Chapter 3: Discrete Random Variables

3.1 The Notion of a Random Variable

3.1

Sample Space:

	Michael \ Carlos	0	1	2
1/4	0	(0,0)	(0,1)	(0,2)
1/2	1	(1,0)	(1,1)	(1,2)
1/4	2	(2,0)	(2,1)	(2,2)
		1/4	1/2	1/4

Probabilities

	0	1	2
0	$1/16$	$1/8$	$1/16$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

Mapping $S \to S_X$

	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$$P[X=0] = P[(0,0)] = 1/16$$

$$P[X=1] = P[\{(1,0),(1,1),(0,1)\}] = \frac{1}{2}$$

$$P[X=2] = 3 \times \frac{1}{16} + 2 \times \frac{1}{8} = \frac{7}{16}.$$

3.2

(a) $S=\{1,2,3,4,5,6\}$ $\quad p_1=p_2=p_3=p_4=p_5=p_6=\frac{1}{6}$

where $p_j=P[\{j\}]$

(b)

S		S_X
1	$\longrightarrow$	0
2	$\longrightarrow$	1
3	$\longrightarrow$	1
4	$\longrightarrow$	2
5	$\longrightarrow$	2
6	$\longrightarrow$	3

(c)

$P[X=0]=p_1=\frac{1}{6}$

$P[X=1]=p_2+p_3=\frac{2}{6}$

$P[X=2]=p_4+p_5=\frac{2}{6}$

$P[X=6]=p_6=\frac{1}{6}$

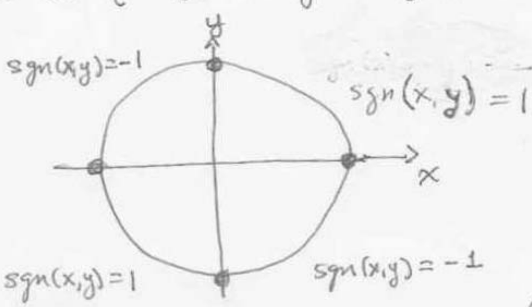
(d) $P[\cancel{X}Y=0]=p_1+p_2=\frac{2}{6}$ $\quad P[Y=1]=p_3+p_4=\frac{2}{6}$ $\quad P[Y=2]=p_5+p_6=\frac{2}{6}$

(e) $X=0$ corresponds to $\{1\}$

$Y=0$ corresponds to $\{1,2\}$

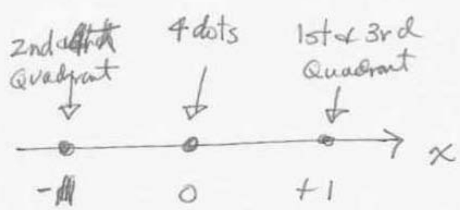
3.3

(a) $S=\{(x,y): x^2+y^2=r^2\}$ where r = radius of clock



Outcomes from S occur uniformly along the circle.

sgn(xy) = 0 at the dots



(c)

$P[X=-1]=P[\text{2nd}+\text{3rd Quad}]=\frac{1}{2}$

$P[X=0]=P[\{(r,0),(0,r),(-r,0),(0,-r)\}]$

$=0$

$P[X=1]=P[\text{1st \& 3rd Quad}]=\frac{1}{2}$

3.4

a) $S = \{0000, 0001, \ldots, 1111\}$

$p_{0000} = p_{0001} = \cdots = p_{1111} = \frac{1}{16}$

b)

S	0000	0001	0010	...	1111
	↓	↓	↓		↓
S_X	0	1	2	...	15

c) $p_0 = p_1 = p_2 = \cdots = p_{15} = \frac{1}{16}$

d) $P[0b_1b_2b_3] = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$ all $b_1b_2b_3$

$P[1b_1b_2b_3] = \frac{3}{4}\frac{1}{8} = \frac{3}{32}$ all $b_1b_2b_3$

$p_0' = p_1' = \cdots = p_7' = \frac{1}{32}$

$p_8' = p_9' = \cdots = p_{15}' = \frac{3}{32}$

3.5

Let A_i = Transmitter #1 sends a signal at time slot i
B_i = " #2 "

A signal gets through if $A_i B_i^c \cup A_i^c B_i$ occurs

Each experiment has 4 outcomes

(a) Experiment i

	A_i	A_i^c
B_i	1/4	1/4
B_i^c	1/4	1/4

Sample Space consists of a Cartesian product of the outcomes of the basic experiment

$S = (s_1, s_2, \ldots)$ where s_i is an outcome from basic experiment

(b) $X(s) = n$
if n is the first occurrence of $A_i B_i^c \cup A_i^c B_i$ in $s_1, s_2 \ldots$

(c) $P[A_i B_i^c \cup A_i^c B_i] = P[A_i B_i^c] + P[A_i^c B_i] = \frac{1}{2} = P[\text{success}]$

$P[X=k] = P[(k-1) \text{ failures, } 1 \text{ success}] = \left(\frac{1}{2}\right)^k$

3.6

$S = \{000, 111, 010, 101, 001, 110, 100, 011\}$

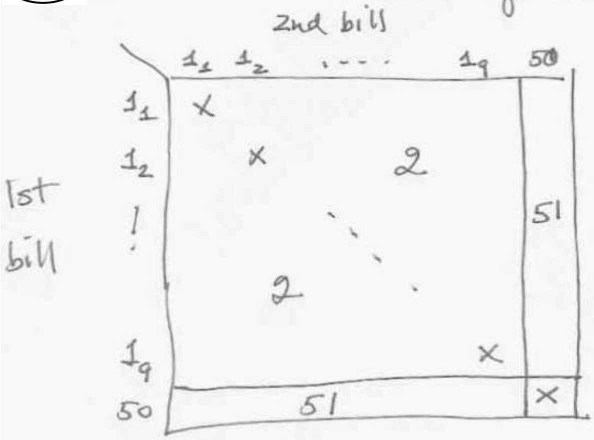
$X(s)$: 2, 2, 3, 3, 4, 4, 4, 4

$P[X=2] = P[\{000, 111\}] = \frac{1}{2}$

$P[X=3] = P[\{010, 101\}] = \frac{1}{4}$

$P[X=4] = P[\{001, 110, 100, 011\}] = \frac{1}{4}$

3.7) Draw 2 bills without replacement.



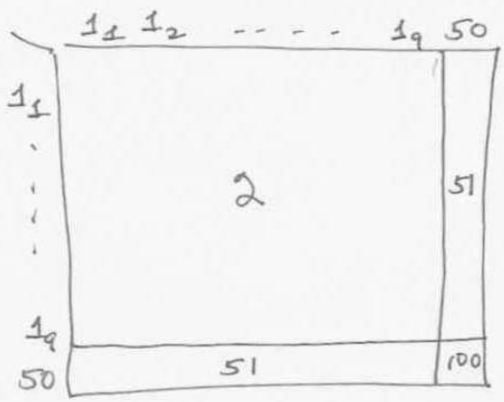
x not allowed

all other outcomes have probability $\frac{1}{9(10)} = \frac{1}{90}$

$$P[X=2] = \frac{9 \cdot 8}{90} = \frac{8}{10} = \frac{4}{5}$$

$$P[X=51] = \frac{9 \cdot 2}{90} = \frac{2}{10} = \frac{1}{5}$$

3.8 . Draw 2 bills with replacement.



all outcomes have probability $\frac{1}{10(10)} = \frac{1}{100}$

$$P[X=2] = \frac{81}{100}$$

$$P[X=51] = \frac{18}{100}$$

$$P[X=100] = \frac{1}{100}$$

3.9 (a) Let m be number of tails $0 \le m \le n$

then number of heads is $n-m$ and the difference is

$$Y = n - m - m = n - 2m \qquad 0 \le m \le n$$

$$\therefore S_Y = \{-n, -n+2, \ldots, n-2, n\}$$

(b) $P[Y=0] = P[n=2m] = P[m=\frac{n}{2}]$ for n even.

$P[Y=k] = P[n-2m=k] = P[m=\frac{n-k}{2}]$ for $n-k$ even

3.10 Let $S = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_{2^m}\}$ be the sequence of m-bit passwords as covered by the hacker.

The target system picks a password at random from S.

$X(\zeta)$ is the index of the selected password.

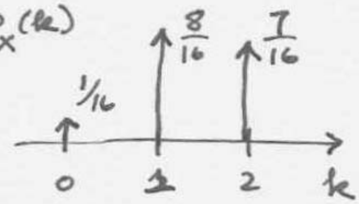
$S_X = \{1, 2, \ldots, 2^m\}$ where the value of X is selected at random from S_X.

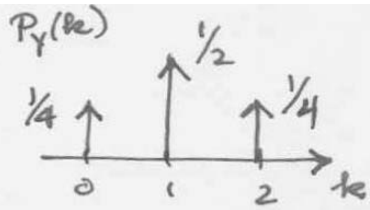
$P[i] = 2^{-m} \quad i \in S_X$.

3.2 Discrete Random Variables And Probability Mass Function

3.11

a)





the max function shifts probability mass to higher values of k

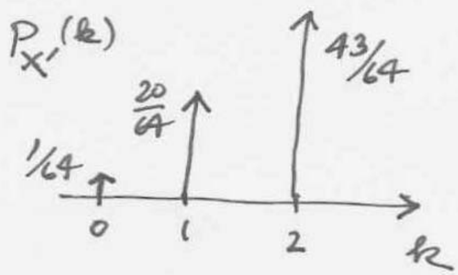
b) If Carlos uses a biased coin:

Michael \ Carlos	0	1	2	
0	0	1	2	1/4
1	1	1	2	1/2
2	2	2	2	1/4
	1/16	6/16	9/16	

$$P[X'=0] = \frac{1}{4}\frac{1}{16} = \frac{1}{64}$$

$$P[X'=1] = \frac{1}{16}\cdot\frac{1}{2} + \frac{6}{16}\cdot\frac{1}{2} + \frac{6}{16}\cdot\frac{1}{4} = \frac{20}{64}$$

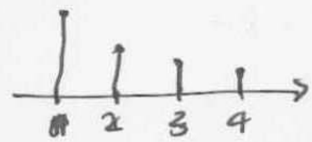
$$P[X'=2] = \frac{43}{64}$$



3.12

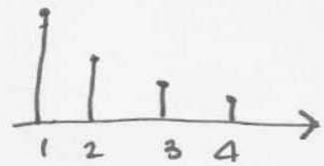
a) $1 = p_1 + p_2 + p_3 + p_4 = p_1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{12} p_1 \quad p_1 = \frac{12}{25}$

$p_1 = \frac{12}{25} \quad p_2 = \frac{6}{25} \quad p_3 = \frac{4}{25} \quad p_4 = \frac{3}{25}$



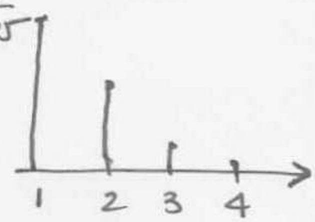
b) $1 = p_1\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{15}{8} p_1$

$p_1 = \frac{8}{15} \quad p_2 = \frac{4}{15} \quad p_3 = \frac{2}{15} \quad p_4 = \frac{1}{15}$



c) $1 = p_1\left(1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{64}\right) = \frac{105}{64} p_1$

$p_1 = \frac{64}{105} \quad p_2 = \frac{32}{105} \quad p_3 = \frac{8}{105} \quad p_4 = \frac{1}{105}$



pmf decays more steeply w/ each example

d) $1 = p_1 \sum_{i=1}^{\infty} \frac{1}{i}$ does not converge so this pmf does not extend to $\{1, 2, \ldots\}$

$1 = p_1 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = p_1 \frac{1}{1 - \frac{1}{2}} \Rightarrow p_1 = \frac{1}{2}.$

this extends to the geometric pmf.

$1 = p_1\left(1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{1+2} + \left(\frac{1}{2}\right)^{1+2+3} + \ldots\right)$

$= p_1 \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j(j+1)/2}$ this is a subseries of the geometric series so it converges

3.13

a) $1 = \sum_{k=1}^{\infty} \frac{c}{k^2} = c \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ is a special case of the zeta function

$= 1.6449 \Rightarrow c = 0.608$

The sum of the first 100 terms gives $1.6349 \Rightarrow c \approx 0.611$

b) $P[X > 4] = 1 - P[X \le 3] = 1 - c\underbrace{\left[1 + \frac{1}{4} + \frac{1}{9}\right]}_{0.8325}$

$= 0.1675$

c) $P[6 \le X \le 8] = c\left[\frac{1}{36} + \frac{1}{49} + \frac{1}{64}\right] = 0.390$

3.14

$P[X \ge 8] = \sum_{k=8}^{15} p_k = \frac{8}{16} = \frac{1}{2}$

$P[Y \ge 8] = \sum_{k=8}^{15} p'_k = 8 \cdot \frac{3}{32} = \frac{24}{32} = \frac{3}{4}$

3.15

		Terminal 2: p	$1-p$
Terminal 1	$\frac{1}{2}$	$\frac{1}{2}p$	$\frac{1}{2}q$
	$\frac{1}{2}$	$\frac{1}{2}p$	$\frac{1}{2}q$

$P_{success} = \frac{1}{2}q + \frac{1}{2}p = \frac{1}{2}$ same

$\therefore$ The pmf of X is unchanged.

$P[\text{Terminal 2 transmitted} \mid \text{success}] = \dfrac{P[\text{success and Terminal 2 transmitted}]}{P[\text{success}]}$

$= \dfrac{\frac{1}{2}p}{\frac{1}{2}} = p$

This suggests that terminal 2 should always transmit (at the expense of terminal 1).

3.16 from problem 3.7b:

(a) $P[X > 2] = 1 - P[X=2] = \frac{1}{5}$

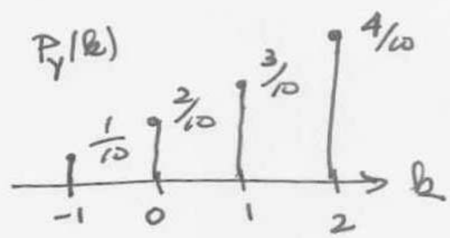
$P[X > 50] = P[X = 51]$
$= \frac{1}{5}$

(b) $P[X > 2] = 1 - P[X = 2] = \frac{19}{100}$

$P[X > 50] = P[X=51] + P[X=100]$
$= \frac{19}{100}$

3.17

(a)
$Y = 0 + 2 = 2$ with prob. $4/10$
$Y = -1 + 2 = 1$ " $3/10$
$Y = -2 + 2 = 0$ " $2/10$
$Y = -3 + 2 = -1$ " $1/10$



(b) $P[Y=2] = 4/10$

(c) $P[Y > 0] = P[Y = 2] + P[Y = 1] = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}$

3.18 Let X be number of transmissions until success.

$$P[X \le k] = \sum_{j=1}^{k} \left(\tfrac{1}{2}\right)^j = \tfrac{1}{2} \sum_{j=0}^{k-1} \left(\tfrac{1}{2}\right)^j = \tfrac{1}{2} \, \frac{1 - \left(\frac{1}{2}\right)^k}{\frac{1}{2}} = 1 - \left(\tfrac{1}{2}\right)^k$$

$1 - \left(\frac{1}{2}\right)^k = 0.99$ $\qquad \left(\frac{1}{2}\right)^k = 0.01$

$k = \frac{\ln 100}{\ln 2} = 6.64 \approx 7$

start sending refresh messages
7×10 seconds before expiry time

3.19

$$
\begin{aligned}
P[\text{decoding error}] &= P[\text{3 or more bit errors}] \\
&= \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5 \\
&= \frac{5!}{2!\,3!} 10^{-3} (.9)^2 + \frac{5!}{4!\,1!} 10^{-4} (.9) + 10^{-5} \\
&= (0.81)(10)(10^{-3}) + (0.9)(5)10^{-4} + 10^{-5} \\
&= 0.00856
\end{aligned}
$$

which is an order of magnitude less than without coding.

3.20

2nd toss

1st toss	1	2	3	4	5	6
1	0	1	2	3	4	5
2	-1	0	1	2	3	4
3	-2	-1	0	1	2	3
4	-3	-2	-1	0	1	2
5	-4	-3	-2	-1	0	1
6	-5	-4	-3	-2	-1	0

$P[X=0] = \frac{6}{36}$

$P[X=1] = \frac{5}{36} = P[X=-1]$

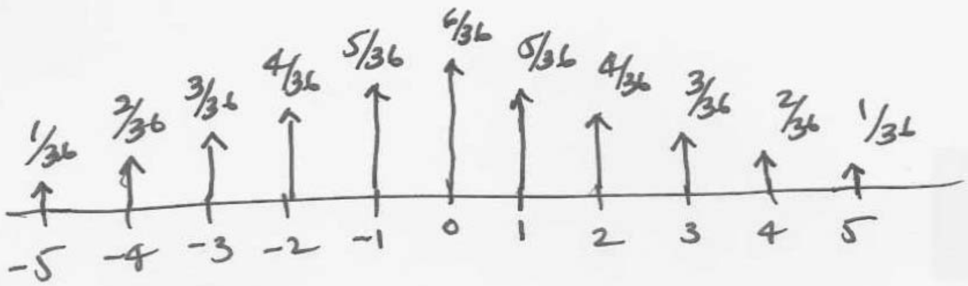
$P[X=2] = \frac{4}{36} = P[X=-2]$

$P[X=3] = \frac{3}{36} = P[X=-3]$

$P[X=4] = \frac{2}{36} = P[X=-4]$

$P[X=5] = \frac{1}{36} = P[X=-5]$

$$P[X=k] = \frac{6-|k|}{36}, \quad |k| \le 5$$



$$P[|X| \le 2] = \frac{6}{36} + 2\frac{5}{36} + 2\cdot\frac{4}{36} = \frac{24}{36}$$

3.3 Expected Value and Moments of Discrete Random Variable

3.21

$$E[X] = 0 \cdot \tfrac{1}{16} + 1 \cdot \tfrac{8}{16} + 2 \cdot \tfrac{7}{16} = \tfrac{22}{16}$$

$$E[Y] = 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{4} = 1$$ which is much less than $E[X]$.

We will use $VAR[X] = E[X^2] - E[X]^2$:

$$E[X^2] = 1^2 \cdot \tfrac{8}{16} + 4 \cdot \tfrac{7}{16} = \tfrac{36}{16}$$

$$E[Y^2] = 1 \cdot \tfrac{1}{2} + 4 \cdot \tfrac{1}{4} = \tfrac{3}{2}$$

$$VAR[X] = \tfrac{36}{16} - \left(\tfrac{22}{16}\right)^2 = \tfrac{82}{256}$$

X has lower variance than Y.

$$VAR[Y] = \tfrac{3}{2} - 1^2 = \tfrac{1}{2}$$

3.22

a) $E[X] = 1\cdot\frac{12}{25} + 2\cdot\frac{6}{25} + 3\cdot\frac{4}{25} + 4\cdot\frac{3}{25} = \frac{48}{25} = 1.92$

$E[X^2] = 1\cdot\frac{12}{25} + 4\cdot\frac{6}{25} + 9\cdot\frac{4}{25} + 16\cdot\frac{3}{25} = \frac{120}{25}$

$VAR[X] = \frac{120}{25} - \left(\frac{48}{25}\right)^2 = \frac{696}{625} = 1.114$

b) $E[X] = 1\cdot\frac{8}{15} + 2\cdot\frac{4}{15} + 3\cdot\frac{2}{15} + 4\cdot\frac{1}{15} = \frac{26}{15} = 1.73$

$E[X^2] = 1\cdot\frac{8}{15} + 4\cdot\frac{4}{15} + 9\frac{2}{15} + 16\frac{1}{15} = \frac{58}{15}$

$VAR[X] = \frac{58}{15} - \left(\frac{26}{15}\right)^2 = \frac{194}{225} = 0.862$

c) $E[X] = 1\cdot\frac{64}{105} + 2\cdot\frac{32}{105} + 3\cdot\frac{8}{105} + 4\frac{1}{105} = \frac{156}{105} = 1.48$

$E[X^2] = 1\cdot\frac{64}{105} + 4\cdot\frac{32}{105} + 9\cdot\frac{8}{105} + 16\frac{1}{105} = \frac{280}{105}$

$VAR[X] = \frac{280}{105} - \left(\frac{156}{105}\right)^2 = \frac{5064}{(105)^2} = 0.459$

The means and variances decrease as we progress through these distributions.

3.23

$$E[X] = \sum_{i=0}^{15} i \frac{1}{16} = \frac{1}{16}\sum_{i=0}^{15} i$$

Let $S_k = 1 + 2 + \ldots + k$

$S_k = k + (k-1) + \ldots + 1$

$$2S_k = k(k+1) \quad \Rightarrow \quad S_k = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

$$\therefore \quad E[X] = \frac{1}{16}\sum_{i=1}^{15} i = \frac{1}{16} \frac{15(16)}{2} = \frac{15}{2} = 7.5.$$

$$E[X^2] = \frac{1}{16}\sum_{i=1}^{15} i^2 \qquad \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

$$= \frac{1}{16}\frac{15(16)(31)}{6} = \frac{155}{2}$$

$$VAR[X] = \frac{155}{2} - \left(\frac{15}{2}\right)^2 = \frac{310-225}{4} = \frac{85}{4}$$

3.24

$$E[X] = 2\,P[X=2] + 3P[X=3] + 4\,P[X=4]$$

$$= 2\cdot\frac{1}{2} + 3\cdot\frac{1}{4} + 4\cdot\frac{1}{4} = 2\tfrac{3}{4} \text{ bits/block}$$

Let $X_1, X_2, \ldots,$ be the codeword lengths for a sequence of source outputs. The average codeword length is

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to E[X] \text{ for large } n$$

$\therefore$ $E[X]$ is the long term average number of bits per block.

3.25 Without replacement

$$E[X] = 2 \cdot \frac{4}{5} + 51 \frac{1}{5} = \frac{59}{5} = 11,80$$

$$E[X^2] = 4 \cdot \frac{4}{5} + 51^2 \cdot \frac{1}{5} = \frac{2617}{5}$$

$$VAR[X] = \frac{2617}{5} - \left(\frac{59}{5}\right)^2 = \frac{9604}{25} = 384.16$$

with replacement:

$$E[X] = 2 \cdot \frac{81}{100} + 51 \frac{18}{100} + 100 \cdot \frac{1}{100} = \frac{1180}{100} = 11.80$$

$$E[X^2] = 4 \cdot \frac{81}{100} + 51^2 \cdot \frac{18}{100} + 10^4 \cdot \frac{1}{100} = \frac{57142}{100}$$

$$VAR[X] = \frac{57142}{100} - \left(\frac{1180}{100}\right)^2 = \frac{43218}{100} = 432.18$$

Means in both draws is the same!

3.26

$$E[Y] = \sum_{j=-5}^{5} j\,P[Y=j]$$

$$= -5\cdot\frac{1}{36} - 4\cdot\frac{2}{36} - 3\cdot\frac{3}{36} - 2\cdot\frac{4}{36} - 1\cdot\frac{5}{36} + 0\cdot\frac{6}{36}$$

$$+1\cdot\frac{5}{36} + 2\cdot\frac{4}{36} + 3\cdot\frac{3}{36} + 4\cdot\frac{2}{36} + 5\cdot\frac{1}{36}$$

$$= 0$$

$$VAR[Y] = E[Y^2] = \sum_{j=-5}^{5} j^2 P[Y=j]$$

$$= \sum_{j=1}^{5} j^2 \left[P[X=j] + P[X=-j]\right]$$

$$= 1\cdot\frac{10}{36} + 4\cdot\frac{8}{36} + 9\cdot\frac{6}{36} + 16\cdot\frac{4}{36} + 25\cdot\frac{2}{36}$$

$$= \frac{185}{36}$$

3.27 $E[X] = \sum_{j=1}^{\infty} j\, P[X=j] = \sum_{j=1}^{\infty} j \frac{c}{j^2} = c\sum_{j=1}^{\infty} \frac{1}{j} = \infty$

mean does not exist.

$$E[X^2] = \sum_{j=1}^{\infty} j^2 \frac{c}{j^2} = c \sum_{j=1}^{\infty} 1 = \infty$$

none of the moments exist

This pmf decays sufficiently fast that probabilities add to 1, but too slowly for moments to exist.

3.28 $E[Y] = -1\cdot\frac{1}{10} + 0\cdot\frac{2}{10} + 1\cdot\frac{3}{10} + 2\cdot\frac{4}{10} = \frac{10}{10} = 1$

$$E[Y^2] = 1\cdot\frac{1}{10} + 1\cdot\frac{3}{10} + 4\cdot\frac{4}{10} = \frac{20}{10} = 2$$

$$VAR[Y] = 2 - 1^2 = 1.$$

3.29 $P[X=j] = \left(\frac{1}{2}\right)^j$

$$E[X] = \sum_{j=1}^{\infty} j\left(\frac{1}{2}\right)^j = \frac{1}{2}\sum_{j=1}^{\infty} j\left(\frac{1}{2}\right)^{j-1} = \frac{1}{2}\sum_{j=0}^{\infty} j\left(\frac{1}{2}\right)^j$$

from geometric series we have

$$\sum_{j=0}^{\infty} \alpha^j = \frac{1}{1-\alpha}$$

$$\therefore \quad \frac{d}{d\alpha}\sum_{j=0}^{\infty} \alpha^j = \sum_{j=0}^{\infty} j\alpha^{j-1} = \frac{1}{(1-\alpha)^2}$$

$$\therefore \quad \sum_{j=0}^{\infty} j\left(\frac{1}{2}\right)^{j-1} = \frac{1}{(1-\frac{1}{2})^2} = 4$$

and $E[X] = \frac{1}{2}\cdot 4 = 2$

3.30 from problem 3.19 a 5-bit codeword is decoded erroneously with probability $P_e = 0.00856$.

In 1000 codeword transmissions we expect only 8.56 to be in error.

In 1000 single bit transmissions, since $p = \frac{1}{10}$ we expect $1000 \cdot \frac{1}{10} = 100$ to be in error.

$\therefore$ Error rate is reduced at expense of slower information transmission rate.

3.31

$$P[X=k] = \binom{n}{k}\left(\tfrac{1}{2}\right)^n$$

$$E[aX^2+bX] = aE[X^2] + bE[X]$$

$$E[X] = \sum_{j=0}^{n} j\binom{n}{j}\left(\tfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right)^n \sum_{j=0}^{n} j\,\frac{n!}{j!\,(n-j)!}$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{j=1}^{n} \frac{n!}{(j-1)!\,(n-j)!} \qquad \text{let } j'=j-1$$

$$= \left(\tfrac{1}{2}\right)^n n \sum_{j'=0}^{n-1} \frac{(n-1)!}{j'!\,(n-1-j')!} = n\left(\tfrac{1}{2}\right)^n \sum_{j'=0}^{n-1}\binom{n-1}{j'}$$

$$= n\left(\tfrac{1}{2}\right)^n 2^{n-1} = \frac{n}{2}.$$

$$E[X^2] = \sum_{j=0}^{n} j^2\binom{n}{j}\left(\tfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right)^n n \sum_{j=1}^{n} j\,\frac{(n-1)!}{(j-1)!\,(n-j)!}$$

$$= n\left(\tfrac{1}{2}\right)^n \sum_{j'=0}^{n-1} (j'+1)\binom{n-1}{j'}$$

$$= n\left(\tfrac{1}{2}\right)\Bigg[\underbrace{\sum_{j'=0}^{n-1} j'\binom{n-1}{j'}\left(\tfrac{1}{2}\right)^{n-1}}_{\substack{(n-1)\frac{1}{2} \\ \text{expected value of} \\ \text{binomial}}} + \underbrace{\sum_{j'=0}^{n-1}\binom{n-1}{j'}\left(\tfrac{1}{2}\right)^{n-1}}_{\substack{1 \\ \text{binomial probs}}}\Bigg]$$

$$= \frac{n}{2}\left[\frac{n}{2}+1\right]$$

$$\therefore E[aX^2+bX] = a\frac{n}{2}\left(\frac{n}{2}+1\right) + b\frac{n}{2} \checkmark \quad \text{average reward.}$$

3.31b

$$E[a^X] = \sum_{j=0}^{n} a^j \binom{n}{j}\left(\tfrac{1}{2}\right)^j = \sum_{j=0}^{n} \binom{n}{j}\left(\tfrac{a}{2}\right)^j$$

$$= \left(1+\tfrac{a}{2}\right)^n$$

3.32 a) $E[g(X)] = E[I_A(X)]$ $\quad A = \{X > 10\}$

$$= \sum_{i=1}^{15} I_A(i)\, P[X=i] = \sum_{i=11}^{15} P[X=i]$$

$$= p_1\left[\sum_{i=11}^{15} \frac{1}{i}\right] = \frac{\sum_{i=11}^{15} 1/i}{\sum_{i=1}^{15} 1/i} = 0.1173$$

b) $E[I_A(X)] = p_1 \sum_{i=11}^{15} 1/2^{(i-1)}$ $\quad$ pmf has faster decay than (a)

$$= \frac{\sum_{i=11}^{15} 1/2^{(i-1)}}{\sum_{i=1}^{15} 1/2^{(i-1)}} = .000946$$

c) $E[I_A(X)] = p_1 \sum_{i=11}^{15} \left(\frac{1}{2}\right)^{j(j-1)/2} = \dfrac{\sum_{i=11}^{15} \left(\frac{1}{2}\right)^{j(j-1)/2}}{\sum_{i=1}^{15} \left(\frac{1}{2}\right)^{j(j-1)/2}}$

$$= 1.69 \times 10^{-17}$$

The last pmf decays much faster than the first two.

3.33

(a) $$E[(X-10)^+] = \sum_{i=1}^{15}(i-10)\,P[X=i] = p_1\sum_{i=1}^{15}(i-10)\frac{1}{i} = 0.33373$$

(b) $$E[(X-10)^+] = p_1\sum_{i=1}^{15}(i-10)\,2^{-(i-1)} = 0.00174$$

(c) $$E[(X-10)^+] = p_1\sum_{i=1}^{15}(i-10)\,2^{-i(i-1)/2} = 1.69\times 10^{-17}$$

3.34

$X_{max} = m$ since casino has 2^m dollars.

(a) $\therefore$ casino can only play up to

(b) $$E[Y] = \sum_{k=1}^{m} 2^k\left(\frac{1}{2}\right)^k = m$$

(c) Player is willing to pay at most m dollars.

3.4 Conditional Probability Mass Function

3.35 a) $P[X=k \mid X>0] = \begin{cases} 8/15 & k=1 \\ 7/15 & k=2 \end{cases}$ since $P[X=k \mid X>0] = \dfrac{P[X=k]}{P[X>0]}$

b) $P[X=k \mid N_m=1] = \dfrac{P[X=k, N_m=1]}{P[N_m=1]}$

$$= \begin{cases} \dfrac{\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{2}\frac{1}{2}}{\frac{1}{2}} = \frac{3}{4} & k=1 \text{ Michael} \\[2ex] \dfrac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{2}} = \frac{1}{4} & k=2 \end{cases}$$

Carlos

	0	1	2	
0				1/4
1	1	1	2	1/2
2				1/4
	1/4	1/2	1/4	

3.35 c) We need to change underlying sample space

C

M	00	01	10	11	
00					1/4
01					1/4
10	1	1	1	2	1/4
11	2	2	2	2	1/4
	1/4	1/4	1/4	1/4	

$P[X=1 \mid M\in\{10,11\}] = \frac{3}{16}/\frac{1}{2} = \frac{6}{16}$

$P[X=1 \mid M\in\{10,11\}] = \frac{5}{16}/\frac{1}{2} = \frac{10}{16}$

d)

N_c

N_m	0	1	2	
0			2	1/4
1			2	1/2
2	2	2	2	1/4
	1/16	6/16	9/16	

$P[N_c=2 \mid X=2] = \dfrac{P[N_c=2, X=2]}{P[X=2]}$

$$= \frac{\frac{9}{16}}{\frac{1}{4}\left(\frac{1}{16}\right) + \frac{1}{4}\frac{6}{16} + \frac{9}{16}} = \frac{36}{43}$$

3.36

a) $$P[X=k \mid X<4] = \frac{P[X=k]}{1-P[X=4]} = \begin{cases} \frac{12}{22} & k=1 \\ 6/22 & k=2 \\ 4/22 & k=3 \end{cases}$$

b) $$P[X=k \mid X<4] = \frac{P[X=k]}{104/105} = \begin{cases} \frac{64}{104} & k=1 \\ \frac{32}{104} & k=2 \\ \frac{8}{104} & k=3 \end{cases}$$

c) $$P[X=k \mid X<4] = \frac{P[X=k]}{14/15} = \begin{cases} 8/14 & k=1 \\ 4/14 & k=2 \\ 2/14 & k=3 \end{cases}$$

3.37

a) $$P[X=k \mid X<8] = \frac{P[X=k]}{P[X<8]} = \frac{1/16}{1/2} = \frac{1}{8} \text{ for } k<8$$

b) $P[X=k \mid \text{1st bit is zero}] = P[X=k \mid X<8]$ same as (a).

c) $$P[X=k \mid \text{4th bit is zero}] = P[X=k \mid X \text{ is even}]$$

$$= \frac{P[X=k, k \text{ even}]}{P[X \text{ is even}]} = \frac{1/16}{1/2} = 1/8 \text{ for } k \text{ even}$$

3.38) "No message gets through" $\Leftrightarrow X > 1$

(a)

$$P[X=k \mid X>1] = \frac{P[X=k]}{P[X>1]} = \frac{(1/2)^k}{1/2} = \left(\frac{1}{2}\right)^{k-1} \text{ for } k>1$$

(b) If $\underbrace{\text{1st transmitter transmitted in slot 1}}_{=C}$, then

collision occurs in time slot 1 with prob 1/2 $\Leftrightarrow X>1$

success " " " " " 1/2 $\Leftrightarrow X=1$

$\therefore P[X=1 \mid C] = 1/2$

for $k>1$

$$P[X=k \mid C] = P[X=k, X>1] = P[X=k \mid X>1]P[X>1]$$

$$= \left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^k \quad k>1$$

$\therefore$ knowledge that C occurred does not change the pmf of X.

3.39

a) $P[X=k \mid X>1] = \left(\frac{1}{2}\right)^{k-1} \quad k=2,3,\ldots$

$$E[X \mid X>1] = \sum_{k=2}^{\infty} k\left(\frac{1}{2}\right)^{k-1} = \sum_{k'=1}^{\infty} (k'+1)\left(\frac{1}{2}\right)^{k'}$$ where $k' = k-1$

$$= \sum_{k'=0}^{\infty} k'\left(\frac{1}{2}\right)^{k'} + \sum_{k'=1}^{\infty}\left(\frac{1}{2}\right)^{k'}$$

$$= \underbrace{E[X]}_{\text{avg. \# as of starting from scratch}} + \underbrace{1}_{\text{1 transmission is certain}} = 3$$

3.39b "message gets through in 1st time slot" = $X=1$

$$P[X=k \mid X=1] = \begin{cases} 0 & k>1 \\ 1 & k=1 \end{cases}$$

$$E[X \mid X=1] = 1 \cdot P[X=1] = 1$$

(c) Let $A = \{X=1\}$ $B = \{X>1\}$ then A & B form a partition

$$E[X] = E[X|A]P[A] + E[X|B]P[B]$$
$$= 1 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{2} = 2$$

note we can use the result of part a to find $E[X]$:

$$E[X] = 1 \cdot \tfrac{1}{2} + (E[X]+1)\tfrac{1}{2} \Rightarrow E[X]=2.$$

(d) $$E[X^2 \mid X>1] = \sum_{k=2}^{\infty} k^2 \left(\tfrac{1}{2}\right)^{k-1} = \sum_{k'=1}^{\infty} (k'+1)^2 \left(\tfrac{1}{2}\right)^{k'}$$

$$= \sum_{k'=1}^{\infty} k'^2 \left(\tfrac{1}{2}\right)^{k'} + 2\sum_{k'=1}^{\infty} k' \left(\tfrac{1}{2}\right)^{k'} + \sum_{k'=1}^{\infty} \left(\tfrac{1}{2}\right)^{k'}$$

$$= E[X^2] + 2E[X] + 1$$
$$= E[X^2] + 5$$

$$E[X^2 \mid X=1] = 1$$

$$\therefore \quad E[X^2] = E[X^2 \mid X=1]\tfrac{1}{2} + E[X^2 \mid X>1]\tfrac{1}{2}$$
$$= \tfrac{1}{2} + [E[X^2]+5]\tfrac{1}{2}$$
$$\Rightarrow E[X^2] = 6$$
$$VAR[X] = E[X^2] - E[X]^2 = 6 - 2^2 = 2$$

3.40

By (3.31b)

$$E[X^2] = \sum_{i=1}^{n} E[X^2/B_i]P[B_i] \quad \text{and} \quad E[X] = \sum_{i=1}^{n} E[X/B_i]P[B_i]$$

$$VAR[X] = E[X^2] - E[X]^2$$

$$= \sum_{i=1}^{n} E[X^2/B_i]P[B_i] - \left(\sum_{i=1}^{n} E[X|B_i]P[B_i]\right)^2$$

$$\neq \sum_{i=1}^{n} \left(E[X^2/B_i] - E[X/B_i]^2\right) P[B_i]$$

3.41

a) $P[X=j \mid 1\text{st draw} = k] \qquad k = 1, 50$

$$P[X=j \mid 1\text{st draw} = 1] = \begin{cases} \frac{8}{9} & j=2 \\ \frac{1}{9} & j=51 \end{cases}$$

$$P[X=j \mid 1\text{st draw} = 50] = \begin{cases} 1 & j=51 \\ 0 & \text{otherwise} \end{cases}$$

b) $E[X \mid 1\text{st draw} = 1] = 2 \cdot \frac{8}{9} + 51 \cdot \frac{1}{9}$

$E[X \mid 1\text{st draw} = 50] = 51$

c) $E[X] = E[X|1] \cdot \frac{9}{10} + E[X|50]\frac{1}{10}$

$$= \frac{67}{9}\frac{9}{10} + \frac{51}{10} = \frac{118}{10}$$

d) $E[X^2|1] = 4 \cdot \frac{8}{9} + (51)^2 \frac{1}{9} \qquad E[X^2|50] = (51)^2$

$$E[X^2] = \left(4 \cdot \frac{8}{9} + \frac{51^2}{9}\right)\frac{9}{10} + \frac{51^2}{10} = \frac{32}{10} + 2\left(\frac{51^2}{10}\right) = \frac{5234}{10}$$

$$VAR[X] = \frac{32}{10} + 2\left(\frac{51^2}{10}\right) - \left(\frac{118}{10}\right)^2 = 384.16$$

3.42 Assume # of heads is k

then $E[Y|k] = n-2k$

$$\therefore \quad E[Y] = \sum_{k=0}^{n} E[Y|k]P[k] = \sum_{k=0}^{n}(n-2k)\binom{n}{k}p^k(1-p)^{n-k}$$

$$= n - 2E[X] = n-2np$$

$$= n(1-2p)$$

Similarly

$$E[Y^2|k] = (n-2k) = n^2 - 4kn + 4k^2$$

$$E[Y^2] = \sum_{k=0}^{n}(n^2-4kn+4k^2)\binom{n}{k}p^k(1-p)^{n-k}$$

$$= n^2 - 4nE[X] + 4E[X^2]$$

$$= n^2 - 4n^2p + 4(npq + (np)^2)$$

$$= n^2 - 4n^2p + 4npq + 4n^2p^2$$

$$VAR[Y] = E[Y^2] - E[Y]^2$$

$$= n^2 - 4n^2p + 4npq + 4n^2p^2 - n^2\underbrace{(1-2p)^2}_{1-4p+4p^2}$$

$$= 4npq$$

3.43

a) If password has not been found after k tries then there remain $2^m - k$ possible passwords.

$$P[X=j \mid X>k] = \begin{cases} \frac{1}{2^m-k} & j = k+1, \ldots, 2^m \\ 0 & \text{otherwise} \end{cases}$$

b)
$$E[X \mid X>k] = \sum_{j=k+1}^{2^m} j \frac{1}{2^m-k} = \frac{1}{2^m-k} \underbrace{\sum_{j=k+1}^{2^m} j}_{\sum_{j=1}^{2^m} j - \sum_{j'=1}^{k} j'}$$

$$= \frac{1}{2^m-k}\left[\frac{2^m(2^m+1)}{2} - \frac{k(k+1)}{2}\right]$$

$$= \frac{1}{2^m-k}\left[\frac{(2^m-k)(2^m+k+1)}{2}\right] = \frac{2^m+k+1}{2}$$

$$= \underbrace{(k+1)}_{\text{minimum}} + \underbrace{\frac{2^m-(k+1)}{2}}_{\text{average additional number of tries}}$$

3.5 Important Discrete Random Variables

3.44

(a) $S = \{1, 2, 3, 4, 5\} \quad A = \{\zeta > 3\}$

$P[I_A = 0] = \frac{3}{5} \quad P[I_A = 1] = \frac{2}{5}$

$E[I_A] = 0 \cdot \frac{3}{5} + 1 \cdot \frac{2}{5} = \frac{2}{5}$

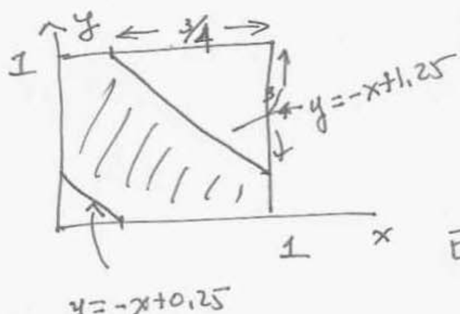
(b) $S = [0, 1] \quad A = \{0.3 < \zeta \leq 0.7\}$

$P[I_A = 0] = P[0 \leq \zeta \leq 0.3] + P[0.7 < \zeta \leq 1] = 0.6$

$P[I_A = 1] = P[0.3 < \zeta \leq 0.7] = 0.4$

$E[I_A] = 0 \cdot 0.6 + 1 \cdot 0.4 = 0.4$

(c)

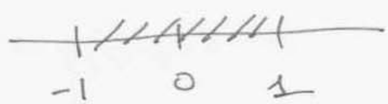


$P[I_A = 0] = \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{10}{16}$

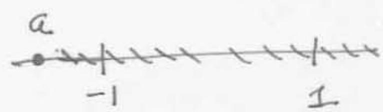
$P[I_A = 1] = 1 - \frac{10}{16} = \frac{6}{16}$

$E[I_A] = 0 \cdot \frac{10}{16} + 1 \cdot \frac{6}{16} = \frac{6}{16}.$

(d) $A = \{\zeta > a\}$



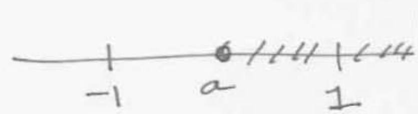
$a < -1$



$P[I_A = 1] = 1$

$E[I_A] = 1$

$-1 < a < 1$



$P[I_A = 1] = \frac{1-a}{2}$

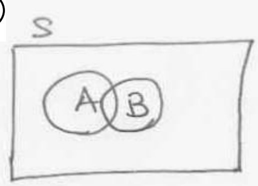
$E[I_A] = \frac{1-a}{2}$

$1 < a$

$P[I_A = 1] = 0$

$E[I_A] = 0$

3.45



ⓐ $I_S = 1$ iff $\xi \in S \Rightarrow I_S = 1$ all ξ

$I_\phi = 1$ iff $\xi \in \phi \Rightarrow I_\phi = 0$ all ξ

ⓑ $I_{A\cap B}(\xi) = 1$ iff $\xi \in A$ and $\xi \cap B \Leftrightarrow I_A(\xi) = 1$ and $I_B(\xi) = 1$

$\Leftrightarrow I_{A\cap B}(\xi) = I_A(\xi) I_B(\xi)$.

$I_{A\cup B}(\xi) = 0$ iff $\xi \notin A\cup B \Leftrightarrow \xi \in A^c \cap B^c \Leftrightarrow I_{A^c}(\xi) I_{B^c}(\xi) = 1$

$\Leftrightarrow (1 - I_A(\xi))(1 - I_B(\xi)) = 1$

$\Leftrightarrow 1 - I_A(\xi) - I_B(\xi) + I_A(\xi) I_B(\xi) = 1$

$\Leftrightarrow I_A(\xi) + I_B(\xi) - I_A(\xi) I_B(\xi) = 0$

$\Leftrightarrow I_A(\xi) + I_B(\xi) - I_{A\cap B}(\xi) = 0 = I_{A\cup B}(\xi)$

ⓒ $E[I_S] = 1 \cdot P[S] = 1$

$E[I_\phi] = 1 \cdot P[\phi] = 0$

$E[I_{A\cap B}] = 1 \cdot P[A\cap B]$

$E[I_{A\cup B}] = E[I_A] + E[I_B] - E[I_{A\cap B}]$

$= P[A] + P[B] - P[A\cap B]$.

3.46 $n = 8 \quad p = 0.25$ Binomial random variable

ⓐ $P[N=0] = \binom{8}{0} p^0 (1-p)^{8-0} = (0.75)^8 = 0.100$

ⓑ $P[N=1] = \binom{n}{1} p(1-p)^7 = 8(0.25)(0.75)^7 = 0.267$

ⓒ $P[N>4] = \sum_{j=5}^{8} \binom{8}{j} (.25)^j (.75)^{8-j} = 0.0273$

ⓓ $P[2<N<6] = \sum_{j=3}^{5} \binom{8}{j} (.25)^j (.75)^{8-j} = 0.3172$

3.47

a) $A_i = \{U_i < 0.25\}$

$A_i^c = \{U_i > 0.25\}$

intersections

$$P[\overbrace{A_1 A_2 A_3 A_4 A_5^c A_6^c A_7^c A_8^c}] = (0.25)^4 (0.75)^4$$

$$= 0.00124$$

b) $P[N=4] = \binom{8}{4}(0.25)^4 (0.75)^4 = 0.0865$

c) $A_i = \{U_i < 0.25\}$

$B_i = \{0.25 < U_i < 0.75\}$

$C_i = \{U_i > 0.75\}$

$$P[A_1 A_2 A_3 B_4 B_5 C_6 C_7 C_8] = (0.25)^3 (0.5)^2 (0.25)^3$$

$$= (0.25)^6 (0.5)^2$$

$$= 6.10 \times 10^{-5}$$

d) $P[N_1 = 3, N_2 = 2, N_3 = 3] = \dfrac{8!}{3!\,2!\,3!} 0.25^3\ 0.5^2\ 0.25^3$

multinomial

$$= 0.0342$$

e) $P[A_1 A_2 A_3 A_4 C_5 C_6 C_7 C_8] = (0.25)^4 (0.25)^4 = 1.526 \times 10^{-5}$

f) $P[N_1 = 4, N_2 = 0, N_3 = 4] = \dfrac{8!}{4!\,0!\,4!} (0.25)^4 (0.5)^0 (0.25)^4$

$$= 0.00107$$

3.48

This Octave program will plot binomial pmf.

```
> n=4;
> x=[0:n];
> p=0.10;
> stem(binomial_pdf(x,n,p))
```

.

3.49

~~3.32~~ a) Let I_k denote the outcome of the kth Benoulli trials. The probability that the single event occurred in the kth trial is:

$$\begin{aligned} P[I_k = 1 | X = 1] &= \frac{P[I_k = 1 \text{ and } I_j = 0 \text{ for all } j \neq k]}{P[X = 1]} \\ &\qquad k\text{th outcome} \\ &= \frac{P[0\ 0 \ldots 1\ 0 \ldots 0]}{P[X = 1]} \\ &= \frac{p(1-p)^{n-1}}{\binom{n}{1} p(1-p)^{n-1}} = \frac{1}{n} \end{aligned}$$

Thus the single event is equally likely to have occurred in any of the n trials.

b) The probability that the two successes occurred in trials j and k is:

$$P[I_j = 1, I_k = 1 | X = 2] = \frac{P[I_j = 1, I_k = 1, I_m = 0 \text{ for all } m \neq j, k]}{P[X = 2]}$$

3.50 a)
$$\frac{p_k}{p_{k-1}} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!}} \frac{p}{q} = \frac{(n-k+1)p}{kq}$$
$$= \frac{(n+1)p - k(1-q)}{kq} = 1 + \frac{(n+1)p-k}{kq}$$

b) First suppose $(n+1)p$ is not an integer, then
for $0 \le k \le [(n+1)p] < (n+1)p$
$$(n+1)p - k > 0$$
so
$$\frac{p_k}{p_{k-1}} = 1 + \frac{(n+1)p-k}{kq} > 1$$
$\Rightarrow p_k$ increases as k increases from 0 to $[(n+1)p]$
for $k > (n+1)p \ge [(n+1)p]$
$$(n+1)p - k < 0$$
so
$$\frac{p_k}{p_{k-1}} = 1 + \frac{(n+1)p-k}{kq} < 1$$
$\Rightarrow p_k$ decreases as k increases beyond $[(n+1)p]$
$\therefore p_k$ attains its maximum at $k_{MAX} = [(n+1)p]$
If $(n+1)p = k_{MAX}$ then above implies that
$$\frac{p_{k_{MAX}}}{p_{k_{MAX}-1}} = 1 \Rightarrow p_{k_{MAX}} = p_{k_{MAX}} - 1$$

3.51

(a) $$(a+b+c)^n = \sum_{k=0}^{n} \binom{n}{k} (a+b)^k c^{n-k}$$

(b) $$= \sum_{k=0}^{n} \binom{n}{k} c^{n-k} \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{n!\, k!}{k!\,(n-k)!\, j!\,(k-j)!} a^j b^{k-j} c^{n-k}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{n!}{j!\,(n-k)!\,(k-j)!} a^j b^{k-j} c^{n-k}$$

$$= \sum_{\substack{j_1\ j_2\ j_3 \\ j_i \geq 0 \\ j_1+j_2+j_3=n}} \frac{n!}{j_1!\, j_2!\, j_3!} a^{j_1} b^{j_2} c^{j_3}$$

(c) $$1 = (p_1+p_2+p_3)^n = \sum_{j_1 j_2 j_3} \frac{n!}{j_1!\, j_2!\, j_3!} p_1^{j_1} p_2^{j_2} p_3^{j_3}$$

3.52

$p = 0.01$ $N = \#$ error free characters until first error

(a) $$P[N=k] = (1-p)^k p \qquad k = 0, 1, 2, \ldots$$

(b) $$E[N] = \sum_{k=0}^{\infty} k(1-p)^k p = (1-p)p \sum_{k=0}^{\infty} k(1-p)^{k-1}.$$

$$= (1-p)p \frac{1}{(1-(1-p))^2} = \frac{1-p}{p} \qquad \text{by Eqn 3.14}$$

(c) $$0.99 = P[N > k_0] = \sum_{k_0+1}^{\infty} (1-p)^k p = p(1-p)^{k_0+1} \sum_{k=0}^{\infty} (1-p)^k$$

$$= (1-p)^{1001} \Rightarrow p = 1 - 0.99^{\frac{1}{1001}}$$

$$= 1.004 \times 10^{-5}$$

3.53 N geometric $n = 1, 2, \ldots$

(a)
$$P[N=k \mid N \le m] = \frac{P[N=k, N \le m]}{P[N \le m]} = \frac{P[N=k]}{P[N \le m]} \quad 1 \le k \le m$$

$$= \frac{p(1-p)^{k-1}}{\sum_{j=1}^{m} p(1-p)^{j-1}} = \frac{p(1-p)^{k-1}}{1-(1-p)^m} \quad 1 \le k \le m$$

(b)
$$P[N \text{ is odd}] = \sum_{j=0}^{\infty} p(1-p)^{2j+1} = p(1-p)\sum_{j=0}^{\infty}\left((1-p)^2\right)^j$$

$$= \frac{p(1-p)}{1-(1-p)^2}$$

3.54
$$P[M \ge k+j \mid M > j] = \frac{P[M \ge k+j, M > j]}{P[M > j]} = \frac{P[M \ge k+j]}{P[M > j]} \quad \text{for } k \ge 1$$

$$= \frac{\sum_{i=k+j}^{\infty} p(1-p)^{i-1}}{\sum_{i'=j+1}^{\infty} p(1-p)^{i'-1}}$$

$$= \frac{(1-p)^{k+j-1}}{(1-p)^j} = (1-p)^{k-1} = P[M \ge k]$$

The probability of k additional trials until the first success is independent of how many failures have already transpired.

3.55

~~3.36~~ The memoryless property states that for $j, k \geq 1$.

$$\begin{aligned} P[M \geq k] &= P[M \geq k+j | M > j] \\ &= \frac{P[M \geq k+j]}{P[M > j]} = \frac{P[M \geq k+j]}{P[M \geq j+1]} \end{aligned}$$

$\Longrightarrow$

$$P[M \geq k+j] = P[M \geq k] P[M \geq j+1]$$

Let

$$a_k = P[M \geq k],$$

then we have

$$(*) \qquad a_{k+j} = a_k a_{j+1} \qquad j \geq 1, k \geq 1$$

where $a_1 = 1$ and $a_2 = 1 - P[M = 1] = 1 - p$.
Equation (*) with $j = 1$ becomes

$$a_{k+1} = a_2 a_k \qquad k \geq 1$$

$$\Rightarrow a_k = a_2^{k-1} \qquad k \geq 1$$

$$\Rightarrow P[M \geq k] = (1-p)^{k-1} \qquad k \geq 1$$

$$\begin{aligned} P[M = k] &= P[M \geq k] - P[M \geq k+1] \\ &= (1-p)^{k-1} - (1-p)^k \\ &= (1-p)^{k-1}(1 - (1-p)) \\ &= (1-p)^{k-1} p \end{aligned}$$

3.56

$C_{unit} = \$50$ $\quad p = \frac{1}{12}$ $\quad N = \#$ failures in 12 months

$C_{repair} = \$20$ $\quad P[N=k] = \binom{12}{k}\left(\frac{1}{12}\right)^k\left(\frac{11}{12}\right)^{12-k} \quad k = 0, 1, \ldots, 12$

$$0.99 = \sum_{j=k_0}^{12} P[N=j] \Rightarrow k_0 = ?$$

k	p_k	$P[N \le k]$
0	0.352	0.352
1	0.384	0.736
2	0.192	0.928
3	0.058	0.986

$\Rightarrow k_0 = 3$

$\Rightarrow$ Charge $\$50 + 3 \times \$20 = \$110.$

Avg cost per player $= \$50 + \$20\, \underbrace{E[N]}_{12\left(\frac{1}{12}\right)} = \$70.$

3.57

$\alpha_S = 48 \quad \alpha_R = 24 \quad \alpha_G = 12 \quad$ slice $= 1/12$

(a) $P[N_G = 0] = \dfrac{\left(\alpha_G \frac{1}{12}\right)^0}{0!} e^{-\alpha_G \frac{1}{12}} = e^{-1} = 0.368$

(b) $P[N_G = 0, N_R \le 2] = P[N_G = 0]\, P[N_R \le 2] = e^{-1} \sum_{k=0}^{2} \dfrac{\left(\alpha_R \frac{1}{12}\right)^k}{k!} e^{-\alpha_R \frac{1}{12}}$

$= e^{-1}\left[e^{-2} + \frac{2}{1!}e^{-2} + \frac{4}{2!}e^{-2}\right]$

$= e^{-3}\left[1 + 2 + 2\right] = 5e^{-3} = 0.249$

(c) $P[N_G = 0, N_R = 0, N_S \ge 5] = P[N_G = 0]\, P[N_R = 0]\, P[N_S \ge 5]$

$= e^{-1} e^{-2} e^{-4} \sum_{k=0}^{5} \dfrac{4^k}{k!} = e^{-3}(0.785)$

$= 0.039$

It's hard to avoid the red and green thingies!

3.58

$$P[X > 4] < 0.9 \Leftrightarrow P[X \leq 4] > 0.1$$

$$P[X \leq 4] = \sum_{k=0}^{4} \frac{\alpha^k}{k!} e^{-\alpha} = \sum_{k=0}^{4} \frac{(5/n)^k}{k!} e^{-5/n}$$

Since $\alpha = \frac{\lambda}{n\mu} = \frac{5}{n}$

If $n = 2$ then $P[X \leq 4] = 0.891$. Therefore ~~one~~ two employees ~~is~~ are sufficient.

$$P[X = 0] = e^{-\alpha} = e^{-2.5} = \cancel{0.05} = 0.082.$$

3.59

$\lambda = 6000$ requests/minute $= 100$ requests/sec

$\alpha = \lambda \frac{1}{10} = 10$ requests/100 ms

(a) $P[N=0] = e^{-10} = 4.54 \times 10^{-5}$

(b) $P[5 \leq N \leq 10] = \sum_{k=5}^{10} \frac{10^k}{k!} e^{-10} = 0.554$

3.60

Use octave to plot pmf.

```
> a = 0.1 ;
> j = [0: 20];
> stem (poisson_pdf (j, a))
```

3.61

$$\mathcal{E}[X] = \sum_{k=0}^{\infty} k \frac{\alpha^k}{k!} e^{-\alpha} = \alpha \sum_{k=1}^{\infty} \frac{\alpha^{k-1}}{(k-1)!} e^{-\alpha} = \alpha \underbrace{\sum_{k'=0}^{\infty} \frac{\alpha^{k'}}{k'!} e^{-\alpha}}_{1} = \alpha$$

$$\begin{aligned} \mathcal{E}[X^2] &= \sum_{k=0}^{\infty} k^2 \frac{\alpha^k}{k!} e^{-\alpha} = \alpha \sum_{k=1}^{\infty} k \frac{\alpha^{k-1}}{(k-1)!} e^{-\alpha} \\ &= \alpha \sum_{k'=0}^{\infty} (k'+1) \frac{\alpha^{k'}}{k'!} e^{-\alpha} = \alpha\{\alpha + 1\} \\ \sigma_X^2 &= \mathcal{E}[X^2] - \mathcal{E}[X]^2 = \alpha\{\alpha + 1 - \alpha\} \\ &= \alpha \end{aligned}$$

3.62

$$\frac{p_k}{p_{k-1}} = \frac{\frac{\alpha^k}{k!}e^{-\alpha}}{\frac{\alpha^{k-1}}{(k-1)!}e^{-\alpha}} = \frac{\alpha}{k}$$

If $\alpha < 1$ then $\frac{p_k}{p_{k-1}} = \frac{\alpha}{k} < 1$ for $k \geq 1$

$\therefore p_k$ decreases as k increases from 0

$\therefore p_k$ attains its maximum at $k = 0$

If $\alpha > 1$ then

for $0 \leq k \leq [\alpha] < \alpha$, $\frac{p_k}{p_{k-1}} = \frac{\alpha}{k} > 1$

$\Rightarrow p_k$ increase from $k = 0$ to $k = [\alpha]$

for $[\alpha] < \alpha < k$, $\frac{p_k}{p_{k-1}} = \frac{\alpha}{k} < 1$

$\Rightarrow p_k$ decreases as k increases beyond $[\alpha]$

$\therefore p_k$ attains its maximum at $k_{\max} = [\alpha]$

If $\alpha = [\alpha]$ then for $k = [\alpha]$

$$\frac{p_k}{p_{k-1}} = 1 \Rightarrow p_{k_{\max}} = p_{k_{\max}} - 1$$

3.63

$n = 10$ $p = 0.1$ $np = 1$

	$k=0$	$k=1$	$k=2$	$k=3$
Binomial	0.3487	0.387	0.1937	0.0574
Poisson	0.3679	0.3679	0.1839	0.0613

$n = 20$ $p = 0.05$ $np = 1$

	$k=0$	$k=1$	$k=2$	$k=3$
Binomial	0.3585	0.3774	0.1887	0.06
Poisson	0.3679	0.3679	0.1839	0.0613

$n = 100$ $p = 0.01$ $np = 1$

	$k=0$	$k=1$	$k=2$	$k=3$
Binomial	0.366	0.3697	0.1849	0.061
Poisson	0.3679	0.3679	0.1839	0.0613

3.64 N Poisson $\alpha = 3$ $R = 2(10^6)$ bps

a) $X = R/N$ "infinite" for $N=0$ R/k for $N=k \geq 1$

$S_X = \{ \infty, 2.0, 1.0, \frac{2}{3}, 0.5, \frac{2}{5}, \frac{2}{7}, \ldots \}$

$P[X = R/k] = P[N=k]$

b) $0.9 = P[N \leq k] = \sum_{j=0}^{k} \frac{(3)^j}{j!} e^{-3}$ $P[N \leq 5] = 0.916$

use Octave
poisson_cdf(k, 3)

c) $X \geq 1 \Leftrightarrow k \leq 2$

$P[N \leq 2] = 0.423$

3.65 $n = 1000 \times 750 = 7.5 \times 10^5$ pixels

$p = 10^{-5}$ $np = 7.5$

$P[\text{display accepted}] = \sum_{k=0}^{15} \binom{n}{k} p^k (1-p)^{n-k} \approx \sum_{k=0}^{15} \frac{(7.5)^k}{k!} e^{-7.5}$

$= 0.9953$

3.66 $n = 10^4$ drives $\quad p = 10^{-3} \quad np = 10^4(10^{-3}) = 10/\text{day}$

a) $P[N=0] \simeq e^{-np} = e^{-10} = 4.54\times10^{-5}$

b) Failure rate in 2 days = 20

$$P[N \le 10] = \sum_{j=0}^{10} \frac{(20)^j}{j!} e^{-20} = 1.08\times10^{-2}$$

c) $$0.99 = P[N \le k] = \sum_{j=0}^{k} \frac{10^j}{j!} e^{-10} \Rightarrow P[N \le \text{[redacted]}] = 0.986$$

3.67 $p = 10^{-6} \quad n = 10^4 \quad np = 10^{-2}$

a) $P[N=0] = e^{-np} = 0.990$

$$P[N \le 3] = \sum_{k=0}^{3} \frac{(0.01)^k}{k!} e^{-np} \approx 1$$

b) Find p so that

$$0.99 = P[N \ge 1] = 1 - P[N=0] = 1 - e^{-np}$$

$$0.01 = e^{-np}$$

$$\Rightarrow p = \frac{\ln 100}{n} = 4.6\times10^{-6}$$

3.68

$$\text{.65} \quad \mathcal{E}[X] = \sum_{k=1}^{n} kP[X=k] = \sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n}\sum_{k=1}^{n} k = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\sigma_X^2 = \mathcal{E}[X^2] - \mathcal{E}[X]^2 = \sum_{k=1}^{n} \frac{k^2}{n} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n^2-1}{12}$$

3.69

3.69) X uniform in $\{-3,-2,\ldots,3,4\}$ $\quad P[X=j]=\frac{1}{8}$

(a) $E[X] = -4+\frac{8+1}{2} = 0.5$

$VAR(X) = \frac{8^2-1}{12} = \frac{63}{12} = \frac{21}{4}$

(b) $E[Y] = E[-2X^2+3] = -2E[X^2]+3$

$= -2[VAR(X)+E[X]^2]+3$

$= -2\left[\frac{21}{4}+(0.5)^2\right]+3 = -8$

$E[Y^2] = E[(-2X^2+3)^2] = E[4X^4-12X^2+9]$

$VAR[Y] = E[Y^2]-E[Y]^2$

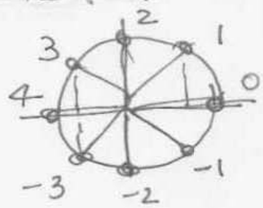
$= 4E[X^4]-12E[X^2]+9-(-8)^2$

$E[X^4] = \frac{1}{8}\left[(-3)^2+(-2)^2+(-1)^2+0^2+1^2+2^2+3^2+4^2\right]$

$= \frac{44}{8} = \frac{11}{2}$

$VAR[Y] = 4\left(\frac{11}{2}\right)-12\,(-11)+9-64 = 99$

(c) $W=\cos\left(\frac{\pi X}{8}\right)$



x	-3	-2	-1	0	1	2	3	4
w	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1

$P\left[W=-\frac{1}{\sqrt{2}}\right]=\frac{2}{8} \quad P[W=0]=\frac{2}{8}$

$P\left[W=\frac{1}{\sqrt{2}}\right]=\frac{2}{8} \quad P[W=1]=\frac{1}{8} \quad P[W=-1]=\frac{1}{8}$

3.70 $p_k = \frac{1}{c_{10}} \frac{1}{k} \quad k = 1, \ldots, 10 \quad c_{10} = 2.93$

$$p_1 = \frac{1}{2.93} = 0.3414$$

$$P[X > 5] = \frac{1}{c_{10}} \left[\frac{1}{6} + \cdots + \frac{1}{10} \right] = 0.2204$$

3.71 $p_k = \frac{1}{c_{1000}} \frac{1}{k} \quad c_{1000} \approx \ln 1000 + 0.57721 = 7.485$

$$P[X \le 10] = \frac{1}{c_{1000}} \sum_{j=1}^{10} \frac{1}{j} = \frac{c_{10}}{c_{1000}} = \frac{2.93}{7.485} = 0.3913$$

$$P[X > 990] = 1 - P[X \le 990] = 1 - \frac{c_{990}}{c_{1000}}$$

$$= 1 - \frac{\ln 990 + 0.57721}{\ln 1000 + 0.57721} = 0.00134$$

3.72 $p_k = \frac{1}{c_L} \frac{1}{k} \qquad \ln p_k = \ln \frac{1}{c_L} + \ln \frac{1}{k}$

$$= -\ln k - \ln c_L$$

$\ln p_k$ is linear in $\ln k$

3.73

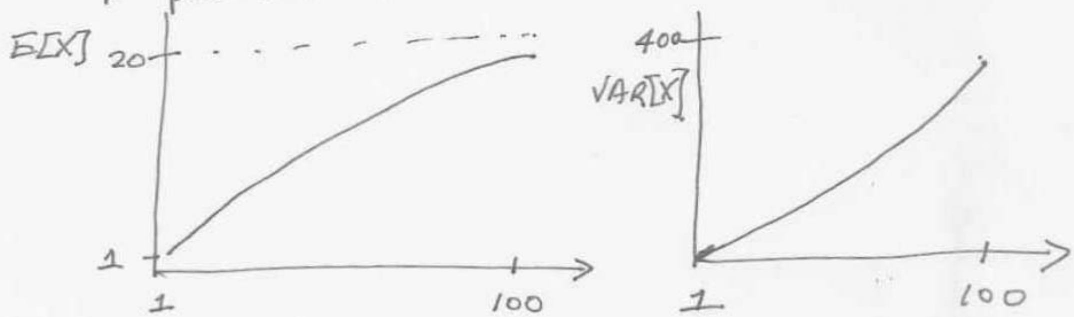
$$E[X] = \frac{L}{c_L} \approx \frac{L}{\ln L + 0.57721} \quad \text{for large } L$$

$$VAR[X] = L^2/c_L^2 = E[X]^2$$

To plot $E[X]$ vs L use Octave

```
>L=[1:100];
>p = L.^(-1);
>cL = cumsum(p);        array of coefficients

> plot (L./cL)          plots means
> plot((L.^2)./(cL.^2))  plots variances
```



3.74

$$p_k = \frac{1}{c_L}\frac{1}{k} \qquad L = 10^4 \qquad c_L = \ln 10^4 + 0.57721 = 9.7876$$

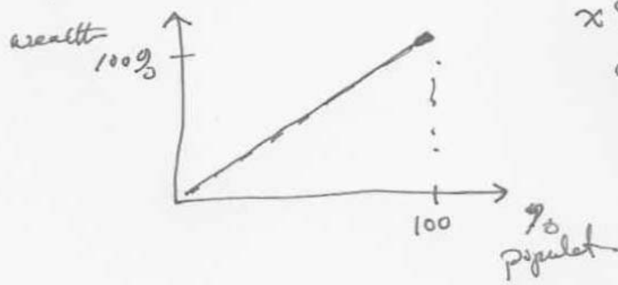
$$0.99 = P[X \le k_0] = \frac{1}{c_{10000}} \sum_{j=1}^{k_0} \frac{1}{j} = \frac{c_{k_0}}{c_{10000}} \approx \frac{\ln k_0 + 0.57721}{9.7876}$$

$$\ln k_0 \approx 0.99(9.7876) - 0.57721 = 9067$$

Zipf decays very slowly!

3.75 Equal ownership

If I is total income then every individual has I/L of the wealth.

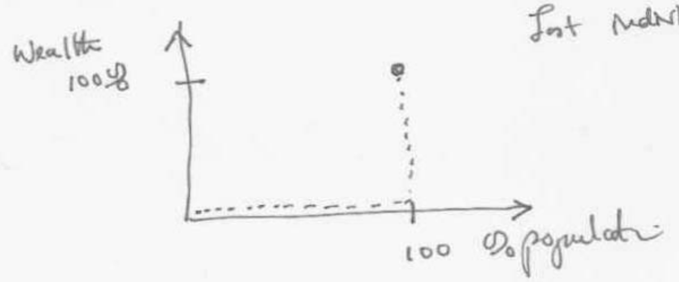


x% of wealth owned by x% of population

Most unequal ownership

0% of wealth owned by x% of population for $x < 100$

Last individual owns 100%



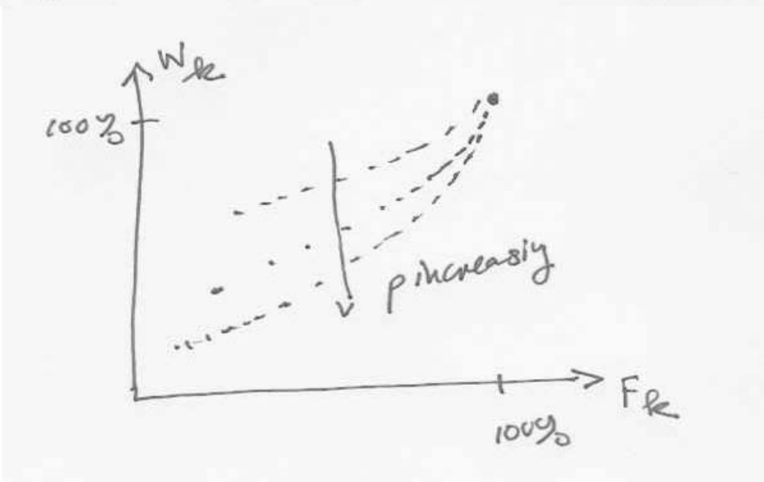
3.76

$$P[X=k] = (1-p)p^{k-1} \qquad k=1,2,\ldots$$

$$F_k = P[X \le k] = c\sum_{j=1}^{k} p^{j-1} = 1-p^k$$

$$w_k = \frac{\sum_{j=1}^{k} j\,c\,p^{j-1}}{\sum_{j'=1}^{\infty} j'\,c\,p^{j'-1}} = \frac{\sum_{j=1}^{k} j\,p^{j-1}}{\sum_{j'=1}^{\infty} j'\,p^{j'-1}} = (1-p^{k+1})-(k+1)p^k(1-p)$$

$$\sum_{j=0}^{m} j\,p^{j-1} = \frac{d}{dp}\,\frac{1-p^{m+1}}{1-p} = \frac{(1-p^{m+1})-(m+1)p^m(1-p)}{1-p^2}$$



3.77 $X = \frac{1}{z_\alpha}\frac{1}{k^\alpha}$ $k = 1, 2, \ldots$

a) $P[X \le k] = \frac{1}{z_\alpha}\sum_{j=1}^{k}\frac{1}{j^\alpha} = \frac{z_{\alpha,k}}{z_\alpha}$ where $z_{\alpha,k} = \sum_{j=1}^{k}\frac{1}{j^\alpha}$

b) z_α is given by the zeta function evaluated at α

Using Octave, we have

> zeta(1.5)

> ans = 2.6124

$z(1.5) = 2.6124$

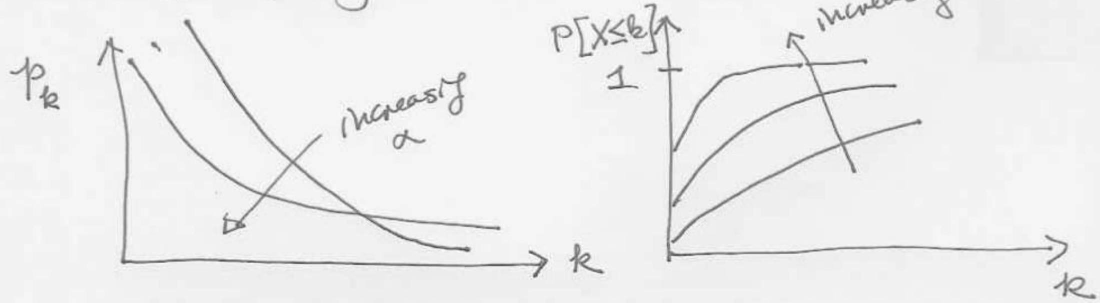
$z(2) = 1.6449$

$z(3) = 1.2021$

If we add the first 100 terms to estimate z_α we have:

$z_{1.5,100} = 2.41$ $z_{2,100} = 1.635$ $z_{3,100} = 1.202$

The series that defines the zeta function decays more slowly as α decreases.



3.6 Generation of Discrete Random Variables

3.78 The following Octave commands will give the requested plots:

(a)

```
x = [0:1:10];
lambda = 0.5;
figure;
plot(x, poisson_pdf(x, lambda));
figure;
plot(x, poisson_cdf(x, lambda));
figure;
plot(x, 1-poisson_cdf(x, lambda));

x = [0:1:20];
lambda = 5;
figure;
plot(x, poisson_pdf(x, lambda));
figure;
plot(x, poisson_cdf(x, lambda));
figure;
plot(x, 1-poisson_cdf(x, lambda));

x = [0:1:100];
lambda = 50;
figure;
plot(x, poisson_pdf(x, lambda));
figure;
plot(x, poisson_cdf(x, lambda));
figure;
plot(x, 1-poisson_cdf(x, lambda));
```

(b)

```
x = [0:1:15];
figure;
plot(x, binomial_pdf(x, 48, 0.1));
figure;
plot(x, binomial_cdf(x, 48, 0.1));
figure;
plot(x, 1-binomial_cdf(x, 48, 0.1));

x = [0:1:30];
figure;
plot(x, binomial_pdf(x, 48, 0.3));
figure;
plot(x, binomial_cdf(x, 48, 0.3));
figure;
plot(x, 1-binomial_cdf(x, 48, 0.3));

x = [0:1:50];
figure;
```

```
plot(x, binomial_pdf(x, 48, 0.5));
figure;
plot(x, binomial_cdf(x, 48, 0.5));
figure;
plot(x, 1-binomial_cdf(x, 48, 0.5));

x = [20:1:50];
figure;
plot(x, binomial_pdf(x, 48, 0.75));
figure;
plot(x, binomial_cdf(x, 48, 0.75));
figure;
plot(x, 1-binomial_cdf(x, 48, 0.75));
```

(c)

```
x = [0:1:10];
n = 100; p = 0.01;
figure;
plot(x, binomial_pdf(x, n, p), "1");
hold on;
plot(x, poisson_pdf(x, n*p), "3");
hold off;
```

3.79 The following Octave commands produce the request plots:

(a)

```
L = 10;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
figure;
plot(k, discrete_pdf(k, k, pk));
figure;
plot(k, discrete_cdf(k, k, pk));
figure;
plot(k, 1-discrete_cdf(k, k, pk));

L = 100;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
figure;
plot(k, discrete_pdf(k, k, pk));
figure;
plot(k, discrete_cdf(k, k, pk));
figure;
plot(k, 1-discrete_cdf(k, k, pk));

L = 1000;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
figure;
plot(k, discrete_pdf(k, k, pk));
figure;
plot(k, discrete_cdf(k, k, pk));
figure;
plot(k, 1-discrete_cdf(k, k, pk));
```

(b)

```
m = 20;
k = [1:1:m];
pk = (1/2).^k;
figure;
semilogy(k, pk);
```

3.80 The following Octave commands will plot the Lorenze curves:

```
L = 10;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Wk = k./L;
Fk = discrete_cdf(k, k, pk);
figure;
plot(Fk, Wk);

L = 100;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Wk = k./L;
Fk = discrete_cdf(k, k, pk);
figure;
plot(Fk, Wk);

L = 1000;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Wk = k./L;
Fk = discrete_cdf(k, k, pk);
figure;
plot(Fk, Wk);
```

3.81 The following Octave commands will plot the requested curves:

```
figure;
hold on;
n = 100; p = 0.1;
k = [0:1:n];
Wk = zeros(1, n+1);
for i = 0:n,
      v = [0:i];
      Wk(i+1) = sum(v.*binomial_pdf(v, n, p))./(n*p);
end;
Fk = binomial_cdf(k, n, p);
plot(Fk, Wk);

n = 100; p = 0.5;
k = [0:1:n];
Wk = zeros(1, n+1);
for i = 0:n,
      v = [0:i];
      Wk(i+1) = sum(v.*binomial_pdf(v, n, p))./(n*p);
end;
Fk = binomial_cdf(k, n, p);
plot(Fk, Wk);

n = 100; p = 0.9;
k = [0:1:n];
Wk = zeros(1, n+1);
for i = 0:n,
      v = [0:i];
      Wk(i+1) = sum(v.*binomial_pdf(v, n, p))./(n*p);
end;
Fk = binomial_cdf(k, n, p);
plot(Fk, Wk);
```

3.82

(a)

(b)

(c)

3.83 The following Octave commands will generate the requested samples of the Zipf random variable and the requested plots.

```
L = 10;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Sk = discrete_rnd(200, k, pk);
figure;
plot(Sk);
figure;
hist(Sk, k);

L = 100;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Sk = discrete_rnd(200, k, pk);
figure;
plot(Sk);
figure;
hist(Sk, k);

L = 1000;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Sk = discrete_rnd(200, k, pk);
figure;
plot(Sk);
figure;
hist(Sk, k);
```

3.84 The following Octave commands generate the samples of the St. Peter's Paradox random variable and the requested plots.

```
m = 20;
k = [1:1:m];
pk = (1/2).^k;
Sk = discrete_rnd(200, k, pk);
figure;
plot(Sk);
figure;
hist(Sk, k);
```

3.85 The following Octave commands generate the requested pairs and plots:

(a)

```
k = [1:10];
pk = ones(1,10)./10;
Sx = discrete_rnd(200, k, pk);
Sy = discrete_rnd(200, k, pk);
figure;
hist(Sx, k);
figure;
hist(Sy, k);
```

(b)

```
Sz = Sx + Sy;
figure;
hist(Sz, [2:20]);
```

(c)

```
Sw = Sx .* Sy;
figure;
hist(Sw, 10);
```

(d)

```
Sv = Sx ./ Sy;
figure;
hist(Sv, 10);
```

3.86 The following Octave commands generate the requested pairs and plots:

(a)

```
Sx = binomial_rnd(8, 0.5, 1, 200);
Sy = binomial_rnd(4, 0.5, 1, 200);
figure;
hist(Sx, [0:8]);
figure;
hist(Sy, [0:4]);
```

(b)

```
Sz = Sx + Sy;
figure;
hist(Sz, [0:12]);
```

3.87 The following Octave commands generate the requested pairs and plots:

(a)

```
Sx = poisson_rnd(5, 1, 200);
Sy = poisson_rnd(10, 1, 200);
figure;
hist(Sx, [0:15]);
figure;
hist(Sy, [0:20]);
```

(b)

```
Sz = Sx + Sy;
figure;
hist(Sz, [0:35]);
```

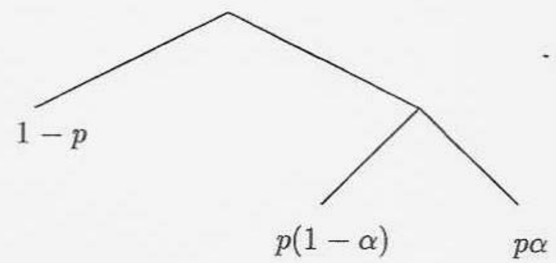
Problems Requiring Cumulative Knowledge

3.88

~~3.153~~ a)

$$
\begin{aligned}
P[\text{pass the test}] &= (1-p) + p(1-\alpha) \\
P[\text{fail the test}] &= p\alpha \\
P[k \text{ items }] &= [(1-p) + p(1-\alpha)]^{k-1}(p\alpha)^1
\end{aligned}
$$

b)



3.89

~~3.155~~ The number of transmissions is a geometric RV. The average number of transmissions is:

$$
\begin{aligned}
\sum_{k=1}^{\infty} kp^{k-1}(1-p) &= (1-p)\sum_{k=1}^{\infty}\frac{dp^k}{dp} \\
&= (1-p)\frac{d}{dp}\sum_{k=1}^{\infty} p^k \\
&= (1-p)\frac{d}{dp}\frac{1}{1-p} \\
&= \frac{1}{1-p}
\end{aligned}
$$

The message transmission takes $\frac{2T}{1-P}$ seconds on the average. The maximum possible rate $= (1-P)/2T$.

3.90) We want to find n so that the nth arrival is after more than 2 minutes 90% of the time:

$$P[N(2) \le n] = 0.90 = \sum_{k=0}^{n} \frac{2^k}{k!} e^{-2}$$

By trial and error we find $n = 5$.

3.91

~~58~~ a)

$$
\begin{aligned}
& P[\text{signal present}|X=k] \\
= & \frac{P[\text{signal present}, X=k]}{P[X=k|\text{signal present}]P[\text{present}] + P[X=k|\text{signal absent}]P[\text{absent}]} \\
= & \frac{\frac{\lambda_1^k}{k!}e^{-\lambda_1}p}{\frac{\lambda_1^k}{k!}e^{-\lambda_1}p + \frac{\lambda_0^k}{k!}e^{-\lambda_0}(1-p)} \\
= & \frac{\lambda_1^k e^{-\lambda_1}p}{\lambda_1^k e^{-\lambda_1}p + \lambda_0^k e^{-\lambda_0}(1-p)}
\end{aligned}
$$

Similarly,

$$P[\text{signal absent}|X=k] = \frac{\lambda_0^k e^{-\lambda_0}(1-p)}{\lambda_1^k e^{-\lambda_1}p + \lambda_0^k e^{-\lambda_0}(1-p)}$$

b) Decide signal present if P[signal present|X=k] > P[signal absent|X=k], i.e.,

$$\lambda_1^k e^{-\lambda_1}p > \lambda_0^k e^{-\lambda_0}(1-p)$$

$$\left(\frac{\lambda_1}{\lambda_0}\right)^k > \frac{1-p}{p}e^{\lambda_1-\lambda_0} \qquad (\lambda_1 > \lambda_0)$$

$$k > \frac{ln\frac{1-p}{p} + \lambda_1 - \lambda_0}{ln\lambda_1 - ln\lambda_0}$$

The threshold T is

$$T = \frac{ln\frac{1-p}{p} + \lambda_1 - \lambda_0}{ln\lambda_1 - ln\lambda_0}$$

c)

$$
\begin{aligned}
P_e = & P[X < T|\text{signal present}]P[\text{present}] + P[X > T|\text{signal absent}]P[\text{absent}] \\
= & p\sum_{k=o}^{\lfloor T\rfloor} \frac{e^{-\lambda_1}\lambda_1^k}{k!} + (1-p)\sum_{k=\lceil T\rceil}^{\infty} \frac{e^{-\lambda_0}\lambda_0^k}{k!}
\end{aligned}
$$

3.92 ~~3.100~~ a)

$$\begin{aligned}
P[\text{prefix has k 0s}] &= P[kM \leq n \leq kM + M - 1] \\
&= \sum_{kM}^{kM+M-1} p^n(1-p) \\
&= (1-p)p^{kM}(1 + p + \ldots + p^{M-1}) \\
&= p^{kM}(1 - p^M) \\
&= \left(\frac{1}{2}\right)^k\left(1 - \frac{1}{2}\right) \\
&= \left(\frac{1}{2}\right)^{k+1}
\end{aligned}$$

b)

$$\begin{aligned}
E[L] &= E[k] + 1 + m \\
&= \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^{k+1} + 1 + m \\
&= m + 2
\end{aligned}$$

c)

$$\begin{aligned}
E[\text{run length (including one 1 at the end)}] &= \sum_0^{\infty}(n+1)p^n(1-p) \\
&= (1-p)\sum_0^{\infty}\frac{d}{dp}p^{n+1} \\
&= (1-p)\frac{d}{dp}\sum_0^{\infty}p^{n+1} \\
&= (1-p)\frac{d}{dp}\frac{p}{1-p} \\
&= \frac{1}{1-p}
\end{aligned}$$

$$\text{Compression ratio} = \frac{\frac{1}{1-p}}{m+2} = \frac{1}{(1-p)(m+2)}$$