

# COMPLETE SOLUTIONS MANUAL

for Stewart's

## MULTIVARIABLE CALCULUS: CONCEPTS AND CONTEXTS FOURTH EDITION

DAN CLEGG  
Palomar College



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## ☐ PREFACE

This *Complete Solutions Manual* contains detailed solutions to all exercises in the text *Multivariable Calculus: Concepts and Contexts*, Fourth Edition (Chapters 8–13 of *Calculus: Concepts and Contexts*, Fourth Edition) by James Stewart. A *Student Solutions Manual* is also available, which contains solutions to the odd-numbered exercises in each chapter section, review section, True-False Quiz, and Focus on Problem Solving section as well as all solutions to the Concept Check questions. (It does not, however, include solutions to any of the projects.)

While I have extended every effort to ensure the accuracy of the solutions presented, I would appreciate correspondence regarding any errors that may exist. Other suggestions or comments are also welcome, and can be sent to me at the email address or mailing address below.

I would like to thank James Stewart for entrusting me with the writing of this manual and offering suggestions, Kathi Townes, Stephanie Kuhns, and Rebekah Steele of TECH-arts for typesetting and producing this manual, and Brian Betsill of TECH-arts for creating the illustrations. Brian Karasek prepared solutions for comparison of accuracy and style in addition to proofreading manuscript; his assistance and suggestions were very helpful and much appreciated. Finally, I would like to thank Richard Stratton and Elizabeth Neustaetter of Brooks/Cole, Cengage Learning for their trust, assistance, and patience.

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## 8 □ INFINITE SEQUENCES AND SERIES

### 8.1 Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.  
 (b) The terms  $a_n$  approach 8 as  $n$  becomes large. In fact, we can make  $a_n$  as close to 8 as we like by taking  $n$  sufficiently large.  
 (c) The terms  $a_n$  become large as  $n$  becomes large. In fact, we can make  $a_n$  as large as we like by taking  $n$  sufficiently large.
2. (a) From Definition 1, a convergent sequence is a sequence for which  $\lim_{n \rightarrow \infty} a_n$  exists. Examples:  $\{1/n\}$ ,  $\{1/2^n\}$   
 (b) A divergent sequence is a sequence for which  $\lim_{n \rightarrow \infty} a_n$  does not exist. Examples:  $\{n\}$ ,  $\{\sin n\}$
3. The first six terms of  $a_n = \frac{n}{2n+1}$  are  $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}$ . It appears that the sequence is approaching  $\frac{1}{2}$ .  

$$\lim_{n \rightarrow \infty} \frac{n}{2n+1} = \lim_{n \rightarrow \infty} \frac{1}{2+1/n} = \frac{1}{2}$$
4.  $\{\cos(n\pi/3)\}_{n=1}^9 = \{\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1\}$ . The sequence does not appear to have a limit. The values will cycle through the first six numbers in the sequence—never approaching a particular number.
5.  $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\}$ . The denominator of the  $n$ th term is the  $n$ th positive odd integer, so  $a_n = \frac{1}{2n-1}$ .
6.  $\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots\}$ . The denominator of the  $n$ th term is the  $(n-1)$ st power of 3, so  $a_n = \frac{1}{3^{n-1}}$ .
7.  $\{2, 7, 12, 17, \dots\}$ . Each term is larger than the preceding one by 5, so  $a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n - 3$ .
8.  $\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \dots\}$ . The numerator of the  $n$ th term is  $n$  and its denominator is  $(n+1)^2$ . Including the alternating signs, we get  $a_n = (-1)^n \frac{n}{(n+1)^2}$ .
9.  $\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \dots\}$ . Each term is  $-\frac{2}{3}$  times the preceding one, so  $a_n = (-\frac{2}{3})^{n-1}$ .
10.  $\{5, 1, 5, 1, 5, 1, \dots\}$ . The average of 5 and 1 is 3, so we can think of the sequence as alternately adding 2 and  $-2$  to 3.  
 Thus,  $a_n = 3 + (-1)^{n+1} \cdot 2$ .
11.  $a_n = \frac{3+5n^2}{n+n^2} = \frac{(3+5n^2)/n^2}{(n+n^2)/n^2} = \frac{5+3/n^2}{1+1/n}$ , so  $a_n \rightarrow \frac{5+0}{1+0} = 5$  as  $n \rightarrow \infty$ . Converges
12.  $a_n = \frac{n^3}{n^3+1} = \frac{n^3/n^3}{(n^3+1)/n^3} = \frac{1}{1+1/n^3}$ , so  $a_n \rightarrow \frac{1}{1+0} = 1$  as  $n \rightarrow \infty$ . Converges
13.  $a_n = 1 - (0.2)^n$ , so  $\lim_{n \rightarrow \infty} a_n = 1 - 0 = 1$  by (7). Converges

14.  $a_n = \frac{n^3}{n+1} = \frac{n^3/n}{(n+1)/n} = \frac{n^2}{1+1/n^2}$ , so  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$  since  $\lim_{n \rightarrow \infty} n^2 = \infty$  and  $\lim_{n \rightarrow \infty} (1+1/n^2) = 1$ . Diverges

15. Because the natural exponential function is continuous at 0, Theorem 5 enables us to write

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} e^{1/n} = e^{\lim_{n \rightarrow \infty} (1/n)} = e^0 = 1. \quad \text{Converges}$$

16.  $a_n = \frac{3^{n+2}}{5^n} = \frac{3^2 3^n}{5^n} = 9\left(\frac{3}{5}\right)^n$ , so  $\lim_{n \rightarrow \infty} a_n = 9 \lim_{n \rightarrow \infty} \left(\frac{3}{5}\right)^n = 9 \cdot 0 = 0$  by (7) with  $r = \frac{3}{5}$ . Converges

17. If  $b_n = \frac{2n\pi}{1+8n}$ , then  $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{(2n\pi)/n}{(1+8n)/n} = \lim_{n \rightarrow \infty} \frac{2\pi}{1/n+8} = \frac{2\pi}{8} = \frac{\pi}{4}$ . Since  $\tan$  is continuous at  $\frac{\pi}{4}$ , by

Theorem 5,  $\lim_{n \rightarrow \infty} \tan\left(\frac{2n\pi}{1+8n}\right) = \tan\left(\lim_{n \rightarrow \infty} \frac{2n\pi}{1+8n}\right) = \tan \frac{\pi}{4} = 1$ . Converges

18. Using the last limit law for sequences and the continuity of the square root function,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{9n+1}} = \sqrt{\lim_{n \rightarrow \infty} \frac{n+1}{9n+1}} = \sqrt{\lim_{n \rightarrow \infty} \frac{1+1/n}{9+1/n}} = \sqrt{\frac{1}{9}} = \frac{1}{3}. \quad \text{Converges}$$

19.  $a_n = \frac{(-1)^{n-1}n}{n^2+1} = \frac{(-1)^{n-1}}{n+1/n}$ , so  $0 \leq |a_n| = \frac{1}{n+1/n} \leq \frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ , so  $a_n \rightarrow 0$  by the Squeeze Theorem and

Theorem 4. Converges

20.  $a_n = \frac{(-1)^n n^3}{n^3 + 2n^2 + 1}$ . Now  $|a_n| = \frac{n^3}{n^3 + 2n^2 + 1} = \frac{1}{1 + \frac{2}{n} + \frac{1}{n^3}} \rightarrow 1$  as  $n \rightarrow \infty$ , but the terms of the sequence  $\{a_n\}$

alternate in sign, so the sequence  $a_1, a_3, a_5, \dots$  converges to  $-1$  and the sequence  $a_2, a_4, a_6, \dots$  converges to  $+1$ .

This shows that the given sequence diverges since its terms don't approach a single real number.

21.  $a_n = \frac{e^n + e^{-n}}{e^{2n} - 1} \cdot \frac{e^{-n}}{e^{-n}} = \frac{1 + e^{-2n}}{e^n - e^{-n}} \rightarrow 0$  as  $n \rightarrow \infty$  because  $1 + e^{-2n} \rightarrow 1$  and  $e^n - e^{-n} \rightarrow \infty$ . Converges

22.  $a_n = \cos(2/n)$ . As  $n \rightarrow \infty$ ,  $2/n \rightarrow 0$ , so  $\cos(2/n) \rightarrow \cos 0 = 1$  because  $\cos$  is continuous. Converges

23.  $a_n = n^2 e^{-n} = \frac{n^2}{e^n}$ . Since  $\lim_{x \rightarrow \infty} \frac{x^2}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2x}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$ , it follows from Theorem 2 that  $\lim_{n \rightarrow \infty} a_n = 0$ . Converges

24.  $2n \rightarrow \infty$  as  $n \rightarrow \infty$ , so since  $\lim_{x \rightarrow \infty} \arctan x = \frac{\pi}{2}$ , we have  $\lim_{n \rightarrow \infty} \arctan 2n = \frac{\pi}{2}$ . Converges

25.  $0 \leq \frac{\cos^2 n}{2^n} \leq \frac{1}{2^n}$  [since  $0 \leq \cos^2 n \leq 1$ ], so since  $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$ ,  $\left\{\frac{\cos^2 n}{2^n}\right\}$  converges to 0 by the Squeeze Theorem.

26.  $a_n = n \cos n\pi = n(-1)^n$ . Since  $|a_n| = n \rightarrow \infty$  as  $n \rightarrow \infty$ , the given sequence diverges.

27.  $y = \left(1 + \frac{2}{x}\right)^x \Rightarrow \ln y = x \ln \left(1 + \frac{2}{x}\right)$ , so

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln(1+2/x)}{1/x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{1+2/x}\right)\left(-\frac{2}{x^2}\right)}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{2}{1+2/x} = 2 \Rightarrow$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^x = \lim_{x \rightarrow \infty} e^{\ln y} = e^2, \text{ so by Theorem 2, } \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = e^2. \quad \text{Convergent}$$

28.  $a_n = \sqrt[n]{2^{1+3n}} = (2^{1+3n})^{1/n} = (2^1 2^{3n})^{1/n} = 2^{1/n} 2^3 = 8 \cdot 2^{1/n}$ , so

$$\lim_{n \rightarrow \infty} a_n = 8 \lim_{n \rightarrow \infty} 2^{1/n} = 8 \cdot 2^{\lim_{n \rightarrow \infty} (1/n)} = 8 \cdot 2^0 = 8 \text{ by Theorem 5, since the function } f(x) = 2^x \text{ is continuous at 0.}$$

Convergent

29.  $a_n = \frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \frac{1}{(2n+1)(2n)} \rightarrow 0 \text{ as } n \rightarrow \infty. \text{ Converges}$

30.  $a_n = \frac{\sin 2n}{1 + \sqrt{n}}$ .  $|a_n| \leq \frac{1}{1 + \sqrt{n}}$  and  $\lim_{n \rightarrow \infty} \frac{1}{1 + \sqrt{n}} = 0$ , so  $\frac{-1}{1 + \sqrt{n}} \leq a_n \leq \frac{1}{1 + \sqrt{n}} \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$  by the

Squeeze Theorem. Converges

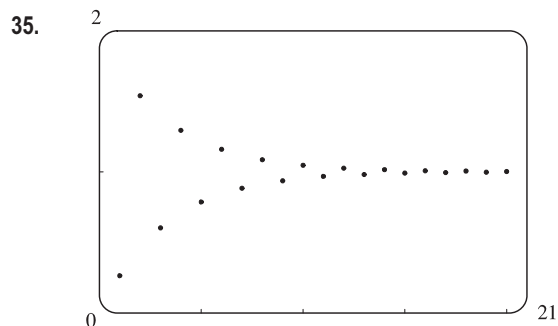
31.  $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$  diverges since the sequence takes on only two values, 0 and 1, and never stays arbitrarily close to either one (or any other value) for  $n$  sufficiently large.

32.  $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2(\ln x)(1/x)}{1} = 2 \lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{H}{=} 2 \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$ , so by Theorem 3,  $\lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n} = 0$ . Convergent

33.  $a_n = \ln(2n^2 + 1) - \ln(n^2 + 1) = \ln\left(\frac{2n^2 + 1}{n^2 + 1}\right) = \ln\left(\frac{2 + 1/n^2}{1 + 1/n^2}\right) \rightarrow \ln 2 \text{ as } n \rightarrow \infty. \text{ Convergent}$

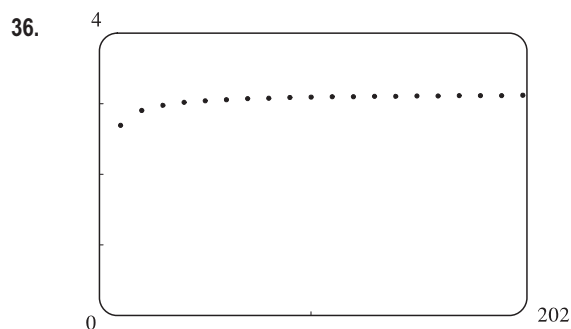
34.  $0 < |a_n| = \frac{3^n}{n!} = \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{3} \cdots \frac{3}{(n-1)} \cdot \frac{3}{n} \leq \frac{3}{1} \cdot \frac{3}{2} \cdot \frac{3}{n} \quad [\text{for } n > 2] = \frac{27}{2n} \rightarrow 0 \text{ as } n \rightarrow \infty$ , so by the Squeeze

Theorem and Theorem 4,  $\{(-3)^n/n!\}$  converges to 0.



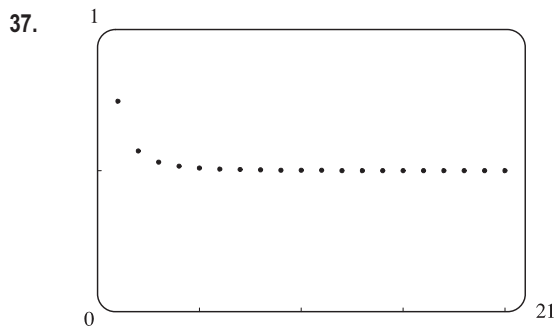
From the graph, it appears that the sequence converges to 1.

$\{(-2/e)^n\}$  converges to 0 by (7), and hence  $\{1 + (-2/e)^n\}$  converges to  $1 + 0 = 1$ .



From the graph, it appears that the sequence converges to a number greater than 3.

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \sqrt{n} \sin\left(\frac{\pi}{\sqrt{n}}\right) = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{\pi}{\sqrt{n}}\right)}{\frac{\pi}{\sqrt{n}}} \cdot \pi \\ &= \lim_{x \rightarrow 0^+} \frac{\sin x}{x} \cdot \pi \quad \left[x = \pi/\sqrt{n}\right] = 1 \cdot \pi = \pi. \end{aligned}$$

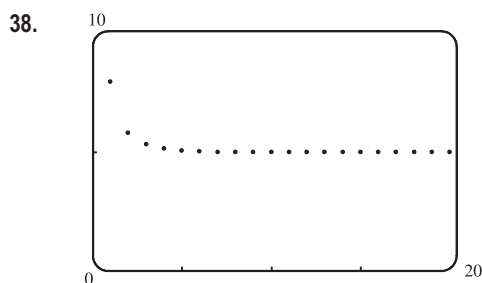


From the graph, it appears that the sequence converges to  $\frac{1}{2}$ .

As  $n \rightarrow \infty$ ,

$$a_n = \sqrt{\frac{3+2n^2}{8n^2+n}} = \sqrt{\frac{3/n^2+2}{8+1/n}} \Rightarrow \sqrt{\frac{0+2}{8+0}} = \sqrt{\frac{1}{4}} = \frac{1}{2},$$

$$\text{so } \lim_{n \rightarrow \infty} a_n = \frac{1}{2}.$$



From the graph, it appears that the sequence converges to 5.

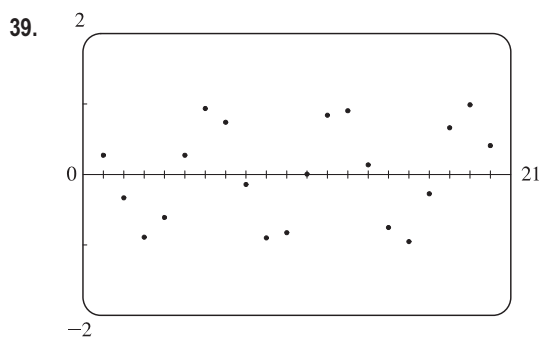
$$\begin{aligned} 5 &= \sqrt[n]{5^n} \leq \sqrt[n]{3^n + 5^n} \leq \sqrt[n]{5^n + 5^n} = \sqrt[n]{2} \sqrt[n]{5^n} \\ &= \sqrt[n]{2} \cdot 5 \rightarrow 5 \text{ as } n \rightarrow \infty \quad \left[ \lim_{n \rightarrow \infty} 2^{1/n} = 2^0 = 1 \right] \end{aligned}$$

Hence,  $a_n \rightarrow 5$  by the Squeeze Theorem.

*Alternate solution:* Let  $y = (3^x + 5^x)^{1/x}$ . Then

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln(3^x + 5^x)}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{3^x \ln 3 + 5^x \ln 5}{3^x + 5^x} = \lim_{x \rightarrow \infty} \frac{\left(\frac{3}{5}\right)^x \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^x + 1} = \ln 5,$$

so  $\lim_{x \rightarrow \infty} y = e^{\ln 5} = 5$ , and so  $\{\sqrt[n]{3^n + 5^n}\}$  converges to 5.



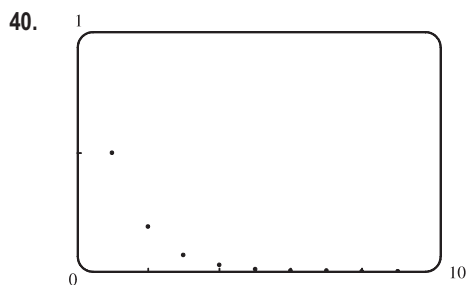
From the graph, it appears that the sequence  $\{a_n\} = \left\{\frac{n^2 \cos n}{1+n^2}\right\}$  is

divergent, since it oscillates between 1 and  $-1$  (approximately). To

prove this, suppose that  $\{a_n\}$  converges to  $L$ . If  $b_n = \frac{n^2}{1+n^2}$ , then

$\{b_n\}$  converges to 1, and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{1} = L$ . But  $\frac{a_n}{b_n} = \cos n$ , so

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  does not exist. This contradiction shows that  $\{a_n\}$  diverges.



From the graph, it appears that the sequence approaches 0.

$$\begin{aligned} 0 < a_n &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n} = \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdots \frac{2n-1}{2n} \\ &\leq \frac{1}{2n} \cdot (1) \cdot (1) \cdots (1) = \frac{1}{2n} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

So by the Squeeze Theorem,  $\left\{\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n}\right\}$  converges to 0.

41. (a)  $a_n = 1000(1.06)^n \Rightarrow a_1 = 1060, a_2 = 1123.60, a_3 = 1191.02, a_4 = 1262.48, \text{ and } a_5 = 1338.23.$

(b)  $\lim_{n \rightarrow \infty} a_n = 1000 \lim_{n \rightarrow \infty} (1.06)^n$ , so the sequence diverges by (7) with  $r = 1.06 > 1$ .

- (b) For two years, use  $2 \cdot 12 = 24$  for  $n$  to get \$70.28.

- (b) Using the recursive formula with  $P_0 = 5000$ , we get  $P_1 = 5100$ ,  $P_2 = 5208$ ,  $P_3 = 5325$  (rounding any portion of a catfish),  $P_4 = 5451$ ,  $P_5 = 5587$ , and  $P_6 = 5734$ , which is the number of catfish in the pond after six months.

- The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point  $a_1$ .

- (b)  $a_1 = 2, a_2 = 4 - a_1 = 4 - 2 = 2, a_3 = 4 - a_2 = 4 - 2 = 2$ . Since all of the terms are 2,  $\lim_{n \rightarrow \infty} a_n = 2$  and hence, the sequence is convergent.

- $$L = 1/(1+L) \Rightarrow L^2 + L - 1 = 0 \Rightarrow L = \frac{-1+\sqrt{5}}{2} \approx 0.618 \text{ (since } L \text{ has to be non-negative if it exists).}$$

47. (a) Let  $a_n$  be the number of rabbit pairs in the  $n$ th month. Clearly  $a_1 = 1 = a_2$ . In the  $n$ th month, each pair that is 2 or more months old (that is,  $a_{n-2}$  pairs) will produce a new pair to add to the  $a_{n-1}$  pairs already present. Thus,
- $$a_n = a_{n-1} + a_{n-2}, \text{ so that } \{a_n\} = \{f_n\}, \text{ the Fibonacci sequence.}$$

(b)  $a_n = \frac{f_{n+1}}{f_n} \Rightarrow a_{n-1} = \frac{f_n}{f_{n-1}} = \frac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \frac{f_{n-2}}{f_{n-1}} = 1 + \frac{1}{f_{n-1}/f_{n-2}} = 1 + \frac{1}{a_{n-2}}$ . If  $L = \lim_{n \rightarrow \infty} a_n$ ,

then  $L = \lim_{n \rightarrow \infty} a_{n-1}$  and  $L = \lim_{n \rightarrow \infty} a_{n-2}$ , so  $L$  must satisfy  $L = 1 + \frac{1}{L} \Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1+\sqrt{5}}{2}$

[since  $L$  must be positive].

48. For  $\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right\}$ ,  $a_1 = 2^{1/2}$ ,  $a_2 = 2^{3/4}$ ,  $a_3 = 2^{7/8}$ ,  $\dots$ , so  $a_n = 2^{(2^n-1)/2^n} = 2^{1-(1/2^n)}$ .

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} 2^{1-(1/2^n)} = 2^1 = 2.$$

*Alternate solution:* Let  $L = \lim_{n \rightarrow \infty} a_n$ . (We could show the limit exists by showing that  $\{a_n\}$  is bounded and increasing.)

Then  $L$  must satisfy  $L = \sqrt{2 \cdot L} \Rightarrow L^2 = 2L \Rightarrow L(L-2) = 0$ .  $L \neq 0$  since the sequence increases, so  $L = 2$ .

49.  $a_n = \frac{1}{2n+3}$  is decreasing since  $a_{n+1} = \frac{1}{2(n+1)+3} = \frac{1}{2n+5} < \frac{1}{2n+3} = a_n$  for each  $n \geq 1$ . The sequence is

bounded since  $0 < a_n \leq \frac{1}{5}$  for all  $n \geq 1$ . Note that  $a_1 = \frac{1}{5}$ .

50.  $a_n = \frac{2n-3}{3n+4}$  defines an increasing sequence since for  $f(x) = \frac{2x-3}{3x+4}$ ,

$$f'(x) = \frac{(3x+4)(2) - (2x-3)(3)}{(3x+4)^2} = \frac{17}{(3x+4)^2} > 0. \text{ The sequence is bounded since } a_n \geq a_1 = -\frac{1}{7} \text{ for } n \geq 1,$$

and  $a_n < \frac{2n-3}{3n} < \frac{2n}{3n} = \frac{2}{3}$  for  $n \geq 1$ .

51. The terms of  $a_n = n(-1)^n$  alternate in sign, so the sequence is not monotonic. The first five terms are  $-1, 2, -3, 4$ , and  $-5$ .

Since  $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} n = \infty$ , the sequence is not bounded.

52.  $a_n = n + \frac{1}{n}$  defines an increasing sequence since the function  $g(x) = x + \frac{1}{x}$  is increasing for  $x > 1$ . [ $g'(x) = 1 - 1/x^2 > 0$

for  $x > 1$ .] The sequence is unbounded since  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ . (It is, however, bounded below by  $a_1 = 2$ .)

53. Since  $\{a_n\}$  is a decreasing sequence,  $a_n > a_{n+1}$  for all  $n \geq 1$ . Because all of its terms lie between 5 and 8,  $\{a_n\}$  is a bounded sequence. By the Monotonic Sequence Theorem,  $\{a_n\}$  is convergent; that is,  $\{a_n\}$  has a limit  $L$ .  $L$  must be less than 8 since  $\{a_n\}$  is decreasing, so  $5 \leq L < 8$ .

54. (a) Let  $P_n$  be the statement that  $a_{n+1} \geq a_n$  and  $a_n \leq 3$ .  $P_1$  is obviously true. We will assume that  $P_n$  is true and

then show that as a consequence  $P_{n+1}$  must also be true.  $a_{n+2} \geq a_{n+1} \Leftrightarrow \sqrt{2+a_{n+1}} \geq \sqrt{2+a_n} \Leftrightarrow$

$$2 + a_{n+1} \geq 2 + a_n \Leftrightarrow a_{n+1} \geq a_n, \text{ which is the induction hypothesis. } a_{n+1} \leq 3 \Leftrightarrow \sqrt{2 + a_n} \leq 3 \Leftrightarrow$$

$$2 + a_n \leq 9 \Leftrightarrow a_n \leq 7, \text{ which is certainly true because we are assuming that } a_n \leq 3. \text{ So } P_n \text{ is true for all } n, \text{ and so}$$

$$a_1 \leq a_n \leq 3 \text{ (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, } \lim_{n \rightarrow \infty} a_n \text{ exists.}$$

$$(b) \text{ If } L = \lim_{n \rightarrow \infty} a_n, \text{ then } \lim_{n \rightarrow \infty} a_{n+1} = L \text{ also, so } L = \sqrt{2 + L} \Rightarrow L^2 = 2 + L \Leftrightarrow L^2 - L - 2 = 0 \Leftrightarrow$$

$$(L + 1)(L - 2) = 0 \Leftrightarrow L = 2 \text{ [since } L \text{ can't be negative].}$$

55.  $a_1 = 1, a_{n+1} = 3 - \frac{1}{a_n}$ . We show by induction that  $\{a_n\}$  is increasing and bounded above by 3. Let  $P_n$  be the proposition

$$\text{that } a_{n+1} > a_n \text{ and } 0 < a_n < 3. \text{ Clearly } P_1 \text{ is true. Assume that } P_n \text{ is true. Then } a_{n+1} > a_n \Rightarrow \frac{1}{a_{n+1}} < \frac{1}{a_n} \Rightarrow$$

$$-\frac{1}{a_{n+1}} > -\frac{1}{a_n}. \text{ Now } a_{n+2} = 3 - \frac{1}{a_{n+1}} > 3 - \frac{1}{a_n} = a_{n+1} \Leftrightarrow P_{n+1}. \text{ This proves that } \{a_n\} \text{ is increasing and bounded}$$

above by 3, so  $1 = a_1 < a_n < 3$ , that is,  $\{a_n\}$  is bounded, and hence convergent by the Monotonic Sequence Theorem.

$$\text{If } L = \lim_{n \rightarrow \infty} a_n, \text{ then } \lim_{n \rightarrow \infty} a_{n+1} = L \text{ also, so } L \text{ must satisfy } L = 3 - 1/L \Rightarrow L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3 \pm \sqrt{5}}{2}.$$

$$\text{But } L > 1, \text{ so } L = \frac{3 + \sqrt{5}}{2}.$$

56.  $a_1 = 2, a_{n+1} = \frac{1}{3 - a_n}$ . We use induction. Let  $P_n$  be the statement that  $0 < a_{n+1} \leq a_n \leq 2$ . Clearly  $P_1$  is true, since

$$a_2 = 1/(3 - 2) = 1. \text{ Now assume that } P_n \text{ is true. Then } a_{n+1} \leq a_n \Rightarrow -a_{n+1} \geq -a_n \Rightarrow 3 - a_{n+1} \geq 3 - a_n \Rightarrow$$

$$a_{n+2} = \frac{1}{3 - a_{n+1}} \leq \frac{1}{3 - a_n} = a_{n+1}. \text{ Also } a_{n+2} > 0 \text{ [since } 3 - a_{n+1} \text{ is positive]} \text{ and } a_{n+1} \leq 2 \text{ by the induction}$$

$$\text{hypothesis, so } P_{n+1} \text{ is true. To find the limit, we use the fact that } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_{n+1} \Rightarrow L = \frac{1}{3 - L} \Rightarrow$$

$$L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3 \pm \sqrt{5}}{2}. \text{ But } L \leq 2, \text{ so we must have } L = \frac{3 - \sqrt{5}}{2}.$$

$$57. (0.8)^n < 0.000001 \Rightarrow \ln(0.8)^n < \ln(0.000001) \Rightarrow n \ln(0.8) < \ln(0.000001) \Rightarrow n > \frac{\ln(0.000001)}{\ln(0.8)} \Rightarrow$$

$n > 61.9$ , so  $n$  must be at least 62 to satisfy the given inequality.

$$58. (a) \text{ If } f \text{ is continuous, then } f(L) = f\left(\lim_{n \rightarrow \infty} a_n\right) = \lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n = L \text{ by Exercise 46(a).}$$

(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that  $L \approx 0.73909$ .

$$59. (a) \text{ Suppose } \{p_n\} \text{ converges to } p. \text{ Then } p_{n+1} = \frac{bp_n}{a + p_n} \Rightarrow \lim_{n \rightarrow \infty} p_{n+1} = \frac{b \lim_{n \rightarrow \infty} p_n}{a + \lim_{n \rightarrow \infty} p_n} \Rightarrow p = \frac{bp}{a + p} \Rightarrow$$

$$p^2 + ap = bp \Rightarrow p(p + a - b) = 0 \Rightarrow p = 0 \text{ or } p = b - a.$$

$$(b) p_{n+1} = \frac{bp_n}{a + p_n} = \frac{\left(\frac{b}{a}\right)p_n}{1 + \frac{p_n}{a}} < \left(\frac{b}{a}\right)p_n \text{ since } 1 + \frac{p_n}{a} > 1.$$

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