x = -\infty$ since $\dfrac{1}{x}$ is positive and $\sec x \to -\infty$ as $x \to (\pi/2)^+$.

38. $\lim\limits_{x\to\pi^-}\cot x = \lim\limits_{x\to\pi^-}\dfrac{\cos x}{\sin x} = -\infty$ since the numerator is negative and the denominator approaches 0 through positive values as $x \to \pi^-$.

39. $\lim\limits_{x\to 2\pi^-} x\csc x = \lim\limits_{x\to 2\pi^-}\dfrac{x}{\sin x} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2\pi^-$.

40. $\lim\limits_{x\to 2^-}\dfrac{x^2-2x}{x^2-4x+4} = \lim\limits_{x\to 2^-}\dfrac{x(x-2)}{(x-2)^2} = \lim\limits_{x\to 2^-}\dfrac{x}{x-2} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2^-$.

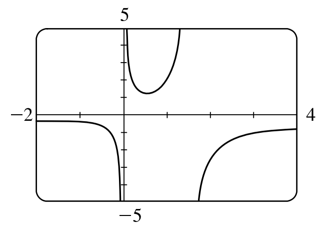
41. $\lim\limits_{x\to 2^+}\dfrac{x^2-2x-8}{x^2-5x+6} = \lim\limits_{x\to 2^+}\dfrac{(x-4)(x+2)}{(x-3)(x-2)} = \infty$ since the numerator is negative and the denominator approaches 0 through negative values as $x \to 2^+$.

42. $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x} - \ln x\right) = \infty$ since $\dfrac{1}{x} \to \infty$ and $\ln x \to -\infty$ as $x \to 0^+$.

43. $\lim\limits_{x\to 0}(\ln x^2 - x^{-2}) = -\infty$ since $\ln x^2 \to -\infty$ and $x^{-2} \to \infty$ as $x \to 0$.

44. (a) The denominator of $y = \dfrac{x^2+1}{3x-2x^2} = \dfrac{x^2+1}{x(3-2x)}$ is equal to zero when $x = 0$ and $x = \frac{3}{2}$ (and the numerator is not), so $x = 0$ and $x = 1.5$ are vertical asymptotes of the function.

(b)



45. (a) $f(x) = \dfrac{1}{x^3-1}$.

x	$f(x)$
0.5	-1.14
0.9	-3.69
0.99	-33.7
0.999	-333.7
0.9999	-3333.7
0.99999	$-33{,}333.7$

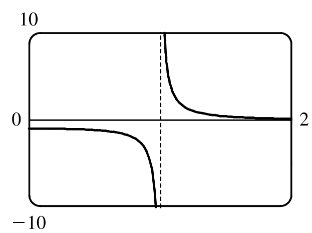
x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

From these calculations, it seems that $\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.

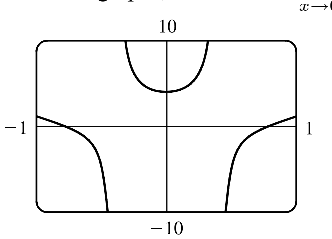
(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of x^3-1, that is, $f(x)$, will be a negative number with large absolute value. So $\lim\limits_{x\to 1^-} f(x) = -\infty$.

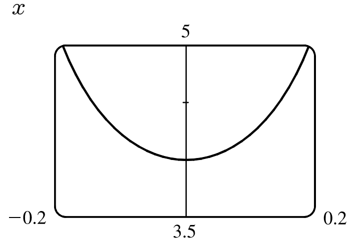
If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim\limits_{x\to 1^+} f(x) = \infty$.

(c) It appears from the graph of f that $\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.



46. (a) From the graphs, it seems that $\lim\limits_{x\to 0} \dfrac{\tan 4x}{x} = 4$.





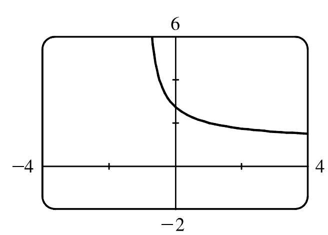
(b)

x	$f(x)$
± 0.1	4.227 932
± 0.01	4.002 135
± 0.001	4.000 021
± 0.0001	4.000 000

47. (a) Let $h(x) = (1+x)^{1/x}$.

x	$h(x)$
-0.001	2.71964
-0.0001	2.71842
-0.00001	2.71830
-0.000001	2.71828
0.000001	2.71828
0.00001	2.71827
0.0001	2.71815
0.001	2.71692

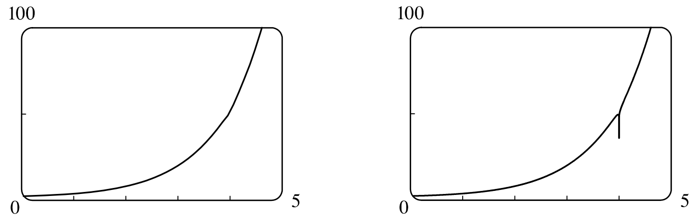
(b)



It appears that $\lim\limits_{x\to 0} (1+x)^{1/x} \approx 2.71828$, which is approximately e.

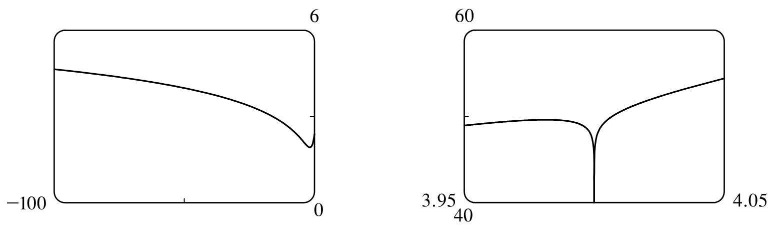
In Section 3.6 we will see that the value of the limit is exactly e.

48. (a)

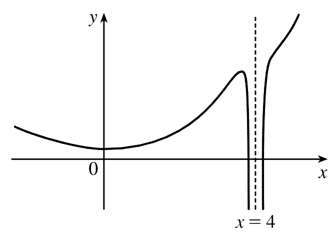


No, because the calculator-produced graph of $f(x) = e^x + \ln|x-4|$ looks like an exponential function, but the graph of f has an infinite discontinuity at $x = 4$. A second graph, obtained by increasing the `numpoints` option in Maple, begins to reveal the discontinuity at $x = 4$.

(b) There isn't a single graph that shows all the features of f. Several graphs are needed since f looks like $\ln|x-4|$ for large negative values of x and like e^x for $x > 5$, but yet has the infinite discontiuity at $x = 4$.



A hand-drawn graph, though distorted, might be better at revealing the main features of this function.



49. For $f(x) = x^2 - (2^x/1000)$:

(a)

x	$f(x)$
1	0.998 000
0.8	0.638 259
0.6	0.358 484
0.4	0.158 680
0.2	0.038 851
0.1	0.008 928
0.05	0.001 465

It appears that $\lim\limits_{x \to 0} f(x) = 0$.

(b)

x	$f(x)$
0.04	0.000 572
0.02	−0.000 614
0.01	−0.000 907
0.005	−0.000 978
0.003	−0.000 993
0.001	−0.001 000

It appears that $\lim\limits_{x \to 0} f(x) = -0.001$.

50. For $h(x) = \dfrac{\tan x - x}{x^3}$:

(a)

x	$h(x)$
1.0	0.557 407 73
0.5	0.370 419 92
0.1	0.334 672 09
0.05	0.333 667 00
0.01	0.333 346 67
0.005	0.333 336 67

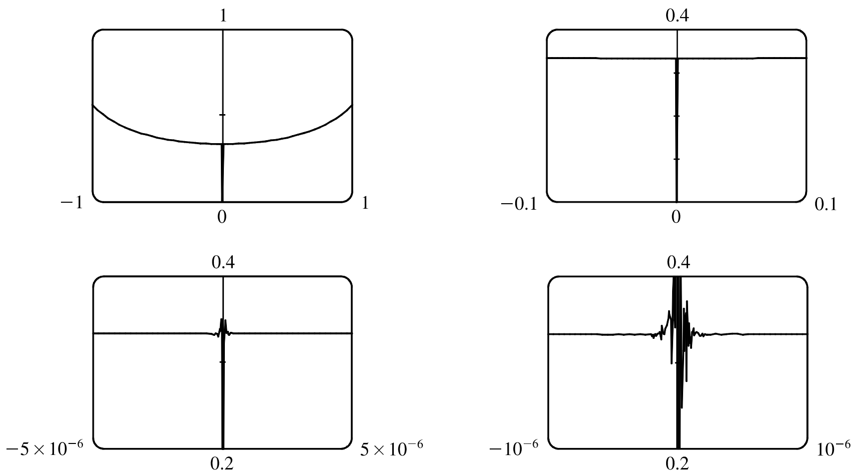
(b) It seems that $\lim\limits_{x \to 0} h(x) = \frac{1}{3}$.

(c)

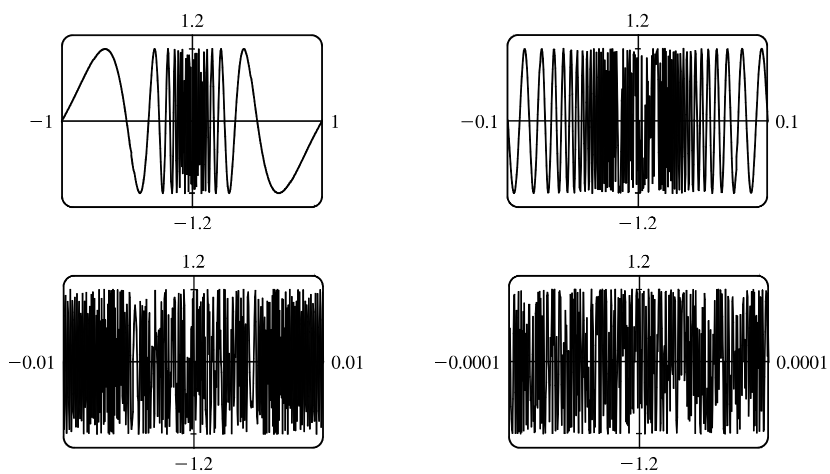
x	$h(x)$
0.001	0.333 333 50
0.0005	0.333 333 44
0.0001	0.333 330 00
0.00005	0.333 336 00
0.00001	0.333 000 00
0.000001	0.000 000 00

Here the values will vary from one calculator to another. Every calculator will eventually give *false values*.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.



51. No matter how many times we zoom in toward the origin, the graphs of $f(x) = \sin(\pi/x)$ appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.

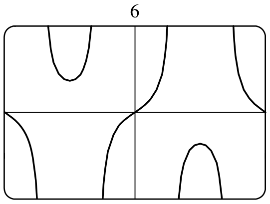


52. (a) For any positive integer n, if $x = \dfrac{1}{n\pi}$, then $f(x) = \tan\dfrac{1}{x} = \tan(n\pi) = 0$. (Remember that the tangent function has period π.)

(b) For any nonnegative number n, if $x = \dfrac{4}{(4n+1)\pi}$, then

$$f(x) = \tan\frac{1}{x} = \tan\frac{(4n+1)\pi}{4} = \tan\left(\frac{4n\pi}{4} + \frac{\pi}{4}\right) = \tan\left(n\pi + \frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

(c) From part (a), $f(x) = 0$ infinitely often as $x \to 0$. From part (b), $f(x) = 1$ infinitely often as $x \to 0$. Thus, $\lim\limits_{x\to 0} \tan\dfrac{1}{x}$ does not exist since $f(x)$ does not get close to a fixed number as $x \to 0$.

53. 

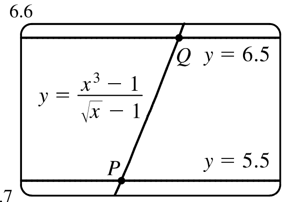
There appear to be vertical asymptotes of the curve $y = \tan(2\sin x)$ at $x \approx \pm 0.90$ and $x \approx \pm 2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x = \frac{\pi}{2} + \pi n$. Thus, we must have $2\sin x = \frac{\pi}{2} + \pi n$, or equivalently, $\sin x = \frac{\pi}{4} + \frac{\pi}{2}n$. Since $-1 \le \sin x \le 1$, we must have $\sin x = \pm\frac{\pi}{4}$ and so $x = \pm\sin^{-1}\frac{\pi}{4}$ (corresponding to $x \approx \pm 0.90$). Just as 150° is the reference angle for 30°, $\pi - \sin^{-1}\frac{\pi}{4}$ is the reference angle for $\sin^{-1}\frac{\pi}{4}$. So $x = \pm\left(\pi - \sin^{-1}\frac{\pi}{4}\right)$ are also equations of vertical asymptotes (corresponding to $x \approx \pm 2.24$).

54. $\lim\limits_{v\to c^-} m = \lim\limits_{v\to c^-} \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$. As $v \to c^-$, $\sqrt{1 - v^2/c^2} \to 0^+$, and $m \to \infty$.

55. (a) Let $y = \dfrac{x^3 - 1}{\sqrt{x} - 1}$.

From the table and the graph, we guess that the limit of y as x approaches 1 is 6.

x	y
0.99	5.925 31
0.999	5.992 50
0.9999	5.999 25
1.01	6.075 31
1.001	6.007 50
1.0001	6.000 75



(b) We need to have $5.5 < \dfrac{x^3 - 1}{\sqrt{x} - 1} < 6.5$. From the graph we obtain the approximate points of intersection $P(0.9314, 5.5)$ and $Q(1.0649, 6.5)$. Now $1 - 0.9314 = 0.0686$ and $1.0649 - 1 = 0.0649$, so by requiring that x be within 0.0649 of 1, we ensure that y is within 0.5 of 6.

2.3 Calculating Limits Using the Limit Laws

1. (a)
$$\begin{aligned}\lim_{x\to 2}[f(x) + 5g(x)] &= \lim_{x\to 2} f(x) + \lim_{x\to 2}[5g(x)] && \text{[Limit Law 1]}\\ &= \lim_{x\to 2} f(x) + 5\lim_{x\to 2} g(x) && \text{[Limit Law 3]}\\ &= 4 + 5(-2) = -6\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to 2}[g(x)]^3 &= \left[\lim_{x\to 2} g(x)\right]^3 && \text{[Limit Law 6]}\\ &= (-2)^3 = -8\end{aligned}$$

(c) $\lim\limits_{x\to 2}\sqrt{f(x)}=\sqrt{\lim\limits_{x\to 2}f(x)}$ [Limit Law 11]

$=\sqrt{4}=2$

(d) $\lim\limits_{x\to 2}\dfrac{3f(x)}{g(x)}=\dfrac{\lim\limits_{x\to 2}[3f(x)]}{\lim\limits_{x\to 2}g(x)}$ [Limit Law 5]

$=\dfrac{3\lim\limits_{x\to 2}f(x)}{\lim\limits_{x\to 2}g(x)}$ [Limit Law 3]

$=\dfrac{3(4)}{-2}=-6$

(e) Because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, $\lim\limits_{x\to 2}\dfrac{g(x)}{h(x)}$, does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(f) $\lim\limits_{x\to 2}\dfrac{g(x)\,h(x)}{f(x)}=\dfrac{\lim\limits_{x\to 2}[g(x)\,h(x)]}{\lim\limits_{x\to 2}f(x)}$ [Limit Law 5]

$=\dfrac{\lim\limits_{x\to 2}g(x)\cdot\lim\limits_{x\to 2}h(x)}{\lim\limits_{x\to 2}f(x)}$ [Limit Law 4]

$=\dfrac{-2\cdot 0}{4}=0$

2. (a) $\lim\limits_{x\to 2}[f(x)+g(x)]=\lim\limits_{x\to 2}f(x)+\lim\limits_{x\to 2}g(x)$ [Limit Law 1]

$=-1+2$

$=1$

(b) $\lim\limits_{x\to 0}f(x)$ exists, but $\lim\limits_{x\to 0}g(x)$ does not exist, so we cannot apply Limit Law 2 to $\lim\limits_{x\to 0}[f(x)-g(x)]$. The limit does not exist.

(c) $\lim\limits_{x\to -1}[f(x)\,g(x)]=\lim\limits_{x\to -1}f(x)\cdot\lim\limits_{x\to -1}g(x)$ [Limit Law 4]

$=1\cdot 2$

$=2$

(d) $\lim\limits_{x\to 3}f(x)=1$, but $\lim\limits_{x\to 3}g(x)=0$, so we cannot apply Limit Law 5 to $\lim\limits_{x\to 3}\dfrac{f(x)}{g(x)}$. The limit does not exist.

Note: $\lim\limits_{x\to 3^-}\dfrac{f(x)}{g(x)}=\infty$ since $g(x)\to 0^+$ as $x\to 3^-$ and $\lim\limits_{x\to 3^+}\dfrac{f(x)}{g(x)}=-\infty$ since $g(x)\to 0^-$ as $x\to 3^+$.

Therefore, the limit does not exist, even as an infinite limit.

(e) $\lim\limits_{x\to 2}\left[x^2f(x)\right]=\lim\limits_{x\to 2}x^2\cdot\lim\limits_{x\to 2}f(x)$ [Limit Law 4]

$=2^2\cdot(-1)$

$=-4$

(f) $f(-1)+\lim\limits_{x\to -1}g(x)$ is undefined since $f(-1)$ is not defined.

3. $\lim\limits_{x\to 3}(5x^3-3x^2+x-6)=\lim\limits_{x\to 3}(5x^3)-\lim\limits_{x\to 3}(3x^2)+\lim\limits_{x\to 3}x-\lim\limits_{x\to 3}6$ [Limit Laws 2 and 1]

$=5\lim\limits_{x\to 3}x^3-3\lim\limits_{x\to 3}x^2+\lim\limits_{x\to 3}x-\lim\limits_{x\to 3}6$ [3]

$=5(3^3)-3(3^2)+3-6$ [9, 8, and 7]

$=105$

4. $$\begin{aligned}\lim_{x\to-1}(x^4-3x)(x^2+5x+3) &= \lim_{x\to-1}(x^4-3x)\lim_{x\to-1}(x^2+5x+3) && \text{[Limit Law 4]}\\ &= \left(\lim_{x\to-1}x^4-\lim_{x\to-1}3x\right)\left(\lim_{x\to-1}x^2+\lim_{x\to-1}5x+\lim_{x\to-1}3\right) && \text{[2, 1]}\\ &= \left(\lim_{x\to-1}x^4-3\lim_{x\to-1}x\right)\left(\lim_{x\to-1}x^2+5\lim_{x\to-1}x+\lim_{x\to-1}3\right) && \text{[3]}\\ &= (1+3)(1-5+3) && \text{[9, 8, and 7]}\\ &= 4(-1)=-4\end{aligned}$$

5. $$\begin{aligned}\lim_{t\to-2}\frac{t^4-2}{2t^2-3t+2} &= \frac{\lim_{t\to-2}(t^4-2)}{\lim_{t\to-2}(2t^2-3t+2)} && \text{[Limit Law 5]}\\ &= \frac{\lim_{t\to-2}t^4-\lim_{t\to-2}2}{2\lim_{t\to-2}t^2-3\lim_{t\to-2}t+\lim_{t\to-2}2} && \text{[1, 2, and 3]}\\ &= \frac{16-2}{2(4)-3(-2)+2} && \text{[9, 7, and 8]}\\ &= \frac{14}{16}=\frac{7}{8}\end{aligned}$$

6. $$\begin{aligned}\lim_{u\to-2}\sqrt{u^4+3u+6} &= \sqrt{\lim_{u\to-2}(u^4+3u+6)} && \text{[11]}\\ &= \sqrt{\lim_{u\to-2}u^4+3\lim_{u\to-2}u+\lim_{u\to-2}6} && \text{[1, 2, and 3]}\\ &= \sqrt{(-2)^4+3(-2)+6} && \text{[9, 8, and 7]}\\ &= \sqrt{16-6+6}=\sqrt{16}=4\end{aligned}$$

7. $$\begin{aligned}\lim_{x\to8}(1+\sqrt[3]{x})(2-6x^2+x^3) &= \lim_{x\to8}(1+\sqrt[3]{x})\cdot\lim_{x\to8}(2-6x^2+x^3) && \text{[Limit Law 4]}\\ &= \left(\lim_{x\to8}1+\lim_{x\to8}\sqrt[3]{x}\right)\cdot\left(\lim_{x\to8}2-6\lim_{x\to8}x^2+\lim_{x\to8}x^3\right) && \text{[1, 2, and 3]}\\ &= \left(1+\sqrt[3]{8}\right)\cdot\left(2-6\cdot8^2+8^3\right) && \text{[7, 10, 9]}\\ &= (3)(130)=390\end{aligned}$$

8. $$\begin{aligned}\lim_{t\to2}\left(\frac{t^2-2}{t^3-3t+5}\right)^2 &= \left(\lim_{t\to2}\frac{t^2-2}{t^3-3t+5}\right)^2 && \text{[Limit Law 6]}\\ &= \left(\frac{\lim_{t\to2}(t^2-2)}{\lim_{t\to2}(t^3-3t+5)}\right)^2 && \text{[5]}\\ &= \left(\frac{\lim_{t\to2}t^2-\lim_{t\to2}2}{\lim_{t\to2}t^3-3\lim_{t\to2}t+\lim_{t\to2}5}\right)^2 && \text{[1, 2, and 3]}\\ &= \left(\frac{4-2}{8-3(2)+5}\right)^2 && \text{[9, 7, and 8]}\\ &= \left(\frac{2}{7}\right)^2=\frac{4}{49}\end{aligned}$$

9. $\lim\limits_{x\to 2}\sqrt{\dfrac{2x^2+1}{3x-2}} = \sqrt{\lim\limits_{x\to 2}\dfrac{2x^2+1}{3x-2}}$ [Limit Law 11]

$= \sqrt{\dfrac{\lim\limits_{x\to 2}(2x^2+1)}{\lim\limits_{x\to 2}(3x-2)}}$ [5]

$= \sqrt{\dfrac{2\lim\limits_{x\to 2}x^2+\lim\limits_{x\to 2}1}{3\lim\limits_{x\to 2}x-\lim\limits_{x\to 2}2}}$ [1, 2, and 3]

$= \sqrt{\dfrac{2(2)^2+1}{3(2)-2}} = \sqrt{\dfrac{9}{4}} = \dfrac{3}{2}$ [9, 8, and 7]

10. (a) The left-hand side of the equation is not defined for $x=2$, but the right-hand side is.

(b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \to 2$, just as in Example 3. Remember that in finding $\lim\limits_{x\to a} f(x)$, we never consider $x = a$.

11. $\lim\limits_{x\to 5}\dfrac{x^2-6x+5}{x-5} = \lim\limits_{x\to 5}\dfrac{(x-5)(x-1)}{x-5} = \lim\limits_{x\to 5}(x-1) = 5-1 = 4$

12. $\lim\limits_{x\to -3}\dfrac{x^2+3x}{x^2-x-12} = \lim\limits_{x\to -3}\dfrac{x(x+3)}{(x-4)(x+3)} = \lim\limits_{x\to -3}\dfrac{x}{x-4} = \dfrac{-3}{-3-4} = \dfrac{3}{7}$

13. $\lim\limits_{x\to 5}\dfrac{x^2-5x+6}{x-5}$ does not exist since $x-5\to 0$, but $x^2-5x+6\to 6$ as $x\to 5$.

14. $\lim\limits_{x\to 4}\dfrac{x^2+3x}{x^2-x-12} = \lim\limits_{x\to 4}\dfrac{x(x+3)}{(x-4)(x+3)} = \lim\limits_{x\to 4}\dfrac{x}{x-4}$. The last limit does not exist since $\lim\limits_{x\to 4^-}\dfrac{x}{x-4} = -\infty$ and $\lim\limits_{x\to 4^+}\dfrac{x}{x-4} = \infty$.

15. $\lim\limits_{t\to -3}\dfrac{t^2-9}{2t^2+7t+3} = \lim\limits_{t\to -3}\dfrac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim\limits_{t\to -3}\dfrac{t-3}{2t+1} = \dfrac{-3-3}{2(-3)+1} = \dfrac{-6}{-5} = \dfrac{6}{5}$

16. $\lim\limits_{x\to -1}\dfrac{2x^2+3x+1}{x^2-2x-3} = \lim\limits_{x\to -1}\dfrac{(2x+1)(x+1)}{(x-3)(x+1)} = \lim\limits_{x\to -1}\dfrac{2x+1}{x-3} = \dfrac{2(-1)+1}{-1-3} = \dfrac{-1}{-4} = \dfrac{1}{4}$

17. $\lim\limits_{h\to 0}\dfrac{(-5+h)^2-25}{h} = \lim\limits_{h\to 0}\dfrac{(25-10h+h^2)-25}{h} = \lim\limits_{h\to 0}\dfrac{-10h+h^2}{h} = \lim\limits_{h\to 0}\dfrac{h(-10+h)}{h} = \lim\limits_{h\to 0}(-10+h) = -10$

18. $\lim\limits_{h\to 0}\dfrac{(2+h)^3-8}{h} = \lim\limits_{h\to 0}\dfrac{(8+12h+6h^2+h^3)-8}{h} = \lim\limits_{h\to 0}\dfrac{12h+6h^2+h^3}{h}$

$= \lim\limits_{h\to 0}(12+6h+h^2) = 12+0+0 = 12$

19. By the formula for the sum of cubes, we have

$$\lim_{x\to -2}\frac{x+2}{x^3+8} = \lim_{x\to -2}\frac{x+2}{(x+2)(x^2-2x+4)} = \lim_{x\to -2}\frac{1}{x^2-2x+4} = \frac{1}{4+4+4} = \frac{1}{12}.$$

20. We use the difference of squares in the numerator and the difference of cubes in the denominator.

$$\lim_{t\to 1}\frac{t^4-1}{t^3-1}=\lim_{t\to 1}\frac{(t^2-1)(t^2+1)}{(t-1)(t^2+t+1)}=\lim_{t\to 1}\frac{(t-1)(t+1)(t^2+1)}{(t-1)(t^2+t+1)}=\lim_{t\to 1}\frac{(t+1)(t^2+1)}{t^2+t+1}=\frac{2(2)}{3}=\frac{4}{3}$$

21. $$\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}=\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}\cdot\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}=\lim_{h\to 0}\frac{\left(\sqrt{9+h}\right)^2-3^2}{h\left(\sqrt{9+h}+3\right)}=\lim_{h\to 0}\frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)}$$
$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{9+h}+3\right)}=\lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}=\frac{1}{3+3}=\frac{1}{6}$$

22. $$\lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2}=\lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2}\cdot\frac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3}=\lim_{u\to 2}\frac{\left(\sqrt{4u+1}\right)^2-3^2}{(u-2)\left(\sqrt{4u+1}+3\right)}$$
$$=\lim_{u\to 2}\frac{4u+1-9}{(u-2)\left(\sqrt{4u+1}+3\right)}=\lim_{u\to 2}\frac{4(u-2)}{(u-2)\left(\sqrt{4u+1}+3\right)}$$
$$=\lim_{u\to 2}\frac{4}{\sqrt{4u+1}+3}=\frac{4}{\sqrt{9}+3}=\frac{2}{3}$$

23. $$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}=\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}\cdot\frac{3x}{3x}=\lim_{x\to 3}\frac{3-x}{3x(x-3)}=\lim_{x\to 3}\frac{-1}{3x}=-\frac{1}{9}$$

24. $$\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=\lim_{h\to 0}\frac{3-(3+h)}{h(3+h)3}=\lim_{h\to 0}\frac{-h}{h(3+h)3}$$
$$=\lim_{h\to 0}\left[-\frac{1}{3(3+h)}\right]=-\frac{1}{\lim\limits_{h\to 0}[3(3+h)]}=-\frac{1}{3(3+0)}=-\frac{1}{9}$$

25. $$\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}=\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}\cdot\frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}=\lim_{t\to 0}\frac{\left(\sqrt{1+t}\right)^2-\left(\sqrt{1-t}\right)^2}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$
$$=\lim_{t\to 0}\frac{(1+t)-(1-t)}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}=\lim_{t\to 0}\frac{2t}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}=\lim_{t\to 0}\frac{2}{\sqrt{1+t}+\sqrt{1-t}}$$
$$=\frac{2}{\sqrt{1}+\sqrt{1}}=\frac{2}{2}=1$$

26. $$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t(t+1)}\right)=\lim_{t\to 0}\frac{t+1-1}{t(t+1)}=\lim_{t\to 0}\frac{1}{t+1}=\frac{1}{0+1}=1$$

27. $$\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}=\lim_{x\to 16}\frac{(4-\sqrt{x})(4+\sqrt{x})}{(16x-x^2)(4+\sqrt{x})}=\lim_{x\to 16}\frac{16-x}{x(16-x)(4+\sqrt{x})}$$
$$=\lim_{x\to 16}\frac{1}{x(4+\sqrt{x})}=\frac{1}{16\left(4+\sqrt{16}\right)}=\frac{1}{16(8)}=\frac{1}{128}$$

28. $$\lim_{x\to 2}\frac{x^2-4x+4}{x^4-3x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x^2-4)(x^2+1)}=\lim_{x\to 2}\frac{(x-2)^2}{(x+2)(x-2)(x^2+1)}$$
$$=\lim_{x\to 2}\frac{x-2}{(x+2)(x^2+1)}=\frac{0}{4\cdot 5}=0$$

29. $\displaystyle \lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim_{t\to 0}\frac{\left(1-\sqrt{1+t}\,\right)\left(1+\sqrt{1+t}\,\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\,\right)}=\lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\,\right)}$

$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\,\right)}=\frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\,\right)}=-\frac{1}{2}$$

30. $\displaystyle \lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}=\lim_{x\to -4}\frac{\left(\sqrt{x^2+9}-5\right)\left(\sqrt{x^2+9}+5\right)}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x^2+9)-25}{(x+4)\left(\sqrt{x^2+9}+5\right)}$

$$=\lim_{x\to -4}\frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x+4)(x-4)}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$

$$=\lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5}=\frac{-4-4}{\sqrt{16+9}+5}=\frac{-8}{5+5}=-\frac{4}{5}$$

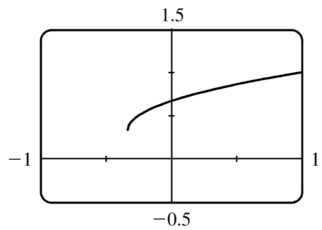
31. $\displaystyle \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$

$$=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$$

32. $\displaystyle \lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\dfrac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}=\lim_{h\to 0}\frac{-h(2x+h)}{hx^2(x+h)^2}$

$$=\lim_{h\to 0}\frac{-(2x+h)}{x^2(x+h)^2}=\frac{-2x}{x^2\cdot x^2}=-\frac{2}{x^3}$$

33. (a)



$\displaystyle \lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$

(b)

x	$f(x)$
-0.001	$0.666\,166\,3$
$-0.000\,1$	$0.666\,616\,7$
$-0.000\,01$	$0.666\,661\,7$
$-0.000\,001$	$0.666\,666\,2$
$0.000\,001$	$0.666\,667\,2$
$0.000\,01$	$0.666\,671\,7$
$0.000\,1$	$0.666\,716\,7$
0.001	$0.667\,166\,3$

The limit appears to be $\frac{2}{3}$.

(c) $\displaystyle \lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}$

$$=\frac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right) \qquad \text{[Limit Law 3]}$$

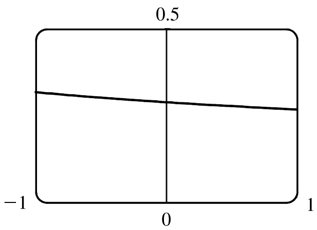
$$=\frac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)}+\lim_{x\to 0}1\right] \qquad \text{[1 and 11]}$$

$$=\frac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right) \qquad \text{[1, 3, and 7]}$$

$$=\frac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right) \qquad \text{[7 and 8]}$$

$$=\frac{1}{3}(1+1)=\frac{2}{3}$$

34. (a)



$\lim_{x\to 0} \dfrac{\sqrt{3+x}-\sqrt{3}}{x} \approx 0.29$

(b)

x	$f(x)$
-0.001	$0.288\,699\,2$
$-0.000\,1$	$0.288\,677\,5$
$-0.000\,01$	$0.288\,675\,4$
$-0.000\,001$	$0.288\,675\,2$
$0.000\,001$	$0.288\,675\,1$
$0.000\,01$	$0.288\,674\,9$
$0.000\,1$	$0.288\,672\,7$
0.001	$0.288\,651\,1$

The limit appears to be approximately 0.2887.

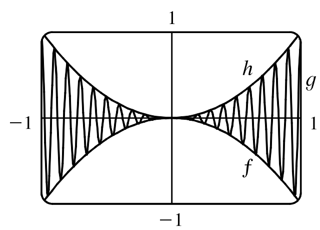
(c)
$$\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)=\lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$$
$$=\frac{\lim_{x\to 0} 1}{\lim_{x\to 0}\sqrt{3+x}+\lim_{x\to 0}\sqrt{3}} \qquad \text{[Limit Laws 5 and 1]}$$
$$=\frac{1}{\sqrt{\lim_{x\to 0}(3+x)}+\sqrt{3}} \qquad \text{[7 and 11]}$$
$$=\frac{1}{\sqrt{3+0}+\sqrt{3}} \qquad \text{[1, 7, and 8]}$$
$$=\frac{1}{2\sqrt{3}}$$

35. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then

$-1\le\cos 20\pi x\le 1 \quad\Rightarrow\quad -x^2\le x^2\cos 20\pi x\le x^2 \quad\Rightarrow\quad f(x)\le g(x)\le h(x)$.

So since $\lim_{x\to 0} f(x)=\lim_{x\to 0} h(x)=0$, by the Squeeze Theorem we have

$\lim_{x\to 0} g(x)=0$.

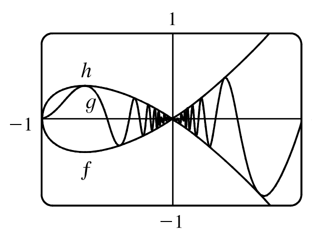


36. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\,\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then

$-1\le\sin(\pi/x)\le 1 \quad\Rightarrow\quad -\sqrt{x^3+x^2}\le\sqrt{x^3+x^2}\sin(\pi/x)\le\sqrt{x^3+x^2} \quad\Rightarrow$

$f(x)\le g(x)\le h(x)$. So since $\lim_{x\to 0} f(x)=\lim_{x\to 0} h(x)=0$, by the Squeeze Theorem

we have $\lim_{x\to 0} g(x)=0$.



37. We have $\lim_{x\to 4}(4x-9)=4(4)-9=7$ and $\lim_{x\to 4}\left(x^2-4x+7\right)=4^2-4(4)+7=7$. Since $4x-9\le f(x)\le x^2-4x+7$

for $x\ge 0$, $\lim_{x\to 4} f(x)=7$ by the Squeeze Theorem.

38. We have $\lim_{x\to 1}(2x)=2(1)=2$ and $\lim_{x\to 1}(x^4-x^2+2)=1^4-1^2+2=2$. Since $2x\le g(x)\le x^4-x^2+2$ for all x,

$\lim_{x\to 1} g(x)=2$ by the Squeeze Theorem.

39. $-1\le\cos(2/x)\le 1 \quad\Rightarrow\quad -x^4\le x^4\cos(2/x)\le x^4$. Since $\lim_{x\to 0}\left(-x^4\right)=0$ and $\lim_{x\to 0} x^4=0$, we have

$\lim_{x\to 0}\left[x^4\cos(2/x)\right]=0$ by the Squeeze Theorem.

40. $-1 \le \sin(\pi/x) \le 1 \quad \Rightarrow \quad e^{-1} \le e^{\sin(\pi/x)} \le e^1 \quad \Rightarrow \quad \sqrt{x}/e \le \sqrt{x}\, e^{\sin(\pi/x)} \le \sqrt{x}\, e$. Since $\lim\limits_{x \to 0^+} (\sqrt{x}/e) = 0$ and $\lim\limits_{x \to 0^+} (\sqrt{x}\, e) = 0$, we have $\lim\limits_{x \to 0^+} \left[\sqrt{x}\, e^{\sin(\pi/x)}\right] = 0$ by the Squeeze Theorem.

41. $|x-3| = \begin{cases} x-3 & \text{if } x-3 \ge 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \ge 3 \\ 3-x & \text{if } x < 3 \end{cases}$

Thus, $\lim\limits_{x \to 3^+} (2x + |x-3|) = \lim\limits_{x \to 3^+} (2x + x - 3) = \lim\limits_{x \to 3^+} (3x-3) = 3(3) - 3 = 6$ and

$\lim\limits_{x \to 3^-} (2x + |x-3|) = \lim\limits_{x \to 3^-} (2x + 3 - x) = \lim\limits_{x \to 3^-} (x+3) = 3 + 3 = 6$. Since the left and right limits are equal, $\lim\limits_{x \to 3} (2x + |x-3|) = 6$.

42. $|x+6| = \begin{cases} x+6 & \text{if } x+6 \ge 0 \\ -(x+6) & \text{if } x+6 < 0 \end{cases} = \begin{cases} x+6 & \text{if } x \ge -6 \\ -(x+6) & \text{if } x < -6 \end{cases}$

We'll look at the one-sided limits.

$$\lim_{x \to -6^+} \frac{2x+12}{|x+6|} = \lim_{x \to -6^+} \frac{2(x+6)}{x+6} = 2 \quad \text{and} \quad \lim_{x \to -6^-} \frac{2x+12}{|x+6|} = \lim_{x \to -6^-} \frac{2(x+6)}{-(x+6)} = -2$$

The left and right limits are different, so $\lim\limits_{x \to -6} \dfrac{2x+12}{|x+6|}$ does not exist.

43. $\left|2x^3 - x^2\right| = \left|x^2(2x-1)\right| = \left|x^2\right| \cdot |2x-1| = x^2\,|2x-1|$

$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \ge 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \ge 0.5 \\ -(2x-1) & \text{if } x < 0.5 \end{cases}$

So $\left|2x^3 - x^2\right| = x^2[-(2x-1)]$ for $x < 0.5$.

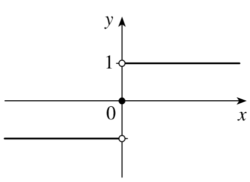
Thus, $\lim\limits_{x \to 0.5^-} \dfrac{2x-1}{|2x^3 - x^2|} = \lim\limits_{x \to 0.5^-} \dfrac{2x-1}{x^2[-(2x-1)]} = \lim\limits_{x \to 0.5^-} \dfrac{-1}{x^2} = \dfrac{-1}{(0.5)^2} = \dfrac{-1}{0.25} = -4$.

44. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x \to -2} \dfrac{2-|x|}{2+x} = \lim\limits_{x \to -2} \dfrac{2-(-x)}{2+x} = \lim\limits_{x \to -2} \dfrac{2+x}{2+x} = \lim\limits_{x \to -2} 1 = 1$.

45. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x \to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{-x}\right) = \lim\limits_{x \to 0^-} \dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

46. Since $|x| = x$ for $x > 0$, we have $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{x}\right) = \lim\limits_{x \to 0^+} 0 = 0$.

47. (a)



(b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\lim\limits_{x \to 0^+} \operatorname{sgn} x = \lim\limits_{x \to 0^+} 1 = 1$.

(ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\lim\limits_{x \to 0^-} \operatorname{sgn} x = \lim\limits_{x \to 0^-} -1 = -1$.

(iii) Since $\lim\limits_{x \to 0^-} \operatorname{sgn} x \ne \lim\limits_{x \to 0^+} \operatorname{sgn} x$, $\lim\limits_{x \to 0} \operatorname{sgn} x$ does not exist.

(iv) Since $|\operatorname{sgn} x| = 1$ for $x \ne 0$, $\lim\limits_{x \to 0} |\operatorname{sgn} x| = \lim\limits_{x \to 0} 1 = 1$.

48. (a) $g(x) = \operatorname{sgn}(\sin x) = \begin{cases} -1 & \text{if } \sin x < 0 \\ 0 & \text{if } \sin x = 0 \\ 1 & \text{if } \sin x > 0 \end{cases}$

(i) $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \operatorname{sgn}(\sin x) = 1$ since $\sin x$ is positive for small positive values of x.

(ii) $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \operatorname{sgn}(\sin x) = -1$ since $\sin x$ is negative for small negative values of x.

(iii) $\lim_{x\to 0} g(x)$ does not exist since $\lim_{x\to 0^+} g(x) \neq \lim_{x\to 0^-} g(x)$.

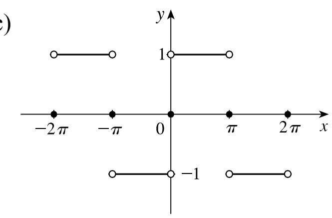
(iv) $\lim_{x\to \pi^+} g(x) = \lim_{x\to \pi^+} \operatorname{sgn}(\sin x) = -1$ since $\sin x$ is negative for values of x slightly greater than π.

(v) $\lim_{x\to \pi^-} g(x) = \lim_{x\to \pi^-} \operatorname{sgn}(\sin x) = 1$ since $\sin x$ is positive for values of x slightly less than π.

(vi) $\lim_{x\to \pi} g(x)$ does not exist since $\lim_{x\to \pi^+} g(x) \neq \lim_{x\to \pi^-} g(x)$.

(b) The sine function changes sign at every integer multiple of π, so the signum function equals 1 on one side and -1 on the other side of $n\pi$, n an integer. Thus, $\lim_{x\to a} g(x)$ does not exist for $a = n\pi$, n an integer.

(c)



49. (a) (i) $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \dfrac{x^2 + x - 6}{|x-2|} = \lim_{x\to 2^+} \dfrac{(x+3)(x-2)}{|x-2|}$

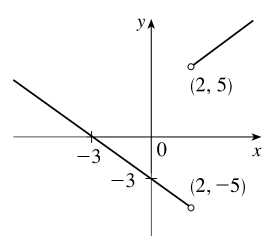
$$= \lim_{x\to 2^+} \frac{(x+3)(x-2)}{x-2} \quad [\text{since } x - 2 > 0 \text{ if } x \to 2^+]$$

$$= \lim_{x\to 2^+} (x+3) = 5$$

(ii) The solution is similar to the solution in part (i), but now $|x-2| = 2 - x$ since $x - 2 < 0$ if $x \to 2^-$. Thus, $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} -(x+3) = -5$.

(b) Since the right-hand and left-hand limits of g at $x = 2$ are not equal, $\lim_{x\to 2} g(x)$ does not exist.

(c)

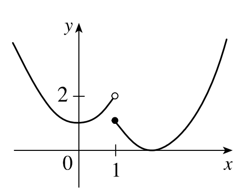


50. (a) $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x-2)^2 & \text{if } x \geq 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 + 1) = 1^2 + 1 = 2$, $\quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-2)^2 = (-1)^2 = 1$

(b) Since the right-hand and left-hand limits of f at $x = 1$ are not equal, $\lim_{x\to 1} f(x)$ does not exist.

(c)



51. For the $\lim_{t\to 2} B(t)$ to exist, the one-sided limits at $t = 2$ must be equal. $\lim_{t\to 2^-} B(t) = \lim_{t\to 2^-} \left(4 - \frac{1}{2}t\right) = 4 - 1 = 3$ and

$\lim_{t\to 2^+} B(t) = \lim_{t\to 2^+} \sqrt{t+c} = \sqrt{2+c}$. Now $3 = \sqrt{2+c} \;\Rightarrow\; 9 = 2 + c \;\Leftrightarrow\; c = 7$.

52. (a) (i) $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x = 1$

(ii) $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (2 - x^2) = 2 - 1^2 = 1$. Since $\lim_{x\to 1^-} g(x) = 1$ and $\lim_{x\to 1^+} g(x) = 1$, we have $\lim_{x\to 1} g(x) = 1$.

Note that the fact $g(1) = 3$ does not affect the value of the limit.

(iii) When $x = 1$, $g(x) = 3$, so $g(1) = 3$.

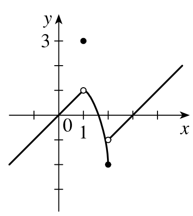
(iv) $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} (2 - x^2) = 2 - 2^2 = 2 - 4 = -2$

(v) $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} (x - 3) = 2 - 3 = -1$

(vi) $\lim_{x\to 2} g(x)$ does not exist since $\lim_{x\to 2^-} g(x) \neq \lim_{x\to 2^+} g(x)$.

(b)

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \le 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$



53. (a) (i) $[\![x]\!] = -2$ for $-2 \le x < -1$, so $\lim_{x\to -2^+} [\![x]\!] = \lim_{x\to -2^+} (-2) = -2$

(ii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim_{x\to -2^-} [\![x]\!] = \lim_{x\to -2^-} (-3) = -3$.

The right and left limits are different, so $\lim_{x\to -2} [\![x]\!]$ does not exist.

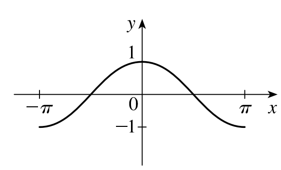
(iii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim_{x\to -2.4} [\![x]\!] = \lim_{x\to -2.4} (-3) = -3$.

(b) (i) $[\![x]\!] = n - 1$ for $n - 1 \le x < n$, so $\lim_{x\to n^-} [\![x]\!] = \lim_{x\to n^-} (n - 1) = n - 1$.

(ii) $[\![x]\!] = n$ for $n \le x < n + 1$, so $\lim_{x\to n^+} [\![x]\!] = \lim_{x\to n^+} n = n$.

(c) $\lim_{x\to a} [\![x]\!]$ exists $\;\Leftrightarrow\;$ a is not an integer.

54. (a) See the graph of $y = \cos x$.

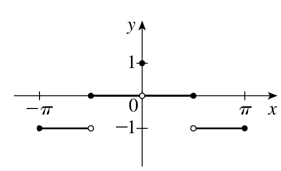


Since $-1 \le \cos x < 0$ on $[-\pi, -\pi/2)$, we have $y = f(x) = [\![\cos x]\!] = -1$ on $[-\pi, -\pi/2)$.

Since $0 \le \cos x < 1$ on $[-\pi/2, 0) \cup (0, \pi/2]$, we have $f(x) = 0$ on $[-\pi/2, 0) \cup (0, \pi/2]$.

Since $-1 \le \cos x < 0$ on $(\pi/2, \pi]$, we have $f(x) = -1$ on $(\pi/2, \pi]$.

Note that $f(0) = 1$.



(b) (i) $\lim_{x \to 0^-} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 0$, so $\lim_{x \to 0} f(x) = 0$.

(ii) As $x \to (\pi/2)^-$, $f(x) \to 0$, so $\lim_{x \to (\pi/2)^-} f(x) = 0$.

(iii) As $x \to (\pi/2)^+$, $f(x) \to -1$, so $\lim_{x \to (\pi/2)^+} f(x) = -1$.

(iv) Since the answers in parts (ii) and (iii) are not equal, $\lim_{x \to \pi/2} f(x)$ does not exist.

(c) $\lim_{x \to a} f(x)$ exists for all a in the open interval $(-\pi, \pi)$ except $a = -\pi/2$ and $a = \pi/2$.

55. The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\lim_{x \to 2^-} f(x) = -1$ and $\lim_{x \to 2^+} f(x) = -1$, so $\lim_{x \to 2} f(x) = -1$. However, $f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$, so $\lim_{x \to 2} f(x) \neq f(2)$.

56. $\lim_{v \to c^-} \left(L_0 \sqrt{1 - \frac{v^2}{c^2}} \right) = L_0 \sqrt{1 - 1} = 0$. As the velocity approaches the speed of light, the length approaches 0.

A left-hand limit is necessary since L is not defined for $v > c$.

57. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Thus, by the Limit Laws,

$$\begin{aligned} \lim_{x \to a} p(x) &= \lim_{x \to a} \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \right) = a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + \cdots + a_n \lim_{x \to a} x^n \\ &= a_0 + a_1 a + a_2 a^2 + \cdots + a_n a^n = p(a) \end{aligned}$$

Thus, for any polynomial p, $\lim_{x \to a} p(x) = p(a)$.

58. Let $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \neq 0$. Then

$$\lim_{x \to a} r(x) = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{\lim_{x \to a} p(x)}{\lim_{x \to a} q(x)} \quad \text{[Limit Law 5]} \quad = \frac{p(a)}{q(a)} \quad \text{[Exercise 57]} \quad = r(a).$$

59. $\lim_{x \to 1} [f(x) - 8] = \lim_{x \to 1} \left[\frac{f(x) - 8}{x - 1} \cdot (x - 1) \right] = \lim_{x \to 1} \frac{f(x) - 8}{x - 1} \cdot \lim_{x \to 1} (x - 1) = 10 \cdot 0 = 0$.

Thus, $\lim_{x \to 1} f(x) = \lim_{x \to 1} \{[f(x) - 8] + 8\} = \lim_{x \to 1} [f(x) - 8] + \lim_{x \to 1} 8 = 0 + 8 = 8$.

Note: The value of $\lim_{x \to 1} \dfrac{f(x) - 8}{x - 1}$ does not affect the answer since it's multiplied by 0. What's important is that $\lim_{x \to 1} \dfrac{f(x) - 8}{x - 1}$ exists.

60. (a) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left[\frac{f(x)}{x^2} \cdot x^2 \right] = \lim_{x \to 0} \frac{f(x)}{x^2} \cdot \lim_{x \to 0} x^2 = 5 \cdot 0 = 0$

(b) $\lim_{x \to 0} \dfrac{f(x)}{x} = \lim_{x \to 0} \left[\frac{f(x)}{x^2} \cdot x \right] = \lim_{x \to 0} \frac{f(x)}{x^2} \cdot \lim_{x \to 0} x = 5 \cdot 0 = 0$

61. Observe that $0 \le f(x) \le x^2$ for all x, and $\lim_{x \to 0} 0 = 0 = \lim_{x \to 0} x^2$. So, by the Squeeze Theorem, $\lim_{x \to 0} f(x) = 0$.

62. Let $f(x) = [\![x]\!]$ and $g(x) = -[\![x]\!]$. Then $\lim\limits_{x\to 3} f(x)$ and $\lim\limits_{x\to 3} g(x)$ do not exist [Example 10]

but $\lim\limits_{x\to 3}[f(x)+g(x)] = \lim\limits_{x\to 3}([\![x]\!] - [\![x]\!]) = \lim\limits_{x\to 3} 0 = 0$.

63. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.59.

Thus, either f or g is 0 for any value of x. Then $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist, but $\lim\limits_{x\to 0}[f(x)g(x)] = \lim\limits_{x\to 0} 0 = 0$.

64.
$$\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \lim_{x\to 2}\left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\cdot\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)$$
$$= \lim_{x\to 2}\left[\frac{\left(\sqrt{6-x}\right)^2-2^2}{\left(\sqrt{3-x}\right)^2-1^2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right] = \lim_{x\to 2}\left(\frac{6-x-4}{3-x-1}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right)$$
$$= \lim_{x\to 2}\frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)} = \lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}$$

65. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need $\lim\limits_{x\to -2}\left(3x^2+ax+a+3\right) = 0 \quad\Leftrightarrow$

$3(-2)^2 + a(-2) + a + 3 = 0 \quad\Leftrightarrow\quad 12 - 2a + a + 3 = 0 \quad\Leftrightarrow\quad a = 15$. With $a = 15$, the limit becomes

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to -2}\frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x\to -2}\frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

66. *Solution 1:* First, we find the coordinates of P and Q as functions of r. Then we can find the equation of the line determined by these two points, and thus find the x-intercept (the point R), and take the limit as $r \to 0$. The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \quad\Leftrightarrow\quad r^2 = 1 + 2x - 1 \quad\Leftrightarrow\quad x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the y-coordinate, we get $\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \quad\Leftrightarrow\quad y^2 = r^2\left(1 - \frac{1}{4}r^2\right) \quad\Leftrightarrow\quad y = r\sqrt{1-\frac{1}{4}r^2}$ (the positive y-value). So the coordinates of Q are $\left(\frac{1}{2}r^2, r\sqrt{1-\frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

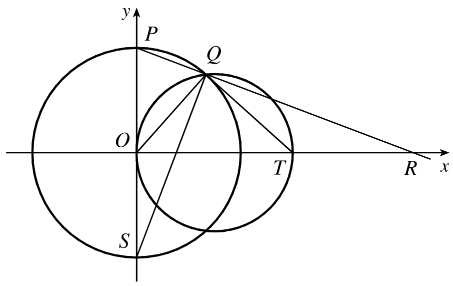
$y - r = \dfrac{r\sqrt{1-\frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0}(x - 0)$. We set $y = 0$ in order to find the x-intercept, and get

$$x = -r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1-\frac{1}{4}r^2}-1\right)} = \frac{-\frac{1}{2}r^2\left(\sqrt{1-\frac{1}{4}r^2}+1\right)}{1-\frac{1}{4}r^2-1} = 2\left(\sqrt{1-\frac{1}{4}r^2}+1\right)$$

Now we take the limit as $r \to 0^+$: $\lim\limits_{r\to 0^+} x = \lim\limits_{r\to 0^+} 2\left(\sqrt{1-\frac{1}{4}r^2}+1\right) = \lim\limits_{r\to 0^+} 2\left(\sqrt{1}+1\right) = 4$.

So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90°$ (subtended by diameter PS). So $\angle SQR = 90° = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90° - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4, 0)$, as above.



2.4 The Precise Definition of a Limit

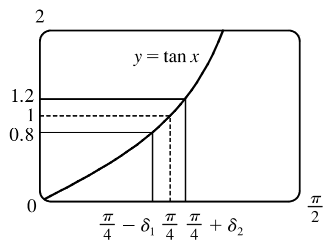
1. If $|f(x) - 1| < 0.2$, then $-0.2 < f(x) - 1 < 0.2 \quad \Rightarrow \quad 0.8 < f(x) < 1.2$. From the graph, we see that the last inequality is true if $0.7 < x < 1.1$, so we can choose $\delta = \min\{1 - 0.7, 1.1 - 1\} = \min\{0.3, 0.1\} = 0.1$ (or any smaller positive number).

2. If $|f(x) - 2| < 0.5$, then $-0.5 < f(x) - 2 < 0.5 \quad \Rightarrow \quad 1.5 < f(x) < 2.5$. From the graph, we see that the last inequality is true if $2.6 < x < 3.8$, so we can take $\delta = \min\{3 - 2.6, 3.8 - 3\} = \min\{0.4, 0.8\} = 0.4$ (or any smaller positive number). Note that $x \neq 3$.

3. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x - 4| < |2.56 - 4| = 1.44$. On the right side, we need $|x - 4| < |5.76 - 4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold—namely, $|x - 4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

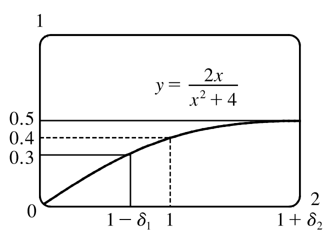
4. The leftmost question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the rightmost question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need $|x - 1| < \left|\frac{1}{\sqrt{2}} - 1\right| \approx 0.292$ (rounding down to be safe). On the right side, we need $|x - 1| < \left|\sqrt{\frac{3}{2}} - 1\right| \approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

5.

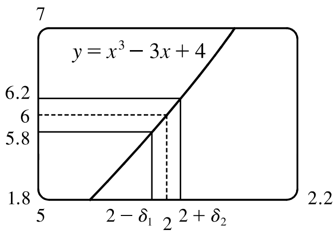


From the graph, we find that $y = \tan x = 0.8$ when $x \approx 0.675$, so $\frac{\pi}{4} - \delta_1 \approx 0.675 \quad \Rightarrow \quad \delta_1 \approx \frac{\pi}{4} - 0.675 \approx 0.1106$. Also, $y = \tan x = 1.2$ when $x \approx 0.876$, so $\frac{\pi}{4} + \delta_2 \approx 0.876 \quad \Rightarrow \quad \delta_2 = 0.876 - \frac{\pi}{4} \approx 0.0906$. Thus, we choose $\delta = 0.0906$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

6.

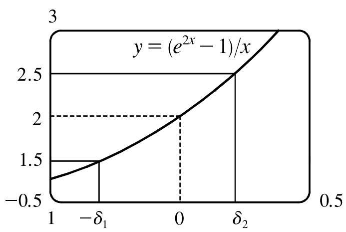


From the graph, we find that $y = 2x/(x^2 + 4) = 0.3$ when $x = \frac{2}{3}$, so $1 - \delta_1 = \frac{2}{3} \quad \Rightarrow \quad \delta_1 = \frac{1}{3}$. Also, $y = 2x/(x^2 + 4) = 0.5$ when $x = 2$, so $1 + \delta_2 = 2 \quad \Rightarrow \quad \delta_2 = 1$. Thus, we choose $\delta = \frac{1}{3}$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

7. 

From the graph with $\varepsilon = 0.2$, we find that $y = x^3 - 3x + 4 = 5.8$ when $x \approx 1.9774$, so $2 - \delta_1 \approx 1.9774 \;\Rightarrow\; \delta_1 \approx 0.0226$. Also, $y = x^3 - 3x + 4 = 6.2$ when $x \approx 2.022$, so $2 + \delta_2 \approx 2.0219 \;\Rightarrow\; \delta_2 \approx 0.0219$. Thus, we choose $\delta = 0.0219$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

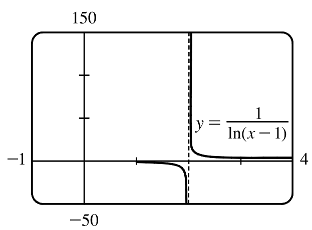
For $\varepsilon = 0.1$, we get $\delta_1 \approx 0.0112$ and $\delta_2 \approx 0.0110$, so we choose $\delta = 0.011$ (or any smaller positive number).

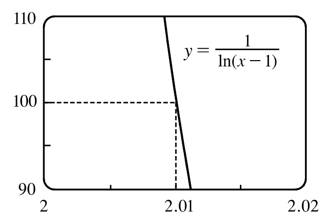
8. 

From the graph with $\varepsilon = 0.5$, we find that $y = (e^{2x} - 1)/x = 1.5$ when $x \approx -0.303$, so $\delta_1 \approx 0.303$. Also, $y = (e^{2x} - 1)/x = 2.5$ when $x \approx 0.215$, so $\delta_2 \approx 0.215$. Thus, we choose $\delta = 0.215$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

For $\varepsilon = 0.1$, we get $\delta_1 \approx 0.052$ and $\delta_2 \approx 0.048$, so we choose $\delta = 0.048$ (or any smaller positive number).

9. (a)

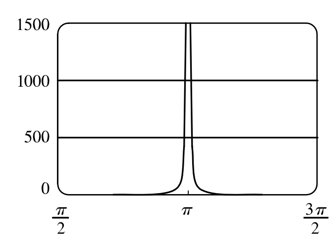




The first graph of $y = \dfrac{1}{\ln(x-1)}$ shows a vertical asymptote at $x = 2$. The second graph shows that $y = 100$ when $x \approx 2.01$ (more accurately, 2.01005). Thus, we choose $\delta = 0.01$ (or any smaller positive number).

(b) From part (a), we see that as x gets closer to 2 from the right, y increases without bound. In symbols, $\displaystyle\lim_{x\to 2^+} \frac{1}{\ln(x-1)} = \infty$.

10. We graph $y = \csc^2 x$ and $y = 500$. The graphs intersect at $x \approx 3.186$, so we choose $\delta = 3.186 - \pi \approx 0.044$. Thus, if $0 < |x - \pi| < 0.044$, then $\csc^2 x > 500$. Similarly, for $M = 1000$, we get $\delta = 3.173 - \pi \approx 0.031$.



11. (a) $A = \pi r^2$ and $A = 1000\text{ cm}^2 \;\Rightarrow\; \pi r^2 = 1000 \;\Rightarrow\; r^2 = \frac{1000}{\pi} \;\Rightarrow\; r = \sqrt{\frac{1000}{\pi}} \quad (r > 0) \quad \approx 17.8412$ cm.

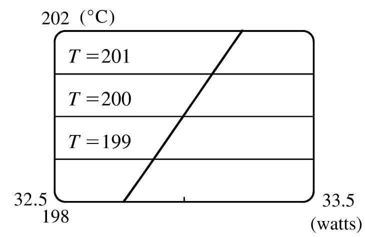
(b) $|A - 1000| \le 5 \;\Rightarrow\; -5 \le \pi r^2 - 1000 \le 5 \;\Rightarrow\; 1000 - 5 \le \pi r^2 \le 1000 + 5 \;\Rightarrow$ $\sqrt{\frac{995}{\pi}} \le r \le \sqrt{\frac{1005}{\pi}} \;\Rightarrow\; 17.7966 \le r \le 17.8858$. $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and $\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

NOT FOR SALE

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

12. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \Rightarrow$

$0.1w^2 + 2.155w + 20 = 200 \Rightarrow$ [by the quadratic formula or from the graph] $w \approx 33.0$ watts $(w > 0)$



(b) From the graph, $199 \leq T \leq 201 \Rightarrow 32.89 < w < 33.11$.

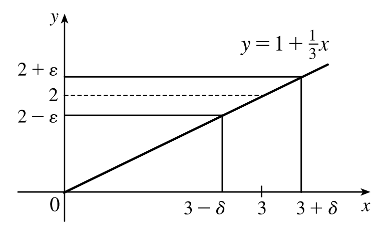
(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

13. (a) $|4x - 8| = 4|x - 2| < 0.1 \Leftrightarrow |x - 2| < \dfrac{0.1}{4}$, so $\delta = \dfrac{0.1}{4} = 0.025$.

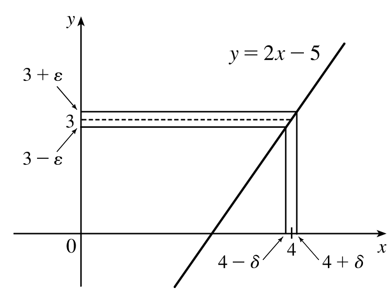
(b) $|4x - 8| = 4|x - 2| < 0.01 \Leftrightarrow |x - 2| < \dfrac{0.01}{4}$, so $\delta = \dfrac{0.01}{4} = 0.0025$.

14. $|(5x - 7) - 3| = |5x - 10| = |5(x - 2)| = 5|x - 2|$. We must have $|f(x) - L| < \varepsilon$, so $5|x - 2| < \varepsilon \Leftrightarrow$ $|x - 2| < \varepsilon/5$. Thus, choose $\delta = \varepsilon/5$. For $\varepsilon = 0.1$, $\delta = 0.02$; for $\varepsilon = 0.05$, $\delta = 0.01$; for $\varepsilon = 0.01$, $\delta = 0.002$.

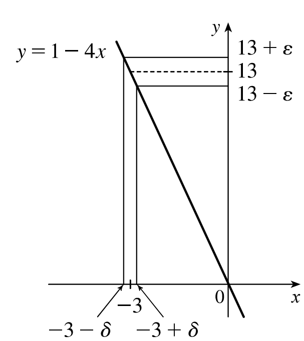
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left|\left(1 + \frac{1}{3}x\right) - 2\right| < \varepsilon$. But $\left|\left(1 + \frac{1}{3}x\right) - 2\right| < \varepsilon \Leftrightarrow \left|\frac{1}{3}x - 1\right| < \varepsilon \Leftrightarrow$ $\left|\frac{1}{3}\right| |x - 3| < \varepsilon \Leftrightarrow |x - 3| < 3\varepsilon$. So if we choose $\delta = 3\varepsilon$, then $0 < |x - 3| < \delta \Rightarrow \left|\left(1 + \frac{1}{3}x\right) - 2\right| < \varepsilon$. Thus, $\lim\limits_{x \to 3}\left(1 + \frac{1}{3}x\right) = 2$ by the definition of a limit.



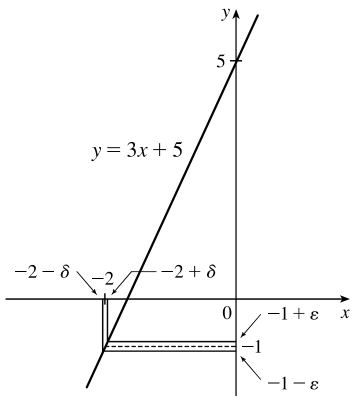
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $|(2x - 5) - 3| < \varepsilon$. But $|(2x - 5) - 3| < \varepsilon \Leftrightarrow |2x - 8| < \varepsilon \Leftrightarrow$ $|2|\,|x - 4| < \varepsilon \Leftrightarrow |x - 4| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x - 4| < \delta \Rightarrow |(2x - 5) - 3| < \varepsilon$. Thus, $\lim\limits_{x \to 4}(2x - 5) = 3$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-3)| < \delta$, then $|(1 - 4x) - 13| < \varepsilon$. But $|(1 - 4x) - 13| < \varepsilon \Leftrightarrow$ $|-4x - 12| < \varepsilon \Leftrightarrow |-4|\,|x + 3| < \varepsilon \Leftrightarrow |x - (-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x - (-3)| < \delta \Rightarrow |(1 - 4x) - 13| < \varepsilon$. Thus, $\lim\limits_{x \to -3}(1 - 4x) = 13$ by the definition of a limit.



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $|(3x + 5) - (-1)| < \varepsilon$. But $|(3x + 5) - (-1)| < \varepsilon \quad \Leftrightarrow$ $|3x + 6| < \varepsilon \quad \Leftrightarrow \quad |3|\,|x + 2| < \varepsilon \quad \Leftrightarrow \quad |x + 2| < \varepsilon/3$. So if we choose $\delta = \varepsilon/3$, then $0 < |x + 2| < \delta \quad \Rightarrow \quad |(3x + 5) - (-1)| < \varepsilon$. Thus, $\lim\limits_{x \to -2} (3x + 5) = -1$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $\left|\dfrac{2 + 4x}{3} - 2\right| < \varepsilon$. But $\left|\dfrac{2 + 4x}{3} - 2\right| < \varepsilon \quad \Leftrightarrow$

$\left|\dfrac{4x - 4}{3}\right| < \varepsilon \quad \Leftrightarrow \quad \left|\frac{4}{3}\right|\,|x - 1| < \varepsilon \quad \Leftrightarrow \quad |x - 1| < \frac{3}{4}\varepsilon$. So if we choose $\delta = \frac{3}{4}\varepsilon$, then $0 < |x - 1| < \delta \quad \Rightarrow$

$\left|\dfrac{2 + 4x}{3} - 2\right| < \varepsilon$. Thus, $\lim\limits_{x \to 1} \dfrac{2 + 4x}{3} = 2$ by the definition of a limit.

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 10| < \delta$, then $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon$. But $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon \quad \Leftrightarrow$ $\left|8 - \frac{4}{5}x\right| < \varepsilon \quad \Leftrightarrow \quad \left|-\frac{4}{5}\right|\,|x - 10| < \varepsilon \quad \Leftrightarrow \quad |x - 10| < \frac{5}{4}\varepsilon$. So if we choose $\delta = \frac{5}{4}\varepsilon$, then $0 < |x - 10| < \delta \quad \Rightarrow$ $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon$. Thus, $\lim\limits_{x \to 10} (3 - \frac{4}{5}x) = -5$ by the definition of a limit.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $\left|\dfrac{x^2 - 2x - 8}{x - 4} - 6\right| < \varepsilon \quad \Leftrightarrow$

$\left|\dfrac{(x - 4)(x + 2)}{x - 4} - 6\right| < \varepsilon \quad \Leftrightarrow \quad |x + 2 - 6| < \varepsilon \quad [x \neq 4] \quad \Leftrightarrow \quad |x - 4| < \varepsilon$. So choose $\delta = \varepsilon$. Then

$0 < |x - 4| < \delta \quad \Rightarrow \quad |x - 4| < \varepsilon \quad \Rightarrow \quad |x + 2 - 6| < \varepsilon \quad \Rightarrow \quad \left|\dfrac{(x - 4)(x + 2)}{x - 4} - 6\right| < \varepsilon \quad [x \neq 4] \quad \Rightarrow$

$\left|\dfrac{x^2 - 2x - 8}{x - 4} - 6\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x \to 4} \dfrac{x^2 - 2x - 8}{x - 4} = 6$.

22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x + 1.5| < \delta$, then $\left|\dfrac{9 - 4x^2}{3 + 2x} - 6\right| < \varepsilon \quad \Leftrightarrow$

$\left|\dfrac{(3 + 2x)(3 - 2x)}{3 + 2x} - 6\right| < \varepsilon \quad \Leftrightarrow \quad |3 - 2x - 6| < \varepsilon \quad [x \neq -1.5] \quad \Leftrightarrow \quad |-2x - 3| < \varepsilon \quad \Leftrightarrow \quad |-2|\,|x + 1.5| < \varepsilon \quad \Leftrightarrow$

$|x + 1.5| < \varepsilon/2$. So choose $\delta = \varepsilon/2$. Then $0 < |x + 1.5| < \delta \quad \Rightarrow \quad |x + 1.5| < \varepsilon/2 \quad \Rightarrow \quad |-2|\,|x + 1.5| < \varepsilon \quad \Rightarrow$

$|-2x - 3| < \varepsilon \quad \Rightarrow \quad |3 - 2x - 6| < \varepsilon \quad \Rightarrow \quad \left|\dfrac{(3 + 2x)(3 - 2x)}{3 + 2x} - 6\right| < \varepsilon \quad [x \neq -1.5] \quad \Rightarrow \quad \left|\dfrac{9 - 4x^2}{3 + 2x} - 6\right| < \varepsilon$.

By the definition of a limit, $\lim\limits_{x \to -1.5} \dfrac{9 - 4x^2}{3 + 2x} = 6$.

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \varepsilon$. So $\delta = \varepsilon$ will work.

24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|c - c| < \varepsilon$. But $|c - c| = 0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^2 - 0\right| < \varepsilon \quad \Leftrightarrow \quad x^2 < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 0| < \delta \quad \Rightarrow \quad \left|x^2 - 0\right| < \varepsilon$. Thus, $\lim\limits_{x \to 0} x^2 = 0$ by the definition of a limit.

26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^3 - 0\right| < \varepsilon \quad \Leftrightarrow \quad |x|^3 < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$. Then $0 < |x - 0| < \delta \quad \Rightarrow \quad \left|x^3 - 0\right| < \delta^3 = \varepsilon$. Thus, $\lim\limits_{x \to 0} x^3 = 0$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\big||x| - 0\big| < \varepsilon$. But $\big||x|\big| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\lim\limits_{x \to 0} |x| = 0$ by the definition of a limit.

28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < x - (-6) < \delta$, then $\left|\sqrt[8]{6 + x} - 0\right| < \varepsilon$. But $\left|\sqrt[8]{6 + x} - 0\right| < \varepsilon \quad \Leftrightarrow \quad \sqrt[8]{6 + x} < \varepsilon \quad \Leftrightarrow \quad 6 + x < \varepsilon^8 \quad \Leftrightarrow \quad x - (-6) < \varepsilon^8$. So if we choose $\delta = \varepsilon^8$, then $0 < x - (-6) < \delta \quad \Rightarrow \quad \left|\sqrt[8]{6 + x} - 0\right| < \varepsilon$. Thus, $\lim\limits_{x \to -6^+} \sqrt[8]{6 + x} = 0$ by the definition of a right-hand limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|\left(x^2 - 4x + 5\right) - 1\right| < \varepsilon \quad \Leftrightarrow \quad \left|x^2 - 4x + 4\right| < \varepsilon \quad \Leftrightarrow \quad \left|(x - 2)^2\right| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 2| < \delta \quad \Leftrightarrow \quad |x - 2| < \sqrt{\varepsilon} \quad \Leftrightarrow \quad \left|(x - 2)^2\right| < \varepsilon$. Thus, $\lim\limits_{x \to 2} \left(x^2 - 4x + 5\right) = 1$ by the definition of a limit.

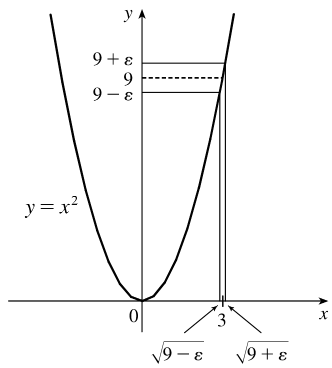
30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|(x^2 + 2x - 7) - 1\right| < \varepsilon$. But $\left|(x^2 + 2x - 7) - 1\right| < \varepsilon \quad \Leftrightarrow \quad \left|x^2 + 2x - 8\right| < \varepsilon \quad \Leftrightarrow \quad |x + 4|\,|x - 2| < \varepsilon$. Thus our goal is to make $|x - 2|$ small enough so that its product with $|x + 4|$ is less than ε. Suppose we first require that $|x - 2| < 1$. Then $-1 < x - 2 < 1 \quad \Rightarrow \quad 1 < x < 3 \quad \Rightarrow \quad 5 < x + 4 < 7 \quad \Rightarrow \quad |x + 4| < 7$, and this gives us $7\,|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \varepsilon/7$. Choose $\delta = \min\{1, \varepsilon/7\}$. Then if $0 < |x - 2| < \delta$, we have $|x - 2| < \varepsilon/7$ and $|x + 4| < 7$, so $\left|(x^2 + 2x - 7) - 1\right| = |(x + 4)(x - 2)| = |x + 4|\,|x - 2| < 7(\varepsilon/7) = \varepsilon$, as desired. Thus, $\lim\limits_{x \to 2}(x^2 + 2x - 7) = 1$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $\left|\left(x^2 - 1\right) - 3\right| < \varepsilon$ or upon simplifying we need $\left|x^2 - 4\right| < \varepsilon$ whenever $0 < |x + 2| < \delta$. Notice that if $|x + 2| < 1$, then $-1 < x + 2 < 1 \quad \Rightarrow \quad -5 < x - 2 < -3 \quad \Rightarrow \quad |x - 2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x + 2| < \delta \quad \Rightarrow \quad |x - 2| < 5$ and $|x + 2| < \varepsilon/5$, so $\left|\left(x^2 - 1\right) - 3\right| = |(x + 2)(x - 2)| = |x + 2|\,|x - 2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to -2}(x^2 - 1) = 3$.

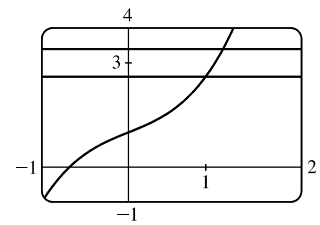
32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|x^3 - 8\right| < \varepsilon$. Now $\left|x^3 - 8\right| = \left|(x - 2)\left(x^2 + 2x + 4\right)\right|$. If $|x - 2| < 1$, that is, $1 < x < 3$, then $x^2 + 2x + 4 \ < \ 3^2 + 2(3) + 4 = 19$ and so $\left|x^3 - 8\right| = |x - 2|\left(x^2 + 2x + 4\right) < 19\,|x - 2|$. So if we take $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$, then $0 < |x - 2| < \delta \quad \Rightarrow \quad \left|x^3 - 8\right| = |x - 2|\left(x^2 + 2x + 4\right) < \frac{\varepsilon}{19} \cdot 19 = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to 2} x^3 = 8$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x-3| < \delta$, then $|x-3| < 2 \Rightarrow -2 < x-3 < 2 \Rightarrow$ $4 < x+3 < 8 \Rightarrow |x+3| < 8$. Also $|x-3| < \frac{\varepsilon}{8}$, so $\left|x^2-9\right| = |x+3|\,|x-3| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim\limits_{x\to 3} x^2 = 9$.

34. From the figure, our choices for δ are $\delta_1 = 3 - \sqrt{9-\varepsilon}$ and $\delta_2 = \sqrt{9+\varepsilon} - 3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1, \delta_2\}$; that is, $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = \sqrt{9+\varepsilon} - 3$.



35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\varepsilon) = \frac{\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{2/3} - 12}{6\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{1/3}}.$$ Thus, $\delta = x(\varepsilon) - 1$.

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093\,272\,342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. *1. Guessing a value for δ* Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| < \varepsilon$ whenever $0 < |x-2| < \delta$. But $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| = \left|\dfrac{2-x}{2x}\right| = \dfrac{|x-2|}{|2x|} < \varepsilon$. We find a positive constant C such that $\dfrac{1}{|2x|} < C \Rightarrow$ $\dfrac{|x-2|}{|2x|} < C\,|x-2|$ and we can make $C\,|x-2| < \varepsilon$ by taking $|x-2| < \dfrac{\varepsilon}{C} = \delta$. We restrict x to lie in the interval $|x-2| < 1 \Rightarrow 1 < x < 3$ so $1 > \dfrac{1}{x} > \dfrac{1}{3} \Rightarrow \dfrac{1}{6} < \dfrac{1}{2x} < \dfrac{1}{2} \Rightarrow \dfrac{1}{|2x|} < \dfrac{1}{2}$. So $C = \dfrac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\varepsilon\}$.

2. Showing that δ works Given $\varepsilon > 0$ we let $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x-2| < \delta$, then $|x-2| < 1 \Rightarrow 1 < x < 3 \Rightarrow$ $\dfrac{1}{|2x|} < \dfrac{1}{2}$ (as in part 1). Also $|x-2| < 2\varepsilon$, so $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| = \dfrac{|x-2|}{|2x|} < \dfrac{1}{2} \cdot 2\varepsilon = \varepsilon$. This shows that $\lim\limits_{x\to 2}(1/x) = \frac{1}{2}$.

37. *1. Guessing a value for δ* Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x-a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \dfrac{|x-a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then

$\dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{|x-a|}{C} < \varepsilon$, and we take $|x-a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a. If $|x-a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x-a < \frac{1}{2}a \;\Rightarrow\; \frac{1}{2}a < x < \frac{3}{2}a \;\Rightarrow\; \sqrt{x}+\sqrt{a} > \sqrt{\frac{1}{2}a}+\sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a}+\sqrt{a}$ is a suitable choice for the constant. So $|x-a| < \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$. If $0 < |x-a| < \delta$, then $|x-a| < \frac{1}{2}a \;\Rightarrow\; \sqrt{x}+\sqrt{a} > \sqrt{\frac{1}{2}a}+\sqrt{a}$ (as in part 1). Also $|x-a| < \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$, so $\left|\sqrt{x}-\sqrt{a}\right| = \dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{\left(\sqrt{a/2}+\sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2}+\sqrt{a}\right)} = \varepsilon$. Therefore, $\lim\limits_{x\to a}\sqrt{x} = \sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim\limits_{t\to 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \;\Rightarrow\; |H(t) - L| < \frac{1}{2} \;\Leftrightarrow\; L - \frac{1}{2} < H(t) < L + \frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L + \frac{1}{2} \;\Rightarrow\; L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L - \frac{1}{2} < 0 \;\Rightarrow\; L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim\limits_{t\to 0} H(t)$ does not exist.

39. Suppose that $\lim\limits_{x\to 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \;\Rightarrow\; |f(x) - L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L \le |L| < \frac{1}{2}$. Now take any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim\limits_{x\to 0} f(x)$ does not exist.

40. First suppose that $\lim\limits_{x\to a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x-a| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$. Then $a - \delta < x < a \;\Rightarrow\; 0 < |x-a| < \delta$ so $|f(x) - L| < \varepsilon$. Thus, $\lim\limits_{x\to a^-} f(x) = L$. Also $a < x < a + \delta \;\Rightarrow\; 0 < |x-a| < \delta$ so $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x\to a^+} f(x) = L$.

Now suppose $\lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to a^-} f(x) = L$, there exists $\delta_1 > 0$ so that $a - \delta_1 < x < a \;\Rightarrow\; |f(x) - L| < \varepsilon$. Since $\lim\limits_{x\to a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a + \delta_2 \;\Rightarrow\; |f(x) - L| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x-a| < \delta \;\Rightarrow\; a - \delta_1 < x < a$ or $a < x < a + \delta_2$ so $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x\to a} f(x) = L$. So we have proved that $\lim\limits_{x\to a} f(x) = L \;\Leftrightarrow\; \lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$.

41. $\dfrac{1}{(x+3)^4} > 10{,}000 \;\Leftrightarrow\; (x+3)^4 < \dfrac{1}{10{,}000} \;\Leftrightarrow\; |x+3| < \dfrac{1}{\sqrt[4]{10{,}000}} \;\Leftrightarrow\; |x-(-3)| < \dfrac{1}{10}$

42. Given $M > 0$, we need $\delta > 0$ such that $0 < |x+3| < \delta \;\Rightarrow\; 1/(x+3)^4 > M$. Now $\dfrac{1}{(x+3)^4} > M \;\Leftrightarrow\; (x+3)^4 < \dfrac{1}{M} \;\Leftrightarrow\; |x+3| < \dfrac{1}{\sqrt[4]{M}}$. So take $\delta = \dfrac{1}{\sqrt[4]{M}}$. Then $0 < |x+3| < \delta = \dfrac{1}{\sqrt[4]{M}} \;\Rightarrow\; \dfrac{1}{(x+3)^4} > M$, so $\lim\limits_{x\to -3} \dfrac{1}{(x+3)^4} = \infty$.

43. Given $M < 0$ we need $\delta > 0$ so that $\ln x < M$ whenever $0 < x < \delta$; that is, $x = e^{\ln x} < e^M$ whenever $0 < x < \delta$. This suggests that we take $\delta = e^M$. If $0 < x < e^M$, then $\ln x < \ln e^M = M$. By the definition of a limit, $\lim\limits_{x \to 0^+} \ln x = -\infty$.

44. (a) Let M be given. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow f(x) > M + 1 - c$. Since $\lim\limits_{x \to a} g(x) = c$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \Rightarrow |g(x) - c| < 1 \Rightarrow g(x) > c - 1$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x - a| < \delta \Rightarrow f(x) + g(x) > (M + 1 - c) + (c - 1) = M$. Thus, $\lim\limits_{x \to a} [f(x) + g(x)] = \infty$.

(b) Let $M > 0$ be given. Since $\lim\limits_{x \to a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow |g(x) - c| < c/2 \Rightarrow g(x) > c/2$. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \Rightarrow f(x) > 2M/c$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta \Rightarrow f(x)\,g(x) > \dfrac{2M}{c}\,\dfrac{c}{2} = M$, so $\lim\limits_{x \to a} f(x)\,g(x) = \infty$.

(c) Let $N < 0$ be given. Since $\lim\limits_{x \to a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow |g(x) - c| < -c/2 \Rightarrow g(x) < c/2$. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \Rightarrow f(x) > 2N/c$. (Note that $c < 0$ and $N < 0 \Rightarrow 2N/c > 0$.) Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta \Rightarrow f(x) > 2N/c \Rightarrow f(x)\,g(x) < \dfrac{2N}{c} \cdot \dfrac{c}{2} = N$, so $\lim\limits_{x \to a} f(x)\,g(x) = -\infty$.

2.5 Continuity

1. From Definition 1, $\lim\limits_{x \to 4} f(x) = f(4)$.

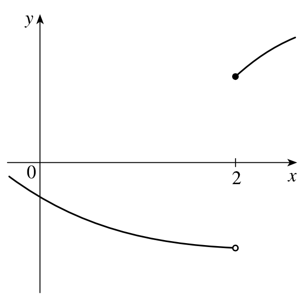
2. The graph of f has no hole, jump, or vertical asymptote.

3. (a) f is discontinuous at -4 since $f(-4)$ is not defined and at -2, 2, and 4 since the limit does not exist (the left and right limits are not the same).

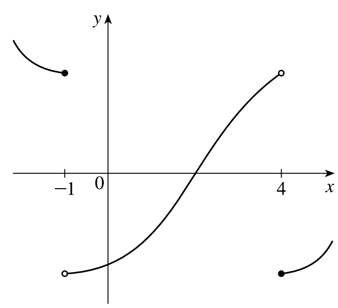
(b) f is continuous from the left at -2 since $\lim\limits_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim\limits_{x \to 2^+} f(x) = f(2)$ and $\lim\limits_{x \to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

4. From the graph of g, we see that g is continuous on the intervals $[-3, -2)$, $(-2, -1)$, $(-1, 0]$, $(0, 1)$, and $(1, 3]$.

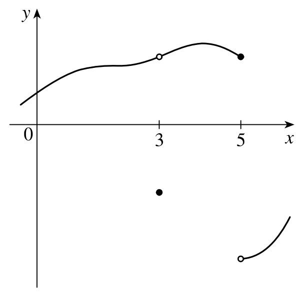
5. The graph of $y = f(x)$ must have a discontinuity at $x = 2$ and must show that $\lim\limits_{x \to 2^+} f(x) = f(2)$.



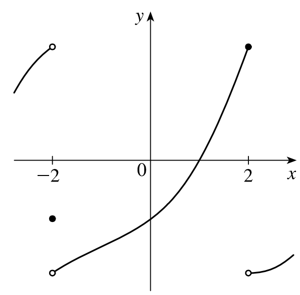
6. The graph of $y = f(x)$ must have discontinuities at $x = -1$ and $x = 4$. It must show that $\lim\limits_{x \to -1^-} f(x) = f(-1)$ and $\lim\limits_{x \to 4^+} f(x) = f(4)$.



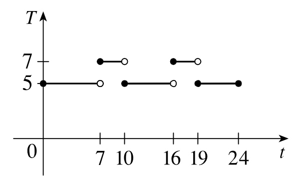
7. The graph of $y = f(x)$ must have a removable discontinuity (a hole) at $x = 3$ and a jump discontinuity at $x = 5$.



8. The graph of $y = f(x)$ must have a discontinuity at $x = -2$ with $\lim\limits_{x \to -2^-} f(x) \neq f(-2)$ and $\lim\limits_{x \to -2^+} f(x) \neq f(-2)$. It must also show that $\lim\limits_{x \to 2^-} f(x) = f(2)$ and $\lim\limits_{x \to 2^+} f(x) \neq f(2)$.



9. (a) The toll is \$7 between 7:00 AM and 10:00 AM and between 4:00 PM and 7:00 PM.



(b) The function T has jump discontinuities at $t = 7$, 10, 16, and 19. Their significance to someone who uses the road is that, because of the sudden jumps in the toll, they may want to avoid the higher rates between $t = 7$ and $t = 10$ and between $t = 16$ and $t = 19$ if feasible.

10. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

11. $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \left(x + 2x^3\right)^4 = \left(\lim\limits_{x \to -1} x + 2 \lim\limits_{x \to -1} x^3\right)^4 = \left[-1 + 2(-1)^3\right]^4 = (-3)^4 = 81 = f(-1)$.

By the definition of continuity, f is continuous at $a = -1$.

12. $\lim\limits_{t \to 2} g(t) = \lim\limits_{t \to 2} \dfrac{t^2 + 5t}{2t + 1} = \dfrac{\lim\limits_{t \to 2}(t^2 + 5t)}{\lim\limits_{t \to 2}(2t + 1)} = \dfrac{\lim\limits_{t \to 2} t^2 + 5 \lim\limits_{t \to 2} t}{2 \lim\limits_{t \to 2} t + \lim\limits_{t \to 2} 1} = \dfrac{2^2 + 5(2)}{2(2) + 1} = \dfrac{14}{5} = g(2)$.

By the definition of continuity, g is continuous at $a = 2$.

13. $\lim\limits_{v\to 1} p(v) = \lim\limits_{v\to 1} 2\sqrt{3v^2+1} = 2\lim\limits_{v\to 1}\sqrt{3v^2+1} = 2\sqrt{\lim\limits_{v\to 1}(3v^2+1)} = 2\sqrt{3\lim\limits_{v\to 1} v^2 + \lim\limits_{v\to 1} 1}$

$$= 2\sqrt{3(1)^2+1} = 2\sqrt{4} = 4 = p(1)$$

By the definition of continuity, p is continuous at $a = 1$.

14. $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}\left(3x^4 - 5x + \sqrt[3]{x^2+4}\right) = 3\lim\limits_{x\to 2} x^4 - 5\lim\limits_{x\to 2} x + \sqrt[3]{\lim\limits_{x\to 2}(x^2+4)}$

$$= 3(2)^4 - 5(2) + \sqrt[3]{2^2+4} = 48 - 10 + 2 = 40 = f(2)$$

By the definition of continuity, f is continuous at $a = 2$.

15. For $a > 4$, we have

$$\begin{aligned} \lim_{x\to a} f(x) &= \lim_{x\to a}(x + \sqrt{x-4}) = \lim_{x\to a} x + \lim_{x\to a}\sqrt{x-4} && \text{[Limit Law 1]} \\ &= a + \sqrt{\lim_{x\to a} x - \lim_{x\to a} 4} && \text{[8, 11, and 2]} \\ &= a + \sqrt{a-4} && \text{[8 and 7]} \\ &= f(a) \end{aligned}$$

So f is continuous at $x = a$ for every a in $(4, \infty)$. Also, $\lim\limits_{x\to 4^+} f(x) = 4 = f(4)$, so f is continuous from the right at 4.

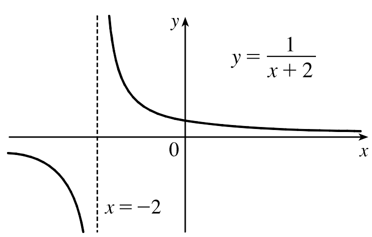
Thus, f is continuous on $[4, \infty)$.

16. For $a < -2$, we have

$$\begin{aligned} \lim_{x\to a} g(x) = \lim_{x\to a}\frac{x-1}{3x+6} &= \frac{\lim_{x\to a}(x-1)}{\lim_{x\to a}(3x+6)} && \text{[Limit Law 5]} \\ &= \frac{\lim_{x\to a} x - \lim_{x\to a} 1}{3\lim_{x\to a} x + \lim_{x\to a} 6} && \text{[2, 1, and 3]} \\ &= \frac{a-1}{3a+6} && \text{[8 and 7]} \end{aligned}$$

Thus, g is continuous at $x = a$ for every a in $(-\infty, -2)$; that is, g is continuous on $(-\infty, -2)$.

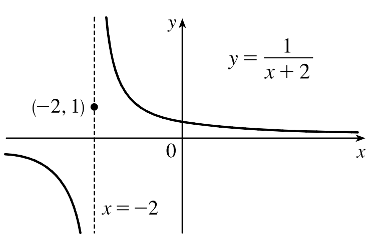
17. $f(x) = \dfrac{1}{x+2}$ is discontinuous at $a = -2$ because $f(-2)$ is undefined.



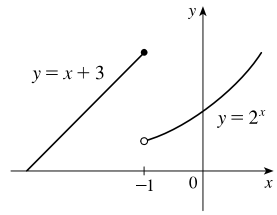
18. $f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$

Here $f(-2) = 1$, but $\lim\limits_{x\to -2^-} f(x) = -\infty$ and $\lim\limits_{x\to -2^+} f(x) = \infty$,

so $\lim\limits_{x\to -2} f(x)$ does not exist and f is discontinuous at -2.



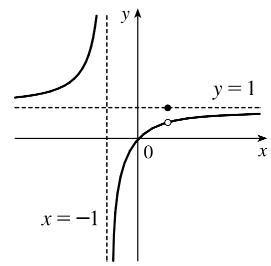
19. $f(x) = \begin{cases} x+3 & \text{if } x \le -1 \\ 2^x & \text{if } x > -1 \end{cases}$



$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-}(x+3) = -1+3 = 2$ and

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} 2^x = 2^{-1} = \frac{1}{2}$. Since the left-hand and the right-hand limits of f at -1 are not equal, $\lim\limits_{x\to -1} f(x)$ does not exist, and f is discontinuous at -1.

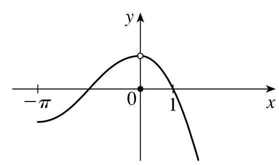
20. $f(x) = \begin{cases} \dfrac{x^2-x}{x^2-1} & \text{if } x \ne 1 \\ 1 & \text{if } x = 1 \end{cases}$



$\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{x^2-x}{x^2-1} = \lim\limits_{x\to 1} \dfrac{x(x-1)}{(x+1)(x-1)} = \lim\limits_{x\to 1} \dfrac{x}{x+1} = \dfrac{1}{2}$,

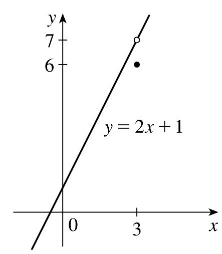
but $f(1) = 1$, so f is discontinous at 1.

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$



$\lim\limits_{x\to 0} f(x) = 1$, but $f(0) = 0 \ne 1$, so f is discontinuous at 0.

22. $f(x) = \begin{cases} \dfrac{2x^2-5x-3}{x-3} & \text{if } x \ne 3 \\ 6 & \text{if } x = 3 \end{cases}$



$\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \dfrac{2x^2-5x-3}{x-3} = \lim\limits_{x\to 3} \dfrac{(2x+1)(x-3)}{x-3} = \lim\limits_{x\to 3}(2x+1) = 7$,

but $f(3) = 6$, so f is discontinuous at 3.

23. $f(x) = \dfrac{x^2-x-2}{x-2} = \dfrac{(x-2)(x+1)}{x-2} = x+1$ for $x \ne 2$. Since $\lim\limits_{x\to 2} f(x) = 2+1 = 3$, define $f(2) = 3$. Then f is continuous at 2.

24. $f(x) = \dfrac{x^3-8}{x^2-4} = \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \dfrac{x^2+2x+4}{x+2}$ for $x \ne 2$. Since $\lim\limits_{x\to 2} f(x) = \dfrac{4+4+4}{2+2} = 3$, define $f(2) = 3$. Then f is continuous at 2.

25. $F(x) = \dfrac{2x^2-x-1}{x^2+1}$ is a rational function, so it is continuous on its domain, $(-\infty, \infty)$, by Theorem 5(b).

26. $G(x) = \dfrac{x^2+1}{2x^2-x-1} = \dfrac{x^2+1}{(2x+1)(x-1)}$ is a rational function, so it is continuous on its domain, $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right) \cup (1, \infty)$, by Theorem 5(b).

27. $x^3 - 2 = 0 \quad \Rightarrow \quad x^3 = 2 \quad \Rightarrow \quad x = \sqrt[3]{2}$, so $Q(x) = \dfrac{\sqrt[3]{x-2}}{x^3 - 2}$ has domain $\left(-\infty, \sqrt[3]{2}\right) \cup \left(\sqrt[3]{2}, \infty\right)$. Now $x^3 - 2$ is continuous everywhere by Theorem 5(a) and $\sqrt[3]{x-2}$ is continuous everywhere by Theorems 5(a), 7, and 9. Thus, Q is continuous on its domain by part 5 of Theorem 4.

28. The domain of $R(t) = \dfrac{e^{\sin t}}{2 + \cos \pi t}$ is $(-\infty, \infty)$ since the denominator is never 0 $[\cos \pi t \geq -1 \quad \Rightarrow \quad 2 + \cos \pi t \geq 1]$. By Theorems 7 and 9, $e^{\sin t}$ and $\cos \pi t$ are continuous on $\mathbb{R}$. By part 1 of Theorem 4, $2 + \cos \pi t$ is continuous on $\mathbb{R}$ and by part 5 of Theorem 4, R is continuous on $\mathbb{R}$.

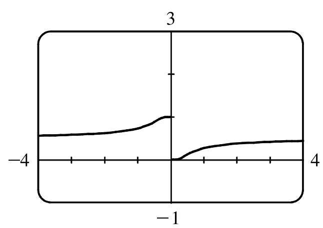
29. By Theorem 5(a), the polynomial $1 + 2t$ is continuous on $\mathbb{R}$. By Theorem 7, the inverse trigonometric function $\arcsin x$ is continuous on its domain, $[-1, 1]$. By Theorem 9, $A(t) = \arcsin(1 + 2t)$ is continuous on its domain, which is $\{t \mid -1 \leq 1 + 2t \leq 1\} = \{t \mid -2 \leq 2t \leq 0\} = \{t \mid -1 \leq t \leq 0\} = [-1, 0]$.

30. By Theorem 7, the trigonometric function $\tan x$ is continuous on its domain, $\left\{x \mid x \neq \frac{\pi}{2} + \pi n\right\}$. By Theorems 5(a), 7, and 9, the composite function $\sqrt{4 - x^2}$ is continuous on its domain $[-2, 2]$. By part 5 of Theorem 4, $B(x) = \dfrac{\tan x}{\sqrt{4 - x^2}}$ is continuous on its domain, $(-2, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 2)$.

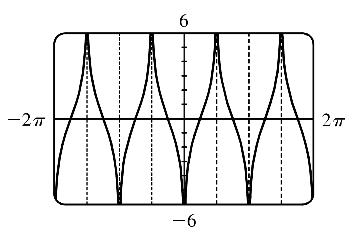
31. $M(x) = \sqrt{1 + \dfrac{1}{x}} = \sqrt{\dfrac{x+1}{x}}$ is defined when $\dfrac{x+1}{x} \geq 0 \quad \Rightarrow \quad x + 1 \geq 0$ and $x > 0$ or $x + 1 \leq 0$ and $x < 0 \quad \Rightarrow \quad x > 0$ or $x \leq -1$, so M has domain $(-\infty, -1] \cup (0, \infty)$. M is the composite of a root function and a rational function, so it is continuous at every number in its domain by Theorems 7 and 9.

32. By Theorems 7 and 9, the composite function e^{-r^2} is continuous on $\mathbb{R}$. By part 1 of Theorem 4, $1 + e^{-r^2}$ is continuous on $\mathbb{R}$. By Theorem 7, the inverse trigonometric function $\tan^{-1}$ is continuous on its domain, $\mathbb{R}$. By Theorem 9, the composite function $N(r) = \tan^{-1}\left(1 + e^{-r^2}\right)$ is continuous on its domain, $\mathbb{R}$.

33. The function $y = \dfrac{1}{1 + e^{1/x}}$ is discontinuous at $x = 0$ because the left- and right-hand limits at $x = 0$ are different.



34. The function $y = \tan^2 x$ is discontinuous at $x = \frac{\pi}{2} + \pi k$, where k is any integer. The function $y = \ln\left(\tan^2 x\right)$ is also discontinuous where $\tan^2 x$ is 0, that is, at $x = \pi k$. So $y = \ln\left(\tan^2 x\right)$ is discontinuous at $x = \frac{\pi}{2} n$, n any integer.



35. Because x is continuous on $\mathbb{R}$ and $\sqrt{20-x^2}$ is continuous on its domain, $-\sqrt{20} \le x \le \sqrt{20}$, the product $f(x) = x\sqrt{20-x^2}$ is continuous on $-\sqrt{20} \le x \le \sqrt{20}$. The number 2 is in that domain, so f is continuous at 2, and $\lim\limits_{x\to 2} f(x) = f(2) = 2\sqrt{16} = 8$.

36. Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function $f(x) = \sin(x + \sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x\to\pi} f(x) = f(\pi) = \sin(\pi + \sin\pi) = \sin\pi = 0$.

37. The function $f(x) = \ln\left(\dfrac{5-x^2}{1+x}\right)$ is continuous throughout its domain because it is the composite of a logarithm function and a rational function. For the domain of f, we must have $\dfrac{5-x^2}{1+x} > 0$, so the numerator and denominator must have the same sign, that is, the domain is $(-\infty, -\sqrt{5}\,] \cup (-1, \sqrt{5}\,]$. The number 1 is in that domain, so f is continuous at 1, and $\lim\limits_{x\to 1} f(x) = f(1) = \ln\dfrac{5-1}{1+1} = \ln 2$.

38. The function $f(x) = 3^{\sqrt{x^2-2x-4}}$ is continuous throughout its domain because it is the composite of an exponential function, a root function, and a polynomial. Its domain is

$$\begin{aligned}\left\{x \mid x^2-2x-4 \ge 0\right\} &= \left\{x \mid x^2-2x+1 \ge 5\right\} = \left\{x \mid (x-1)^2 \ge 5\right\} \\ &= \left\{x \,\middle|\, |x-1| \ge \sqrt{5}\right\} = (-\infty, 1-\sqrt{5}\,] \cup [1+\sqrt{5}, \infty)\end{aligned}$$

The number 4 is in that domain, so f is continuous at 4, and $\lim\limits_{x\to 4} f(x) = f(4) = 3^{\sqrt{16-8-4}} = 3^2 = 9$.

39. $f(x) = \begin{cases} 1-x^2 & \text{if } x \le 1 \\ \ln x & \text{if } x > 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial $1-x^2$ on $(-\infty, 1]$, f is continuous on $(-\infty, 1]$.

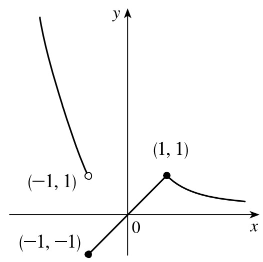
By Theorem 7, since $f(x)$ equals the logarithm function $\ln x$ on $(1, \infty)$, f is continuous on $(1, \infty)$.

At $x = 1$, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(1-x^2) = 1-1^2 = 0$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \ln x = \ln 1 = 0$. Thus, $\lim\limits_{x\to 1} f(x)$ exists and equals 0. Also, $f(1) = 1-1^2 = 0$. Thus, f is continuous at $x = 1$. We conclude that f is continuous on $(-\infty, \infty)$.

40. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \ge \pi/4 \end{cases}$

By Theorem 7, the trigonometric functions are continuous. Since $f(x) = \sin x$ on $(-\infty, \pi/4)$ and $f(x) = \cos x$ on $(\pi/4, \infty)$, f is continuous on $(-\infty, \pi/4) \cup (\pi/4, \infty)$. $\lim\limits_{x\to(\pi/4)^-} f(x) = \lim\limits_{x\to(\pi/4)^-} \sin x = \sin\frac{\pi}{4} = 1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim\limits_{x\to(\pi/4)^+} f(x) = \lim\limits_{x\to(\pi/4)^+} \cos x = 1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim\limits_{x\to(\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty, \infty)$.

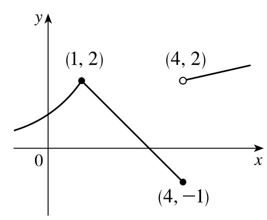
41. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ 1/x & \text{if } x \ge 1 \end{cases}$



f is continuous on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, where it is a polynomial, a polynomial, and a rational function, respectively.

Now $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} x^2 = 1$ and $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} x = -1$, so f is discontinuous at -1. Since $f(-1) = -1$, f is continuous from the right at -1. Also, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{1}{x} = 1 = f(1)$, so f is continuous at 1.

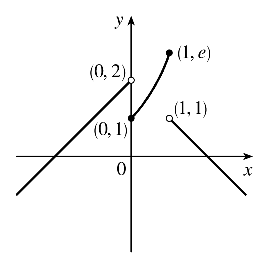
42. $f(x) = \begin{cases} 2^x & \text{if } x \le 1 \\ 3 - x & \text{if } 1 < x \le 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$



f is continuous on $(-\infty, 1)$, $(1, 4)$, and $(4, \infty)$, where it is an exponential, a polynomial, and a root function, respectively.

Now $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 2^x = 2$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (3 - x) = 2$. Since $f(1) = 2$ we have continuity at 1. Also, $\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (3 - x) = -1 = f(4)$ and $\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} \sqrt{x} = 2$, so f is discontinuous at 4, but it is continuous from the left at 4.

43. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$



f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0, 1)$ since it is an exponential.

Now $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x + 2) = 2$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^x = 1$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0. Also $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} e^x = e$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2 - x) = 1$, so f is discontinuous at 1. Since $f(1) = e$, f is continuous from the left at 1.

44. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r = R$.

$\lim\limits_{r \to R^-} F(r) = \lim\limits_{r \to R^-} \dfrac{GMr}{R^3} = \dfrac{GM}{R^2}$ and $\lim\limits_{r \to R^+} F(r) = \lim\limits_{r \to R^+} \dfrac{GM}{r^2} = \dfrac{GM}{R^2}$, so $\lim\limits_{r \to R} F(r) = \dfrac{GM}{R^2}$. Since $F(R) = \dfrac{GM}{R^2}$, F is continuous at R. Therefore, F is a continuous function of r.

45. $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \ge 2 \end{cases}$

f is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \left(cx^2 + 2x\right) = 4c + 4$ and

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}(x^3 - cx) = 8 - 2c$. So f is continuous $\Leftrightarrow$ $4c + 4 = 8 - 2c$ $\Leftrightarrow$ $6c = 4$ $\Leftrightarrow$ $c = \frac{2}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$.

46. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$

At $x = 2$: $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2^-} \dfrac{(x+2)(x-2)}{x-2} = \lim\limits_{x\to 2^-}(x+2) = 2 + 2 = 4$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}(ax^2 - bx + 3) = 4a - 2b + 3$

We must have $4a - 2b + 3 = 4$, or $\boldsymbol{4a - 2b = 1}$ **(1)**.

At $x = 3$: $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-}(ax^2 - bx + 3) = 9a - 3b + 3$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+}(2x - a + b) = 6 - a + b$

We must have $9a - 3b + 3 = 6 - a + b$, or $\boldsymbol{10a - 4b = 3}$ **(2)**.

Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{array}{rcr} -8a + 4b &=& -2 \\ 10a - 4b &=& 3 \\ \hline 2a \phantom{{}-4b} &=& 1 \end{array}$$

So $a = \frac{1}{2}$. Substituting $\frac{1}{2}$ for a in **(1)** gives us $-2b = -1$, so $b = \frac{1}{2}$ as well. Thus, for f to be continuous on $(-\infty, \infty)$, $a = b = \frac{1}{2}$.

47. If f and g are continuous and $g(2) = 6$, then $\lim\limits_{x\to 2}[3f(x) + f(x)\,g(x)] = 36$ $\Rightarrow$

$3\lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} f(x) \cdot \lim\limits_{x\to 2} g(x) = 36$ $\Rightarrow$ $3f(2) + f(2)\cdot 6 = 36$ $\Rightarrow$ $9f(2) = 36$ $\Rightarrow$ $f(2) = 4$.

48. (a) $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$, so $(f \circ g)(x) = f(g(x)) = f(1/x^2) = 1\,/(1/x^2) = x^2$.

(b) The domain of $f \circ g$ is the set of numbers x in the domain of g (all nonzero reals) such that $g(x)$ is in the domain of f (also all nonzero reals). Thus, the domain is $\left\{x \;\middle|\; x \neq 0 \text{ and } \dfrac{1}{x^2} \neq 0\right\} = \{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$. Since $f \circ g$ is the composite of two rational functions, it is continuous throughout its domain; that is, everywhere except $x = 0$.

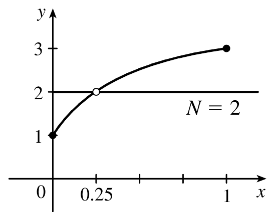
49. (a) $f(x) = \dfrac{x^4-1}{x-1} = \dfrac{(x^2+1)(x^2-1)}{x-1} = \dfrac{(x^2+1)(x+1)(x-1)}{x-1} = (x^2+1)(x+1)$ [or $x^3 + x^2 + x + 1$]

for $x \neq 1$. The discontinuity is removable and $g(x) = x^3 + x^2 + x + 1$ agrees with f for $x \neq 1$ and is continuous on $\mathbb{R}$.

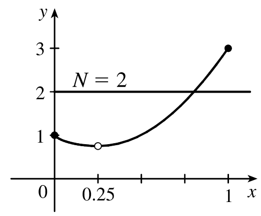
(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x-2} = \dfrac{x(x^2-x-2)}{x-2} = \dfrac{x(x-2)(x+1)}{x-2} = x(x+1)$ [or $x^2 + x$] for $x \neq 2$. The discontinuity is removable and $g(x) = x^2 + x$ agrees with f for $x \neq 2$ and is continuous on $\mathbb{R}$.

(c) $\lim\limits_{x\to\pi^-} f(x) = \lim\limits_{x\to\pi^-} [\![\sin x]\!] = \lim\limits_{x\to\pi^-} 0 = 0$ and $\lim\limits_{x\to\pi^+} f(x) = \lim\limits_{x\to\pi^+} [\![\sin x]\!] = \lim\limits_{x\to\pi^+}(-1) = -1$, so $\lim\limits_{x\to\pi} f(x)$ does not exist. The discontinuity at $x = \pi$ is a jump discontinuity.

50.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

51. $f(x) = x^2 + 10\sin x$ is continuous on the interval $[31, 32]$, $f(31) \approx 957$, and $f(32) \approx 1030$. Since $957 < 1000 < 1030$, there is a number c in $(31, 32)$ such that $f(c) = 1000$ by the Intermediate Value Theorem. *Note:* There is also a number c in $(-32, -31)$ such that $f(c) = 1000$.

52. Suppose that $f(3) < 6$. By the Intermediate Value Theorem applied to the continuous function f on the closed interval $[2, 3]$, the fact that $f(2) = 8 > 6$ and $f(3) < 6$ implies that there is a number c in $(2, 3)$ such that $f(c) = 6$. This contradicts the fact that the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. Hence, our supposition that $f(3) < 6$ was incorrect. It follows that $f(3) \geq 6$. But $f(3) \neq 6$ because the only solutions of $f(x) = 6$ are $x = 1$ and $x = 4$. Therefore, $f(3) > 6$.

53. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

54. The equation $\ln x = x - \sqrt{x}$ is equivalent to the equation $\ln x - x + \sqrt{x} = 0$. $f(x) = \ln x - x + \sqrt{x}$ is continuous on the interval $[2, 3]$, $f(2) = \ln 2 - 2 + \sqrt{2} \approx 0.107$, and $f(3) = \ln 3 - 3 + \sqrt{3} \approx -0.169$. Since $f(2) > 0 > f(3)$, there is a number c in $(2, 3)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x - x + \sqrt{x} = 0$, or $\ln x = x - \sqrt{x}$, in the interval $(2, 3)$.

55. The equation $e^x = 3 - 2x$ is equivalent to the equation $e^x + 2x - 3 = 0$. $f(x) = e^x + 2x - 3$ is continuous on the interval $[0, 1]$, $f(0) = -2$, and $f(1) = e - 1 \approx 1.72$. Since $-2 < 0 < e - 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^x + 2x - 3 = 0$, or $e^x = 3 - 2x$, in the interval $(0, 1)$.

56. The equation $\sin x = x^2 - x$ is equivalent to the equation $\sin x - x^2 + x = 0$. $f(x) = \sin x - x^2 + x$ is continuous on the interval $[1, 2]$, $f(1) = \sin 1 \approx 0.84$, and $f(2) = \sin 2 - 2 \approx -1.09$. Since $\sin 1 > 0 > \sin 2 - 2$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - x^2 + x = 0$, or $\sin x = x^2 - x$, in the interval $(1, 2)$.

57. (a) $f(x) = \cos x - x^3$ is continuous on the interval $[0, 1]$, $f(0) = 1 > 0$, and $f(1) = \cos 1 - 1 \approx -0.46 < 0$. Since $1 > 0 > -0.46$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x^3 = 0$, or $\cos x = x^3$, in the interval $(0, 1)$.

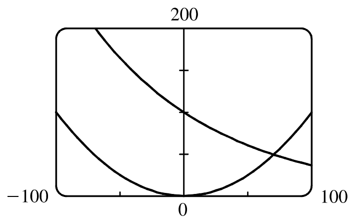
(b) $f(0.86) \approx 0.016 > 0$ and $f(0.87) \approx -0.014 < 0$, so there is a root between 0.86 and 0.87, that is, in the interval $(0.86, 0.87)$.

58. (a) $f(x) = \ln x - 3 + 2x$ is continuous on the interval $[1, 2]$, $f(1) = -1 < 0$, and $f(2) = \ln 2 + 1 \approx 1.7 > 0$. Since $-1 < 0 < 1.7$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x - 3 + 2x = 0$, or $\ln x = 3 - 2x$, in the interval $(1, 2)$.

(b) $f(1.34) \approx -0.03 < 0$ and $f(1.35) \approx 0.0001 > 0$, so there is a root between 1.34 and 1.35, that is, in the interval $(1.34, 1.35)$.

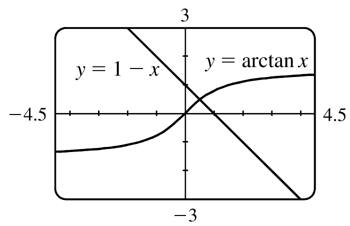
59. (a) Let $f(x) = 100e^{-x/100} - 0.01x^2$. Then $f(0) = 100 > 0$ and $f(100) = 100e^{-1} - 100 \approx -63.2 < 0$. So by the Intermediate Value Theorem, there is a number c in $(0, 100)$ such that $f(c) = 0$. This implies that $100e^{-c/100} = 0.01c^2$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 70.347$, correct to three decimal places.



60. (a) Let $f(x) = \arctan x + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = \frac{\pi}{4} > 0$. So by the Intermediate Value Theorem, there is a number c in $(0, 1)$ such that $f(c) = 0$. This implies that $\arctan c = 1 - c$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 0.520$, correct to three decimal places.



61. Let $f(x) = \sin x^3$. Then f is continuous on $[1, 2]$ since f is the composite of the sine function and the cubing function, both of which are continuous on $\mathbb{R}$. The zeros of the sine are at $n\pi$, so we note that $0 < 1 < \pi < \frac{3}{2}\pi < 2\pi < 8 < 3\pi$, and that the pertinent cube roots are related by $1 < \sqrt[3]{\frac{3}{2}\pi}$ [call this value A] < 2. [By observation, we might notice that $x = \sqrt[3]{\pi}$ and $x = \sqrt[3]{2\pi}$ are zeros of f.]

Now $f(1) = \sin 1 > 0$, $f(A) = \sin \frac{3}{2}\pi = -1 < 0$, and $f(2) = \sin 8 > 0$. Applying the Intermediate Value Theorem on $[1, A]$ and then on $[A, 2]$, we see there are numbers c and d in $(1, A)$ and $(A, 2)$ such that $f(c) = f(d) = 0$. Thus, f has at least two x-intercepts in $(1, 2)$.

62. Let $f(x) = x^2 - 3 + 1/x$. Then f is continuous on $(0, 2]$ since f is a rational function whose domain is $(0, \infty)$. By inspection, we see that $f\left(\frac{1}{4}\right) = \frac{17}{16} > 0$, $f(1) = -1 < 0$, and $f(2) = \frac{3}{2} > 0$. Appling the Intermediate Value Theorem on $\left[\frac{1}{4}, 1\right]$ and then on $[1, 2]$, we see there are numbers c and d in $\left(\frac{1}{4}, 1\right)$ and $(1, 2)$ such that $f(c) = f(d) = 0$. Thus, f has at least two x-intercepts in $(0, 2)$.

63. ($\Rightarrow$) If f is continuous at a, then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h \to 0} f(a+h) = f\Big(\lim_{h \to 0}(a+h)\Big) = f(a).$$

($\Leftarrow$) Let $\varepsilon > 0$. Since $\lim_{h \to 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \quad \Rightarrow$

$|f(a+h) - f(a)| < \varepsilon$. So if $0 < |x - a| < \delta$, then $|f(x) - f(a)| = |f(a + (x-a)) - f(a)| < \varepsilon$.

Thus, $\lim_{x \to a} f(x) = f(a)$ and so f is continuous at a.

64. $\lim_{h \to 0} \sin(a+h) = \lim_{h \to 0}(\sin a \cos h + \cos a \sin h) = \lim_{h \to 0}(\sin a \cos h) + \lim_{h \to 0}(\cos a \sin h)$

$$= \Big(\lim_{h \to 0} \sin a\Big)\Big(\lim_{h \to 0} \cos h\Big) + \Big(\lim_{h \to 0} \cos a\Big)\Big(\lim_{h \to 0} \sin h\Big) = (\sin a)(1) + (\cos a)(0) = \sin a$$

65. As in the previous exercise, we must show that $\lim_{h \to 0} \cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\lim_{h \to 0} \cos(a+h) = \lim_{h \to 0}(\cos a \cos h - \sin a \sin h) = \lim_{h \to 0}(\cos a \cos h) - \lim_{h \to 0}(\sin a \sin h)$$

$$= \Big(\lim_{h \to 0} \cos a\Big)\Big(\lim_{h \to 0} \cos h\Big) - \Big(\lim_{h \to 0} \sin a\Big)\Big(\lim_{h \to 0} \sin h\Big) = (\cos a)(1) - (\sin a)(0) = \cos a$$

66. (a) Since f is continuous at a, $\lim_{x \to a} f(x) = f(a)$. Thus, using the Constant Multiple Law of Limits, we have

$\lim_{x \to a}(cf)(x) = \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cf(a) = (cf)(a)$. Therefore, cf is continuous at a.

(b) Since f and g are continuous at a, $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$. Since $g(a) \neq 0$, we can use the Quotient Law

of Limits: $\lim_{x \to a}\left(\dfrac{f}{g}\right)(x) = \lim_{x \to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{f(a)}{g(a)} = \left(\dfrac{f}{g}\right)(a)$. Thus, $\dfrac{f}{g}$ is continuous at a.

67. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a - \delta, a + \delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1, there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x \to a} f(x) \neq f(a)$. [In fact, $\lim_{x \to a} f(x)$ does not even exist.]

68. $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$ is continuous at 0. To see why, note that $-|x| \leq g(x) \leq |x|$, so by the Squeeze Theorem $\lim_{x \to 0} g(x) = 0 = g(0)$. But g is continuous nowhere else. For if $a \neq 0$ and $\delta > 0$, the interval $(a - \delta, a + \delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a) = 0$ or a, there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|g(x) - g(a)| > |a|/2$. Thus, $\lim_{x \to a} g(x) \neq g(a)$.

69. If there is such a number, it satisfies the equation $x^3 + 1 = x \quad \Leftrightarrow \quad x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

70. $\dfrac{a}{x^3 + 2x^2 - 1} + \dfrac{b}{x^3 + x - 2} = 0 \quad \Rightarrow \quad a(x^3 + x - 2) + b(x^3 + 2x^2 - 1) = 0$. Let $p(x)$ denote the left side of the last equation. Since p is continuous on $[-1, 1]$, $p(-1) = -4a < 0$, and $p(1) = 2b > 0$, there exists a c in $(-1, 1)$ such that

$p(c) = 0$ by the Intermediate Value Theorem. Note that the only root of either denominator that is in $(-1, 1)$ is $(-1 + \sqrt{5})/2 = r$, but $p(r) = (3\sqrt{5} - 9)a/2 \neq 0$. Thus, c is not a root of either denominator, so $p(c) = 0 \quad \Rightarrow$ $x = c$ is a root of the given equation.

71. $f(x) = x^4 \sin(1/x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$ since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since $-1 \leq \sin(1/x) \leq 1$, we have $-x^4 \leq x^4 \sin(1/x) \leq x^4$. Because $\lim\limits_{x \to 0}(-x^4) = 0$ and $\lim\limits_{x \to 0} x^4 = 0$, the Squeeze Theorem gives us $\lim\limits_{x \to 0}(x^4 \sin(1/x)) = 0$, which equals $f(0)$. Thus, f is continuous at 0 and, hence, on $(-\infty, \infty)$.

72. (a) $\lim\limits_{x \to 0^+} F(x) = 0$ and $\lim\limits_{x \to 0^-} F(x) = 0$, so $\lim\limits_{x \to 0} F(x) = 0$, which is $F(0)$, and hence F is continuous at $x = a$ if $a = 0$. For $a > 0$, $\lim\limits_{x \to a} F(x) = \lim\limits_{x \to a} x = a = F(a)$. For $a < 0$, $\lim\limits_{x \to a} F(x) = \lim\limits_{x \to a}(-x) = -a = F(a)$. Thus, F is continuous at $x = a$; that is, continuous everywhere.

(b) Assume that f is continuous on the interval I. Then for $a \in I$, $\lim\limits_{x \to a} |f(x)| = \left|\lim\limits_{x \to a} f(x)\right| = |f(a)|$ by Theorem 8. (If a is an endpoint of I, use the appropriate one-sided limit.) So $|f|$ is continuous on I.

(c) No, the converse is false. For example, the function $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at $x = 0$, but $|f(x)| = 1$ is continuous on $\mathbb{R}$.

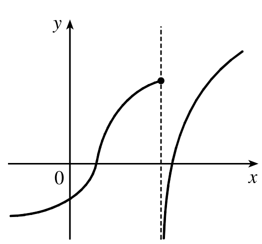
73. Define $u(t)$ to be the monk's distance from the monastery, as a function of time t (in hours), on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$.
So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \quad \Leftrightarrow$ $u(t_0) = d(t_0)$. So at time t_0 after 7:00 AM, the monk will be at the same place on both days.

2.6 Limits at Infinity; Horizontal Asymptotes

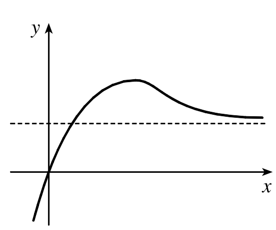
1. (a) As x becomes large, the values of $f(x)$ approach 5.

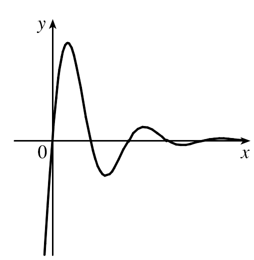
(b) As x becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

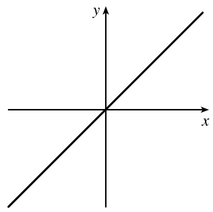


The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

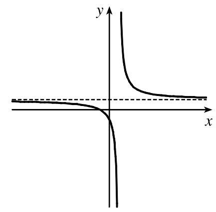




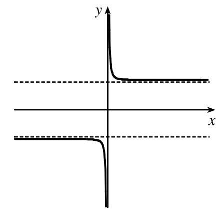
(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote



One horizontal asymptote



Two horizontal asymptotes

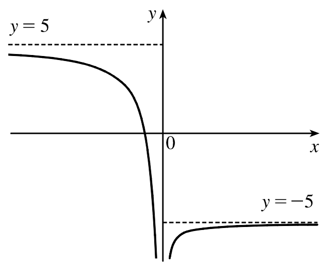
3. (a) $\lim_{x\to\infty} f(x) = -2$ (b) $\lim_{x\to-\infty} f(x) = 2$ (c) $\lim_{x\to1} f(x) = \infty$

(d) $\lim_{x\to3} f(x) = -\infty$ (e) Vertical: $x = 1$, $x = 3$; horizontal: $y = -2$, $y = 2$

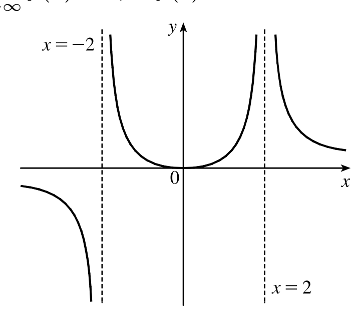
4. (a) $\lim_{x\to\infty} g(x) = 2$ (b) $\lim_{x\to-\infty} g(x) = -1$ (c) $\lim_{x\to0} g(x) = -\infty$

(d) $\lim_{x\to2^-} g(x) = -\infty$ (e) $\lim_{x\to2^+} g(x) = \infty$ (f) Vertical: $x = 0$, $x = 2$; horizontal: $y = -1$, $y = 2$

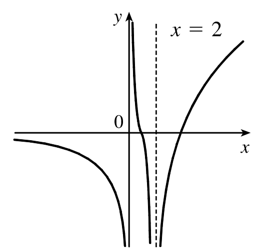
5. $\lim_{x\to0} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = 5$, $\lim_{x\to\infty} f(x) = -5$



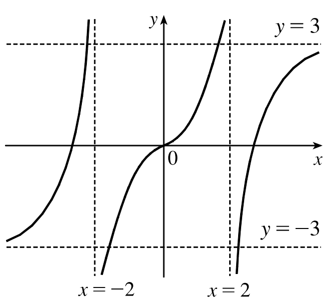
6. $\lim_{x\to2} f(x) = \infty$, $\lim_{x\to-2^+} f(x) = \infty$, $\lim_{x\to-2^-} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 0$, $f(0) = 0$



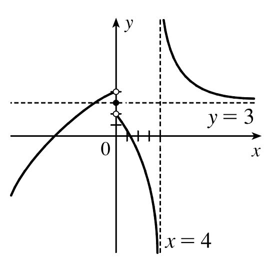
7. $\lim_{x\to2} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to0^+} f(x) = \infty$, $\lim_{x\to0^-} f(x) = -\infty$



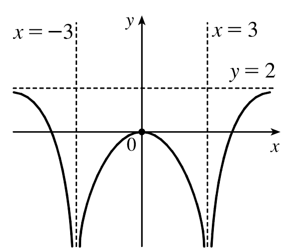
8. $\lim_{x\to\infty} f(x) = 3$, $\lim_{x\to2^-} f(x) = \infty$, $\lim_{x\to2^+} f(x) = -\infty$, f is odd



9. $f(0) = 3$, $\lim_{x\to0^-} f(x) = 4$, $\lim_{x\to0^+} f(x) = 2$, $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to4^-} f(x) = -\infty$, $\lim_{x\to4^+} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 3$



10. $\lim_{x\to3} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 2$, $f(0) = 0$, f is even



11. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$. It appears that $\lim\limits_{x\to\infty} \left(x^2/2^x\right) = 0$.

12. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10{,}000]$ by $[0, 0.2]$, we estimate that $\lim\limits_{x\to\infty} f(x) = 0.14$ (to two decimal places.)

(b)

x	$f(x)$
10,000	0.135 308
100,000	0.135 333
1,000,000	0.135 335

From the table, we estimate that $\lim\limits_{x\to\infty} f(x) = 0.1353$ (to four decimal places.)

13.
$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^2-7}{5x^2+x-3} &= \lim_{x\to\infty} \frac{(2x^2-7)/x^2}{(5x^2+x-3)/x^2} && \text{[Divide both the numerator and denominator by } x^2 \text{ (the highest power of } x \text{ that appears in the denominator)]}\\
&= \frac{\lim\limits_{x\to\infty}(2-7/x^2)}{\lim\limits_{x\to\infty}(5+1/x-3/x^2)} && \text{[Limit Law 5]}\\
&= \frac{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty}(7/x^2)}{\lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty}(1/x) - \lim\limits_{x\to\infty}(3/x^2)} && \text{[Limit Laws 1 and 2]}\\
&= \frac{2 - 7\lim\limits_{x\to\infty}(1/x^2)}{5 + \lim\limits_{x\to\infty}(1/x) - 3\lim\limits_{x\to\infty}(1/x^2)} && \text{[Limit Laws 7 and 3]}\\
&= \frac{2-7(0)}{5+0+3(0)} && \text{[Theorem 2.6.5]}\\
&= \frac{2}{5}
\end{aligned}$$

14.
$$\begin{aligned}
\lim_{x\to\infty} \sqrt{\frac{9x^3+8x-4}{3-5x+x^3}} &= \sqrt{\lim_{x\to\infty} \frac{9x^3+8x-4}{3-5x+x^3}} && \text{[Limit Law 11]}\\
&= \sqrt{\lim_{x\to\infty} \frac{9+8/x^2-4/x^3}{3/x^3-5/x^2+1}} && \text{[Divide by } x^3\text{]}\\
&= \sqrt{\frac{\lim\limits_{x\to\infty}(9+8/x^2-4/x^3)}{\lim\limits_{x\to\infty}(3/x^3-5/x^2+1)}} && \text{[Limit Law 5]}\\
&= \sqrt{\frac{\lim\limits_{x\to\infty} 9 + \lim\limits_{x\to\infty}(8/x^2) - \lim\limits_{x\to\infty}(4/x^3)}{\lim\limits_{x\to\infty}(3/x^3) - \lim\limits_{x\to\infty}(5/x^2) + \lim\limits_{x\to\infty} 1}} && \text{[Limit Laws 1 and 2]}\\
&= \sqrt{\frac{9 + 8\lim\limits_{x\to\infty}(1/x^2) - 4\lim\limits_{x\to\infty}(1/x^3)}{3\lim\limits_{x\to\infty}(1/x^3) - 5\lim\limits_{x\to\infty}(1/x^2) + 1}} && \text{[Limit Laws 7 and 3]}\\
&= \sqrt{\frac{9+8(0)-4(0)}{3(0)-5(0)+1}} && \text{[Theorem 2.6.5]}\\
&= \sqrt{\frac{9}{1}} = \sqrt{9} = 3
\end{aligned}$$

15. $\displaystyle \lim_{x\to\infty} \frac{3x-2}{2x+1} = \lim_{x\to\infty} \frac{(3x-2)/x}{(2x+1)/x} = \lim_{x\to\infty} \frac{3-2/x}{2+1/x} = \frac{\lim\limits_{x\to\infty} 3 - 2\lim\limits_{x\to\infty} 1/x}{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} 1/x} = \frac{3-2(0)}{2+0} = \frac{3}{2}$

16. $\displaystyle \lim_{x\to\infty} \frac{1-x^2}{x^3-x+1} = \lim_{x\to\infty} \frac{(1-x^2)/x^3}{(x^3-x+1)/x^3} = \lim_{x\to\infty} \frac{1/x^3-1/x}{1-1/x^2+1/x^3}$

$$= \frac{\lim\limits_{x\to\infty} 1/x^3 - \lim\limits_{x\to\infty} 1/x}{\lim\limits_{x\to\infty} 1 - \lim\limits_{x\to\infty} 1/x^2 + \lim\limits_{x\to\infty} 1/x^3} = \frac{0-0}{1-0+0} = 0$$

17. $\displaystyle \lim_{x\to-\infty} \frac{x-2}{x^2+1} = \lim_{x\to-\infty} \frac{(x-2)/x^2}{(x^2+1)/x^2} = \lim_{x\to-\infty} \frac{1/x-2/x^2}{1+1/x^2} = \frac{\lim\limits_{x\to-\infty} 1/x - 2\lim\limits_{x\to-\infty} 1/x^2}{\lim\limits_{x\to-\infty} 1 + \lim\limits_{x\to-\infty} 1/x^2} = \frac{0-2(0)}{1+0} = 0$

18. $\displaystyle \lim_{x\to-\infty} \frac{4x^3+6x^2-2}{2x^3-4x+5} = \lim_{x\to-\infty} \frac{(4x^3+6x^2-2)/x^3}{(2x^3-4x+5)/x^3} = \lim_{x\to-\infty} \frac{4+6/x-2/x^3}{2-4/x^2+5/x^3} = \frac{4+0-0}{2-0+0} = 2$

19. $\displaystyle \lim_{t\to\infty} \frac{\sqrt{t}+t^2}{2t-t^2} = \lim_{t\to\infty} \frac{(\sqrt{t}+t^2)/t^2}{(2t-t^2)/t^2} = \lim_{t\to\infty} \frac{1/t^{3/2}+1}{2/t-1} = \frac{0+1}{0-1} = -1$

20. $\displaystyle \lim_{t\to\infty} \frac{t-t\sqrt{t}}{2t^{3/2}+3t-5} = \lim_{t\to\infty} \frac{\left(t-t\sqrt{t}\right)/t^{3/2}}{\left(2t^{3/2}+3t-5\right)/t^{3/2}} = \lim_{t\to\infty} \frac{1/t^{1/2}-1}{2+3/t^{1/2}-5/t^{3/2}} = \frac{0-1}{2+0-0} = -\frac{1}{2}$

21. $\displaystyle \lim_{x\to\infty} \frac{(2x^2+1)^2}{(x-1)^2(x^2+x)} = \lim_{x\to\infty} \frac{(2x^2+1)^2/x^4}{[(x-1)^2(x^2+x)]/x^4} = \lim_{x\to\infty} \frac{[(2x^2+1)/x^2]^2}{[(x^2-2x+1)/x^2][(x^2+x)/x^2]}$

$$= \lim_{x\to\infty} \frac{(2+1/x^2)^2}{(1-2/x+1/x^2)(1+1/x)} = \frac{(2+0)^2}{(1-0+0)(1+0)} = 4$$

22. $\displaystyle \lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}} = \lim_{x\to\infty} \frac{x^2/x^2}{\sqrt{x^4+1}/x^2} = \lim_{x\to\infty} \frac{1}{\sqrt{(x^4+1)/x^4}}$ $\quad$ [since $x^2 = \sqrt{x^4}$ for $x > 0$]

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^4}} = \frac{1}{\sqrt{1+0}} = 1$$

23. $\displaystyle \lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to\infty} \frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3} = \frac{\lim\limits_{x\to\infty} \sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to\infty} (2/x^3-1)}$ $\quad$ [since $x^3 = \sqrt{x^6}$ for $x > 0$]

$$= \frac{\lim\limits_{x\to\infty} \sqrt{1/x^6+4}}{\lim\limits_{x\to\infty} (2/x^3) - \lim\limits_{x\to\infty} 1} = \frac{\sqrt{\lim\limits_{x\to\infty} (1/x^6) + \lim\limits_{x\to\infty} 4}}{0-1}$$

$$= \frac{\sqrt{0+4}}{-1} = \frac{2}{-1} = -2$$

24. $\displaystyle \lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to-\infty} \frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3} = \frac{\lim\limits_{x\to-\infty} -\sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to-\infty} (2/x^3-1)}$ $\quad$ [since $x^3 = -\sqrt{x^6}$ for $x < 0$]

$$= \frac{\lim\limits_{x\to-\infty} -\sqrt{1/x^6+4}}{2\lim\limits_{x\to-\infty} (1/x^3) - \lim\limits_{x\to-\infty} 1} = \frac{-\sqrt{\lim\limits_{x\to-\infty} (1/x^6) + \lim\limits_{x\to-\infty} 4}}{2(0)-1}$$

$$= \frac{-\sqrt{0+4}}{-1} = \frac{-2}{-1} = 2$$

25. $\displaystyle \lim_{x\to\infty} \frac{\sqrt{x+3x^2}}{4x-1} = \lim_{x\to\infty} \frac{\sqrt{x+3x^2}/x}{(4x-1)/x} = \frac{\lim_{x\to\infty} \sqrt{(x+3x^2)/x^2}}{\lim_{x\to\infty} (4-1/x)}$ [since $x = \sqrt{x^2}$ for $x > 0$]

$$= \frac{\lim_{x\to\infty} \sqrt{1/x+3}}{\lim_{x\to\infty} 4 - \lim_{x\to\infty} (1/x)} = \frac{\sqrt{\lim_{x\to\infty} (1/x) + \lim_{x\to\infty} 3}}{4-0} = \frac{\sqrt{0+3}}{4} = \frac{\sqrt{3}}{4}$$

26. $\displaystyle \lim_{x\to\infty} \frac{x+3x^2}{4x-1} = \lim_{x\to\infty} \frac{(x+3x^2)/x}{(4x-1)/x} = \lim_{x\to\infty} \frac{1+3x}{4-1/x}$

$= \infty$ since $1+3x \to \infty$ and $4-1/x \to 4$ as $x \to \infty$.

27. $\displaystyle \lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right) = \lim_{x\to\infty} \frac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{\left(\sqrt{9x^2+x}\right)^2 - (3x)^2}{\sqrt{9x^2+x}+3x}$

$$= \lim_{x\to\infty} \frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x\to\infty} \frac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x} = \lim_{x\to\infty} \frac{1}{\sqrt{9+1/x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

28. $\displaystyle \lim_{x\to-\infty} \left(\sqrt{4x^2+3x}+2x\right) = \lim_{x\to-\infty} \left(\sqrt{4x^2+3x}+2x\right)\left[\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}\right]$

$$= \lim_{x\to-\infty} \frac{(4x^2+3x)-(2x)^2}{\sqrt{4x^2+3x}-2x} = \lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+3x}-2x}$$

$$= \lim_{x\to-\infty} \frac{3x/x}{\left(\sqrt{4x^2+3x}-2x\right)/x} = \lim_{x\to-\infty} \frac{3}{-\sqrt{4+3/x}-2} \quad \left[\text{since } x = -\sqrt{x^2} \text{ for } x < 0\right]$$

$$= \frac{3}{-\sqrt{4+0}-2} = -\frac{3}{4}$$

29. $\displaystyle \lim_{x\to\infty} \left(\sqrt{x^2+ax} - \sqrt{x^2+bx}\right) = \lim_{x\to\infty} \frac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$

$$= \lim_{x\to\infty} \frac{(x^2+ax)-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}} = \lim_{x\to\infty} \frac{[(a-b)x]/x}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)/\sqrt{x^2}}$$

$$= \lim_{x\to\infty} \frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}} = \frac{a-b}{\sqrt{1+0}+\sqrt{1+0}} = \frac{a-b}{2}$$

30. For $x > 0$, $\sqrt{x^2+1} > \sqrt{x^2} = x$. So as $x \to \infty$, we have $\sqrt{x^2+1} \to \infty$, that is, $\lim_{x\to\infty} \sqrt{x^2+1} = \infty$.

31. $\displaystyle \lim_{x\to\infty} \frac{x^4-3x^2+x}{x^3-x+2} = \lim_{x\to\infty} \frac{(x^4-3x^2+x)/x^3}{(x^3-x+2)/x^3}$ $\left[\begin{array}{c}\text{divide by the highest power} \\ \text{of } x \text{ in the denominator}\end{array}\right]$ $\displaystyle = \lim_{x\to\infty} \frac{x-3/x+1/x^2}{1-1/x^2+2/x^3} = \infty$

since the numerator increases without bound and the denominator approaches 1 as $x \to \infty$.

32. $\lim_{x\to\infty} (e^{-x} + 2\cos 3x)$ does not exist. $\lim_{x\to\infty} e^{-x} = 0$, but $\lim_{x\to\infty} (2\cos 3x)$ does not exist because the values of $2\cos 3x$ oscillate between the values of -2 and 2 infinitely often, so the given limit does not exist.

33. $\lim\limits_{x\to-\infty}(x^2+2x^7)=\lim\limits_{x\to-\infty}x^7\left(\dfrac{1}{x^5}+2\right)$ [factor out the largest power of x] $=-\infty$ because $x^7\to-\infty$ and $1/x^5+2\to2$ as $x\to-\infty$.

Or: $\lim\limits_{x\to-\infty}\left(x^2+2x^7\right)=\lim\limits_{x\to-\infty}x^2\left(1+2x^5\right)=-\infty$.

34. $\lim\limits_{x\to-\infty}\dfrac{1+x^6}{x^4+1}=\lim\limits_{x\to-\infty}\dfrac{(1+x^6)/x^4}{(x^4+1)/x^4}$ $\left[\begin{array}{c}\text{divide by the highest power}\\ \text{of } x \text{ in the denominator}\end{array}\right]$ $=\lim\limits_{x\to-\infty}\dfrac{1/x^4+x^2}{1+1/x^4}=\infty$ since the numerator increases without bound and the denominator approaches 1 as $x\to-\infty$.

35. Let $t=e^x$. As $x\to\infty$, $t\to\infty$. $\lim\limits_{x\to\infty}\arctan(e^x)=\lim\limits_{t\to\infty}\arctan t=\frac{\pi}{2}$ by (3).

36. Divide numerator and denominator by e^{3x}: $\lim\limits_{x\to\infty}\dfrac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}=\lim\limits_{x\to\infty}\dfrac{1-e^{-6x}}{1+e^{-6x}}=\dfrac{1-0}{1+0}=1$

37. $\lim\limits_{x\to\infty}\dfrac{1-e^x}{1+2e^x}=\lim\limits_{x\to\infty}\dfrac{(1-e^x)/e^x}{(1+2e^x)/e^x}=\lim\limits_{x\to\infty}\dfrac{1/e^x-1}{1/e^x+2}=\dfrac{0-1}{0+2}=-\dfrac{1}{2}$

38. Since $0\le\sin^2x\le1$, we have $0\le\dfrac{\sin^2x}{x^2+1}\le\dfrac{1}{x^2+1}$. We know that $\lim\limits_{x\to\infty}0=0$ and $\lim\limits_{x\to\infty}\dfrac{1}{x^2+1}=0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty}\dfrac{\sin^2x}{x^2+1}=0$.

39. Since $-1\le\cos x\le1$ and $e^{-2x}>0$, we have $-e^{-2x}\le e^{-2x}\cos x\le e^{-2x}$. We know that $\lim\limits_{x\to\infty}(-e^{-2x})=0$ and $\lim\limits_{x\to\infty}\left(e^{-2x}\right)=0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty}(e^{-2x}\cos x)=0$.

40. Let $t=\ln x$. As $x\to0^+$, $t\to-\infty$. $\lim\limits_{x\to0^+}\tan^{-1}(\ln x)=\lim\limits_{t\to-\infty}\tan^{-1}t=-\frac{\pi}{2}$ by (4).

41. $\lim\limits_{x\to\infty}[\ln(1+x^2)-\ln(1+x)]=\lim\limits_{x\to\infty}\ln\dfrac{1+x^2}{1+x}=\ln\left(\lim\limits_{x\to\infty}\dfrac{1+x^2}{1+x}\right)=\ln\left(\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}+x}{\frac{1}{x}+1}\right)=\infty$, since the limit in parentheses is ∞.

42. $\lim\limits_{x\to\infty}[\ln(2+x)-\ln(1+x)]=\lim\limits_{x\to\infty}\ln\left(\dfrac{2+x}{1+x}\right)=\lim\limits_{x\to\infty}\ln\left(\dfrac{2/x+1}{1/x+1}\right)=\ln\dfrac{1}{1}=\ln1=0$

43. (a) (i) $\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}\dfrac{x}{\ln x}=0$ since $x\to0^+$ and $\ln x\to-\infty$ as $x\to0^+$.

(ii) $\lim\limits_{x\to1^-}f(x)=\lim\limits_{x\to1^-}\dfrac{x}{\ln x}=-\infty$ since $x\to1$ and $\ln x\to0^-$ as $x\to1^-$.

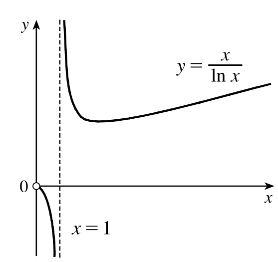
(iii) $\lim\limits_{x\to1^+}f(x)=\lim\limits_{x\to1^+}\dfrac{x}{\ln x}=\infty$ since $x\to1$ and $\ln x\to0^+$ as $x\to1^+$.

(b)

x	$f(x)$
10,000	1085.7
100,000	8685.9
1,000,000	72,382.4

It appears that $\lim\limits_{x\to\infty}f(x)=\infty$.

(c)



44. (a) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\left(\dfrac{2}{x} - \dfrac{1}{\ln x}\right) = 0$

since $\dfrac{2}{x} \to 0$ and $\dfrac{1}{\ln x} \to 0$ as $x \to \infty$.

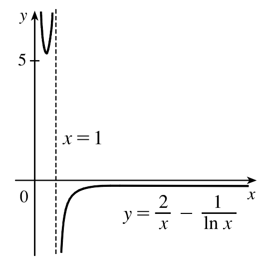
(b) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\left(\dfrac{2}{x} - \dfrac{1}{\ln x}\right) = \infty$

since $\dfrac{2}{x} \to \infty$ and $\dfrac{1}{\ln x} \to 0$ as $x \to 0^+$.

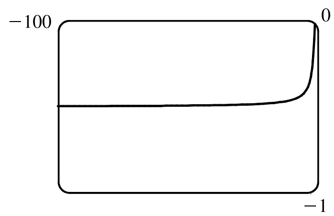
(c) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}\left(\dfrac{2}{x} - \dfrac{1}{\ln x}\right) = \infty$ since $\dfrac{2}{x} \to 2$ and $\dfrac{1}{\ln x} \to -\infty$ as $x \to 1^-$.

(d) $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}\left(\dfrac{2}{x} - \dfrac{1}{\ln x}\right) = -\infty$ since $\dfrac{2}{x} \to 2$ and $\dfrac{1}{\ln x} \to \infty$ as $x \to 1^+$.

(e)



45. (a)



From the graph of $f(x) = \sqrt{x^2+x+1} + x$, we estimate the value of $\lim\limits_{x\to-\infty} f(x)$ to be -0.5.

(b)

x	$f(x)$
$-10{,}000$	$-0.499\,962\,5$
$-100{,}000$	$-0.499\,996\,2$
$-1{,}000{,}000$	$-0.499\,999\,6$

From the table, we estimate the limit to be -0.5.

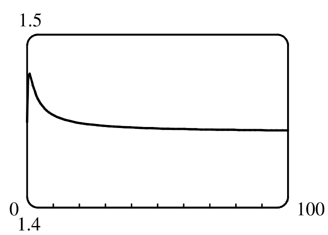
(c) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right) = \lim\limits_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)\left[\dfrac{\sqrt{x^2+x+1}-x}{\sqrt{x^2+x+1}-x}\right] = \lim\limits_{x\to-\infty}\dfrac{\left(x^2+x+1\right)-x^2}{\sqrt{x^2+x+1}-x}$

$$= \lim_{x\to-\infty}\frac{(x+1)(1/x)}{\left(\sqrt{x^2+x+1}-x\right)(1/x)} = \lim_{x\to-\infty}\frac{1+(1/x)}{-\sqrt{1+(1/x)+(1/x^2)}-1}$$

$$= \frac{1+0}{-\sqrt{1+0+0}-1} = -\frac{1}{2}$$

Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x, with $x < 0$, we get $\dfrac{1}{x}\sqrt{x^2+x+1} = -\dfrac{1}{\sqrt{x^2}}\sqrt{x^2+x+1} = -\sqrt{1+(1/x)+(1/x^2)}$.

46. (a)



From the graph of $f(x) = \sqrt{3x^2+8x+6} - \sqrt{3x^2+3x+1}$, we estimate (to one decimal place) the value of $\lim\limits_{x\to\infty} f(x)$ to be 1.4.

(b)

x	$f(x)$
10,000	1.443 39
100,000	1.443 38
1,000,000	1.443 38

From the table, we estimate (to four decimal places) the limit to be 1.4434.

(c) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\left(\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}\right)\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$

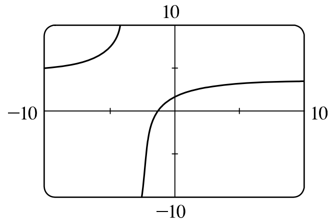
$$= \lim_{x\to\infty} \frac{\left(3x^2+8x+6\right)-\left(3x^2+3x+1\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}} = \lim_{x\to\infty} \frac{(5x+5)(1/x)}{\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)(1/x)}$$

$$= \lim_{x\to\infty} \frac{5+5/x}{\sqrt{3+8/x+6/x^2}+\sqrt{3+3/x+1/x^2}} = \frac{5}{\sqrt{3}+\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376$$

47. $\lim\limits_{x\to\pm\infty} \dfrac{5+4x}{x+3} = \lim\limits_{x\to\pm\infty} \dfrac{(5+4x)/x}{(x+3)/x} = \lim\limits_{x\to\pm\infty} \dfrac{5/x+4}{1+3/x} = \dfrac{0+4}{1+0} = 4$, so

$y=4$ is a horizontal asymptote. $y = f(x) = \dfrac{5+4x}{x+3}$, so $\lim\limits_{x\to -3^+} f(x) = -\infty$

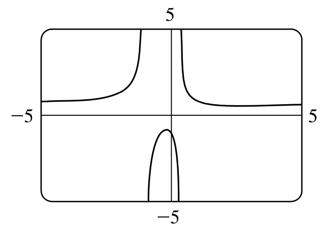
since $5+4x \to -7$ and $x+3 \to 0^+$ as $x \to -3^+$. Thus, $x=-3$ is a vertical asymptote. The graph confirms our work.



48. $\lim\limits_{x\to\pm\infty} \dfrac{2x^2+1}{3x^2+2x-1} = \lim\limits_{x\to\pm\infty} \dfrac{(2x^2+1)/x^2}{(3x^2+2x-1)/x^2}$

$$= \lim_{x\to\pm\infty} \frac{2+1/x^2}{3+2/x-1/x^2} = \frac{2}{3}$$

so $y = \dfrac{2}{3}$ is a horizontal asymptote. $y = f(x) = \dfrac{2x^2+1}{3x^2+2x-1} = \dfrac{2x^2+1}{(3x-1)(x+1)}$.



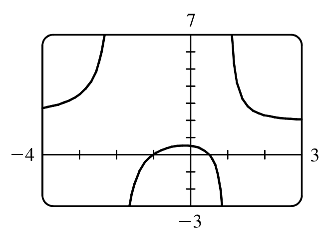
The denominator is zero when $x = \frac{1}{3}$ and -1, but the numerator is nonzero, so $x = \frac{1}{3}$ and $x=-1$ are vertical asymptotes. The graph confirms our work.

49. $\lim\limits_{x\to\pm\infty} \dfrac{2x^2+x-1}{x^2+x-2} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{2x^2+x-1}{x^2}}{\dfrac{x^2+x-2}{x^2}} = \lim\limits_{x\to\pm\infty} \dfrac{2+\dfrac{1}{x}-\dfrac{1}{x^2}}{1+\dfrac{1}{x}-\dfrac{2}{x^2}} = \dfrac{\lim\limits_{x\to\pm\infty}\left(2+\dfrac{1}{x}-\dfrac{1}{x^2}\right)}{\lim\limits_{x\to\pm\infty}\left(1+\dfrac{1}{x}-\dfrac{2}{x^2}\right)}$

$$= \frac{\lim\limits_{x\to\pm\infty} 2 + \lim\limits_{x\to\pm\infty} \dfrac{1}{x} - \lim\limits_{x\to\pm\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\pm\infty} 1 + \lim\limits_{x\to\pm\infty} \dfrac{1}{x} - 2\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2}} = \frac{2+0-0}{1+0-2(0)} = 2,$$ so $y=2$ is a horizontal asymptote.

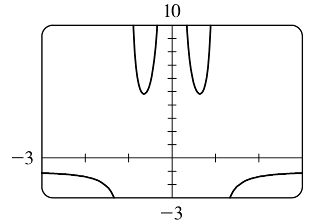
$y = f(x) = \dfrac{2x^2+x-1}{x^2+x-2} = \dfrac{(2x-1)(x+1)}{(x+2)(x-1)}$, so $\lim\limits_{x\to -2^-} f(x) = \infty$,

$\lim\limits_{x\to -2^+} f(x) = -\infty$, $\lim\limits_{x\to 1^-} f(x) = -\infty$, and $\lim\limits_{x\to 1^+} f(x) = \infty$. Thus, $x=-2$ and $x=1$ are vertical asymptotes. The graph confirms our work.



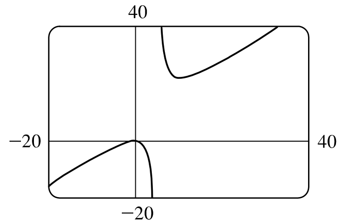
50. $\lim\limits_{x\to\pm\infty} \dfrac{1+x^4}{x^2-x^4} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{1+x^4}{x^4}}{\dfrac{x^2-x^4}{x^4}} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{1}{x^4}+1}{\dfrac{1}{x^2}-1} = \dfrac{\lim\limits_{x\to\pm\infty}\left(\dfrac{1}{x^4}+1\right)}{\lim\limits_{x\to\pm\infty}\left(\dfrac{1}{x^2}-1\right)} = \dfrac{\lim\limits_{x\to\pm\infty} \dfrac{1}{x^4} + \lim\limits_{x\to\pm\infty} 1}{\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2} - \lim\limits_{x\to\pm\infty} 1}$

$$= \frac{0+1}{0-1} = -1,$$ so $y=-1$ is a horizontal asymptote.



$y = f(x) = \dfrac{1+x^4}{x^2 - x^4} = \dfrac{1+x^4}{x^2(1-x^2)} = \dfrac{1+x^4}{x^2(1+x)(1-x)}$. The denominator is zero when $x = 0, -1$, and 1, but the numerator is nonzero, so $x = 0$, $x = -1$, and $x = 1$ are vertical asymptotes. Notice that as $x \to 0$, the numerator and denominator are both positive, so $\lim\limits_{x\to 0} f(x) = \infty$. The graph confirms our work.

51. $y = f(x) = \dfrac{x^3 - x}{x^2 - 6x + 5} = \dfrac{x(x^2-1)}{(x-1)(x-5)} = \dfrac{x(x+1)(x-1)}{(x-1)(x-5)} = \dfrac{x(x+1)}{x-5} = g(x)$ for $x \neq 1$.

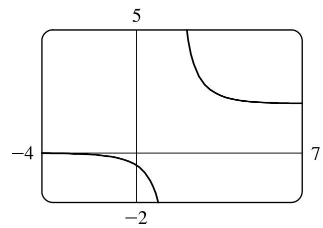


The graph of g is the same as the graph of f with the exception of a hole in the graph of f at $x = 1$. By long division, $g(x) = \dfrac{x^2+x}{x-5} = x + 6 + \dfrac{30}{x-5}$.

As $x \to \pm\infty$, $g(x) \to \pm\infty$, so there is no horizontal asymptote. The denominator of g is zero when $x = 5$. $\lim\limits_{x\to 5^-} g(x) = -\infty$ and $\lim\limits_{x\to 5^+} g(x) = \infty$, so $x = 5$ is a vertical asymptote. The graph confirms our work.

52. $\lim\limits_{x\to\infty} \dfrac{2e^x}{e^x - 5} = \lim\limits_{x\to\infty} \dfrac{2e^x}{e^x - 5} \cdot \dfrac{1/e^x}{1/e^x} = \lim\limits_{x\to\infty} \dfrac{2}{1 - (5/e^x)} = \dfrac{2}{1-0} = 2$, so $y = 2$ is a horizontal asymptote.

$\lim\limits_{x\to -\infty} \dfrac{2e^x}{e^x - 5} = \dfrac{2(0)}{0-5} = 0$, so $y = 0$ is a horizontal asymptote. The denominator is zero (and the numerator isn't) when $e^x - 5 = 0 \quad\Rightarrow\quad e^x = 5 \quad\Rightarrow\quad x = \ln 5$.

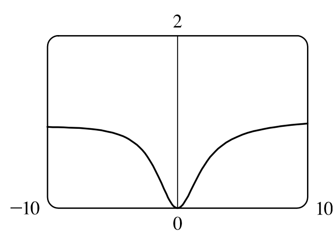


$\lim\limits_{x\to(\ln 5)^+} \dfrac{2e^x}{e^x - 5} = \infty$ since the numerator approaches 10 and the denominator approaches 0 through positive values as $x \to (\ln 5)^+$. Similarly,

$\lim\limits_{x\to(\ln 5)^-} \dfrac{2e^x}{e^x - 5} = -\infty$. Thus, $x = \ln 5$ is a vertical asymptote. The graph confirms our work.

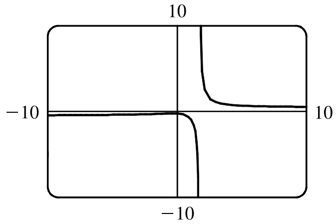
53. From the graph, it appears $y = 1$ is a horizontal asymptote.

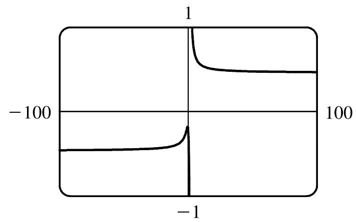
$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000} &= \lim_{x\to\pm\infty} \frac{\dfrac{3x^3 + 500x^2}{x^3}}{\dfrac{x^3 + 500x^2 + 100x + 2000}{x^3}} \\
&= \lim_{x\to\pm\infty} \frac{3 + (500/x)}{1 + (500/x) + (100/x^2) + (2000/x^3)} \\
&= \frac{3+0}{1+0+0+0} = 3, \quad \text{so } y = 3 \text{ is a horizontal asymptote.}
\end{aligned}$$



The discrepancy can be explained by the choice of the viewing window. Try $[-100{,}000, 100{,}000]$ by $[-1, 4]$ to get a graph that lends credibility to our calculation that $y = 3$ is a horizontal asymptote.

54. (a)





From the graph, it appears at first that there is only one horizontal asymptote, at $y \approx 0$, and a vertical asymptote at $x \approx 1.7$. However, if we graph the function with a wider and shorter viewing rectangle, we see that in fact there seem to be two horizontal asymptotes: one at $y \approx 0.5$ and one at $y \approx -0.5$. So we estimate that

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \approx 0.5 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \approx -0.5$$

(b) $f(1000) \approx 0.4722$ and $f(10{,}000) \approx 0.4715$, so we estimate that $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} \approx 0.47$.

$f(-1000) \approx -0.4706$ and $f(-10{,}000) \approx -0.4713$, so we estimate that $\lim\limits_{x\to-\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} \approx -0.47$.

(c) $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} = \lim\limits_{x\to\infty} \dfrac{\sqrt{2+1/x^2}}{3-5/x}$ $\quad$ [since $\sqrt{x^2} = x$ for $x > 0$] $\quad = \dfrac{\sqrt{2}}{3} \approx 0.471404$.

For $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the numerator by x, with $x < 0$, we get $\dfrac{1}{x}\sqrt{2x^2+1} = -\dfrac{1}{\sqrt{x^2}}\sqrt{2x^2+1} = -\sqrt{2+1/x^2}$. Therefore,

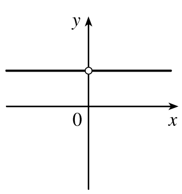
$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+1/x^2}}{3-5/x} = -\frac{\sqrt{2}}{3} \approx -0.471404.$$

55. Divide the numerator and the denominator by the highest power of x in $Q(x)$.

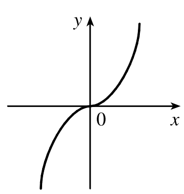
(a) If $\deg P < \deg Q$, then the numerator $\to 0$ but the denominator doesn't. So $\lim\limits_{x\to\infty} [P(x)/Q(x)] = 0$.

(b) If $\deg P > \deg Q$, then the numerator $\to \pm\infty$ but the denominator doesn't, so $\lim\limits_{x\to\infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

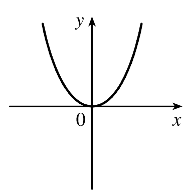
56.



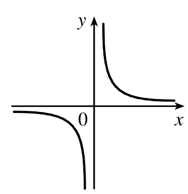
(i) $n = 0$



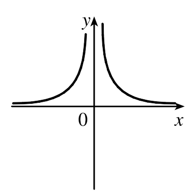
(ii) $n > 0$ (n odd)



(iii) $n > 0$ (n even)



(iv) $n < 0$ (n odd)



(v) $n < 0$ (n even)

From these sketches we see that

(a) $\lim\limits_{x\to 0^+} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ \infty & \text{if } n < 0 \end{cases}$ $\qquad$ (b) $\lim\limits_{x\to 0^-} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ -\infty & \text{if } n < 0,\ n \text{ odd} \\ \infty & \text{if } n < 0,\ n \text{ even} \end{cases}$

(c) $\lim\limits_{x\to\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \\ 0 & \text{if } n < 0 \end{cases}$ (d) $\lim\limits_{x\to-\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ -\infty & \text{if } n > 0,\ n \text{ odd} \\ \infty & \text{if } n > 0,\ n \text{ even} \\ 0 & \text{if } n < 0 \end{cases}$

57. Let's look for a rational function.

(1) $\lim\limits_{x\to\pm\infty} f(x) = 0 \quad\Rightarrow\quad$ degree of numerator $<$ degree of denominator

(2) $\lim\limits_{x\to 0} f(x) = -\infty \quad\Rightarrow\quad$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim\limits_{x\to 3^-} f(x) = \infty$ and $\lim\limits_{x\to 3^+} f(x) = -\infty \quad\Rightarrow\quad$ vertical asymptote at $x = 3$; there is a factor of $(x-3)$ in the denominator.

(4) $f(2) = 0 \quad\Rightarrow\quad$ 2 is an x-intercept; there is at least one factor of $(x-2)$ in the numerator.

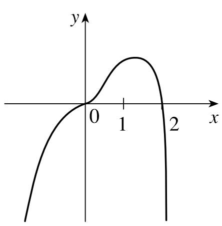
Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us $f(x) = \dfrac{2-x}{x^2(x-3)}$ as one possibility.

58. Since the function has vertical asymptotes $x = 1$ and $x = 3$, the denominator of the rational function we are looking for must have factors $(x-1)$ and $(x-3)$. Because the horizontal asymptote is $y = 1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility is $f(x) = \dfrac{x^2}{(x-1)(x-3)}$.

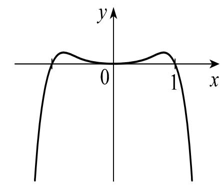
59. (a) We must first find the function f. Since f has a vertical asymptote $x = 4$ and x-intercept $x = 1$, $x - 4$ is a factor of the denominator and $x - 1$ is a factor of the numerator. There is a removable discontinuity at $x = -1$, so $x - (-1) = x + 1$ is a factor of both the numerator and denominator. Thus, f now looks like this: $f(x) = \dfrac{a(x-1)(x+1)}{(x-4)(x+1)}$, where a is still to be determined. Then $\lim\limits_{x\to-1} f(x) = \lim\limits_{x\to-1} \dfrac{a(x-1)(x+1)}{(x-4)(x+1)} = \lim\limits_{x\to-1} \dfrac{a(x-1)}{x-4} = \dfrac{a(-1-1)}{(-1-4)} = \dfrac{2}{5}a$, so $\dfrac{2}{5}a = 2$, and $a = 5$. Thus $f(x) = \dfrac{5(x-1)(x+1)}{(x-4)(x+1)}$ is a ratio of quadratic functions satisfying all the given conditions and $f(0) = \dfrac{5(-1)(1)}{(-4)(1)} = \dfrac{5}{4}$.

(b) $\lim\limits_{x\to\infty} f(x) = 5 \lim\limits_{x\to\infty} \dfrac{x^2-1}{x^2-3x-4} = 5 \lim\limits_{x\to\infty} \dfrac{(x^2/x^2)-(1/x^2)}{(x^2/x^2)-(3x/x^2)-(4/x^2)} = 5\dfrac{1-0}{1-0-0} = 5(1) = 5$

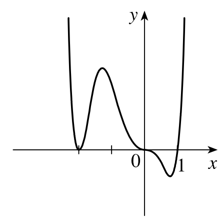
60. $y = f(x) = 2x^3 - x^4 = x^3(2-x)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0 and 2. There are sign changes at 0 and 2 (odd exponents on x and $2-x$). As $x\to\infty$, $f(x)\to-\infty$ because $x^3\to\infty$ and $2-x\to-\infty$. As $x\to-\infty$, $f(x)\to-\infty$ because $x^3\to-\infty$ and $2-x\to\infty$. Note that the graph of f near $x = 0$ flattens out (looks like $y = x^3$).



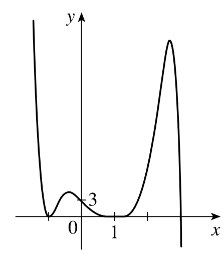
61. $y = f(x) = x^4 - x^6 = x^4(1 - x^2) = x^4(1 + x)(1 - x)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -1, and 1 [found by solving $f(x) = 0$ for x]. Since $x^4 > 0$ for $x \neq 0$, f doesn't change sign at $x = 0$. The function does change sign at $x = -1$ and $x = 1$. As $x \to \pm\infty$, $f(x) = x^4(1 - x^2)$ approaches $-\infty$ because $x^4 \to \infty$ and $(1 - x^2) \to -\infty$.



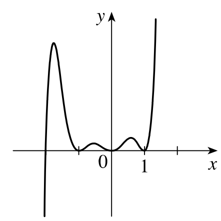
62. $y = f(x) = x^3(x + 2)^2(x - 1)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -2, and 1. There are sign changes at 0 and 1 (odd exponents on x and $x - 1$). There is no sign change at -2. Also, $f(x) \to \infty$ as $x \to \infty$ because all three factors are large. And $f(x) \to \infty$ as $x \to -\infty$ because $x^3 \to -\infty$, $(x + 2)^2 \to \infty$, and $(x - 1) \to -\infty$. Note that the graph of f at $x = 0$ flattens out (looks like $y = -x^3$).



63. $y = f(x) = (3 - x)(1 + x)^2(1 - x)^4$. The y-intercept is $f(0) = 3(1)^2(1)^4 = 3$. The x-intercepts are 3, -1, and 1. There is a sign change at 3, but not at -1 and 1. When x is large positive, $3 - x$ is negative and the other factors are positive, so $\lim\limits_{x\to\infty} f(x) = -\infty$. When x is large negative, $3 - x$ is positive, so $\lim\limits_{x\to-\infty} f(x) = \infty$.

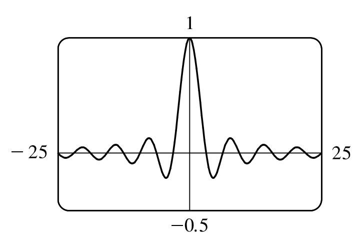


64. $y = f(x) = x^2(x^2 - 1)^2(x + 2) = x^2(x + 1)^2(x - 1)^2(x + 2)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -1, 1, and -2. There is a sign change at -2, but not at 0, -1, and 1. When x is large positive, all the factors are positive, so $\lim\limits_{x\to\infty} f(x) = \infty$. When x is large negative, only $x + 2$ is negative, so $\lim\limits_{x\to-\infty} f(x) = -\infty$.

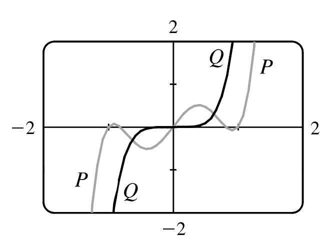


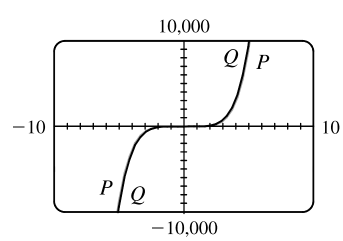
65. (a) Since $-1 \leq \sin x \leq 1$ for all x, $-\dfrac{1}{x} \leq \dfrac{\sin x}{x} \leq \dfrac{1}{x}$ for $x > 0$. As $x \to \infty$, $-1/x \to 0$ and $1/x \to 0$, so by the Squeeze Theorem, $(\sin x)/x \to 0$. Thus, $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$.

(b) From part (a), the horizontal asymptote is $y = 0$. The function $y = (\sin x)/x$ crosses the horizontal asymptote whenever $\sin x = 0$; that is, at $x = \pi n$ for every integer n. Thus, the graph crosses the asymptote *an infinite number of times.*



66. (a) In both viewing rectangles, $\lim\limits_{x\to\infty} P(x) = \lim\limits_{x\to\infty} Q(x) = \infty$ and $\lim\limits_{x\to-\infty} P(x) = \lim\limits_{x\to-\infty} Q(x) = -\infty$. In the larger viewing rectangle, P and Q become less distinguishable.





(b) $\lim\limits_{x\to\infty} \dfrac{P(x)}{Q(x)} = \lim\limits_{x\to\infty} \dfrac{3x^5 - 5x^3 + 2x}{3x^5} = \lim\limits_{x\to\infty} \left(1 - \dfrac{5}{3} \cdot \dfrac{1}{x^2} + \dfrac{2}{3} \cdot \dfrac{1}{x^4}\right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \quad \Rightarrow$

P and Q have the same end behavior.

67. $\lim\limits_{x\to\infty} \dfrac{5\sqrt{x}}{\sqrt{x-1}} \cdot \dfrac{1/\sqrt{x}}{1/\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{5}{\sqrt{1-(1/x)}} = \dfrac{5}{\sqrt{1-0}} = 5$ and

$\lim\limits_{x\to\infty} \dfrac{10e^x - 21}{2e^x} \cdot \dfrac{1/e^x}{1/e^x} = \lim\limits_{x\to\infty} \dfrac{10 - (21/e^x)}{2} = \dfrac{10-0}{2} = 5$. Since $\dfrac{10e^x - 21}{2e^x} < f(x) < \dfrac{5\sqrt{x}}{\sqrt{x-1}}$,

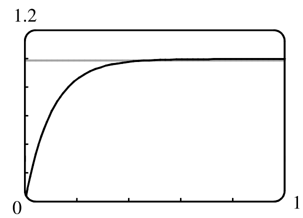
we have $\lim\limits_{x\to\infty} f(x) = 5$ by the Squeeze Theorem.

68. (a) After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000 + 25t)$ liters of water and $25t \cdot 30 = 750t$ grams of salt. Therefore, the salt concentration at time t will be

$C(t) = \dfrac{750t}{5000 + 25t} = \dfrac{30t}{200 + t}\ \dfrac{\text{g}}{\text{L}}$.

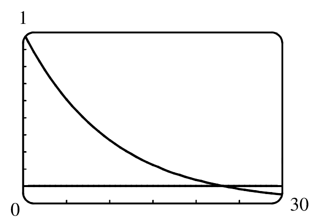
(b) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty} \dfrac{30t}{200+t} = \lim\limits_{t\to\infty} \dfrac{30t/t}{200/t + t/t} = \dfrac{30}{0+1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

69. (a) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty} v^*\left(1 - e^{-gt/v^*}\right) = v^*(1-0) = v^*$



(b) We graph $v(t) = 1 - e^{-9.8t}$ and $v(t) = 0.99v^*$, or in this case, $v(t) = 0.99$. Using an intersect feature or zooming in on the point of intersection, we find that $t \approx 0.47$ s.

70. (a) $y = e^{-x/10}$ and $y = 0.1$ intersect at $x_1 \approx 23.03$.

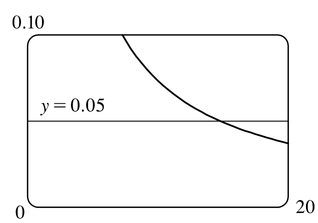


If $x > x_1$, then $e^{-x/10} < 0.1$.

(b) $e^{-x/10} < 0.1 \quad \Rightarrow \quad -x/10 < \ln 0.1 \quad \Rightarrow$

$x > -10 \ln \frac{1}{10} = -10 \ln 10^{-1} = 10 \ln 10 \approx 23.03$

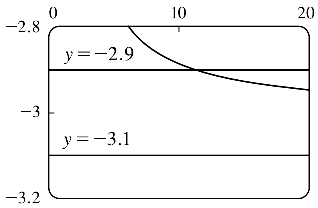
71. Let $g(x) = \dfrac{3x^2 + 1}{2x^2 + x + 1}$ and $f(x) = |g(x) - 1.5|$. Note that

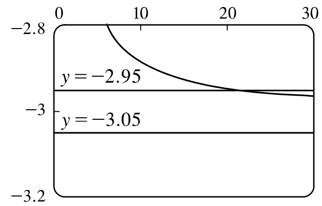


$\lim\limits_{x\to\infty} g(x) = \frac{3}{2}$ and $\lim\limits_{x\to\infty} f(x) = 0$. We are interested in finding the x-value at which $f(x) < 0.05$. From the graph, we find that $x \approx 14.804$, so we choose $N = 15$ (or any larger number).

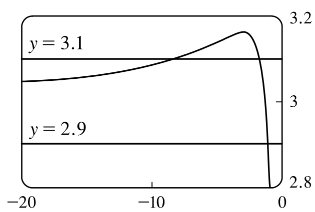
72. We want to find a value of N such that $x > N \quad \Rightarrow \quad \left|\dfrac{1-3x}{\sqrt{x^2+1}} - (-3)\right| < \varepsilon$, or equivalently,

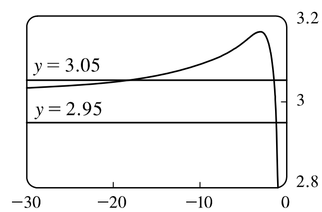
$-3 - \varepsilon < \dfrac{1-3x}{\sqrt{x^2+1}} < -3 + \varepsilon$. When $\varepsilon = 0.1$, we graph $y = f(x) = \dfrac{1-3x}{\sqrt{x^2+1}}$, $y = -3.1$, and $y = -2.9$. From the graph, we find that $f(x) = -2.9$ at about $x = 11.283$, so we choose $N = 12$ (or any larger number). Similarly for $\varepsilon = 0.05$, we find that $f(x) = -2.95$ at about $x = 21.379$, so we choose $N = 22$ (or any larger number).



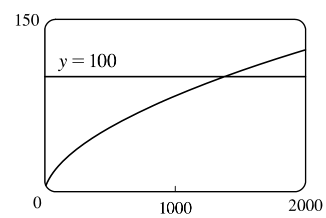


73. We want a value of N such that $x < N \Rightarrow \left|\dfrac{1-3x}{\sqrt{x^2+1}} - 3\right| < \varepsilon$, or equivalently, $3 - \varepsilon < \dfrac{1-3x}{\sqrt{x^2+1}} < 3 + \varepsilon$. When $\varepsilon = 0.1$, we graph $y = f(x) = \dfrac{1-3x}{\sqrt{x^2+1}}$, $y = 3.1$, and $y = 2.9$. From the graph, we find that $f(x) = 3.1$ at about $x = -8.092$, so we choose $N = -9$ (or any lesser number). Similarly for $\varepsilon = 0.05$, we find that $f(x) = 3.05$ at about $x = -18.338$, so we choose $N = -19$ (or any lesser number).





74. We want to find a value of N such that $x > N \quad \Rightarrow \quad \sqrt{x \ln x} > 100$. We graph $y = f(x) = \sqrt{x \ln x}$ and $y = 100$. From the graph, we find that $f(x) = 100$ at about $x = 1382.773$, so we choose $N = 1383$ (or any larger number).



75. (a) $1/x^2 < 0.0001 \quad \Leftrightarrow \quad x^2 > 1/0.0001 = 10\,000 \quad \Leftrightarrow \quad x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \quad \Leftrightarrow \quad x^2 > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\sqrt{\varepsilon}} \quad \Rightarrow \quad \left|\dfrac{1}{x^2} - 0\right| = \dfrac{1}{x^2} < \varepsilon$, so $\lim\limits_{x \to \infty} \dfrac{1}{x^2} = 0$.

76. (a) $1/\sqrt{x} < 0.0001 \quad \Leftrightarrow \quad \sqrt{x} > 1/0.0001 = 10^4 \quad \Leftrightarrow \quad x > 10^8$

(b) If $\varepsilon > 0$ is given, then $1/\sqrt{x} < \varepsilon \quad \Leftrightarrow \quad \sqrt{x} > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\varepsilon^2$. Let $N = 1/\varepsilon^2$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\varepsilon^2} \quad \Rightarrow \quad \left|\dfrac{1}{\sqrt{x}} - 0\right| = \dfrac{1}{\sqrt{x}} < \varepsilon$, so $\lim\limits_{x \to \infty} \dfrac{1}{\sqrt{x}} = 0$.

77. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \quad \Leftrightarrow \quad x < -1/\varepsilon$.
Take $N = -1/\varepsilon$. Then $x < N \quad \Rightarrow \quad x < -1/\varepsilon \quad \Rightarrow \quad |(1/x) - 0| = -1/x < \varepsilon$, so $\lim\limits_{x \to -\infty} (1/x) = 0$.

78. Given $M > 0$, we need $N > 0$ such that $x > N \quad \Rightarrow \quad x^3 > M$. Now $x^3 > M \quad \Leftrightarrow \quad x > \sqrt[3]{M}$, so take $N = \sqrt[3]{M}$. Then $x > N = \sqrt[3]{M} \quad \Rightarrow \quad x^3 > M$, so $\lim\limits_{x \to \infty} x^3 = \infty$.

79. Given $M > 0$, we need $N > 0$ such that $x > N \Rightarrow e^x > M$. Now $e^x > M \Leftrightarrow x > \ln M$, so take $N = \max(1, \ln M)$. (This ensures that $N > 0$.) Then $x > N = \max(1, \ln M) \Rightarrow e^x > \max(e, M) \geq M$, so $\lim_{x \to \infty} e^x = \infty$.

80. Definition Let f be a function defined on some interval $(-\infty, a)$. Then $\lim_{x \to -\infty} f(x) = -\infty$ means that for every negative number M there is a corresponding negative number N such that $f(x) < M$ whenever $x < N$. Now we use the definition to prove that $\lim_{x \to -\infty} (1 + x^3) = -\infty$. Given a negative number M, we need a negative number N such that $x < N \Rightarrow 1 + x^3 < M$. Now $1 + x^3 < M \Leftrightarrow x^3 < M - 1 \Leftrightarrow x < \sqrt[3]{M-1}$. Thus, we take $N = \sqrt[3]{M-1}$ and find that $x < N \Rightarrow 1 + x^3 < M$. This proves that $\lim_{x \to -\infty} (1 + x^3) = -\infty$.

81. (a) Suppose that $\lim_{x \to \infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \Leftrightarrow 0 < 1/x < 1/N \Leftrightarrow 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that $\lim_{t \to 0^+} f(1/t) = L = \lim_{x \to \infty} f(x)$.

Now suppose that $\lim_{x \to -\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \Leftrightarrow 1/N < 1/x < 0 \Leftrightarrow 1/N < t < 0$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\lim_{t \to 0^-} f(1/t) = L = \lim_{x \to -\infty} f(x)$.

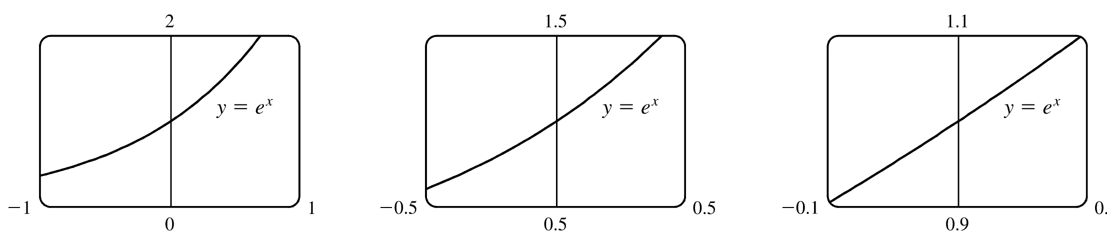
(b)
$$\begin{aligned}
\lim_{x \to 0^+} x \sin \frac{1}{x} &= \lim_{t \to 0^+} t \sin \frac{1}{t} && [\text{let } x = t] \\
&= \lim_{y \to \infty} \frac{1}{y} \sin y && [\text{part (a) with } y = 1/t] \\
&= \lim_{x \to \infty} \frac{\sin x}{x} && [\text{let } y = x] \\
&= 0 && [\text{by Exercise 65}]
\end{aligned}$$

2.7 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$.

2. The curve looks more like a line as the viewing rectangle gets smaller.



3. (a) (i) Using Definition 1 with $f(x) = 4x - x^2$ and $P(1, 3)$,

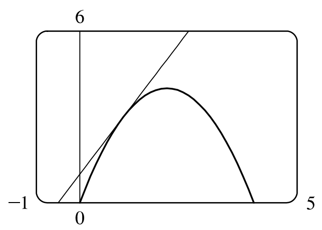
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{(4x - x^2) - 3}{x - 1} = \lim_{x \to 1} \frac{-(x^2 - 4x + 3)}{x - 1} = \lim_{x \to 1} \frac{-(x - 1)(x - 3)}{x - 1}$$
$$= \lim_{x \to 1} (3 - x) = 3 - 1 = 2$$

(ii) Using Equation 2 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[4(1 + h) - (1 + h)^2\right] - 3}{h}$$
$$= \lim_{h \to 0} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h \to 0} \frac{-h^2 + 2h}{h} = \lim_{h \to 0} \frac{h(-h + 2)}{h} = \lim_{h \to 0} (-h + 2) = 2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \;\Rightarrow\; y - f(1) = f'(1)(x - 1) \;\Rightarrow\; y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(c)



The graph of $y = 2x + 1$ is tangent to the graph of $y = 4x - x^2$ at the point $(1, 3)$. Now zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistiguishable.

4. (a) (i) Using Definition 1 with $f(x) = x - x^3$ and $P(1, 0)$,

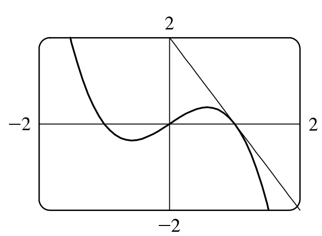
$$m = \lim_{x \to 1} \frac{f(x) - 0}{x - 1} = \lim_{x \to 1} \frac{x - x^3}{x - 1} = \lim_{x \to 1} \frac{x(1 - x^2)}{x - 1} = \lim_{x \to 1} \frac{x(1 + x)(1 - x)}{x - 1}$$
$$= \lim_{x \to 1} [-x(1 + x)] = -1(2) = -2$$

(ii) Using Equation 2 with $f(x) = x - x^3$ and $P(1, 0)$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[(1 + h) - (1 + h)^3\right] - 0}{h}$$
$$= \lim_{h \to 0} \frac{1 + h - (1 + 3h + 3h^2 + h^3)}{h} = \lim_{h \to 0} \frac{-h^3 - 3h^2 - 2h}{h} = \lim_{h \to 0} \frac{h(-h^2 - 3h - 2)}{h}$$
$$= \lim_{h \to 0} (-h^2 - 3h - 2) = -2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \;\Rightarrow\; y - f(1) = f'(1)(x - 1) \;\Rightarrow\; y - 0 = -2(x - 1)$, or $y = -2x + 2$.

(c)



The graph of $y = -2x + 2$ is tangent to the graph of $y = x - x^3$ at the point $(1, 0)$. Now zoom in toward the point $(1, 0)$ until the cubic and the tangent line are indistinguishable.

5. Using (1) with $f(x) = 4x - 3x^2$ and $P(2, -4)$ [we could also use (2)],

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 2} \frac{(4x - 3x^2) - (-4)}{x - 2} = \lim_{x \to 2} \frac{-3x^2 + 4x + 4}{x - 2}$$
$$= \lim_{x \to 2} \frac{(-3x - 2)(x - 2)}{x - 2} = \lim_{x \to 2}(-3x - 2) = -3(2) - 2 = -8$$

Tangent line: $y - (-4) = -8(x - 2) \quad \Leftrightarrow \quad y + 4 = -8x + 16 \quad \Leftrightarrow \quad y = -8x + 12$.

6. Using (2) with $f(x) = x^3 - 3x + 1$ and $P(2, 3)$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{(2 + h)^3 - 3(2 + h) + 1 - 3}{h}$$
$$= \lim_{h \to 0} \frac{8 + 12h + 6h^2 + h^3 - 6 - 3h - 2}{h} = \lim_{h \to 0} \frac{9h + 6h^2 + h^3}{h} = \lim_{h \to 0} \frac{h(9 + 6h + h^2)}{h}$$
$$= \lim_{h \to 0}(9 + 6h + h^2) = 9$$

Tangent line: $y - 3 = 9(x - 2) \quad \Leftrightarrow \quad y - 3 = 9x - 18 \quad \Leftrightarrow \quad y = 9x - 15$

7. Using (1), $m = \lim\limits_{x \to 1} \dfrac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim\limits_{x \to 1} \dfrac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim\limits_{x \to 1} \dfrac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim\limits_{x \to 1} \dfrac{1}{\sqrt{x} + 1} = \dfrac{1}{2}$.

Tangent line: $y - 1 = \frac{1}{2}(x - 1) \quad \Leftrightarrow \quad y = \frac{1}{2}x + \frac{1}{2}$

8. Using (1) with $f(x) = \dfrac{2x + 1}{x + 2}$ and $P(1, 1)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{\dfrac{2x + 1}{x + 2} - 1}{x - 1} = \lim_{x \to 1} \frac{\dfrac{2x + 1 - (x + 2)}{x + 2}}{x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x + 2)}$$
$$= \lim_{x \to 1} \frac{1}{x + 2} = \frac{1}{1 + 2} = \frac{1}{3}$$

Tangent line: $y - 1 = \frac{1}{3}(x - 1) \quad \Leftrightarrow \quad y - 1 = \frac{1}{3}x - \frac{1}{3} \quad \Leftrightarrow \quad y = \frac{1}{3}x + \frac{2}{3}$

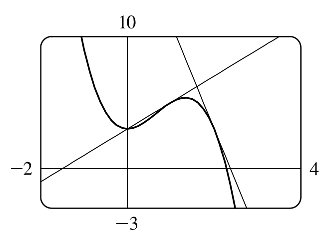
9. (a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{3 + 4(a + h)^2 - 2(a + h)^3 - (3 + 4a^2 - 2a^3)}{h}$$
$$= \lim_{h \to 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h}$$
$$= \lim_{h \to 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h}$$
$$= \lim_{h \to 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h} = \lim_{h \to 0} \frac{h(8a + 4h - 6a^2 - 6ah - 2h^2)}{h}$$
$$= \lim_{h \to 0}(8a + 4h - 6a^2 - 6ah - 2h^2) = 8a - 6a^2$$

(b) At $(1, 5)$: $m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is $y - 5 = 2(x - 1) \quad \Leftrightarrow \quad y = 2x + 3$.

At $(2, 3)$: $m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent line is $y - 3 = -8(x - 2) \quad \Leftrightarrow \quad y = -8x + 19$.

(c)



10. (a) Using (1),

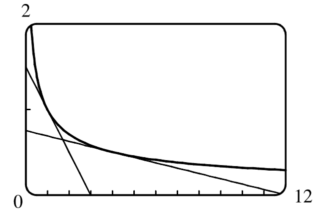
$$m = \lim_{x \to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} = \lim_{x \to a} \frac{\dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}}}{x - a} = \lim_{x \to a} \frac{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})}{\sqrt{ax}\,(x - a)(\sqrt{a} + \sqrt{x})} = \lim_{x \to a} \frac{a - x}{\sqrt{ax}\,(x - a)(\sqrt{a} + \sqrt{x})}$$

$$= \lim_{x \to a} \frac{-1}{\sqrt{ax}\,(\sqrt{a} + \sqrt{x})} = \frac{-1}{\sqrt{a^2}\,(2\sqrt{a})} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2} \quad [a > 0]$$

(b) At $(1, 1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 1) \quad \Leftrightarrow \quad y = -\frac{1}{2}x + \frac{3}{2}$.

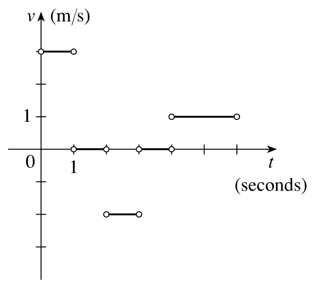
At $\left(4, \frac{1}{2}\right)$: $m = -\frac{1}{16}$, so an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x - 4) \quad \Leftrightarrow \quad y = -\frac{1}{16}x + \frac{3}{4}$.

(c)



11. (a) The particle is moving to the right when s is increasing; that is, on the intervals $(0, 1)$ and $(4, 6)$. The particle is moving to the left when s is decreasing; that is, on the interval $(2, 3)$. The particle is standing still when s is constant; that is, on the intervals $(1, 2)$ and $(3, 4)$.

(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval $(0, 1)$, the slope is $\dfrac{3 - 0}{1 - 0} = 3$. On the interval $(2, 3)$, the slope is $\dfrac{1 - 3}{3 - 2} = -2$. On the interval $(4, 6)$, the slope is $\dfrac{3 - 1}{6 - 4} = 1$.



12. (a) **Runner A** runs the entire 100-meter race at the same velocity since the slope of the position function is constant. **Runner B** starts the race at a slower velocity than runner A, but finishes the race at a faster velocity.

(b) The distance between the runners is the greatest at the time when the largest vertical line segment fits between the two graphs—this appears to be somewhere between 9 and 10 seconds.

(c) The runners had the same velocity when the slopes of their respective position functions are equal—this also appears to be at about 9.5 s. Note that the answers for parts (b) and (c) must be the same for these graphs because as soon as the velocity for runner B overtakes the velocity for runner A, the distance between the runners starts to decrease.

13. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \to 2} \frac{\left(40t - 16t^2\right) - 16}{t - 2} = \lim_{t \to 2} \frac{-16t^2 + 40t - 16}{t - 2} = \lim_{t \to 2} \frac{-8\left(2t^2 - 5t + 2\right)}{t - 2}$$

$$= \lim_{t \to 2} \frac{-8(t - 2)(2t - 1)}{t - 2} = -8 \lim_{t \to 2}(2t - 1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

14. (a) Let $H(t) = 10t - 1.86t^2$.

$$\begin{aligned} v(1) &= \lim_{h\to 0} \frac{H(1+h) - H(1)}{h} = \lim_{h\to 0} \frac{\left[10(1+h) - 1.86(1+h)^2\right] - (10 - 1.86)}{h} \\ &= \lim_{h\to 0} \frac{10 + 10h - 1.86(1 + 2h + h^2) - 10 + 1.86}{h} \\ &= \lim_{h\to 0} \frac{10 + 10h - 1.86 - 3.72h - 1.86h^2 - 10 + 1.86}{h} \\ &= \lim_{h\to 0} \frac{6.28h - 1.86h^2}{h} = \lim_{h\to 0}(6.28 - 1.86h) = 6.28 \end{aligned}$$

The velocity of the rock after one second is 6.28 m/s.

(b) $$\begin{aligned} v(a) &= \lim_{h\to 0} \frac{H(a+h) - H(a)}{h} = \lim_{h\to 0} \frac{\left[10(a+h) - 1.86(a+h)^2\right] - (10a - 1.86a^2)}{h} \\ &= \lim_{h\to 0} \frac{10a + 10h - 1.86(a^2 + 2ah + h^2) - 10a + 1.86a^2}{h} \\ &= \lim_{h\to 0} \frac{10a + 10h - 1.86a^2 - 3.72ah - 1.86h^2 - 10a + 1.86a^2}{h} = \lim_{h\to 0} \frac{10h - 3.72ah - 1.86h^2}{h} \\ &= \lim_{h\to 0} \frac{h(10 - 3.72a - 1.86h)}{h} = \lim_{h\to 0}(10 - 3.72a - 1.86h) = 10 - 3.72a \end{aligned}$$

The velocity of the rock when $t = a$ is $(10 - 3.72a)$ m/s.

(c) The rock will hit the surface when $H = 0 \iff 10t - 1.86t^2 = 0 \iff t(10 - 1.86t) = 0 \iff t = 0$ or $1.86t = 10$.

The rock hits the surface when $t = 10/1.86 \approx 5.4$ s.

(d) The velocity of the rock when it hits the surface is $v\left(\frac{10}{1.86}\right) = 10 - 3.72\left(\frac{10}{1.86}\right) = 10 - 20 = -10$ m/s.

15. $$\begin{aligned} v(a) &= \lim_{h\to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\dfrac{a^2 - (a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2} \\ &= \lim_{h\to 0} \frac{-(2ah + h^2)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-h(2a+h)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-(2a+h)}{a^2(a+h)^2} = \frac{-2a}{a^2 \cdot a^2} = \frac{-2}{a^3} \text{ m/s} \end{aligned}$$

So $v(1) = \dfrac{-2}{1^3} = -2$ m/s, $v(2) = \dfrac{-2}{2^3} = -\dfrac{1}{4}$ m/s, and $v(3) = \dfrac{-2}{3^3} = -\dfrac{2}{27}$ m/s.

16. (a) The average velocity between times t and $t + h$ is

$$\begin{aligned} \frac{s(t+h) - s(t)}{(t+h) - t} &= \frac{\frac{1}{2}(t+h)^2 - 6(t+h) + 23 - \left(\frac{1}{2}t^2 - 6t + 23\right)}{h} \\ &= \frac{\frac{1}{2}t^2 + th + \frac{1}{2}h^2 - 6t - 6h + 23 - \frac{1}{2}t^2 + 6t - 23}{h} \\ &= \frac{th + \frac{1}{2}h^2 - 6h}{h} = \frac{h\left(t + \frac{1}{2}h - 6\right)}{h} = \left(t + \tfrac{1}{2}h - 6\right) \text{ ft/s} \end{aligned}$$

(i) $[4, 8]$: $t = 4$, $h = 8 - 4 = 4$, so the average velocity is $4 + \frac{1}{2}(4) - 6 = 0$ ft/s.

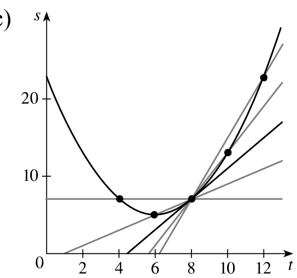
(ii) $[6, 8]$: $t = 6$, $h = 8 - 6 = 2$, so the average velocity is $6 + \frac{1}{2}(2) - 6 = 1$ ft/s.

(iii) $[8, 10]$: $t = 8$, $h = 10 - 8 = 2$, so the average velocity is $8 + \frac{1}{2}(2) - 6 = 3$ ft/s.

(iv) $[8, 12]$: $t = 8$, $h = 12 - 8 = 4$, so the average velocity is $8 + \frac{1}{2}(4) - 6 = 4$ ft/s.

(b) $v(t) = \lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h} = \lim\limits_{h \to 0} \left(t + \frac{1}{2}h - 6\right)$

$= t - 6, \quad \text{so } v(8) = 2 \text{ ft/s}.$

(c) 

17. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

18. (a) On $[20, 60]$: $\dfrac{f(60) - f(20)}{60 - 20} = \dfrac{700 - 300}{40} = \dfrac{400}{40} = 10$

(b) Pick any interval that has the same y-value at its endpoints. $[0, 57]$ is such an interval since $f(0) = 600$ and $f(57) = 600$.

(c) On $[40, 60]$: $\dfrac{f(60) - f(40)}{60 - 40} = \dfrac{700 - 200}{20} = \dfrac{500}{20} = 25$

On $[40, 70]$: $\dfrac{f(70) - f(40)}{70 - 40} = \dfrac{900 - 200}{30} = \dfrac{700}{30} = 23\frac{1}{3}$

Since $25 > 23\frac{1}{3}$, the average rate of change on $[40, 60]$ is larger.

(d) $\dfrac{f(40) - f(10)}{40 - 10} = \dfrac{200 - 400}{30} = \dfrac{-200}{30} = -6\frac{2}{3}$

This value represents the slope of the line segment from $(10, f(10))$ to $(40, f(40))$.

19. (a) The tangent line at $x = 50$ appears to pass through the points $(43, 200)$ and $(60, 640)$, so

$f'(50) \approx \dfrac{640 - 200}{60 - 43} = \dfrac{440}{17} \approx 26.$

(b) The tangent line at $x = 10$ is steeper than the tangent line at $x = 30$, so it is larger in magnitude, but less in numerical value, that is, $f'(10) < f'(30)$.

(c) The slope of the tangent line at $x = 60$, $f'(60)$, is greater than the slope of the line through $(40, f(40))$ and $(80, f(80))$.

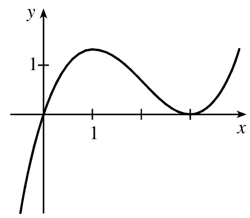
So yes, $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$.

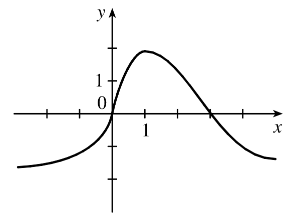
20. Since $g(5) = -3$, the point $(5, -3)$ is on the graph of g. Since $g'(5) = 4$, the slope of the tangent line at $x = 5$ is 4. Using the point-slope form of a line gives us $y - (-3) = 4(x - 5)$, or $y = 4x - 23$.

21. For the tangent line $y = 4x - 5$: when $x = 2$, $y = 4(2) - 5 = 3$ and its slope is 4 (the coefficient of x). At the point of tangency, these values are shared with the curve $y = f(x)$; that is, $f(2) = 3$ and $f'(2) = 4$.

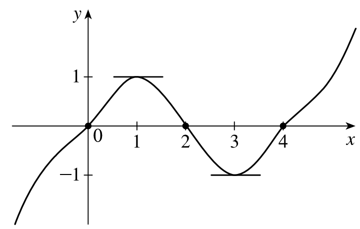
22. Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

23. We begin by drawing a curve through the origin with a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Last, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.

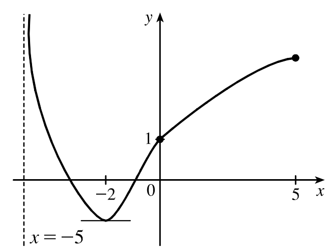




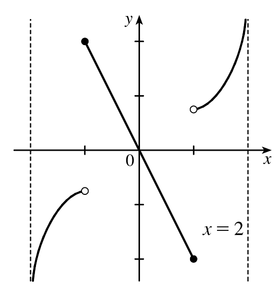
24. We begin by drawing a curve through the origin with a slope of 1 to satisfy $g(0) = 0$ and $g'(0) = 1$. We round off our figure at $x = 1$ to satisfy $g'(1) = 0$, and then pass through $(2, 0)$ with slope -1 to satisfy $g(2) = 0$ and $g'(2) = -1$. We round the figure at $x = 3$ to satisfy $g'(3) = 0$, and then pass through $(4, 0)$ with slope 1 to satisfy $g(4) = 0$ and $g'(4) = 1$. Finally we extend the curve on both ends to satisfy $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to-\infty} g(x) = -\infty$.



25. We begin by drawing a curve through $(0, 1)$ with a slope of 1 to satisfy $g(0) = 1$ and $g'(0) = 1$. We round off our figure at $x = -2$ to satisfy $g'(-2) = 0$. As $x \to -5^+$, $y \to \infty$, so we draw a vertical asymptote at $x = -5$. As $x \to 5^-$, $y \to 3$, so we draw a dot at $(5, 3)$ [the dot could be open or closed].



26. We begin by drawing an odd function (symmetric with respect to the origin) through the origin with slope -2 to satisfy $f'(0) = -2$. Now draw a curve starting at $x = 1$ and increasing without bound as $x \to 2^-$ since $\lim_{x\to 2^-} f(x) = \infty$. Lastly, reflect the last curve through the origin (rotate 180°) since f is an odd function.



27. Using (4) with $f(x) = 3x^2 - x^3$ and $a = 1$,

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{[3(1+h)^2 - (1+h)^3] - 2}{h} \\ &= \lim_{h\to 0} \frac{(3 + 6h + 3h^2) - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h\to 0} \frac{3h - h^3}{h} = \lim_{h\to 0} \frac{h(3 - h^2)}{h} \\ &= \lim_{h\to 0} (3 - h^2) = 3 - 0 = 3 \end{aligned}$$

Tangent line: $y - 2 = 3(x - 1) \quad\Leftrightarrow\quad y - 2 = 3x - 3 \quad\Leftrightarrow\quad y = 3x - 1$

28. Using (5) with $g(x) = x^4 - 2$ and $a = 1$,

$$g'(1) = \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1} \frac{(x^4 - 2) - (-1)}{x - 1} = \lim_{x\to 1} \frac{x^4 - 1}{x - 1} = \lim_{x\to 1} \frac{(x^2 + 1)(x^2 - 1)}{x - 1}$$
$$= \lim_{x\to 1} \frac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = \lim_{x\to 1} [(x^2 + 1)(x + 1)] = 2(2) = 4$$

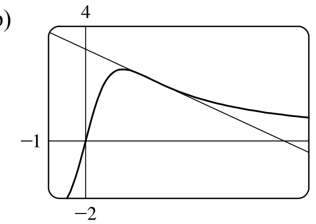
Tangent line: $y - (-1) = 4(x - 1) \quad \Leftrightarrow \quad y + 1 = 4x - 4 \quad \Leftrightarrow \quad y = 4x - 5$

29. (a) Using (4) with $F(x) = 5x/(1 + x^2)$ and the point $(2, 2)$, we have

$$F'(2) = \lim_{h\to 0} \frac{F(2 + h) - F(2)}{h} = \lim_{h\to 0} \frac{\dfrac{5(2 + h)}{1 + (2 + h)^2} - 2}{h}$$
$$= \lim_{h\to 0} \frac{\dfrac{5h + 10}{h^2 + 4h + 5} - 2}{h} = \lim_{h\to 0} \frac{\dfrac{5h + 10 - 2(h^2 + 4h + 5)}{h^2 + 4h + 5}}{h}$$
$$= \lim_{h\to 0} \frac{-2h^2 - 3h}{h(h^2 + 4h + 5)} = \lim_{h\to 0} \frac{h(-2h - 3)}{h(h^2 + 4h + 5)} = \lim_{h\to 0} \frac{-2h - 3}{h^2 + 4h + 5} = \frac{-3}{5}$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = -\frac{3}{5}(x - 2)$ or $y = -\frac{3}{5}x + \frac{16}{5}$.

(b)

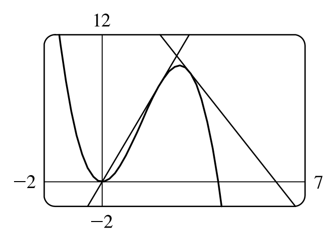


30. (a) Using (4) with $G(x) = 4x^2 - x^3$, we have

$$G'(a) = \lim_{h\to 0} \frac{G(a + h) - G(a)}{h} = \lim_{h\to 0} \frac{[4(a + h)^2 - (a + h)^3] - (4a^2 - a^3)}{h}$$
$$= \lim_{h\to 0} \frac{4a^2 + 8ah + 4h^2 - (a^3 + 3a^2h + 3ah^2 + h^3) - 4a^2 + a^3}{h}$$
$$= \lim_{h\to 0} \frac{8ah + 4h^2 - 3a^2h - 3ah^2 - h^3}{h} = \lim_{h\to 0} \frac{h(8a + 4h - 3a^2 - 3ah - h^2)}{h}$$
$$= \lim_{h\to 0} (8a + 4h - 3a^2 - 3ah - h^2) = 8a - 3a^2$$

At the point $(2, 8)$, $G'(2) = 16 - 12 = 4$, and an equation of the tangent line is $y - 8 = 4(x - 2)$, or $y = 4x$. At the point $(3, 9)$, $G'(3) = 24 - 27 = -3$, and an equation of the tangent line is $y - 9 = -3(x - 3)$, or $y = -3x + 18$.

(b)



31. Use (4) with $f(x) = 3x^2 - 4x + 1$.

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h\to 0} \frac{[3(a + h)^2 - 4(a + h) + 1] - (3a^2 - 4a + 1)]}{h}$$
$$= \lim_{h\to 0} \frac{3a^2 + 6ah + 3h^2 - 4a - 4h + 1 - 3a^2 + 4a - 1}{h} = \lim_{h\to 0} \frac{6ah + 3h^2 - 4h}{h}$$
$$= \lim_{h\to 0} \frac{h(6a + 3h - 4)}{h} = \lim_{h\to 0} (6a + 3h - 4) = 6a - 4$$

32. Use (4) with $f(t) = 2t^3 + t$.

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{[2(a+h)^3 + (a+h)] - (2a^3 + a)}{h} \\ &= \lim_{h \to 0} \frac{2a^3 + 6a^2h + 6ah^2 + 2h^3 + a + h - 2a^3 - a}{h} = \lim_{h \to 0} \frac{6a^2h + 6ah^2 + 2h^3 + h}{h} \\ &= \lim_{h \to 0} \frac{h(6a^2 + 6ah + 2h^2 + 1)}{h} = \lim_{h \to 0} (6a^2 + 6ah + 2h^2 + 1) = 6a^2 + 1 \end{aligned}$$

33. Use (4) with $f(t) = (2t+1)/(t+3)$.

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\dfrac{2(a+h)+1}{(a+h)+3} - \dfrac{2a+1}{a+3}}{h} \\ &= \lim_{h \to 0} \frac{(2a+2h+1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)} \\ &= \lim_{h \to 0} \frac{(2a^2 + 6a + 2ah + 6h + a + 3) - (2a^2 + 2ah + 6a + a + h + 3)}{h(a+h+3)(a+3)} \\ &= \lim_{h \to 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h \to 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2} \end{aligned}$$

34. Use (4) with $f(x) = x^{-2} = 1/x^2$.

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h \to 0} \frac{\dfrac{a^2 - (a+h)^2}{a^2(a+h)^2}}{h} \\ &= \lim_{h \to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2} = \lim_{h \to 0} \frac{-2ah - h^2}{ha^2(a+h)^2} = \lim_{h \to 0} \frac{h(-2a-h)}{ha^2(a+h)^2} \\ &= \lim_{h \to 0} \frac{-2a-h}{a^2(a+h)^2} = \frac{-2a}{a^2(a^2)} = \frac{-2}{a^3} \end{aligned}$$

35. Use (4) with $f(x) = \sqrt{1-2x}$.

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h} \cdot \frac{\sqrt{1-2(a+h)} + \sqrt{1-2a}}{\sqrt{1-2(a+h)} + \sqrt{1-2a}} \\ &= \lim_{h \to 0} \frac{\left(\sqrt{1-2(a+h)}\right)^2 - \left(\sqrt{1-2a}\right)^2}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} = \lim_{h \to 0} \frac{(1-2a-2h) - (1-2a)}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} \\ &= \lim_{h \to 0} \frac{-2h}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} = \lim_{h \to 0} \frac{-2}{\sqrt{1-2(a+h)} + \sqrt{1-2a}} \\ &= \frac{-2}{\sqrt{1-2a} + \sqrt{1-2a}} = \frac{-2}{2\sqrt{1-2a}} = \frac{-1}{\sqrt{1-2a}} \end{aligned}$$

36. Use (4) with $f(x) = \dfrac{4}{\sqrt{1-x}}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{4}{\sqrt{1-(a+h)}} - \dfrac{4}{\sqrt{1-a}}}{h}$$

$$= 4\lim_{h\to 0} \frac{\dfrac{\sqrt{1-a}-\sqrt{1-a-h}}{\sqrt{1-a-h}\,\sqrt{1-a}}}{h} = 4\lim_{h\to 0} \frac{\sqrt{1-a}-\sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}}$$

$$= 4\lim_{h\to 0} \frac{\sqrt{1-a}-\sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}} \cdot \frac{\sqrt{1-a}+\sqrt{1-a-h}}{\sqrt{1-a}+\sqrt{1-a-h}} = 4\lim_{h\to 0} \frac{(\sqrt{1-a})^2-(\sqrt{1-a-h})^2}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$= 4\lim_{h\to 0} \frac{(1-a)-(1-a-h)}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})} = 4\lim_{h\to 0} \frac{h}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$= 4\lim_{h\to 0} \frac{1}{\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})} = 4\cdot\frac{1}{\sqrt{1-a}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a})}$$

$$= \frac{4}{(1-a)(2\sqrt{1-a})} = \frac{2}{(1-a)^1(1-a)^{1/2}} = \frac{2}{(1-a)^{3/2}}$$

37. By (4), $\displaystyle\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} = f'(9)$, where $f(x) = \sqrt{x}$ and $a = 9$.

38. By (4), $\displaystyle\lim_{h\to 0} \frac{e^{-2+h}-e^{-2}}{h} = f'(-2)$, where $f(x) = e^x$ and $a = -2$.

39. By Equation 5, $\displaystyle\lim_{x\to 2} \frac{x^6-64}{x-2} = f'(2)$, where $f(x) = x^6$ and $a = 2$.

40. By Equation 5, $\displaystyle\lim_{x\to 1/4} \frac{\dfrac{1}{x}-4}{x-\dfrac{1}{4}} = f'(4)$, where $f(x) = \dfrac{1}{x}$ and $a = \dfrac{1}{4}$.

41. By (4), $\displaystyle\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By (4), $\displaystyle\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h} = f'(0)$, where $f(x) = \cos(\pi+x)$ and $a = 0$.

42. By Equation 5, $\displaystyle\lim_{\theta\to\pi/6} \frac{\sin\theta-\frac{1}{2}}{\theta-\dfrac{\pi}{6}} = f'\left(\frac{\pi}{6}\right)$, where $f(\theta) = \sin\theta$ and $a = \dfrac{\pi}{6}$.

43. $\displaystyle v(4) = f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{\left[80(4+h)-6(4+h)^2\right]-\left[80(4)-6(4)^2\right]}{h}$

$$= \lim_{h\to 0} \frac{(320+80h-96-48h-6h^2)-(320-96)}{h} = \lim_{h\to 0} \frac{32h-6h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(32-6h)}{h} = \lim_{h\to 0}(32-6h) = 32 \text{ m/s}$$

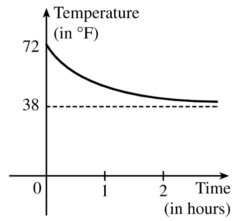
The speed when $t = 4$ is $|32| = 32$ m/s.

44. $v(4) = f'(4) = \lim\limits_{h \to 0} \dfrac{f(4+h) - f(4)}{h} = \lim\limits_{h \to 0} \dfrac{\left(10 + \dfrac{45}{4+h+1}\right) - \left(10 + \dfrac{45}{4+1}\right)}{h} = \lim\limits_{h \to 0} \dfrac{\dfrac{45}{5+h} - 9}{h}$

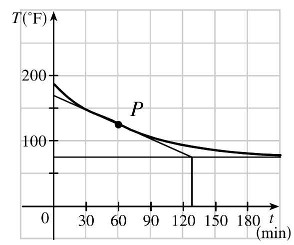
$$= \lim_{h \to 0} \frac{45 - 9(5+h)}{h(5+h)} = \lim_{h \to 0} \frac{-9h}{h(5+h)} = \lim_{h \to 0} \frac{-9}{5+h} = -\frac{9}{5} \text{ m/s.}$$

The speed when $t = 4$ is $\left|-\frac{9}{5}\right| = \frac{9}{5}$ m/s.

45. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



46. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t = 1$ h seems to be about $\dfrac{75 - 168}{132 - 0} \approx -0.7\,{}^\circ\text{F/min}$.



47. (a) (i) $[1.0, 2.0]$: $\dfrac{C(2) - C(1)}{2 - 1} = \dfrac{0.18 - 0.33}{1} = -0.15\ \dfrac{\text{mg/mL}}{\text{h}}$

(ii) $[1.5, 2.0]$: $\dfrac{C(2) - C(1.5)}{2 - 1.5} = \dfrac{0.18 - 0.24}{0.5} = \dfrac{-0.06}{0.5} = -0.12\ \dfrac{\text{mg/mL}}{\text{h}}$

(iii) $[2.0, 2.5]$: $\dfrac{C(2.5) - C(2)}{2.5 - 2} = \dfrac{0.12 - 0.18}{0.5} = \dfrac{-0.06}{0.5} = -0.12\ \dfrac{\text{mg/mL}}{\text{h}}$

(iv) $[2.0, 3.0]$: $\dfrac{C(3) - C(2)}{3 - 2} = \dfrac{0.07 - 0.18}{1} = -0.11\ \dfrac{\text{mg/mL}}{\text{h}}$

(b) We estimate the instantaneous rate of change at $t = 2$ by averaging the average rates of change for $[1.5, 2.0]$ and $[2.0, 2.5]$:

$\dfrac{-0.12 + (-0.12)}{2} = -0.12\ \dfrac{\text{mg/mL}}{\text{h}}$. After 2 hours, the BAC is decreasing at a rate of 0.12 (mg/mL)/h.

48. (a) (i) $[2006, 2008]$: $\dfrac{N(2008) - N(2006)}{2008 - 2006} = \dfrac{16{,}680 - 12{,}440}{2} = \dfrac{4240}{2} = 2120$ locations/year

(ii) $[2008, 2010]$: $\dfrac{N(2010) - N(2008)}{2010 - 2008} = \dfrac{16{,}858 - 16{,}680}{2} = \dfrac{178}{2} = 89$ locations/year.

The rate of growth decreased over the period from 2006 to 2010.

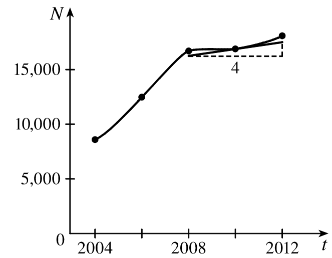
(b) $[2010, 2012]$: $\dfrac{N(2012) - N(2010)}{2012 - 2010} = \dfrac{18{,}066 - 16{,}858}{2} = \dfrac{1208}{2} = 604$ locations/year.

Using that value and the value from part (a)(ii), we have $\dfrac{89 + 604}{2} = \dfrac{693}{2} = 346.5$ locations/year.

(c) The tangent segment has endpoints $(2008, 16{,}250)$ and $(2012, 17{,}500)$.

An estimate of the instantaneous rate of growth in 2010 is

$$\frac{17{,}500 - 16{,}250}{2012 - 2008} = \frac{1250}{4} = 312.5 \text{ locations/year.}$$



49. (a) $[1990, 2005]$: $\dfrac{84{,}077 - 66{,}533}{2005 - 1990} = \dfrac{17{,}544}{15} = 1169.6$ thousands of barrels per day per year. This means that oil consumption rose by an average of 1169.6 thousands of barrels per day each year from 1990 to 2005.

(b) $[1995, 2000]$: $\dfrac{76{,}784 - 70{,}099}{2000 - 1995} = \dfrac{6685}{5} = 1337$

$[2000, 2005]$: $\dfrac{84{,}077 - 76{,}784}{2005 - 2000} = \dfrac{7293}{5} = 1458.6$

An estimate of the instantaneous rate of change in 2000 is $\frac{1}{2}(1337 + 1458.6) = 1397.8$ thousands of barrels per day per year.

50. (a) (i) $[4, 11]$: $\dfrac{V(11) - V(4)}{11 - 4} = \dfrac{9.4 - 53}{7} = \dfrac{-43.6}{7} \approx -6.23 \; \dfrac{\text{RNA copies/mL}}{\text{day}}$

(ii) $[8, 11]$: $\dfrac{V(11) - V(8)}{11 - 8} = \dfrac{9.4 - 18}{3} = \dfrac{-8.6}{3} \approx -2.87 \; \dfrac{\text{RNA copies/mL}}{\text{day}}$

(iii) $[11, 15]$: $\dfrac{V(15) - V(11)}{15 - 11} = \dfrac{5.2 - 9.4}{4} = \dfrac{-4.2}{4} = -1.05 \; \dfrac{\text{RNA copies/mL}}{\text{day}}$

(iv) $[11, 22]$: $\dfrac{V(22) - V(11)}{22 - 11} = \dfrac{3.6 - 9.4}{11} = \dfrac{-5.8}{11} \approx -0.53 \; \dfrac{\text{RNA copies/mL}}{\text{day}}$

(b) An estimate of $V'(11)$ is the average of the answers from part (a)(ii) and (iii).

$$V'(11) \approx \tfrac{1}{2}[-2.87 + (-1.05)] = -1.96 \; \frac{\text{RNA copies/mL}}{\text{day}}.$$

$V'(11)$ measures the instantaneous rate of change of patient 303's viral load 11 days after ABT-538 treatment began.

51. (a) (i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105) - C(100)}{105 - 100} = \dfrac{6601.25 - 6500}{5} = \$20.25/\text{unit}.$

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101) - C(100)}{101 - 100} = \dfrac{6520.05 - 6500}{1} = \$20.05/\text{unit}.$

(b) $$\frac{C(100 + h) - C(100)}{h} = \frac{\left[5000 + 10(100 + h) + 0.05(100 + h)^2\right] - 6500}{h} = \frac{20h + 0.05h^2}{h}$$
$$= 20 + 0.05h,\ h \neq 0$$

So the instantaneous rate of change is $\displaystyle\lim_{h \to 0} \frac{C(100 + h) - C(100)}{h} = \lim_{h \to 0}(20 + 0.05h) = \$20/\text{unit}.$

52. $\Delta V = V(t+h) - V(t) = 100{,}000\left(1 - \dfrac{t+h}{60}\right)^2 - 100{,}000\left(1 - \dfrac{t}{60}\right)^2$

$$= 100{,}000\left[\left(1 - \frac{t+h}{30} + \frac{(t+h)^2}{3600}\right) - \left(1 - \frac{t}{30} + \frac{t^2}{3600}\right)\right] = 100{,}000\left(-\frac{h}{30} + \frac{2th}{3600} + \frac{h^2}{3600}\right)$$

$$= \frac{100{,}000}{3600}h\left(-120 + 2t + h\right) = \frac{250}{9}h\left(-120 + 2t + h\right)$$

Dividing ΔV by h and then letting $h \to 0$, we see that the instantaneous rate of change is $\frac{500}{9}(t - 60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\overline{3}$	$100{,}000$
10	$-2777.\overline{7}$	$69{,}444.\overline{4}$
20	$-2222.\overline{2}$	$44{,}444.\overline{4}$
30	$-1666.\overline{6}$	$25{,}000$
40	$-1111.\overline{1}$	$11{,}111.\overline{1}$
50	$-555.\overline{5}$	$2{,}777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

53. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

54. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

55. (a) $H'(58)$ is the rate at which the daily heating cost changes with respect to temperature when the outside temperature is $58\,^\circ\mathrm{F}$. The units are dollars/$\,^\circ\mathrm{F}$.

(b) If the outside temperature increases, the building should require less heating, so we would expect $H'(58)$ to be negative.

56. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).

(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

57. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\mathrm{mg/L})/^\circ\mathrm{C}$.

(b) For $T = 16^\circ\mathrm{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \dfrac{6 - 14}{32 - 0} = -\dfrac{8}{32} = -0.25\ (\mathrm{mg/L})/^\circ\mathrm{C}$. This means that as the temperature increases past $16^\circ\mathrm{C}$, the oxygen solubility is decreasing at a rate of $0.25\ (\mathrm{mg/L})/^\circ\mathrm{C}$.

58. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are $(\text{cm/s})/^\circ\text{C}$.

(b) For $T = 15^\circ\text{C}$, it appears the tangent line to the curve goes through the points $(10, 25)$ and $(20, 32)$. So $S'(15) \approx \dfrac{32-25}{20-10} = 0.7\ (\text{cm/s})/^\circ\text{C}$. This tells us that at $T = 15^\circ\text{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of 0.7 $(\text{cm/s})/^\circ\text{C}$. In a similar fashion for $T = 25^\circ\text{C}$, we can use the points $(20, 35)$ and $(25, 25)$ to obtain $S'(25) \approx \dfrac{25-35}{25-20} = -2\ (\text{cm/s})/^\circ\text{C}$. As it gets warmer than 20°C, the maximum sustainable speed decreases rapidly.

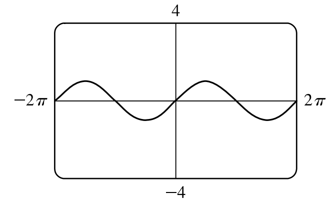
59. Since $f(x) = x\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h\sin(1/h) - 0}{h} = \lim\limits_{h\to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 2.2.4.)

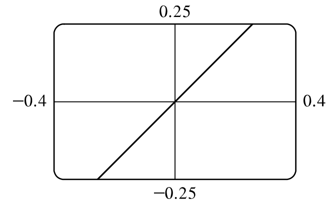
60. Since $f(x) = x^2\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^2\sin(1/h) - 0}{h} = \lim\limits_{h\to 0} h\sin(1/h)$. Since $-1 \le \sin\dfrac{1}{h} \le 1$, we have $-|h| \le |h|\sin\dfrac{1}{h} \le |h| \;\;\Rightarrow\;\; -|h| \le h\sin\dfrac{1}{h} \le |h|$. Because $\lim\limits_{h\to 0}(-|h|) = 0$ and $\lim\limits_{h\to 0}|h| = 0$, we know that $\lim\limits_{h\to 0}\left(h\sin\dfrac{1}{h}\right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

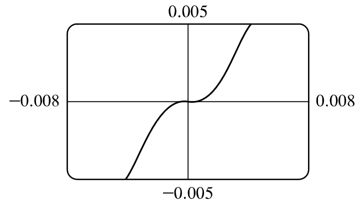
61. (a) The slope at the origin appears to be 1.



(b) The slope at the origin still appears to be 1.

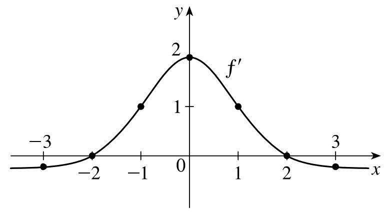


(c) Yes, the slope at the origin now appears to be 0.



2.8 The Derivative as a Function

1. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

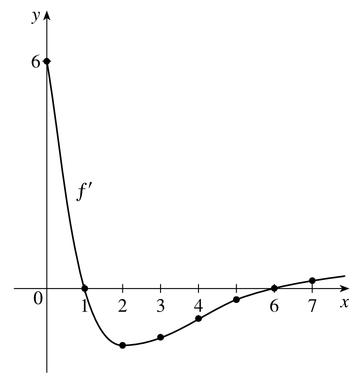


(a) $f'(-3) \approx -0.2$

(b) $f'(-2) \approx 0$ (c) $f'(-1) \approx 1$ (d) $f'(0) \approx 2$

(e) $f'(1) \approx 1$ (f) $f'(2) \approx 0$ (g) $f'(3) \approx -0.2$

2. Your answers may vary depending on your estimates.



(a) *Note:* By estimating the slopes of tangent lines on the graph of f, it appears that $f'(0) \approx 6$.

(b) $f'(1) \approx 0$

(c) $f'(2) \approx -1.5$ (d) $f'(3) \approx -1.3$ (e) $f'(4) \approx -0.8$

(f) $f'(5) \approx -0.3$ (g) $f'(6) \approx 0$ (h) $f'(7) \approx 0.2$

3. (a)$' =$ II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

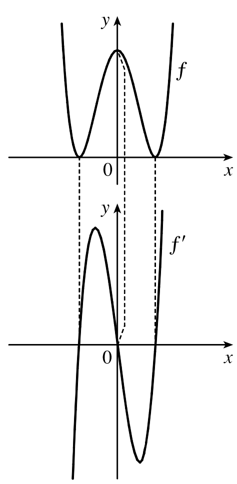
(b)$' =$ IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$' =$ I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

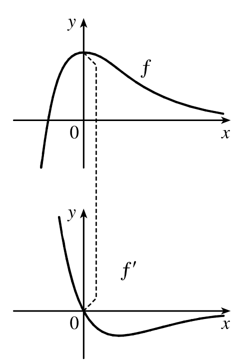
(d)$' =$ III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 4–11: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f—there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

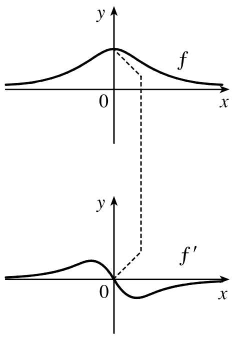
4.



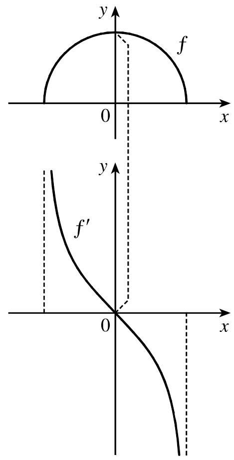
5.



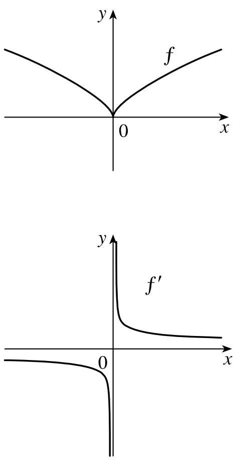
6.



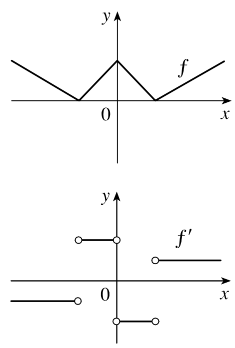
7.



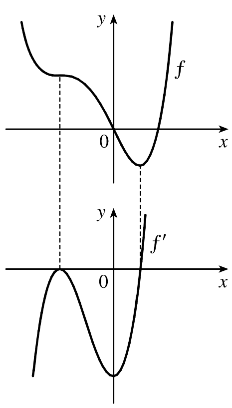
9.



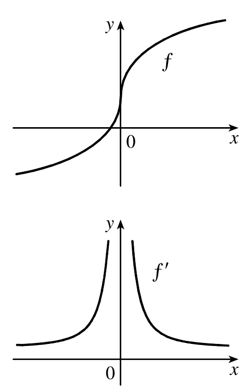
9.



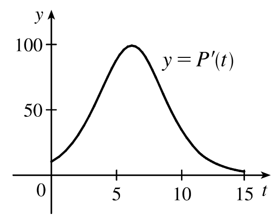
10.



11.

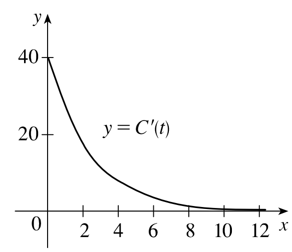


12. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y-values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y-values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



13. (a) $C'(t)$ is the instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours.

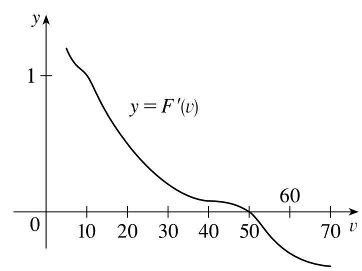
(b) The graph of $C'(t)$ tells us that the rate of change of percentage of full capacity is decreasing and approaching 0.



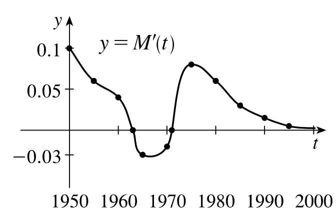
14. (a) $F'(v)$ is the instantaneous rate of change of fuel economy with respect to speed.

(b) Graphs will vary depending on estimates of F', but will change from positive to negative at about $v = 50$.

(c) To save on gas, drive at the speed where F is a maximum and F' is 0, which is about 50 mi/h.

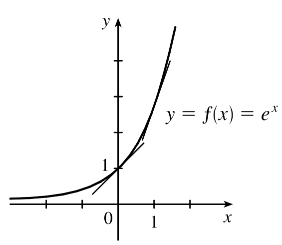


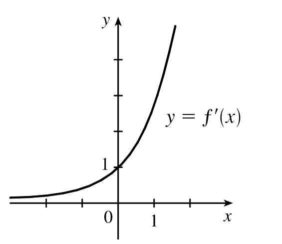
15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.



16. See Figure 3.3.1.

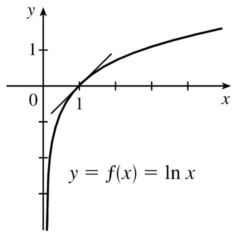
17.

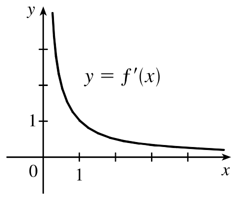




The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7. As x decreases, the slope gets closer to 0. Since the graphs are so similar, we might guess that $f'(x) = e^x$.

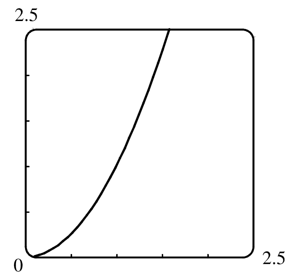
18.





As x increases toward 1, $f'(x)$ decreases from very large numbers to 1. As x becomes large, $f'(x)$ gets closer to 0. As a guess, $f'(x) = 1/x^2$ or $f'(x) = 1/x$ makes sense.

19. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.



(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

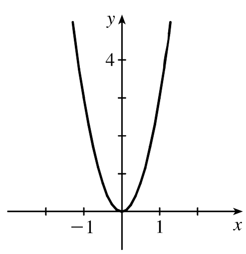
(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.

(d) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{\left(x^2 + 2hx + h^2\right) - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x$$

20. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

(b) By symmetry, $f'(-x) = f'(x)$. So $f'\left(-\frac{1}{2}\right) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

(c)



(d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$. Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

(e) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2$$

21. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[3(x+h)-8]-(3x-8)}{h} = \lim\limits_{h\to 0} \dfrac{3x+3h-8-3x+8}{h}$

$= \lim\limits_{h\to 0} \dfrac{3h}{h} = \lim\limits_{h\to 0} 3 = 3$

Domain of f = domain of $f' = \mathbb{R}$.

22. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[m(x+h)+b]-(mx+b)}{h} = \lim\limits_{h\to 0} \dfrac{mx+mh+b-mx-b}{h}$

$= \lim\limits_{h\to 0} \dfrac{mh}{h} = \lim\limits_{h\to 0} m = m$

Domain of f = domain of $f' = \mathbb{R}$.

23. $f'(t) = \lim\limits_{h\to 0} \dfrac{f(t+h)-f(t)}{h} = \lim\limits_{h\to 0} \dfrac{\left[2.5(t+h)^2+6(t+h)\right]-\left(2.5t^2+6t\right)}{h}$

$= \lim\limits_{h\to 0} \dfrac{2.5(t^2+2th+h^2)+6t+6h-2.5t^2-6t}{h} = \lim\limits_{h\to 0} \dfrac{2.5t^2+5th+2.5h^2+6h-2.5t^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{5th+2.5h^2+6h}{h} = \lim\limits_{h\to 0} \dfrac{h\,(5t+2.5h+6)}{h} = \lim\limits_{h\to 0} (5t+2.5h+6)$

$= 5t+6$

Domain of f = domain of $f' = \mathbb{R}$.

24. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\left[4+8(x+h)-5(x+h)^2\right]-(4+8x-5x^2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{4+8x+8h-5(x^2+2xh+h^2)-4-8x+5x^2}{h} = \lim\limits_{h\to 0} \dfrac{8h-5x^2-10xh-5h^2+5x^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{8h-10xh-5h^2}{h} = \lim\limits_{h\to 0} \dfrac{h(8-10x-5h)}{h} = \lim\limits_{h\to 0} (8-10x-5h)$

$= 8-10x$

Domain of f = domain of $f' = \mathbb{R}$.

25. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[(x+h)^2-2(x+h)^3]-(x^2-2x^3)}{h}$

$= \lim\limits_{h\to 0} \dfrac{x^2+2xh+h^2-2x^3-6x^2h-6xh^2-2h^3-x^2+2x^3}{h}$

$= \lim\limits_{h\to 0} \dfrac{2xh+h^2-6x^2h-6xh^2-2h^3}{h} = \lim\limits_{h\to 0} \dfrac{h(2x+h-6x^2-6xh-2h^2)}{h}$

$= \lim\limits_{h\to 0} (2x+h-6x^2-6xh-2h^2) = 2x-6x^2$

Domain of f = domain of $f' = \mathbb{R}$.

26. $g'(t) = \lim\limits_{h\to 0} \dfrac{g(t+h)-g(t)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{1}{\sqrt{t+h}} - \dfrac{1}{\sqrt{t}}}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\,\sqrt{t}}}{h} = \lim\limits_{h\to 0} \left(\dfrac{\sqrt{t}-\sqrt{t+h}}{h\,\sqrt{t+h}\,\sqrt{t}} \cdot \dfrac{\sqrt{t}+\sqrt{t+h}}{\sqrt{t}+\sqrt{t+h}}\right)$

$= \lim\limits_{h\to 0} \dfrac{t-(t+h)}{h\,\sqrt{t+h}\,\sqrt{t}\,\left(\sqrt{t}+\sqrt{t+h}\right)} = \lim\limits_{h\to 0} \dfrac{-h}{h\,\sqrt{t+h}\,\sqrt{t}\,\left(\sqrt{t}+\sqrt{t+h}\right)} = \lim\limits_{h\to 0} \dfrac{-1}{\sqrt{t+h}\,\sqrt{t}\,\left(\sqrt{t}+\sqrt{t+h}\right)}$

$= \dfrac{-1}{\sqrt{t}\,\sqrt{t}\,\left(\sqrt{t}+\sqrt{t}\right)} = \dfrac{-1}{t\left(2\sqrt{t}\right)} = -\dfrac{1}{2t^{3/2}}$

Domain of g = domain of $g' = (0, \infty)$.

27. $g'(x) = \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h}\left[\dfrac{\sqrt{9-(x+h)}+\sqrt{9-x}}{\sqrt{9-(x+h)}+\sqrt{9-x}}\right]$

$$= \lim_{h\to 0} \frac{[9-(x+h)]-(9-x)}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]} = \lim_{h\to 0} \frac{-h}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}} = \frac{-1}{2\sqrt{9-x}}$$

Domain of $g = (-\infty, 9]$, domain of $g' = (-\infty, 9)$.

28. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{(x+h)^2-1}{2(x+h)-3} - \dfrac{x^2-1}{2x-3}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{[(x+h)^2-1](2x-3)-[2(x+h)-3](x^2-1)}{[2(x+h)-3](2x-3)}}{h}$$

$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2-1)(2x-3)-(2x+2h-3)(x^2-1)}{h[2(x+h)-3](2x-3)}$$

$$= \lim_{h\to 0} \frac{(2x^3+4x^2h+2xh^2-2x-3x^2-6xh-3h^2+3)-(2x^3+2x^2h-3x^2-2x-2h+3)}{h(2x+2h-3)(2x-3)}$$

$$= \lim_{h\to 0} \frac{4x^2h+2xh^2-6xh-3h^2-2x^2h+2h}{h(2x+2h-3)(2x-3)} = \lim_{h\to 0} \frac{h(2x^2+2xh-6x-3h+2)}{h(2x+2h-3)(2x-3)}$$

$$= \lim_{h\to 0} \frac{2x^2+2xh-6x-3h+2}{(2x+2h-3)(2x-3)} = \frac{2x^2-6x+2}{(2x-3)^2}$$

Domain of f = domain of $f' = (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$.

29. $G'(t) = \lim_{h\to 0} \dfrac{G(t+h)-G(t)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1-2(t+h)}{3+(t+h)} - \dfrac{1-2t}{3+t}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{[1-2(t+h)](3+t)-[3+(t+h)](1-2t)}{[3+(t+h)](3+t)}}{h}$$

$$= \lim_{h\to 0} \frac{3+t-6t-2t^2-6h-2ht-(3-6t+t-2t^2+h-2ht)}{h[3+(t+h)](3+t)} = \lim_{h\to 0} \frac{-6h-h}{h(3+t+h)(3+t)}$$

$$= \lim_{h\to 0} \frac{-7h}{h(3+t+h)(3+t)} = \lim_{h\to 0} \frac{-7}{(3+t+h)(3+t)} = \frac{-7}{(3+t)^2}$$

Domain of G = domain of $G' = (-\infty, -3) \cup (-3, \infty)$.

30. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^{3/2}-x^{3/2}}{h} = \lim_{h\to 0} \dfrac{[(x+h)^{3/2}-x^{3/2}][(x+h)^{3/2}+x^{3/2}]}{h\left[(x+h)^{3/2}+x^{3/2}\right]}$

$$= \lim_{h\to 0} \frac{(x+h)^3-x^3}{h[(x+h)^{3/2}+x^{3/2}]} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h[(x+h)^{3/2}+x^{3/2}]} = \lim_{h\to 0} \frac{h\left(3x^2+3xh+h^2\right)}{h[(x+h)^{3/2}+x^{3/2}]}$$

$$= \lim_{h\to 0} \frac{3x^2+3xh+h^2}{(x+h)^{3/2}+x^{3/2}} = \frac{3x^2}{2x^{3/2}} = \tfrac{3}{2}x^{1/2}$$

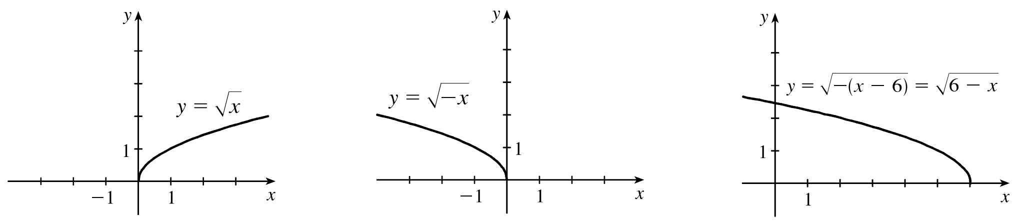
Domain of f = domain of $f' = [0, \infty)$. Strictly speaking, the domain of f' is $(0, \infty)$ because the limit that defines $f'(0)$ does not exist (as a two-sided limit). But the right-hand derivative (in the sense of Exercise 64) does exist at 0, so in that sense one could regard the domain of f' to be $[0, \infty)$.

31. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{(x+h)^4 - x^4}{h} = \lim\limits_{h \to 0} \dfrac{\left(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4\right) - x^4}{h}$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h \to 0} \left(4x^3 + 6x^2h + 4xh^2 + h^3\right) = 4x^3$$

Domain of f = domain of $f' = \mathbb{R}$.

32. (a)



(b) Note that the third graph in part (a) has small negative values for its slope, f'; but as $x \to 6^-$, $f' \to -\infty$.

See the graph in part (d).

(c) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

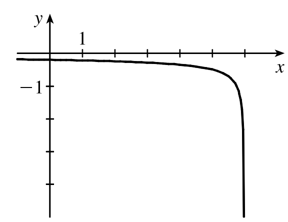
$$= \lim_{h \to 0} \frac{\sqrt{6-(x+h)} - \sqrt{6-x}}{h} \left[\frac{\sqrt{6-(x+h)} + \sqrt{6-x}}{\sqrt{6-(x+h)} + \sqrt{6-x}}\right]$$

$$= \lim_{h \to 0} \frac{[6-(x+h)] - (6-x)}{h\left[\sqrt{6-(x+h)} + \sqrt{6-x}\right]} = \lim_{h \to 0} \frac{-h}{h\left(\sqrt{6-x-h} + \sqrt{6-x}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{6-x-h} + \sqrt{6-x}} = \frac{-1}{2\sqrt{6-x}}$$

Domain of $f = (-\infty, 6]$, domain of $f' = (-\infty, 6)$.

(d)



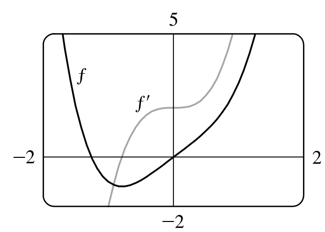
33. (a) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{[(x+h)^4 + 2(x+h)] - (x^4 + 2x)}{h}$

$$= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2x + 2h - x^4 - 2x}{h}$$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2h}{h} = \lim_{h \to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3 + 2)}{h}$$

$$= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3 + 2) = 4x^3 + 2$$

(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope.

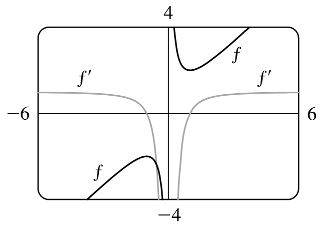


34. (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[(x+h) + 1/(x+h)] - (x + 1/x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{(x+h)^2+1}{x+h} - \dfrac{x^2+1}{x}}{h}$

$$= \lim_{h\to 0} \frac{x[(x+h)^2+1] - (x+h)(x^2+1)}{h(x+h)x} = \lim_{h\to 0} \frac{(x^3 + 2hx^2 + xh^2 + x) - (x^3 + x + hx^2 + h)}{h(x+h)x}$$

$$= \lim_{h\to 0} \frac{hx^2 + xh^2 - h}{h(x+h)x} = \lim_{h\to 0} \frac{h(x^2 + xh - 1)}{h(x+h)x} = \lim_{h\to 0} \frac{x^2 + xh - 1}{(x+h)x} = \frac{x^2-1}{x^2}, \text{ or } 1 - \frac{1}{x^2}$$

(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope. Both functions are discontinuous at $x = 0$.



35. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent unemployed per year.

(b) To find $U'(t)$, we use $\lim\limits_{h\to 0} \dfrac{U(t+h) - U(t)}{h} \approx \dfrac{U(t+h) - U(t)}{h}$ for small values of h.

For 2003: $U'(2003) \approx \dfrac{U(2004) - U(2003)}{2004 - 2003} = \dfrac{5.5 - 6.0}{1} = -0.5$

For 2004: We estimate $U'(2004)$ by using $h = -1$ and $h = 1$, and then average the two results to obtain a final estimate.

$h = -1 \quad \Rightarrow \quad U'(2004) \approx \dfrac{U(2003) - U(2004)}{2003 - 2004} = \dfrac{6.0 - 5.5}{-1} = -0.5;$

$h = 1 \quad \Rightarrow \quad U'(2004) \approx \dfrac{U(2005) - U(2004)}{2005 - 2004} = \dfrac{5.1 - 5.5}{1} = -0.4.$

So we estimate that $U'(2004) \approx \frac{1}{2}[-0.5 + (-0.4)] = -0.45$.

t	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
$U'(t)$	-0.50	-0.45	-0.45	-0.25	0.60	2.35	1.90	-0.20	-0.75	-0.80

36. (a) $N'(t)$ is the rate at which the number of minimally invasive cosmetic surgery procedures performed in the United States is changing with respect to time. Its units are thousands of surgeries per year.

(b) To find $N'(t)$, we use $\lim\limits_{h\to 0} \dfrac{N(t+h) - N(t)}{h} \approx \dfrac{N(t+h) - N(t)}{h}$ for small values of h.

For 2000: $N'(2000) \approx \dfrac{N(2002) - N(2000)}{2002 - 2000} = \dfrac{4897 - 5500}{2} = -301.5$

For 2002: We estimate $N'(2002)$ by using $h = -2$ and $h = 2$, and then average the two results to obtain a final estimate.

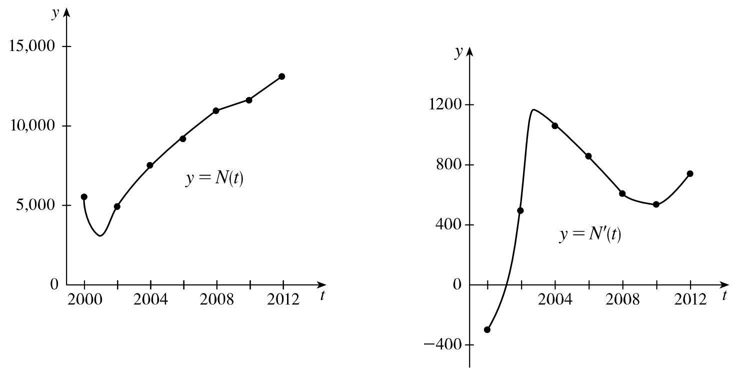
$h = -2 \quad \Rightarrow \quad N'(2002) \approx \dfrac{N(2000) - N(2002)}{2000 - 2002} = \dfrac{5500 - 4897}{-2} = -301.5$

$h = 2 \quad \Rightarrow \quad N'(2002) \approx \dfrac{N(2004) - N(2002)}{2004 - 2002} = \dfrac{7470 - 4897}{2} = 1286.5$

So we estimate that $N'(2002) \approx \frac{1}{2}[-301.5 + 1286.5] = 492.5$.

t	2000	2002	2004	2006	2008	2010	2012
$N'(t)$	-301.5	492.5	1060.25	856.75	605.75	534.5	737

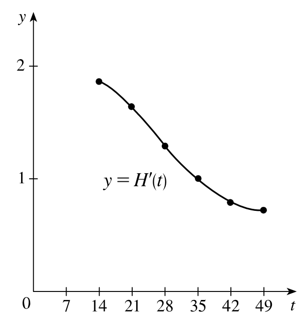
(c)



(d) We could get more accurate values for $N'(t)$ by obtaining data for more values of t.

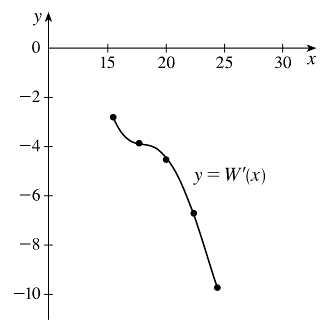
37. As in Exercise 35, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values.

t	14	21	28	35	42	49
$H(t)$	41	54	64	72	78	83
$H'(t)$	$\frac{13}{7}$	$\frac{23}{14}$	$\frac{18}{14}$	$\frac{14}{14}$	$\frac{11}{14}$	$\frac{5}{7}$



38. As in Exercise 35, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values. The units for $W'(x)$ are grams per degree (g/°C).

x	15.5	17.7	20.0	22.4	24.4
$W(x)$	37.2	31.0	19.8	9.7	−9.8
$W'(x)$	−2.82	−3.87	−4.53	−6.73	−9.75



39. (a) dP/dt is the rate at which the percentage of the city's electrical power produced by solar panels changes with respect to time t, measured in percentage points per year.

(b) 2 years after January 1, 2000 (January 1, 2002), the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.

40. dN/dp is the rate at which the number of people who travel by car to another state for a vacation changes with respect to the price of gasoline. If the price of gasoline goes up, we would expect fewer people to travel, so we would expect dN/dp to be negative.

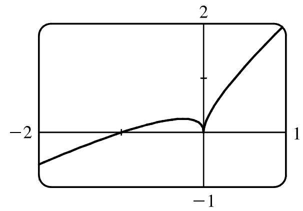
41. f is not differentiable at $x = -4$, because the graph has a corner there, and at $x = 0$, because there is a discontinuity there.

42. f is not differentiable at $x = -1$, because there is a discontinuity there, and at $x = 2$, because the graph has a corner there.

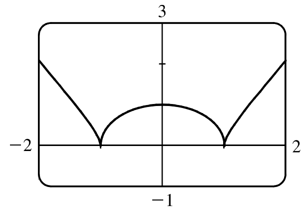
43. f is not differentiable at $x = 1$, because f is not defined there, and at $x = 5$, because the graph has a vertical tangent there.

44. f is not differentiable at $x = -2$ and $x = 3$, because the graph has corners there, and at $x = 1$, because there is a discontinuity there.

45. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



46. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so f is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = \pm 1$.



47. Call the curve with the positive y-intercept g and the other curve h. Notice that g has a maximum (horizontal tangent) at $x = 0$, but $h \neq 0$, so h cannot be the derivative of g. Also notice that where g is positive, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is negative since f' is below the x-axis there and $f''(1)$ is positive since f is concave upward at $x = 1$. Therefore, $f''(1)$ is greater than $f'(-1)$.

48. Call the curve with the smallest positive x-intercept g and the other curve h. Notice that where g is positive in the first quadrant, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is positive since f' is above the x-axis there and $f''(1)$ appears to be zero since f has an inflection point at $x = 1$. Therefore, $f'(1)$ is greater than $f''(-1)$.

49. $a = f, b = f', c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

50. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a, so $b' = a$. We conclude that $d = f, c = f', b = f''$, and $a = f'''$.

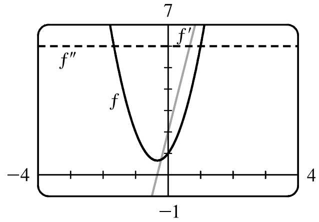
51. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

52. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

53.
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[3(x+h)^2+2(x+h)+1]-(3x^2+2x+1)}{h} \\ &= \lim_{h\to 0}\frac{(3x^2+6xh+3h^2+2x+2h+1)-(3x^2+2x+1)}{h} = \lim_{h\to 0}\frac{6xh+3h^2+2h}{h} \\ &= \lim_{h\to 0}\frac{h(6x+3h+2)}{h} = \lim_{h\to 0}(6x+3h+2) = 6x+2 \end{aligned}$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{[6(x+h)+2] - (6x+2)}{h} = \lim_{h\to 0} \frac{(6x+6h+2) - (6x+2)}{h}$$

$$= \lim_{h\to 0} \frac{6h}{h} = \lim_{h\to 0} 6 = 6$$



We see from the graph that our answers are reasonable because the graph of f' is that of a linear function and the graph of f'' is that of a constant function.

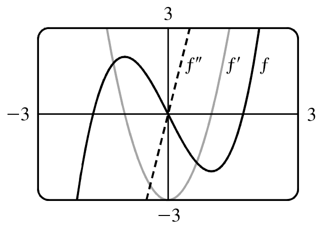
54. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3 - 3(x+h)] - (x^3 - 3x)}{h}$

$$= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h) - (x^3 - 3x)}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h}$$

$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} = \lim_{h\to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)^2 - 3] - (3x^2 - 3)}{h} = \lim_{h\to 0} \frac{(3x^2 + 6xh + 3h^2 - 3) - (3x^2 - 3)}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2}{h} = \lim_{h\to 0} \frac{h(6x + 3h)}{h} = \lim_{h\to 0} (6x + 3h) = 6x$$



We see from the graph that our answers are reasonable because the graph of f' is that of an even function (f is an odd function) and the graph of f'' is that of an odd function. Furthermore, $f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent.

55. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[2(x+h)^2 - (x+h)^3\right] - (2x^2 - x^3)}{h}$

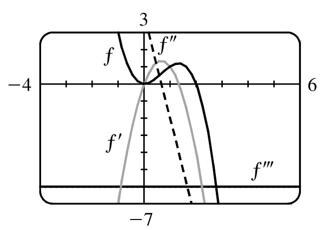
$$= \lim_{h\to 0} \frac{h(4x + 2h - 3x^2 - 3xh - h^2)}{h} = \lim_{h\to 0} (4x + 2h - 3x^2 - 3xh - h^2) = 4x - 3x^2$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{\left[4(x+h) - 3(x+h)^2\right] - (4x - 3x^2)}{h} = \lim_{h\to 0} \frac{h(4 - 6x - 3h)}{h}$$

$$= \lim_{h\to 0} (4 - 6x - 3h) = 4 - 6x$$

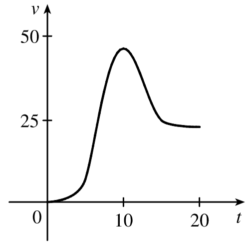
$$f'''(x) = \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h\to 0} \frac{[4 - 6(x+h)] - (4 - 6x)}{h} = \lim_{h\to 0} \frac{-6h}{h} = \lim_{h\to 0} (-6) = -6$$

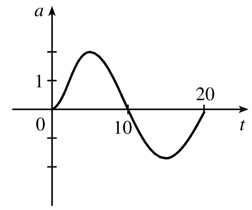
$$f^{(4)}(x) = \lim_{h\to 0} \frac{f'''(x+h) - f'''(x)}{h} = \lim_{h\to 0} \frac{-6 - (-6)}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} (0) = 0$$



The graphs are consistent with the geometric interpretations of the derivatives because f' has zeros where f has a local minimum and a local maximum, f'' has a zero where f' has a local maximum, and f''' is a constant function equal to the slope of f''.

56. (a) Since we estimate the velocity to be a maximum at $t = 10$, the acceleration is 0 at $t = 10$.





(b) Drawing a tangent line at $t = 10$ on the graph of a, a appears to decrease by 10 ft/s^2 over a period of 20 s. So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5\ (\text{ft/s}^2)/\text{s}$ or ft/s^3.

57. (a) Note that we have factored $x - a$ as the difference of two cubes in the third step.

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{x-a} = \lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$$= \lim_{x\to a}\frac{1}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \tfrac{1}{3}a^{-2/3}$$

(b) $f'(0) = \lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0}\dfrac{\sqrt[3]{h}-0}{h} = \lim\limits_{h\to 0}\dfrac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

(c) $\lim\limits_{x\to 0}|f'(x)| = \lim\limits_{x\to 0}\dfrac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

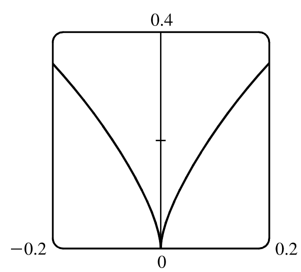
58. (a) $g'(0) = \lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{x^{2/3}-0}{x} = \lim\limits_{x\to 0}\dfrac{1}{x^{1/3}}$, which does not exist.

(b) $$g'(a) = \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = \lim_{x\to a}\frac{x^{2/3}-a^{2/3}}{x-a} = \lim_{x\to a}\frac{(x^{1/3}-a^{1/3})(x^{1/3}+a^{1/3})}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$$= \lim_{x\to a}\frac{x^{1/3}+a^{1/3}}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{ or } \tfrac{2}{3}a^{-1/3}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and $\lim\limits_{x\to 0}|g'(x)| = \lim\limits_{x\to 0}\dfrac{2}{3\,|x|^{1/3}} = \infty$. This shows that g has a vertical tangent line at $x = 0$.

(d)



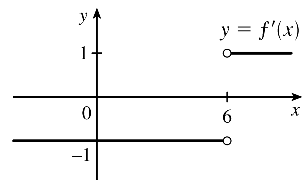
59. $f(x) = |x-6| = \begin{cases} x-6 & \text{if } x-6 \geq 6 \\ -(x-6) & \text{if } x-6 < 0 \end{cases} = \begin{cases} x-6 & \text{if } x \geq 6 \\ 6-x & \text{if } x < 6 \end{cases}$

So the right-hand limit is $\lim\limits_{x\to 6^+}\dfrac{f(x)-f(6)}{x-6} = \lim\limits_{x\to 6^+}\dfrac{|x-6|-0}{x-6} = \lim\limits_{x\to 6^+}\dfrac{x-6}{x-6} = \lim\limits_{x\to 6^+} 1 = 1$, and the left-hand limit is $\lim\limits_{x\to 6^-}\dfrac{f(x)-f(6)}{x-6} = \lim\limits_{x\to 6^-}\dfrac{|x-6|-0}{x-6} = \lim\limits_{x\to 6^-}\dfrac{6-x}{x-6} = \lim\limits_{x\to 6^-}(-1) = -1$. Since these limits are not equal,

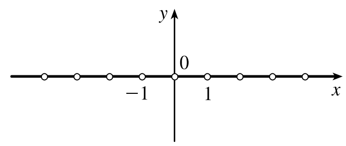
$f'(6) = \lim_{x \to 6} \dfrac{f(x) - f(6)}{x - 6}$ does not exist and f is not differentiable at 6.

However, a formula for f' is $f'(x) = \begin{cases} 1 & \text{if } x > 6 \\ -1 & \text{if } x < 6 \end{cases}$

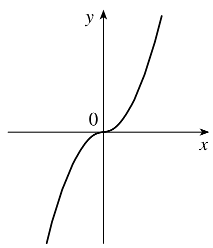
Another way of writing the formula is $f'(x) = \dfrac{x - 6}{|x - 6|}$.



60. $f(x) = [\![x]\!]$ is not continuous at any integer n, so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a, so $f'(a) = 0$. Thus, $f'(x) = 0$, x not an integer.



61. (a) $f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



(b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$. [See Exercise 19(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x\,|x|}{x} = \lim_{x \to 0} |x| = 0.$$

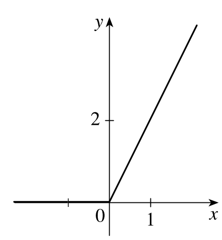
So f is differentiable at 0. Thus, f is differentiable for all x.

(c) From part (b), we have $f'(x) = \left\{ \begin{array}{ll} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{array} \right\} = 2\,|x|$.

62. (a) $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

so $f(x) = x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$.

Graph the line $y = 2x$ for $x \geq 0$ and graph $y = 0$ (the x-axis) for $x < 0$.



(b) g is not differentiable at $x = 0$ because the graph has a corner there, but is differentiable at all other values; that is, g is differentiable on $(-\infty, 0) \cup (0, \infty)$.

(c) $g(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \Rightarrow \quad g'(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

Another way of writing the formula is $g'(x) = 1 + \operatorname{sgn} x$ for $x \neq 0$.

63. (a) If f is even, then

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x + h) - f(-x)}{h} = \lim_{h \to 0} \frac{f[-(x - h)] - f(-x)}{h} \\ &= \lim_{h \to 0} \frac{f(x - h) - f(x)}{h} = -\lim_{h \to 0} \frac{f(x - h) - f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= -\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = -f'(x) \end{aligned}$$

Therefore, f' is odd.

(b) If f is odd, then

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{f[-(x-h)]-f(-x)}{h} \\ &= \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) \end{aligned}$$

Therefore, f' is even.

64. (a) $f'_-(4) = \lim\limits_{h\to 0^-}\dfrac{f(4+h)-f(4)}{h} = \lim\limits_{h\to 0^-}\dfrac{5-(4+h)-1}{h}$

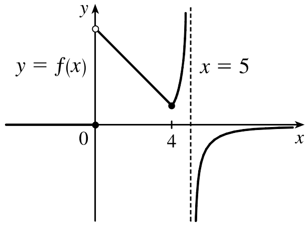
$= \lim\limits_{h\to 0^-}\dfrac{-h}{h} = -1$

and

$f'_+(4) = \lim\limits_{h\to 0^+}\dfrac{f(4+h)-f(4)}{h} = \lim\limits_{h\to 0^+}\dfrac{\dfrac{1}{5-(4+h)}-1}{h}$

$= \lim\limits_{h\to 0^+}\dfrac{1-(1-h)}{h(1-h)} = \lim\limits_{h\to 0^+}\dfrac{1}{1-h} = 1$

(b)



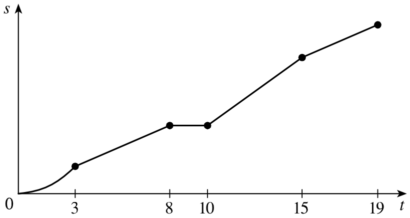
(c) $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5-x & \text{if } 0 < x < 4 \\ 1/(5-x) & \text{if } x \ge 4 \end{cases}$

At 4 we have $\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^-}(5-x) = 1$ and $\lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^+}\dfrac{1}{5-x} = 1$, so $\lim\limits_{x\to 4} f(x) = 1 = f(4)$ and f is continuous at 4. Since $f(5)$ is not defined, f is discontinuous at 5. These expressions show that f is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ and $(5, \infty)$. Since $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}(5-x) = 5 \ne 0 = \lim\limits_{x\to 0^-} f(x)$, $\lim\limits_{x\to 0} f(x)$ does not exist, so f is discontinuous (and therefore not differentiable) at 0.

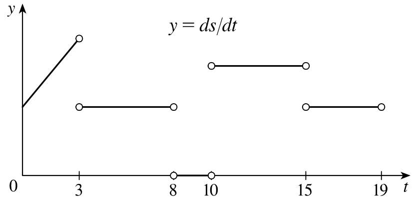
(d) From (a), f is not differentiable at 4 since $f'_-(4) \ne f'_+(4)$, and from (c), f is not differentiable at 0 or 5.

65. These graphs are idealizations conveying the spirit of the problem. In reality, changes in speed are not instantaneous, so the graph in (a) would not have corners and the graph in (b) would be continuous.

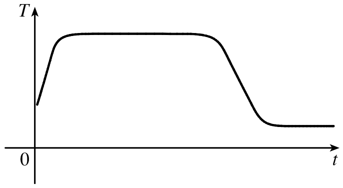
(a)



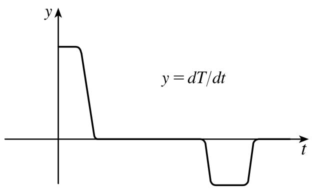
(b)

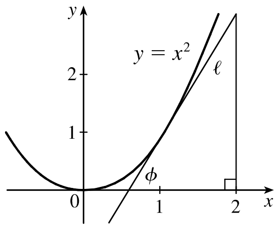


66. (a)



(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

(c) 

67. 

In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent to angle ϕ. Then the slope of the tangent line ℓ is $m = \Delta y/\Delta x = \tan\phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 19) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63°$.

2 Review

TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don't).

2. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

3. True. Limit Law 5 applies.

4. False. $\dfrac{x^2 - 9}{x - 3}$ is not defined when $x = 3$, but $x + 3$ is.

5. True. $\displaystyle \lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)} = \lim_{x\to 3}(x+3)$

6. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as x approaches 5. (The denominator approaches 0 and the numerator doesn't.)

7. False. Consider $\displaystyle \lim_{x\to 5} \frac{x(x-5)}{x-5}$ or $\displaystyle \lim_{x\to 5} \frac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 2.2.3, we know that the latter limit exists (and it is equal to 1).

8. False. If $f(x) = 1/x$, $g(x) = -1/x$, and $a = 0$, then $\displaystyle \lim_{x\to 0} f(x)$ does not exist, $\displaystyle \lim_{x\to 0} g(x)$ does not exist, but $\displaystyle \lim_{x\to 0} [f(x) + g(x)] = \lim_{x\to 0} 0 = 0$ exists.

9. True. Suppose that $\displaystyle \lim_{x\to a} [f(x) + g(x)]$ exists. Now $\displaystyle \lim_{x\to a} f(x)$ exists and $\displaystyle \lim_{x\to a} g(x)$ does not exist, but $\displaystyle \lim_{x\to a} g(x) = \lim_{x\to a} \{[f(x) + g(x)] - f(x)\} = \lim_{x\to a} [f(x) + g(x)] - \lim_{x\to a} f(x)$ [by Limit Law 2], which exists, and we have a contradiction. Thus, $\displaystyle \lim_{x\to a} [f(x) + g(x)]$ does not exist.

10. False. Consider $\lim_{x\to 6}[f(x)g(x)] = \lim_{x\to 6}\left[(x-6)\,\dfrac{1}{x-6}\right]$. It exists (its value is 1) but $f(6) = 0$ and $g(6)$ does not exist, so $f(6)g(6) \neq 1$.

11. True. A polynomial is continuous everywhere, so $\lim_{x\to b} p(x)$ exists and is equal to $p(b)$.

12. False. Consider $\lim_{x\to 0}[f(x) - g(x)] = \lim_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

13. True. See Figure 2.6.8.

14. False. Consider $f(x) = \sin x$ for $x \geq 0$. $\lim_{x\to\infty} f(x) \neq \pm\infty$ and f has no horizontal asymptote.

15. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

16. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ -1 & \text{if } x = 3 \end{cases}$ There is no number $c \in [0, 3]$ with $f(c) = 0$.

17. True. Use Theorem 2.5.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.

18. True. Use the Intermediate Value Theorem with $a = -1$, $b = 1$, and $N = \pi$, since $3 < \pi < 4$.

19. True, by the definition of a limit with $\varepsilon = 1$.

20. False. For example, let $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

Then $f(x) > 1$ for all x, but $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x^2 + 1\right) = 1$.

21. False. See the note after Theorem 2.8.4.

22. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim_{x\to r} f(x) = f(r)$.

23. False. $\dfrac{d^2y}{dx^2}$ is the second derivative while $\left(\dfrac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\dfrac{d^2y}{dx^2} = 0$, but $\left(\dfrac{dy}{dx}\right)^2 = 1$.

24. True. $f(x) = x^{10} - 10x^2 + 5$ is continuous on the interval $[0, 2]$, $f(0) = 5$, $f(1) = -4$, and $f(2) = 989$. Since $-4 < 0 < 5$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^{10} - 10x^2 + 5 = 0$ in the interval $(0, 1)$. Similarly, there is a root in $(1, 2)$.

25. True. See Exercise 2.5.72(b).

26. False See Exercise 2.5.72(b).

EXERCISES

1. (a) (i) $\lim_{x\to 2^+} f(x) = 3$ (ii) $\lim_{x\to -3^+} f(x) = 0$

(iii) $\lim_{x\to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim_{x\to 4} f(x) = 2$

(v) $\lim_{x\to 0} f(x) = \infty$ (vi) $\lim_{x\to 2^-} f(x) = -\infty$

(vii) $\lim_{x\to \infty} f(x) = 4$ (viii) $\lim_{x\to -\infty} f(x) = -1$

(b) The equations of the horizontal asymptotes are $y = -1$ and $y = 4$.

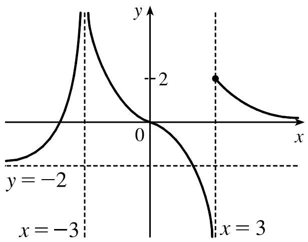
(c) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.

(d) f is discontinuous at $x = -3$, 0, 2, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

2. $\lim_{x\to -\infty} f(x) = -2$, $\lim_{x\to \infty} f(x) = 0$, $\lim_{x\to -3} f(x) = \infty$,

$\lim_{x\to 3^-} f(x) = -\infty$, $\lim_{x\to 3^+} f(x) = 2$,

f is continuous from the right at 3



3. Since the exponential function is continuous, $\lim_{x\to 1} e^{x^3-x} = e^{1-1} = e^0 = 1$.

4. Since rational functions are continuous, $\lim_{x\to 3} \dfrac{x^2-9}{x^2+2x-3} = \dfrac{3^2-9}{3^2+2(3)-3} = \dfrac{0}{12} = 0$.

5. $\lim_{x\to -3} \dfrac{x^2-9}{x^2+2x-3} = \lim_{x\to -3} \dfrac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to -3} \dfrac{x-3}{x-1} = \dfrac{-3-3}{-3-1} = \dfrac{-6}{-4} = \dfrac{3}{2}$

6. $\lim_{x\to 1^+} \dfrac{x^2-9}{x^2+2x-3} = -\infty$ since $x^2+2x-3 \to 0^+$ as $x \to 1^+$ and $\dfrac{x^2-9}{x^2+2x-3} < 0$ for $1 < x < 3$.

7. $\lim_{h\to 0} \dfrac{(h-1)^3+1}{h} = \lim_{h\to 0} \dfrac{\left(h^3-3h^2+3h-1\right)+1}{h} = \lim_{h\to 0} \dfrac{h^3-3h^2+3h}{h} = \lim_{h\to 0} \left(h^2-3h+3\right) = 3$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\begin{aligned}\lim_{h\to 0} \frac{(h-1)^3+1}{h} &= \lim_{h\to 0} \frac{(h-1)^3+1^3}{h} = \lim_{h\to 0} \frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h} \\ &= \lim_{h\to 0} \left[(h-1)^2-h+2\right] = 1-0+2 = 3\end{aligned}$$

8. $\lim_{t\to 2} \dfrac{t^2-4}{t^3-8} = \lim_{t\to 2} \dfrac{(t+2)(t-2)}{(t-2)(t^2+2t+4)} = \lim_{t\to 2} \dfrac{t+2}{t^2+2t+4} = \dfrac{2+2}{4+4+4} = \dfrac{4}{12} = \dfrac{1}{3}$

9. $\lim_{r\to 9} \dfrac{\sqrt{r}}{(r-9)^4} = \infty$ since $(r-9)^4 \to 0^+$ as $r \to 9$ and $\dfrac{\sqrt{r}}{(r-9)^4} > 0$ for $r \neq 9$.

10. $\displaystyle\lim_{v\to 4^+}\frac{4-v}{|4-v|}=\lim_{v\to 4^+}\frac{4-v}{-(4-v)}=\lim_{v\to 4^+}\frac{1}{-1}=-1$

11. $\displaystyle\lim_{u\to 1}\frac{u^4-1}{u^3+5u^2-6u}=\lim_{u\to 1}\frac{(u^2+1)(u^2-1)}{u(u^2+5u-6)}=\lim_{u\to 1}\frac{(u^2+1)(u+1)(u-1)}{u(u+6)(u-1)}=\lim_{u\to 1}\frac{(u^2+1)(u+1)}{u(u+6)}=\frac{2(2)}{1(7)}=\frac{4}{7}$

12. $\displaystyle\lim_{x\to 3}\frac{\sqrt{x+6}-x}{x^3-3x^2}=\lim_{x\to 3}\left[\frac{\sqrt{x+6}-x}{x^2(x-3)}\cdot\frac{\sqrt{x+6}+x}{\sqrt{x+6}+x}\right]=\lim_{x\to 3}\frac{(\sqrt{x+6})^2-x^2}{x^2(x-3)(\sqrt{x+6}+x)}$

$$=\lim_{x\to 3}\frac{x+6-x^2}{x^2(x-3)\left(\sqrt{x+6}+x\right)}=\lim_{x\to 3}\frac{-(x^2-x-6)}{x^2(x-3)\left(\sqrt{x+6}+x\right)}=\lim_{x\to 3}\frac{-(x-3)(x+2)}{x^2(x-3)\left(\sqrt{x+6}+x\right)}$$

$$=\lim_{x\to 3}\frac{-(x+2)}{x^2\left(\sqrt{x+6}+x\right)}=-\frac{5}{9(3+3)}=-\frac{5}{54}$$

13. Since x is positive, $\sqrt{x^2}=|x|=x$. Thus,

$$\lim_{x\to\infty}\frac{\sqrt{x^2-9}}{2x-6}=\lim_{x\to\infty}\frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/x}=\lim_{x\to\infty}\frac{\sqrt{1-9/x^2}}{2-6/x}=\frac{\sqrt{1-0}}{2-0}=\frac{1}{2}$$

14. Since x is negative, $\sqrt{x^2}=|x|=-x$. Thus,

$$\lim_{x\to-\infty}\frac{\sqrt{x^2-9}}{2x-6}=\lim_{x\to-\infty}\frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/(-x)}=\lim_{x\to-\infty}\frac{\sqrt{1-9/x^2}}{-2+6/x}=\frac{\sqrt{1-0}}{-2+0}=-\frac{1}{2}$$

15. Let $t=\sin x$. Then as $x\to\pi^-$, $\sin x\to 0^+$, so $t\to 0^+$. Thus, $\displaystyle\lim_{x\to\pi^-}\ln(\sin x)=\lim_{t\to 0^+}\ln t=-\infty$.

16. $\displaystyle\lim_{x\to-\infty}\frac{1-2x^2-x^4}{5+x-3x^4}=\lim_{x\to-\infty}\frac{(1-2x^2-x^4)/x^4}{(5+x-3x^4)/x^4}=\lim_{x\to-\infty}\frac{1/x^4-2/x^2-1}{5/x^4+1/x^3-3}=\frac{0-0-1}{0+0-3}=\frac{-1}{-3}=\frac{1}{3}$

17. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right)=\lim_{x\to\infty}\left[\frac{\sqrt{x^2+4x+1}-x}{1}\cdot\frac{\sqrt{x^2+4x+1}+x}{\sqrt{x^2+4x+1}+x}\right]=\lim_{x\to\infty}\frac{(x^2+4x+1)-x^2}{\sqrt{x^2+4x+1}+x}$

$$=\lim_{x\to\infty}\frac{(4x+1)/x}{(\sqrt{x^2+4x+1}+x)/x}\qquad\left[\text{divide by } x=\sqrt{x^2} \text{ for } x>0\right]$$

$$=\lim_{x\to\infty}\frac{4+1/x}{\sqrt{1+4/x+1/x^2}+1}=\frac{4+0}{\sqrt{1+0+0}+1}=\frac{4}{2}=2$$

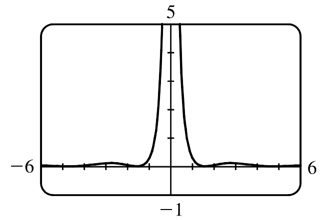
18. Let $t=x-x^2=x(1-x)$. Then as $x\to\infty$, $t\to-\infty$, and $\displaystyle\lim_{x\to\infty}e^{x-x^2}=\lim_{t\to-\infty}e^t=0$.

19. Let $t=1/x$. Then as $x\to 0^+$, $t\to\infty$, and $\displaystyle\lim_{x\to 0^+}\tan^{-1}(1/x)=\lim_{t\to\infty}\tan^{-1}t=\frac{\pi}{2}$.

20. $\displaystyle\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)=\lim_{x\to 1}\left[\frac{1}{x-1}+\frac{1}{(x-1)(x-2)}\right]=\lim_{x\to 1}\left[\frac{x-2}{(x-1)(x-2)}+\frac{1}{(x-1)(x-2)}\right]$

$$=\lim_{x\to 1}\left[\frac{x-1}{(x-1)(x-2)}\right]=\lim_{x\to 1}\frac{1}{x-2}=\frac{1}{1-2}=-1$$

21. From the graph of $y = (\cos^2 x)/x^2$, it appears that $y = 0$ is the horizontal asymptote and $x = 0$ is the vertical asymptote. Now $0 \le (\cos x)^2 \le 1 \quad \Rightarrow$ $\dfrac{0}{x^2} \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2} \quad \Rightarrow \quad 0 \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2}$. But $\lim\limits_{x\to\pm\infty} 0 = 0$ and $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2} = 0$, so by the Squeeze Theorem, $\lim\limits_{x\to\pm\infty} \dfrac{\cos^2 x}{x^2} = 0$.



Thus, $y = 0$ is the horizontal asymptote. $\lim\limits_{x\to 0} \dfrac{\cos^2 x}{x^2} = \infty$ because $\cos^2 x \to 1$ and $x^2 \to 0^+$ as $x \to 0$, so $x = 0$ is the vertical asymptote.

22. From the graph of $y = f(x) = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$, it appears that there are 2 horizontal asymptotes and possibly 2 vertical asymptotes. To obtain a different form for f, let's multiply and divide it by its conjugate.

$$f_1(x) = \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right) \frac{\sqrt{x^2+x+1} + \sqrt{x^2-x}}{\sqrt{x^2+x+1} + \sqrt{x^2-x}} = \frac{(x^2+x+1) - (x^2-x)}{\sqrt{x^2+x+1} + \sqrt{x^2-x}}$$
$$= \frac{2x+1}{\sqrt{x^2+x+1} + \sqrt{x^2-x}}$$

Now

$$\lim_{x\to\infty} f_1(x) = \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1} + \sqrt{x^2-x}}$$
$$= \lim_{x\to\infty} \frac{2 + (1/x)}{\sqrt{1 + (1/x) + (1/x^2)} + \sqrt{1 - (1/x)}} \qquad \left[\text{since } \sqrt{x^2} = x \text{ for } x > 0\right]$$
$$= \frac{2}{1+1} = 1,$$

so $y = 1$ is a horizontal asymptote. For $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the denominator by x, with $x < 0$, we get

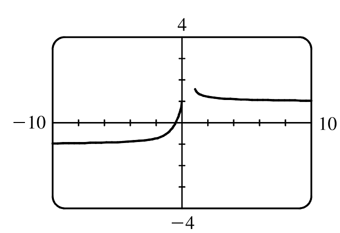
$$\frac{\sqrt{x^2+x+1} + \sqrt{x^2-x}}{x} = -\frac{\sqrt{x^2+x+1} + \sqrt{x^2-x}}{\sqrt{x^2}} = -\left[\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}}\right]$$

Therefore,

$$\lim_{x\to-\infty} f_1(x) = \lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2+x+1} + \sqrt{x^2-x}} = \lim_{x\to\infty} \frac{2 + (1/x)}{-\left[\sqrt{1 + (1/x) + (1/x^2)} + \sqrt{1 - (1/x)}\right]}$$
$$= \frac{2}{-(1+1)} = -1,$$

so $y = -1$ is a horizontal asymptote.

The domain of f is $(-\infty, 0] \cup [1, \infty)$. As $x \to 0^-$, $f(x) \to 1$, so $x = 0$ is *not* a vertical asymptote. As $x \to 1^+$, $f(x) \to \sqrt{3}$, so $x = 1$ is *not* a vertical asymptote and hence there are no vertical asymptotes.



23. Since $2x-1 \le f(x) \le x^2$ for $0<x<3$ and $\lim\limits_{x\to1}(2x-1)=1=\lim\limits_{x\to1}x^2$, we have $\lim\limits_{x\to1}f(x)=1$ by the Squeeze Theorem.

24. Let $f(x)=-x^2$, $g(x)=x^2\cos(1/x^2)$ and $h(x)=x^2$. Then since $\left|\cos(1/x^2)\right| \le 1$ for $x\neq0$, we have $f(x)\le g(x)\le h(x)$ for $x\neq0$, and so $\lim\limits_{x\to0}f(x)=\lim\limits_{x\to0}h(x)=0 \;\Rightarrow\; \lim\limits_{x\to0}g(x)=0$ by the Squeeze Theorem.

25. Given $\varepsilon>0$, we need $\delta>0$ such that if $0<|x-2|<\delta$, then $|(14-5x)-4|<\varepsilon$. But $|(14-5x)-4|<\varepsilon \;\Leftrightarrow\; |-5x+10|<\varepsilon \;\Leftrightarrow\; |-5|\,|x-2|<\varepsilon \;\Leftrightarrow\; |x-2|<\varepsilon/5$. So if we choose $\delta=\varepsilon/5$, then $0<|x-2|<\delta \;\Rightarrow\; |(14-5x)-4|<\varepsilon$. Thus, $\lim\limits_{x\to2}(14-5x)=4$ by the definition of a limit.

26. Given $\varepsilon>0$ we must find $\delta>0$ so that if $0<|x-0|<\delta$, then $|\sqrt[3]{x}-0|<\varepsilon$. Now $|\sqrt[3]{x}-0|=|\sqrt[3]{x}|<\varepsilon \;\Rightarrow\; |x|=|\sqrt[3]{x}|^3<\varepsilon^3$. So take $\delta=\varepsilon^3$. Then $0<|x-0|=|x|<\varepsilon^3 \;\Rightarrow\; |\sqrt[3]{x}-0|=|\sqrt[3]{x}|=\sqrt[3]{|x|}<\sqrt[3]{\varepsilon^3}=\varepsilon$.
Therefore, by the definition of a limit, $\lim\limits_{x\to0}\sqrt[3]{x}=0$.

27. Given $\varepsilon>0$, we need $\delta>0$ so that if $0<|x-2|<\delta$, then $\left|x^2-3x-(-2)\right|<\varepsilon$. First, note that if $|x-2|<1$, then $-1<x-2<1$, so $0<x-1<2 \;\Rightarrow\; |x-1|<2$. Now let $\delta=\min\{\varepsilon/2,1\}$. Then $0<|x-2|<\delta \;\Rightarrow\; \left|x^2-3x-(-2)\right|=|(x-2)(x-1)|=|x-2|\,|x-1|<(\varepsilon/2)(2)=\varepsilon$.
Thus, $\lim\limits_{x\to2}(x^2-3x)=-2$ by the definition of a limit.

28. Given $M>0$, we need $\delta>0$ such that if $0<x-4<\delta$, then $2/\sqrt{x-4}>M$. This is true $\;\Leftrightarrow\; \sqrt{x-4}<2/M \;\Leftrightarrow\; x-4<4/M^2$. So if we choose $\delta=4/M^2$, then $0<x-4<\delta \;\Rightarrow\; 2/\sqrt{x-4}>M$. So by the definition of a limit, $\lim\limits_{x\to4^+}\left(2/\sqrt{x-4}\right)=\infty$.

29. (a) $f(x)=\sqrt{-x}$ if $x<0$, $f(x)=3-x$ if $0\le x<3$, $f(x)=(x-3)^2$ if $x>3$.

(i) $\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}(3-x)=3$

(ii) $\lim\limits_{x\to0^-}f(x)=\lim\limits_{x\to0^-}\sqrt{-x}=0$

(iii) Because of (i) and (ii), $\lim\limits_{x\to0}f(x)$ does not exist.

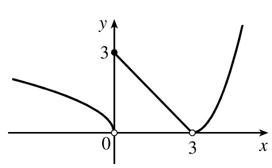
(iv) $\lim\limits_{x\to3^-}f(x)=\lim\limits_{x\to3^-}(3-x)=0$

(v) $\lim\limits_{x\to3^+}f(x)=\lim\limits_{x\to3^+}(x-3)^2=0$

(vi) Because of (iv) and (v), $\lim\limits_{x\to3}f(x)=0$.

(b) f is discontinuous at 0 since $\lim\limits_{x\to0}f(x)$ does not exist.
f is discontinuous at 3 since $f(3)$ does not exist.

(c)



30. (a) $g(x)=2x-x^2$ if $0\le x\le2$, $g(x)=2-x$ if $2<x\le3$, $g(x)=x-4$ if $3<x<4$, $g(x)=\pi$ if $x\ge4$.
Therefore, $\lim\limits_{x\to2^-}g(x)=\lim\limits_{x\to2^-}\left(2x-x^2\right)=0$ and $\lim\limits_{x\to2^+}g(x)=\lim\limits_{x\to2^+}(2-x)=0$. Thus, $\lim\limits_{x\to2}g(x)=0=g(2)$, so g is continuous at 2. $\lim\limits_{x\to3^-}g(x)=\lim\limits_{x\to3^-}(2-x)=-1$ and $\lim\limits_{x\to3^+}g(x)=\lim\limits_{x\to3^+}(x-4)=-1$. Thus,

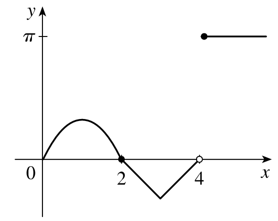
$\lim_{x\to 3} g(x) = -1 = g(3)$, so g is continuous at 3.

$\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (x-4) = 0$ and $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \pi = \pi$.

Thus, $\lim_{x\to 4} g(x)$ does not exist, so g is discontinuous at 4. But

$\lim_{x\to 4^+} g(x) = \pi = g(4)$, so g is continuous from the right at 4.

(b)



31. $\sin x$ and e^x are continuous on $\mathbb{R}$ by Theorem 2.5.7. Since e^x is continuous on $\mathbb{R}$, $e^{\sin x}$ is continuous on $\mathbb{R}$ by Theorem 2.5.9. Lastly, x is continuous on $\mathbb{R}$ since it's a polynomial and the product $xe^{\sin x}$ is continuous on its domain $\mathbb{R}$ by Theorem 2.5.4.

32. $x^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0, \infty)$ by Theorem 2.5.7, so the composition $\sqrt{x^2-9}$ is continuous on $\{x \mid x^2 - 9 \geq 0\} = (-\infty, -3] \cup [3, \infty)$ by Theorem 2.5.9. Note that $x^2 - 2 \neq 0$ on this set and so the quotient function $g(x) = \dfrac{\sqrt{x^2-9}}{x^2-2}$ is continuous on its domain, $(-\infty, -3] \cup [3, \infty)$ by Theorem 2.5.4.

33. $f(x) = x^5 - x^3 + 3x - 5$ is continuous on the interval $[1, 2]$, $f(1) = -2$, and $f(2) = 25$. Since $-2 < 0 < 25$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^3 + 3x - 5 = 0$ in the interval $(1, 2)$.

34. $f(x) = \cos\sqrt{x} - e^x + 2$ is continuous on the interval $[0, 1]$, $f(0) = 2$, and $f(1) \approx -0.2$. Since $-0.2 < 0 < 2$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos\sqrt{x} - e^x + 2 = 0$, or $\cos\sqrt{x} = e^x - 2$, in the interval $(0, 1)$.

35. (a) The slope of the tangent line at $(2, 1)$ is

$$\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{9-2x^2-1}{x-2} = \lim_{x\to 2} \frac{8-2x^2}{x-2} = \lim_{x\to 2} \frac{-2(x^2-4)}{x-2} = \lim_{x\to 2} \frac{-2(x-2)(x+2)}{x-2}$$
$$= \lim_{x\to 2} [-2(x+2)] = -2 \cdot 4 = -8$$

(b) An equation of this tangent line is $y - 1 = -8(x-2)$ or $y = -8x + 17$.

36. For a general point with x-coordinate a, we have

$$m = \lim_{x\to a} \frac{2/(1-3x) - 2/(1-3a)}{x-a} = \lim_{x\to a} \frac{2(1-3a) - 2(1-3x)}{(1-3a)(1-3x)(x-a)} = \lim_{x\to a} \frac{6(x-a)}{(1-3a)(1-3x)(x-a)}$$
$$= \lim_{x\to a} \frac{6}{(1-3a)(1-3x)} = \frac{6}{(1-3a)^2}$$

For $a = 0$, $m = 6$ and $f(0) = 2$, so an equation of the tangent line is $y - 2 = 6(x-0)$ or $y = 6x + 2$. For $a = -1$, $m = \frac{3}{8}$ and $f(-1) = \frac{1}{2}$, so an equation of the tangent line is $y - \frac{1}{2} = \frac{3}{8}(x+1)$ or $y = \frac{3}{8}x + \frac{7}{8}$.

37. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1+h]$ is

$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{1 + 2(1+h) + (1+h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10+h}{4}$$

[continued]

So for the following intervals the average velocities are:

(i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = (10 + 2)/4 = 3$ m/s (ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = (10 + 1)/4 = 2.75$ m/s

(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = (10 + 0.5)/4 = 2.625$ m/s (iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = (10 + 0.1)/4 = 2.525$ m/s

(b) When $t = 1$, the instantaneous velocity is $\lim\limits_{h \to 0} \dfrac{s(1+h) - s(1)}{h} = \lim\limits_{h \to 0} \dfrac{10+h}{4} = \dfrac{10}{4} = 2.5$ m/s.

38. (a) When V increases from 200 in^3 to 250 in^3, we have $\Delta V = 250 - 200 = 50$ in^3, and since $P = 800/V$,

$\Delta P = P(250) - P(200) = \dfrac{800}{250} - \dfrac{800}{200} = 3.2 - 4 = -0.8$ lb/in^2. So the average rate of change

is $\dfrac{\Delta P}{\Delta V} = \dfrac{-0.8}{50} = -0.016 \, \dfrac{\text{lb/in}^2}{\text{in}^3}$.

(b) Since $V = 800/P$, the instantaneous rate of change of V with respect to P is

$$\lim_{h \to 0} \frac{\Delta V}{\Delta P} = \lim_{h \to 0} \frac{V(P+h) - V(P)}{h} = \lim_{h \to 0} \frac{800/(P+h) - 800/P}{h} = \lim_{h \to 0} \frac{800\,[P - (P+h)]}{h(P+h)P}$$

$$= \lim_{h \to 0} \frac{-800}{(P+h)P} = -\frac{800}{P^2}$$

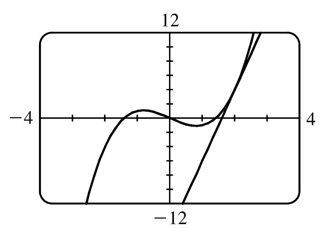
which is inversely proportional to the square of P.

39. (a) $f'(2) = \lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 2} \dfrac{x^3 - 2x - 4}{x - 2}$

$$= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 2)}{x - 2} = \lim_{x \to 2} (x^2 + 2x + 2) = 10$$

(b) $y - 4 = 10(x - 2)$ or $y = 10x - 16$

(c)



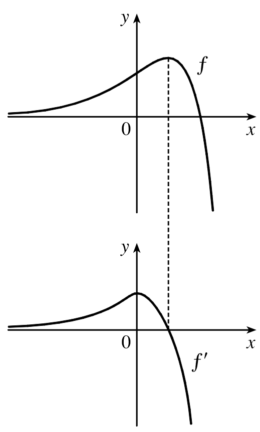
40. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.

41. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

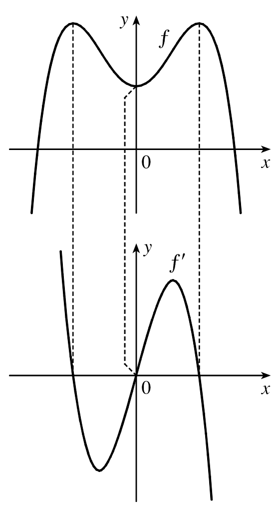
(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

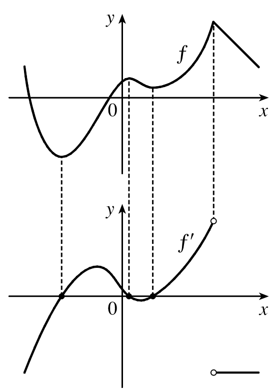
42.



43.



44.



45. (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{3-5(x+h)}-\sqrt{3-5x}}{h} \dfrac{\sqrt{3-5(x+h)}+\sqrt{3-5x}}{\sqrt{3-5(x+h)}+\sqrt{3-5x}}$

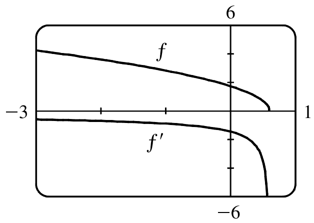
$$= \lim_{h\to 0} \frac{[3-5(x+h)]-(3-5x)}{h\left(\sqrt{3-5(x+h)}+\sqrt{3-5x}\right)} = \lim_{h\to 0} \frac{-5}{\sqrt{3-5(x+h)}+\sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

(b) Domain of f: (the radicand must be nonnegative) $3-5x \geq 0 \quad \Rightarrow$

$5x \leq 3 \quad \Rightarrow \quad x \in \left(-\infty, \frac{3}{5}\right]$

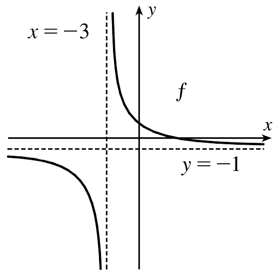
Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero;

$x \in \left(-\infty, \frac{3}{5}\right)$

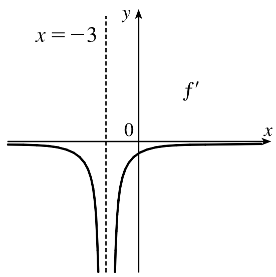


(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.

46. (a) As $x \to \pm\infty$, $f(x) = (4-x)/(3+x) \to -1$, so there is a horizontal asymptote at $y=-1$. As $x \to -3^+$, $f(x) \to \infty$, and as $x \to -3^-$, $f(x) \to -\infty$. Thus, there is a vertical asymptote at $x=-3$.



(b) Note that f is decreasing on $(-\infty, -3)$ and $(-3, \infty)$, so f' is negative on those intervals. As $x \to \pm\infty$, $f' \to 0$. As $x \to -3^-$ and as $x \to -3^+$, $f' \to -\infty$.



(c) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{4-(x+h)}{3+(x+h)} - \dfrac{4-x}{3+x}}{h} = \lim\limits_{h\to 0} \dfrac{(3+x)\,[4-(x+h)]-(4-x)\,[3+(x+h)]}{h\,[3+(x+h)]\,(3+x)}$

$$= \lim_{h\to 0} \frac{(12-3x-3h+4x-x^2-hx)-(12+4x+4h-3x-x^2-hx)}{h[3+(x+h)](3+x)}$$

$$= \lim_{h\to 0} \frac{-7h}{h\,[3+(x+h)]\,(3+x)} = \lim_{h\to 0} \frac{-7}{[3+(x+h)]\,(3+x)} = -\frac{7}{(3+x)^2}$$

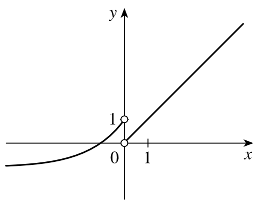
(d) The graphing device confirms our graph in part (b).

47. f is not differentiable: at $x=-4$ because f is not continuous, at $x=-1$ because f has a corner, at $x=2$ because f is not continuous, and at $x=5$ because f has a vertical tangent.

48. The graph of a has tangent lines with positive slope for $x<0$ and negative slope for $x>0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a. The graph of c has horizontal tangent lines to the left and right of the x-axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c. Therefore, a is the graph of f, c is the graph of f', and b is the graph of f''.

49. Domain: $(-\infty, 0) \cup (0, \infty)$; $\lim_{x\to 0^-} f(x) = 1$; $\lim_{x\to 0^+} f(x) = 0$;

$f'(x) > 0$ for all x in the domain; $\lim_{x\to -\infty} f'(x) = 0$; $\lim_{x\to \infty} f'(x) = 1$



50. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h\to 0} \frac{P(t+h) - P(t)}{h} \approx \frac{P(t+h) - P(t)}{h}$ for small values of h.

For 1950: $P'(1950) \approx \frac{P(1960) - P(1950)}{1960 - 1950} = \frac{35.7 - 31.1}{10} = 0.46$

For 1960: We estimate $P'(1960)$ by using $h = -10$ and $h = 10$, and then average the two results to obtain a final estimate.

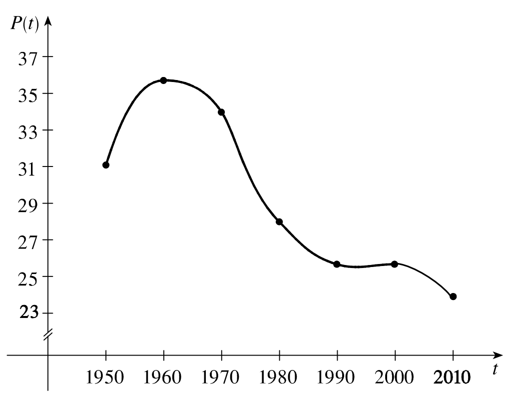
$h = -10 \quad \Rightarrow \quad P'(1960) \approx \frac{P(1950) - P(1960)}{1950 - 1960} = \frac{31.1 - 35.7}{-10} = 0.46$

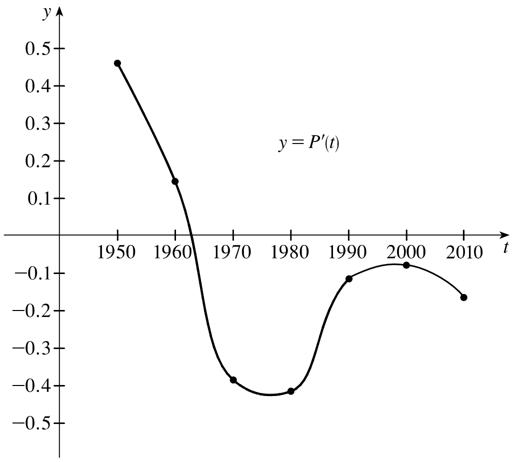
$h = 10 \quad \Rightarrow \quad P'(1960) \approx \frac{P(1970) - P(1960)}{1970 - 1960} = \frac{34.0 - 35.7}{10} = -0.17$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46 + (-0.17)] = 0.145$.

t	1950	1960	1970	1980	1990	2000	2010
$P'(t)$	0.460	0.145	−0.385	−0.415	−0.115	−0.085	−0.170

(c)





(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, 1995, and 2005.

51. $B'(t)$ is the rate at which the number of US \$20 bills in circulation is changing with respect to time. Its units are billions of bills per year. We use a symmetric difference quotient to estimate $B'(2000)$.

$B'(2000) \approx \frac{B(2005) - B(1995)}{2005 - 1995} = \frac{5.77 - 4.21}{10} = 0.156$ billions of bills per year (or 156 million bills per year).

52. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$, $F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

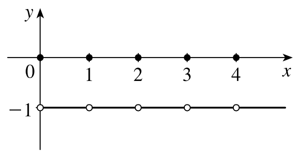
(c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

53. $|f(x)| \le g(x) \quad \Leftrightarrow \quad -g(x) \le f(x) \le g(x)$ and $\lim\limits_{x \to a} g(x) = 0 = \lim\limits_{x \to a} -g(x)$.

Thus, by the Squeeze Theorem, $\lim\limits_{x \to a} f(x) = 0$.

54. (a) Note that f is an even function since $f(x) = f(-x)$. Now for any integer n, $[\![n]\!] + [\![-n]\!] = n - n = 0$, and for any real number k which is not an integer, $[\![k]\!] + [\![-k]\!] = [\![k]\!] + (-[\![k]\!] - 1) = -1$. So $\lim\limits_{x \to a} f(x)$ exists (and is equal to -1) for all values of a.



(b) f is discontinuous at all integers.

PROBLEMS PLUS

1. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1}\frac{t^2-1}{t^3-1} = \lim_{t\to 1}\frac{(t-1)(t+1)}{(t-1)\left(t^2+t+1\right)} = \lim_{t\to 1}\frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x}+1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$.

2. First rationalize the numerator: $\displaystyle\lim_{x\to 0}\frac{\sqrt{ax+b}-2}{x}\cdot\frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim_{x\to 0}\frac{ax+b-4}{x\left(\sqrt{ax+b}+2\right)}$. Now since the denominator approaches 0 as $x \to 0$, the limit will exist only if the numerator also approaches 0 as $x \to 0$. So we require that $a(0)+b-4=0 \;\Rightarrow\; b=4$. So the equation becomes $\displaystyle\lim_{x\to 0}\frac{a}{\sqrt{ax+4}+2} = 1 \;\Rightarrow\; \frac{a}{\sqrt{4}+2} = 1 \;\Rightarrow\; a = 4$.

Therefore, $a = b = 4$.

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$.

Therefore, $\displaystyle\lim_{x\to 0}\frac{|2x-1|-|2x+1|}{x} = \lim_{x\to 0}\frac{-(2x-1)-(2x+1)}{x} = \lim_{x\to 0}\frac{-4x}{x} = \lim_{x\to 0}(-4) = -4.$

4. Let R be the midpoint of OP, so the coordinates of R are $\left(\frac{1}{2}x, \frac{1}{2}x^2\right)$ since the coordinates of P are $\left(x, x^2\right)$. Let $Q = (0, a)$.

Since the slope $m_{OP} = \dfrac{x^2}{x} = x$, $m_{QR} = -\dfrac{1}{x}$ (negative reciprocal). But $m_{QR} = \dfrac{\frac{1}{2}x^2 - a}{\frac{1}{2}x - 0} = \dfrac{x^2-2a}{x}$, so we conclude that $-1 = x^2 - 2a \;\Rightarrow\; 2a = x^2+1 \;\Rightarrow\; a = \frac{1}{2}x^2 + \frac{1}{2}$. As $x \to 0$, $a \to \frac{1}{2}$, and the limiting position of Q is $\left(0, \frac{1}{2}\right)$.

5. (a) For $0 < x < 1$, $[\![x]\!] = 0$, so $\dfrac{[\![x]\!]}{x} = 0$, and $\displaystyle\lim_{x\to 0^+}\frac{[\![x]\!]}{x} = 0$. For $-1 < x < 0$, $[\![x]\!] = -1$, so $\dfrac{[\![x]\!]}{x} = \dfrac{-1}{x}$, and

$\displaystyle\lim_{x\to 0^-}\frac{[\![x]\!]}{x} = \lim_{x\to 0^-}\left(\frac{-1}{x}\right) = \infty$. Since the one-sided limits are not equal, $\displaystyle\lim_{x\to 0}\frac{[\![x]\!]}{x}$ does not exist.

(b) For $x > 0$, $1/x - 1 \le [\![1/x]\!] \le 1/x \;\Rightarrow\; x(1/x-1) \le x[\![1/x]\!] \le x(1/x) \;\Rightarrow\; 1 - x \le x[\![1/x]\!] \le 1$.

As $x \to 0^+$, $1 - x \to 1$, so by the Squeeze Theorem, $\displaystyle\lim_{x\to 0^+} x[\![1/x]\!] = 1$.

For $x < 0$, $1/x - 1 \le [\![1/x]\!] \le 1/x \;\Rightarrow\; x(1/x-1) \ge x[\![1/x]\!] \ge x(1/x) \;\Rightarrow\; 1 - x \ge x[\![1/x]\!] \ge 1$.

As $x \to 0^-$, $1 - x \to 1$, so by the Squeeze Theorem, $\displaystyle\lim_{x\to 0^-} x[\![1/x]\!] = 1$.

Since the one-sided limits are equal, $\displaystyle\lim_{x\to 0} x[\![1/x]\!] = 1$.

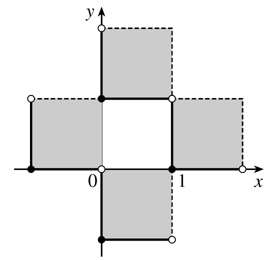
6. (a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$. Since $[\![x]\!]^2$ and $[\![y]\!]^2$ are positive integers or 0, there are only 4 cases:

Case (i): $[\![x]\!] = 1, [\![y]\!] = 0 \quad \Rightarrow 1 \le x < 2$ and $0 \le y < 1$

Case (ii): $[\![x]\!] = -1, [\![y]\!] = 0 \Rightarrow -1 \le x < 0$ and $0 \le y < 1$

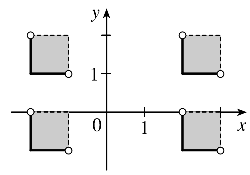
Case (iii): $[\![x]\!] = 0, [\![y]\!] = 1 \quad \Rightarrow 0 \le x < 1$ and $1 \le y < 2$

Case (iv): $[\![x]\!] = 0, [\![y]\!] = -1 \Rightarrow 0 \le x < 1$ and $-1 \le y < 0$

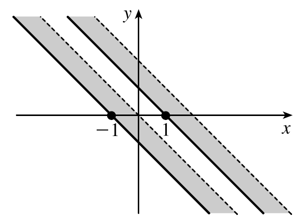


(b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$. The only integral solution of $n^2 - m^2 = 3$ is $n = \pm 2$ and $m = \pm 1$. So the graph is

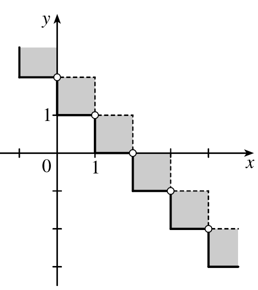
$$\{(x, y) \mid [\![x]\!] = \pm 2,\ [\![y]\!] = \pm 1\} = \left\{ (x, y) \,\middle|\, \begin{array}{l} 2 \le x \le 3 \ \text{ or } \ -2 \le x < 1, \\ 1 \le y < 2 \ \text{ or } \ -1 \le y < 0 \end{array} \right\}.$$



(c) $[\![x + y]\!]^2 = 1 \quad \Rightarrow \quad [\![x + y]\!] = \pm 1 \quad \Rightarrow \quad 1 \le x + y < 2$ or $-1 \le x + y < 0$



(d) For $n \le x < n + 1$, $[\![x]\!] = n$. Then $[\![x]\!] + [\![y]\!] = 1 \quad \Rightarrow \quad [\![y]\!] = 1 - n \quad \Rightarrow$ $1 - n \le y < 2 - n$. Choosing integer values for n produces the graph.

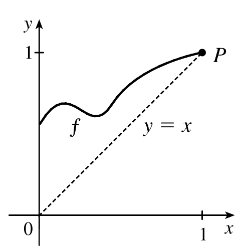


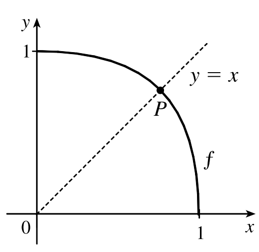
7. f is continuous on $(-\infty, a)$ and (a, ∞). To make f continuous on $\mathbb{R}$, we must have continuity at a. Thus,

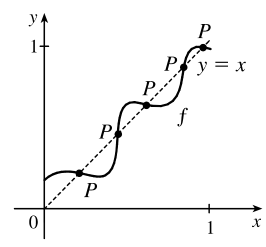
$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \quad \Rightarrow \quad \lim_{x \to a^+} x^2 = \lim_{x \to a^-} (x + 1) \quad \Rightarrow \quad a^2 = a + 1 \quad \Rightarrow \quad a^2 - a - 1 = 0 \quad \Rightarrow$$

[by the quadratic formula] $\quad a = \left(1 \pm \sqrt{5}\right)/2 \approx 1.618$ or -0.618.

8. (a) Here are a few possibilities:







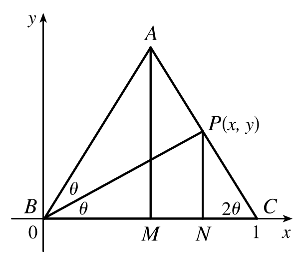
(b) The "obstacle" is the line $x = y$ (see diagram). Any intersection of the graph of f with the line $y = x$ constitutes a fixed point, and if the graph of the function does not cross the line somewhere in $(0, 1)$, then it must either start at $(0, 0)$ (in which case 0 is a fixed point) or finish at $(1, 1)$ (in which case 1 is a fixed point).

(c) Consider the function $F(x) = f(x) - x$, where f is any continuous function with domain $[0, 1]$ and range in $[0, 1]$. We shall prove that f has a fixed point. Now if $f(0) = 0$ then we are done: f has a fixed point (the number 0), which is what we are trying to prove. So assume $f(0) \neq 0$. For the same reason we can assume that $f(1) \neq 1$. Then $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there exists some number c in the interval $(0, 1)$ such that $F(c) = f(c) - c = 0$. So $f(c) = c$, and therefore f has a fixed point.

9. $$\begin{cases} \lim\limits_{x\to a}[f(x)+g(x)] = 2 \\ \lim\limits_{x\to a}[f(x)-g(x)] = 1 \end{cases} \Rightarrow \begin{cases} \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) = 2 & \textbf{(1)} \\ \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x) = 1 & \textbf{(2)} \end{cases}$$

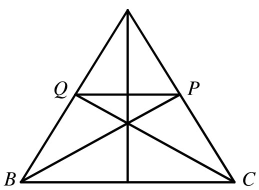
Adding equations **(1)** and **(2)** gives us $2\lim\limits_{x\to a} f(x) = 3 \;\Rightarrow\; \lim\limits_{x\to a} f(x) = \frac{3}{2}$. From equation **(1)**, $\lim\limits_{x\to a} g(x) = \frac{1}{2}$. Thus, $\lim\limits_{x\to a}[f(x)\,g(x)] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4}$.

10. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular from P, as shown. We see from $\angle NCP$ that $\tan 2\theta = \dfrac{y}{1-x}$, and from $\angle NBP$ that $\tan\theta = y/x$. Using the double-angle formula for tangents, we get $\dfrac{y}{1-x} = \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2(y/x)}{1-(y/x)^2}$. After a bit of simplification, this becomes $\dfrac{1}{1-x} = \dfrac{2x}{x^2-y^2} \;\Leftrightarrow\; y^2 = x(3x-2)$.

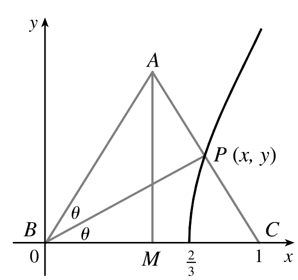


As the altitude AM decreases in length, the point P will approach the x-axis, that is, $y \to 0$, so the limiting location of P must be one of the roots of the equation $x(3x-2) = 0$. Obviously it is not $x = 0$ (the point P can never be to the left of the altitude AM, which it would have to be in order to approach 0) so it must be $3x - 2 = 0$, that is, $x = \frac{2}{3}$.

Solution 2: We add a few lines to the original diagram, as shown. Now note that $\angle BPQ = \angle PBC$ (alternate angles; $QP \parallel BC$ by symmetry) and similarly $\angle CQP = \angle QCB$. So ΔBPQ and ΔCQP are isosceles, and the line segments BQ, QP and PC are all of equal length. As $|AM| \to 0$, P and Q approach points on the base, and the point P is seen to approach a position two-thirds of the way between B and C, as above.



(b) The equation $y^2 = x(3x-2)$ calculated in part (a) is the equation of the curve traced out by P. Now as $|AM| \to \infty$, $2\theta \to \frac{\pi}{2}$, $\theta \to \frac{\pi}{4}$, $x \to 1$, and since $\tan\theta = y/x$, $y \to 1$. Thus, P only traces out the part of the curve with $0 \le y < 1$.



11. (a) Consider $G(x) = T(x + 180°) - T(x)$. Fix any number a. If $G(a) = 0$, we are done: Temperature at $a =$ Temperature at $a + 180°$. If $G(a) > 0$, then $G(a + 180°) = T(a + 360°) - T(a + 180°) = T(a) - T(a + 180°) = -G(a) < 0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a + 180°]$. If $G(a) < 0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

12. $$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{(x+h)f(x+h) - xf(x)}{h} = \lim_{h\to 0} \left[\frac{xf(x+h) - xf(x)}{h} + \frac{hf(x+h)}{h}\right]$$
$$= x \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x+h) = xf'(x) + f(x)$$

because f is differentiable and therefore continuous.

13. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0 + 0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 \quad \Rightarrow \quad f(0) = 2f(0)$.

Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

(b) $$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\left[f(0) + f(h) + 0^2h + 0h^2\right] - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{x\to 0} \frac{f(x)}{x} = 1$$

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[f(x) + f(h) + x^2h + xh^2\right] - f(x)}{h} = \lim_{h\to 0} \frac{f(h) + x^2h + xh^2}{h}$$
$$= \lim_{h\to 0} \left[\frac{f(h)}{h} + x^2 + xh\right] = 1 + x^2$$

14. We are given that $|f(x)| \le x^2$ for all x. In particular, $|f(0)| \le 0$, but $|a| \ge 0$ for all a. The only conclusion is that $f(0) = 0$. Now $\left|\dfrac{f(x) - f(0)}{x - 0}\right| = \left|\dfrac{f(x)}{x}\right| = \dfrac{|f(x)|}{|x|} \le \dfrac{x^2}{|x|} = \dfrac{|x^2|}{|x|} = |x| \quad \Rightarrow \quad -|x| \le \dfrac{f(x) - f(0)}{x - 0} \le |x|$.

But $\lim_{x\to 0} (-|x|) = 0 = \lim_{x\to 0} |x|$, so by the Squeeze Theorem, $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = 0$. So by the definition of a derivative, f is differentiable at 0 and, furthermore, $f'(0) = 0$.