1 Functions and Limits

1.1 Four Ways to Represent a Function

Suggested Time and Emphasis

1 class   Essential material

Points to Stress

1. Understanding the interplay between the four ways of representing a function (verbally, numerically, visually, algebraically) perhaps using the concepts of increasing and decreasing functions as an example.
2. Finding the domain and range of a function, regardless of representation.
3. Investigating even and odd functions.
4. Working with piecewise defined functions.

Quiz Questions

- **TEXT QUESTION** Why does the author assert that "the $\sqrt{x}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$"?

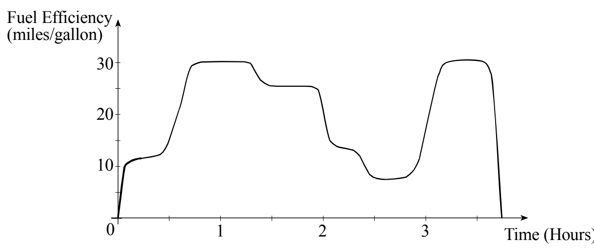
  ANSWER The calculator gives an approximation to the square root.

- **DRILL QUESTION** Fill in the blanks: $|x| = \begin{cases} \_\_\_ & \text{if } x \geq 0 \\ \_\_\_ & \text{if } x < 0 \end{cases}$

  ANSWER x, $-x$

Materials for Lecture

- Draw a graph of electrical power consumption in the classroom versus time on a typical weekday, pointing out important features throughout, and using the vocabulary of this section as much as possible.
- Draw a graph of fuel efficiency versus time on a trip, such as the one below. Lead a discussion of what could have happened on the trip.



- In 1984, United States President Ronald Reagan proposed a plan to change the United States personal income tax system. According to his plan, the income tax would be 15% on the first $19,300 earned, 25% on the next $18,800, and 35% on all income above and beyond that. Describe this situation to the class, and have them graph (marginal) tax rate and tax owed versus income for incomes ranging from $0 to $80,000. Then have them try to come up with equations describing this situation.

- In the year 2000, Presidential candidate Steve Forbes proposed a "flat tax" model: 0% on the first \$36,000 and 17% on the rest. Have the students do the same analysis, and compare the two models. As an extension, perhaps have the students look at a current tax table and draw similar graphs.

Workshop/Discussion

- Present graphs of even and odd functions, such as $\sin x$, $\cos x + x^2$, and $\cos(\sin x)$, and check with the standard algebraic tests.
- Start with a table of values for the function $f(x) = \frac{1}{4}x^2 + x$:

x	0	1	2	3	4
$f(x)$	0	1.25	3	5.25	8

  First, have the class describe the behavior of the function in words, trying to elicit the information that the function is increasing, and that its rate of increase is also increasing. Then, have them try to extrapolate the function in both directions, debating whether or not the function is always positive and increasing. Plot the points and connect the dots, then have them try to concoct a formula (not necessarily expecting them to succeed).
- Discuss the domain and range of a function such as $f(x) = \begin{cases} \sqrt{x} & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

  Also talk about why f is neither increasing nor decreasing for $x > 0$. Stress that when dealing with new sorts of functions, it becomes important to know the precise mathematical definitions of such terms.
- Define "difference quotient" as done in the text. Define $f(x) = x^3$, and show that $\dfrac{f(a+h) - f(a)}{h} = 3a^2 + 3ah + h^2$. This example both reviews algebra skills and foreshadows future calculations.

Group Work 1: Every Picture Tells a Story

Put the students in groups of four, and have them work on the exercise. If there are questions, encourage them to ask each other before asking you. After going through the correct matching with them, have each group tell their story to the class and see if it fits the remaining graph.

ANSWERS

1. (b) **2.** (a) **3.** (c) **4.** The roast beef was cooked in the morning and put in the refrigerator in the afternoon.

Group Work 2: Finding a Formula

Make sure that the students know the equation of a circle with radius r, and that they remember the notation for piecewise-defined functions. Split the students into groups of four. In each group, have half of the students work on each problem first, and then have them check each other's work. If the students find these problems difficult, have them work together on each problem.

ANSWERS

1. $f(x) = \begin{cases} -x-2 & \text{if } x \le -2 \\ x+2 & \text{if } -2 < x \le 0 \\ 2 & \text{if } x > 0 \end{cases}$ **2.** $g(x) = \begin{cases} x+4 & \text{if } x \le -2 \\ 2 & \text{if } -2 < x \le 0 \\ \sqrt{4-x^2} & \text{if } 0 < x \le 2 \\ x-2 & \text{if } x > 2 \end{cases}$

Homework Problems

CORE EXERCISES 3, 7, 15, 23, 25, 27, 31, 33, 39, 41, 43, 69, 73, 77

SAMPLE ASSIGNMENT 2, 3, 7, 9, 15, 23, 25, 27, 29, 31, 33, 39, 41, 43, 47, 49, 53, 61, 63, 64, 69, 73, 75, 77

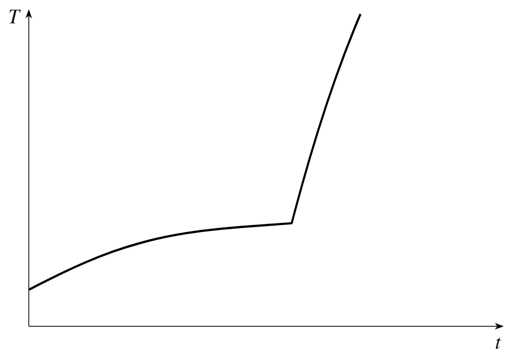
EXERCISE	D	A	N	G
2	×	×		
3				×
7				×
9				×
15				×
23			×	×
25		×		
27		×		
29		×		
31		×		
33		×		
39		×		×

EXERCISE	D	A	N	G
41		×		×
43		×		×
47		×		×
49		×		×
53		×		
61		×		
63		×		
64		×		
69				×
73		×		×
75		×		×
77		×		×

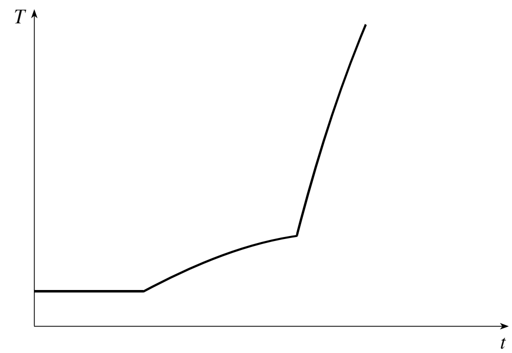
GROUP WORK 1, SECTION 1.1

Every Picture Tells a Story

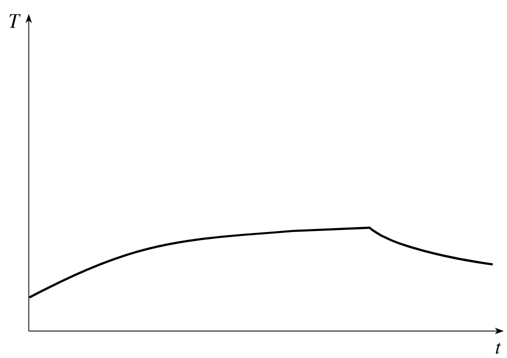
One of the skills you will be learning in this course is the ability to take a description of a real-world occurrence, and translate it into mathematics. Conversely, given a mathematical description of a phenomenon, you will learn how to describe what is happening in plain language. Here follow four graphs of temperature versus time and three stories. Match the stories with the graphs. When finished, write a similar story that would correspond to the final graph.



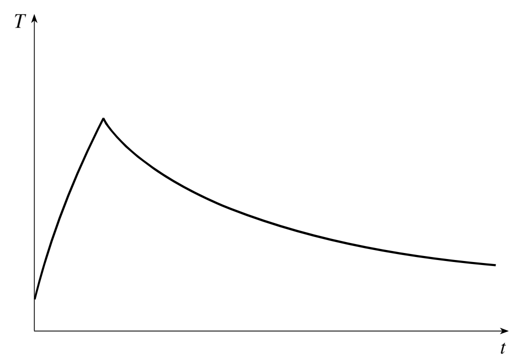
Graph 1



Graph 2



Graph 3



Graph 4

(a) I took my roast beef out of the freezer at noon, and left it on the counter to thaw. Then I cooked it in the oven when I got home.

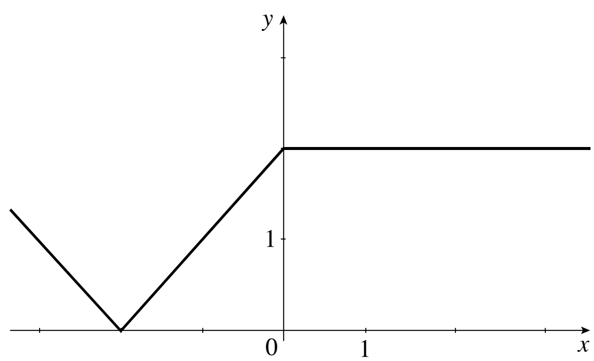
(b) I took my roast beef out of the freezer this morning, and left it on the counter to thaw. Then I cooked it in the oven when I got home.

(c) I took my roast beef out of the freezer this morning, and left it on the counter to thaw. I forgot about it, and went out for Chinese food on my way home from work. I put it in the refrigerator when I finally got home.

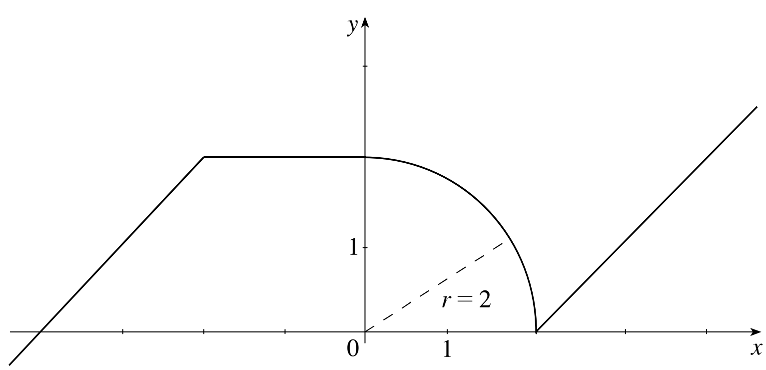
GROUP WORK 2, SECTION 1.1
Finding a Formula

Find formulas for the following functions:

1.



2.



Mathematical Models: A Catalog of Essential Functions

Suggested Time and Emphasis

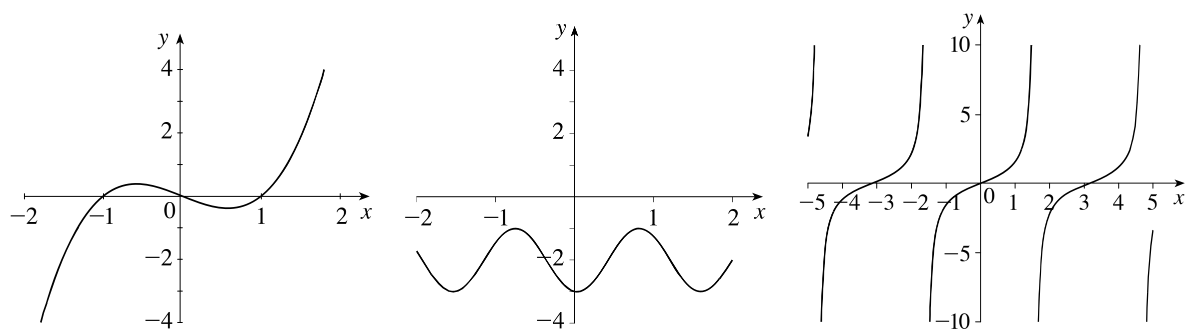
1 class Recommended material

Points to Stress

1. The modeling process: developing, analyzing, and interpreting a mathematical model.
2. Classes of functions: linear, power, rational, algebraic, and trigonometric functions. Include the special characteristics of each class of functions.

Quiz Questions

- **TEXT QUESTION** What is the difference between a power function x^n with $n = 3$ and a cubic function?
  ANSWER A cubic function can have lower order terms, whereas a power function has just one term.
- **DRILL QUESTION** Classify each function graphed below as a power function, root function, polynomial, rational function, algebraic function, or trigonometric function. Explain your reasoning.



ANSWER Polynomial, trigonometric, trigonometric

Materials for Lecture

- Show that linear functions have constant differences in y-values for equally spaced x-values. This example illustrates the point:

Linear function (difference=1.2)

x	$f(x)$
−2	−2.0
0	−0.8
2	0.4
4	1.6

- Discuss the shape, symmetries, and general "flatness" near 0 of the power functions x^n for various values of n. Similarly discuss $\sqrt[n]{x}$ for n even and n odd. A blackline master is provided at the end of this section, before the group work handouts.
- If Exercises 24–28 are to be assigned, Exercise 23 can be done in class, discussing part (c) in the context of the technology available to the students.

Workshop/Discussion

- Have the students graph 2^x, $\sin x$, $\sin 2^x$, and $2^{\sin x}$. Discuss why the latter two look the way that they do, then discuss the relationship among the graphs of $f(x) = 2^x$, $g(x) = (0.5)^x$, $h(x) = 2^{-x}$, and $k(x) = 4^x$.
- Figure 17 shows examples of a noncontinuous function and a nondifferentiable function, both expressible as simple formulas. Discuss these curves with the students, trying to get them to describe the ideas of a break in a graph and a cusp.

Group Work 1: Rounding the Bases

On the board, review how to compute the percentage error when estimating π by $\frac{22}{7}$. (Answer: 0.04%) Have them work on the problem in groups. If a group finishes early, have them look at $h(7)$ and $h(10)$ to see how fast the error grows. Exponential functions will be covered in more detail in Chapter 6.

ANSWERS **1.** 17.811434627, 17, 4.56% **2.** 220.08649875, 201, 8.67% **3.** 45.4314240633, 32, 29.56%

Group Work 2: The Small Shall Grow Large

If a group finishes early, ask them to similarly compare x^3 and x^4.

ANSWERS **1.** $x^6 \geq x^8$ for $-1 \leq x \leq 1$ **2.** $x^3 \geq x^5$ for $-\infty < x \leq -1, 0 \leq x \leq 1$ **3.** $x^3 \geq x^{105}$ for $-\infty < x \leq -1, 0 \leq x \leq 1$. If the exponents are both even, the answer is the same as for Problem 1, if the exponents are both odd, the answer is the same as for Problem 2.

Group Work 3: Fun with Fourier

This activity should be given before Fourier series are discussed in class. This activity will get students looking at combinations of sine curves, while at the same time foreshadowing the concepts of infinite series and Fourier series.

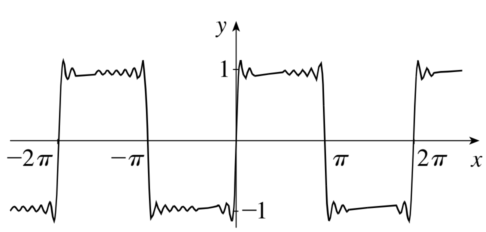
ANSWERS

1. No

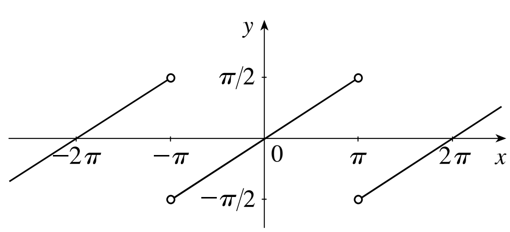
2. $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$

3. $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \frac{1}{9}\sin 9x\right)$

4. $\dfrac{4}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7} + \dfrac{\sin 9x}{9} + \dfrac{\sin 11x}{11} + \dfrac{\sin 13x}{13} + \dfrac{\sin 15x}{15} + \dfrac{\sin 17x}{17} + \dfrac{\sin 19x}{19}\right)$



5.

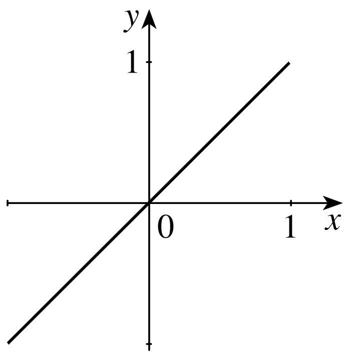


Homework Problems

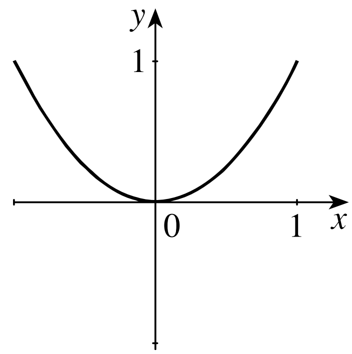
CORE EXERCISES 1, 3, 13, 15, 19, 21, 27

SAMPLE ASSIGNMENT 1, 3, 4, 5, 8, 11, 13, 15, 17, 19, 21, 27

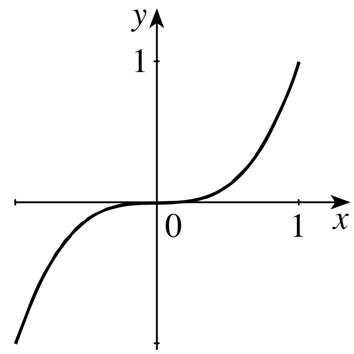
EXERCISE	D	A	N	G
1	×			
3				×
4				×
5	×	×		×
8		×		×
11	×	×		
13				×
15		×		
17		×		
19				×
21	×		×	×
27		×	×	



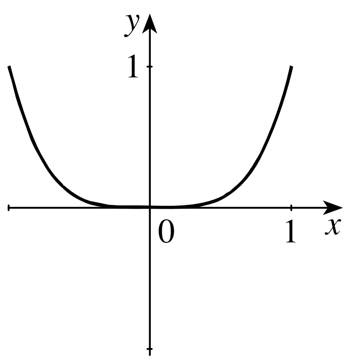
x



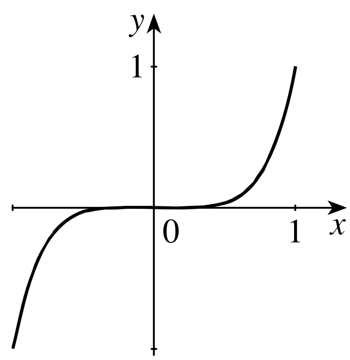
x^2



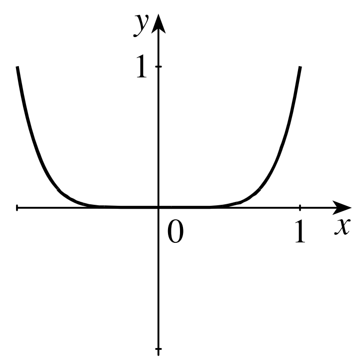
x^3



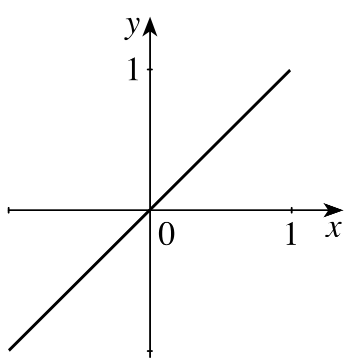
x^4



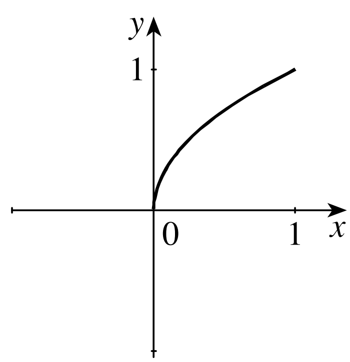
x^5



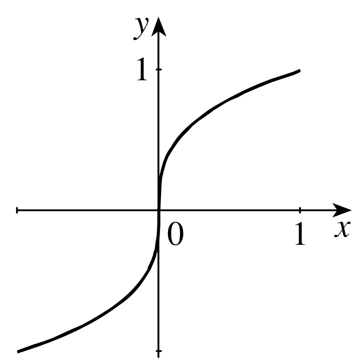
x^6



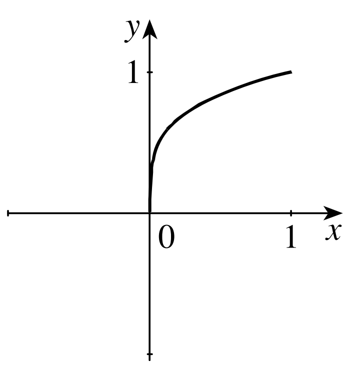
x



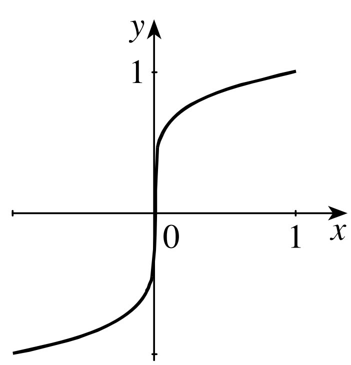
$\sqrt{x}$



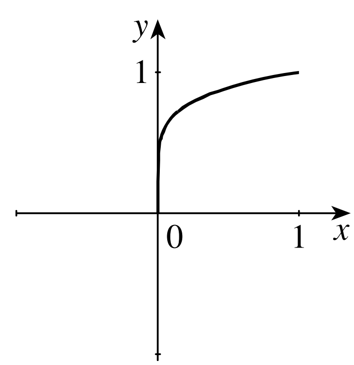
$\sqrt[3]{x}$



$\sqrt[4]{x}$



$\sqrt[5]{x}$



$\sqrt[6]{x}$

GROUP WORK 1, SECTION 1.2

Rounding the Bases

1. For computational efficiency and speed, we often round off constants in equations. For example, consider the linear function

$$f(x) = 3.137619523x + 2.123337012$$

In theory, it is very easy and quick to find $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$. In practice, most people doing this computation would probably substitute

$$f(x) = 3x + 2$$

unless a very accurate answer is called for. For example, compute $f(5)$ both ways to see the difference.

The actual value of $f(5)$: __________

The "rounding" estimate: __________

The percentage error: __________

2. Now consider

$$g(x) = 1.12755319x^3 + 3.125694x^2 + 1$$

Again, one is tempted to substitute $g(x) = x^3 + 3x^2 + 1$.

The actual value of $g(5)$: __________

The "rounding" estimate: __________

The percentage error: __________

3. It turns out to be very dangerous to similarly round off exponential functions, due to the nature of their growth. For example, let's look at the function

$$h(x) = (2.145217198123)^x$$

One may be tempted to substitute $h(x) = 2^x$ for this one. Once again, look at the difference between these two functions.

The actual value of $h(5)$: __________

The "rounding" estimate: __________

The percentage error: __________

GROUP WORK 2, SECTION 1.2

The Small Shall Grow Large

1. For what values of x is $x^6 \geq x^8$? For what values is $x^6 \leq x^8$?

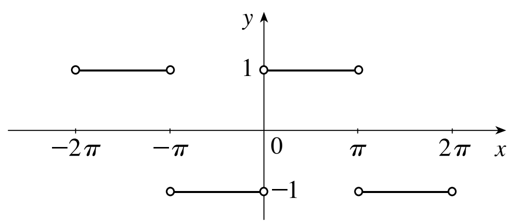
2. For what values of x is $x^3 \geq x^5$?

3. For what values of x is $x^3 \geq x^{105}$? Can you generalize your results?

GROUP WORK 3, SECTION 1.2

Fun with Fourier

The following function $S(x)$ is called a "square wave".



1. Can you find a function on your calculator which has the given graph?

2. Select the function from among the following which gives the best approximation to the square wave: $\frac{4}{\pi}\sin x$, $\frac{4}{\pi}(\sin x + \sin 3x)$, and $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$.

3. Select the function from among the following which gives the best approximation to the square wave: $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$, $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right)$, $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x\right)$, and $\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \frac{1}{9}\sin 9x\right)$.

4. A *Fourier approximation* of a function is an approximation of the form

$$F(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots + a_n \cos nx + b_n \sin nx$$

You have just discovered the Fourier approximation to $S(x)$ with five terms. Find the Fourier approximation to $S(x)$ with ten terms, and sketch its graph.

5. The following expressions are Fourier approximations to a different function, $T(x)$:

$$\begin{aligned}
T(x) &\approx \sin x \\
T(x) &\approx \sin x - \tfrac{1}{2}\sin 2x \\
T(x) &\approx \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x \\
T(x) &\approx \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x \\
T(x) &\approx \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \tfrac{1}{5}\sin 5x
\end{aligned}$$

Sketch $T(x)$.

1.3 New Functions from Old Functions

Suggested Time and Emphasis

1 class Essential material

Points to Stress

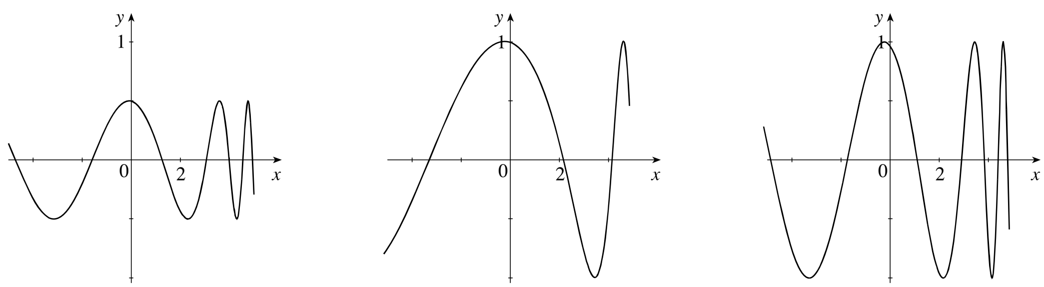
1. The mechanics and geometry of transforming functions.

2. The mechanics and geometry of adding, subtracting, multiplying, and dividing functions.

3. The mechanics and geometry of composing functions.

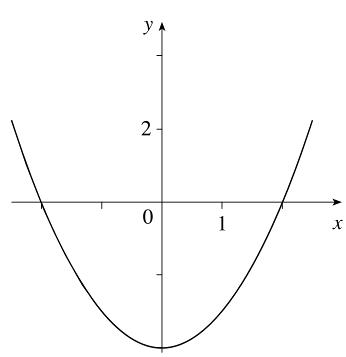
Quiz Questions

- **TEXT QUESTION** Label the following graphs: $f(x)$, $\frac{1}{2}f(x)$, $f\left(\frac{1}{2}x\right)$.

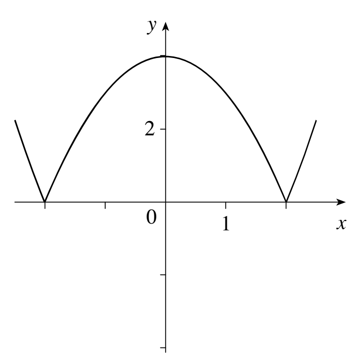


ANSWER $\frac{1}{2}f(x)$, $f\left(\frac{1}{2}x\right)$, $f(x)$

- **DRILL QUESTION** How can we construct the graph of $y = |f(x)|$ from the graph of $y = f(x)$? Explain in words, and demonstrate with the graph of $y = x^2 - 4$.
ANSWER We leave the positive values of $f(x)$ alone, and reflect the negative values about the x-axis.



$y = x^2 - 4$

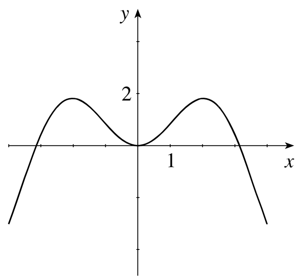


$y = \left|x^2 - 4\right|$

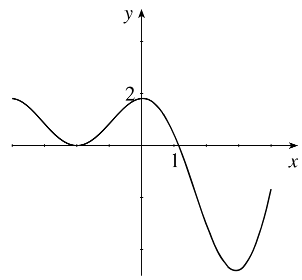
Materials for Lecture

- Using $f(x) = x \sin x$, explore graphs of $f(x+2)$, $f(x)+2$, $-f(x)$, $f(-x)$, $|f(x)|$. Note why $f(-x) = f(x)$.

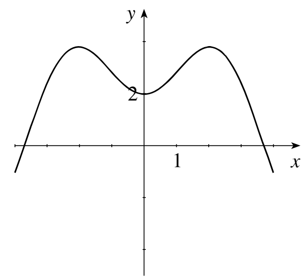
  ANSWER



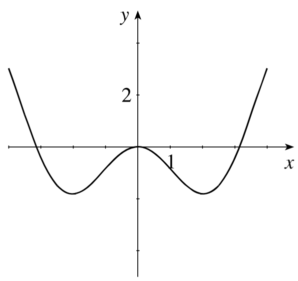
$y = f(x)$



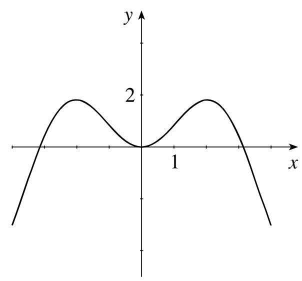
$y = f(x+2)$



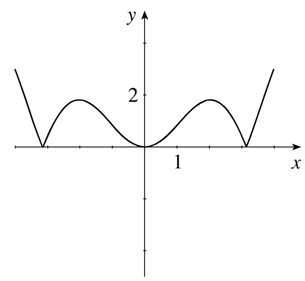
$y = f(x)+2$



$y = -f(x)$



$y = f(-x)$



$y = |f(x)|$

  $f(-x) = f(x)$ because f is even.

- Graph $f(x) = \sin\left(\sqrt{x}\right)$ and $g(x) = \sqrt{\sin x}$. Draw the relevant "arrow diagrams" and then write them in the forms $l \circ k$ and $k \circ l$. Then discuss reasons for the differences in their graphs.

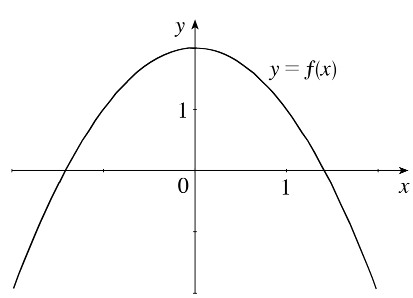
  ANSWER See the text for sample arrow diagrams. f is a sine function whose argument grows larger more and more slowly as we move away from the origin. g is a root function whose argument oscillates, causing g to oscillate as well.

Workshop/Discussion

- Using $f(x) = 1/x^2$ and $g(x) = \cos x$, compute the domains of $f+g$, f/g, g/f, $f \circ g$, and $g \circ f$, and the range of $g \circ f$. Pay particular attention to the domain of g/f, as many students will think it is $\mathbb{R}$.

  ANSWER $f+g$ has domain $\{x \mid x \neq 0\}$, f/g has domain $\left\{x \mid x \neq 0, x \neq \frac{\pi}{2} + k\pi\right\}$, g/f has domain $\{x \mid x \neq 0\}$, $f \circ g$ has domain $\left\{x \mid x \neq 0, x \neq \frac{\pi}{2} + k\pi\right\}$, $g \circ f$ has domain $\{x \mid x \neq 0\}$, and $g \circ f$ has range $[-1, 1]$.

- Do the following problem with the students:



From the graph of $y = f(x) = -x^2 + 2$ shown above, compute $f \circ f$ at $x = -1$, 0, and 1. First do it graphically (as in Exercises 51 and 52), then algebraically.

- TEC Have the students graph the following functions, where $f(x) = x + x^2$, first guessing what each graph will look like, and then using a calculator to confirm their guesses.

1. $f(2x)$	**4.** $-2f(x)$	**7.** $f(x) + 2$
2. $f\left(\frac{1}{2}x\right)$	**5.** $f(x-2)$	**8.** $2f(2x)$
3. $2f(x)$	**6.** $f(x) - 2$	**9.** $2f\left(\frac{1}{2}x\right)$

- After doing a few basic examples of composition, it is possible to foreshadow the idea of inverses, which is covered in Chapter 6. Let $f(x) = 2x^3 + 3$ and $g(x) = x^2 - x$. Compute $f \circ g$ and $g \circ f$ for your students. Then ask them to come up with a function $h(x)$ with the property that $(f \circ h)(x) = x$. They may not be used to the idea of finding examples by themselves; important hints they might need are "Don't give up," "When in doubt, just try something and see what happens," and "I'm not expecting you to get it in fifteen seconds." If the class is really stuck, have them try $f(x) = 2x^3$ to get a feel for how the game is played. Once they have determined that $h(x) = \sqrt[3]{\frac{x-3}{2}}$, have them first compute $(h \circ f)(x)$, then conjecture whether $(f \circ g)(x) = x$ implies $(g \circ f)(x) = x$ in general.

Group Work 1: Which is the Original?

ANSWERS **1.** $2f(x+2)$, $2f(x)$, $f(2x)$, $f(x+2)$, $f(x)$ **2.** $2f(x)$, $f(x)$, $f(x+2)$, $f(2x)$, $2f(x+2)$

Group Work 2: Label Label Label, I Made It Out of Clay

Some of these transformations were not covered directly in the book. If the students are urged not to give up, and to use the process of elimination and testing individual points, they should be able to complete this activity.

ANSWERS **1.** (d) **2.** (a) **3.** (f) **4.** (e) **5.** (i) **6.** (j) **7.** (b) **8.** (c) **9.** (g) **10.** (h)

Group Work 3: It's More Fun to Compute

Each group gets one copy of the graph. During each round, one representative from each group stands, and one of the questions below is asked. The representatives write their answer down, and all display their answers at the same time. Each representative has the choice of consulting with their group or not. A correct solo answer is worth two points, and a correct answer after a consult is worth one point.

ANSWERS **1.** 0 **2.** 0 **3.** 1 **4.** 5 **5.** 1 **6.** 1 **7.** 1 **8.** 0 **9.** 2 **10.** 1 **11.** 1 **12.** 1

Homework Problems

CORE EXERCISES 3, 13, 15, 17, 19, 29, 31, 37, 41, 51, 53

SAMPLE ASSIGNMENT 2, 3, 5, 9, 11, 13, 15, 17, 19, 21, 29, 31, 35, 37, 39, 41, 50, 51, 53, 55

EXERCISE	D	A	N	G
2	×			
3				×
5				×
9				×
11				×
13				×
15				×
17				×
19				×
21				×

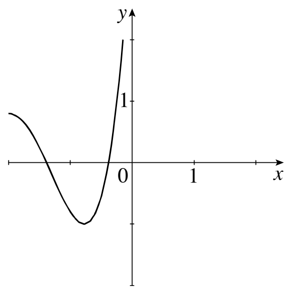
EXERCISE	D	A	N	G
29		×		
31		×		
35		×		
37		×		
39		×		
41		×		
50		×	×	
51				×
53	×	×		
55	×	×		

GROUP WORK 1, SECTION 1.3

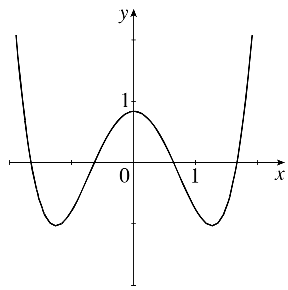
Which is the Original?

Below are five graphs. One is the graph of a function $f(x)$ and the others include the graphs of $2f(x)$, $f(2x)$, $f(x+2)$, and $2f(x+2)$. Determine which is the graph of $f(x)$ and match the other functions with their graphs.

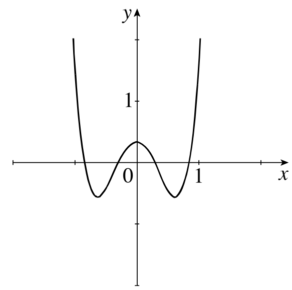
1.



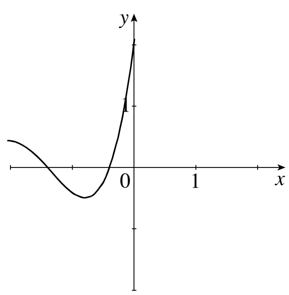
Graph 1



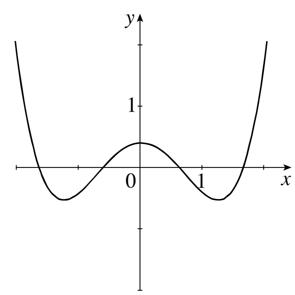
Graph 2



Graph 3

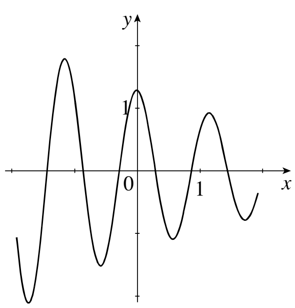


Graph 4

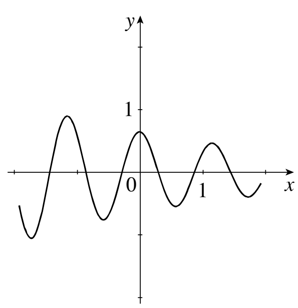


Graph 5

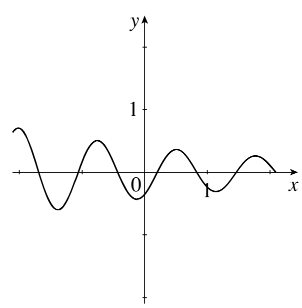
2.



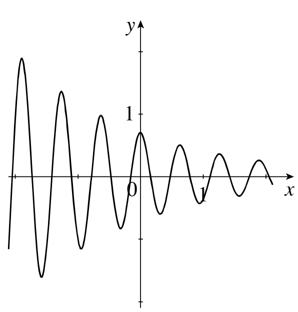
Graph 1



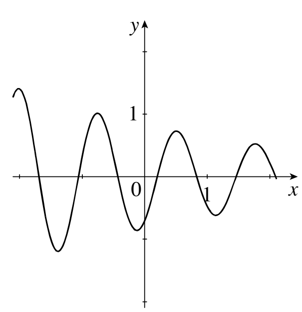
Graph 2



Graph 3



Graph 4

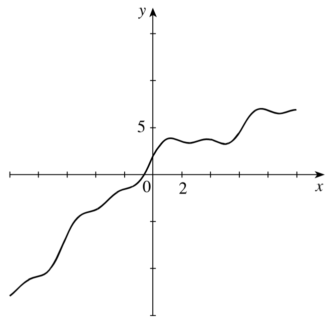


Graph 5

GROUP WORK 2, SECTION 1.3

Label Label Label, I Made it Out of Clay

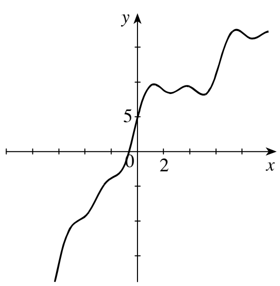
This is a graph of the function $f(x)$:



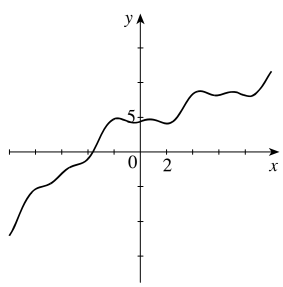
Give each graph below the correct label from the following:

(a) $f(x+3)$ (b) $f(x-3)$ (c) $f(2x)$ (d) $2f(x)$ (e) $|f(x)|$

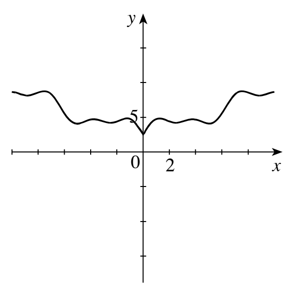
(f) $f(|x|)$ (g) $2f(x)-1$ (h) $f(2x)+2$ (i) $f(x)-x$ (j) $1/f(x)$



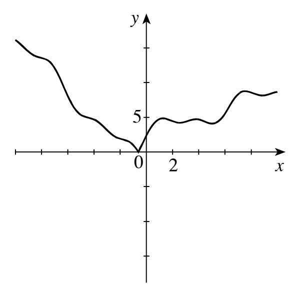
Graph 1



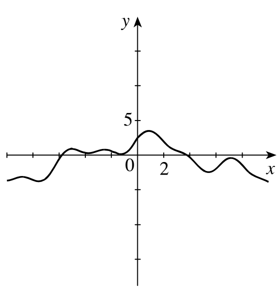
Graph 2



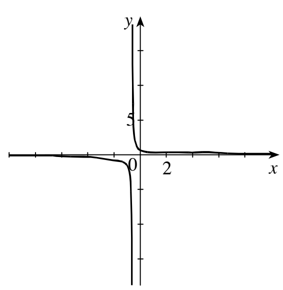
Graph 3



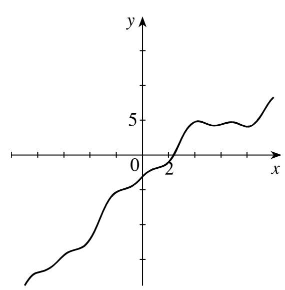
Graph 4



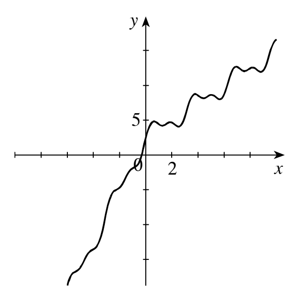
Graph 5



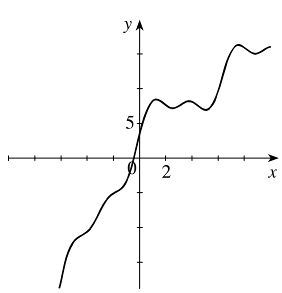
Graph 6



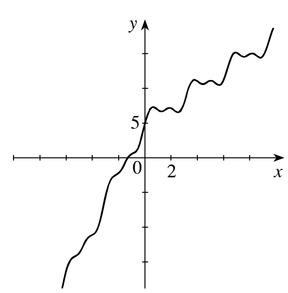
Graph 7



Graph 8



Graph 9



Graph 10

GROUP WORK 3, SECTION 1.3

It's More Fun to Compute

Using the graph below, find the following quantities.

1. $(f \circ g)(5)$

2. $(g \circ f)(5)$

3. $(f \circ g)(0)$

4. $(f \circ f)(5)$

5. $(g \circ g)(5)$

6. $(g \circ g)(-3)$

7. $(g \circ g)(-1)$

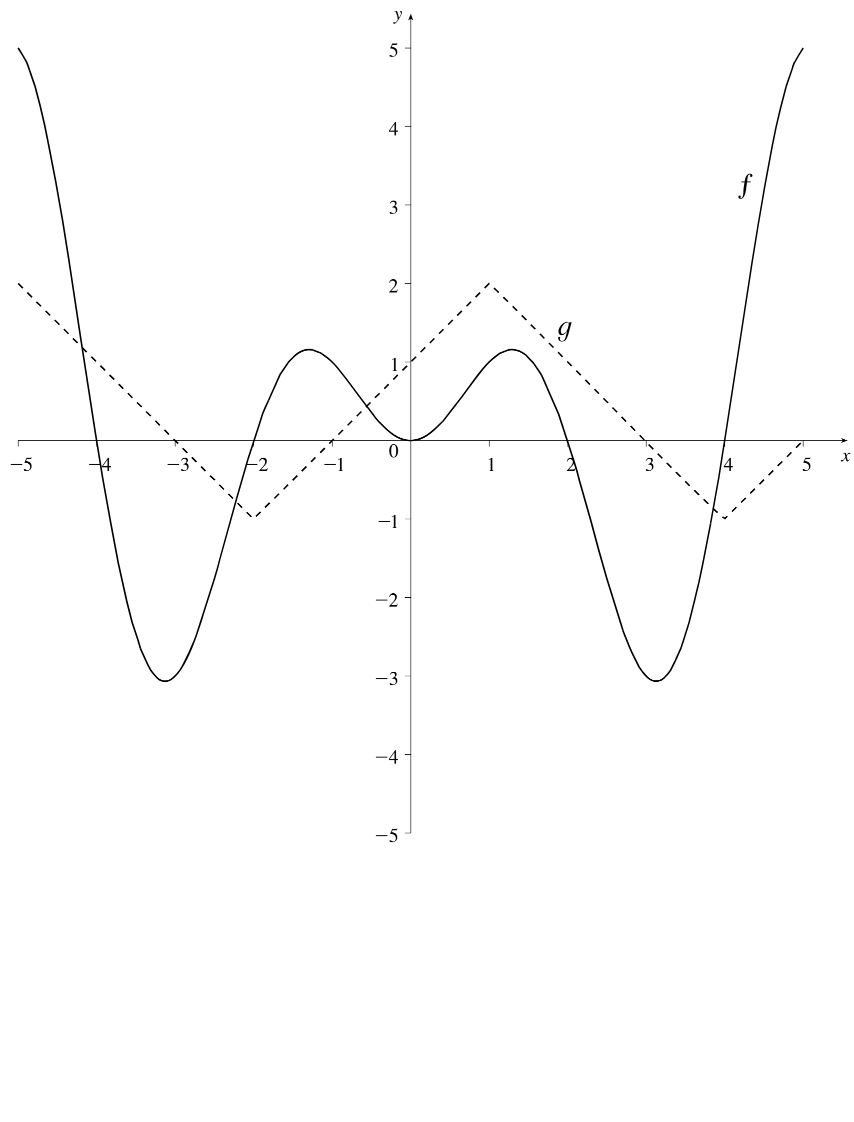
8. $(f \circ g)(1)$

9. $(g \circ f)(1)$

10. $(f \circ f \circ g)(4)$

11. $(g \circ f \circ f)(4)$

12. $(f \circ g \circ f)(4)$



1.4 The Tangent and Velocity Problems

Suggested Time and Emphasis

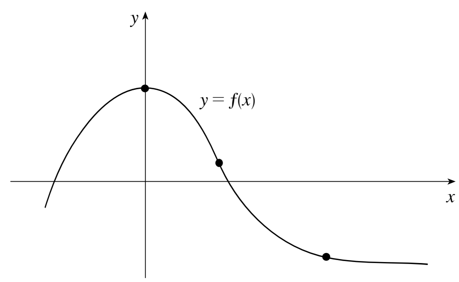
$\frac{1}{2}$–1 class   Essential material

Points to Stress

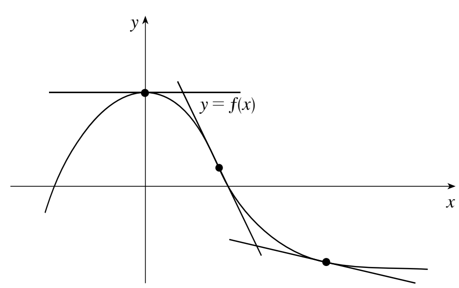
1. The tangent line viewed as the limit of secant lines.
2. The concepts of average versus instantaneous velocity, described numerically, graphically, and in physical terms.
3. The tangent line as the line obtained by "zooming in" on a smooth function; local linearity.
4. Approximating the slope of the tangent line using slopes of secant lines.

Quiz Questions

- **TEXT QUESTION** Geometrically, what is "the line tangent to a curve" at a particular point?
  ANSWER There are different correct ones. Examples include the best linear approximation to a curve at a point, or the result of repeated "zooming in" on a curve.
- **DRILL QUESTION** Draw the line tangent to the following curve at each of the indicated points:



ANSWER



Materials for Lecture

- Point out that if a car is driving along a curve, the headlights will point along the direction of the tangent line.
- Discuss the phrase "instantaneous velocity." Ask the class for a definition, such as, "It is the limit of average velocities." Use this discussion to shape a more precise definition of a limit.
- Illustrate that many functions such as x^2 and $x - 2\sin x$ look locally linear, and discuss the relationship of this property to the concept of the tangent line. Then pose the question, "What does a secant line to a linear function look like?"

- Show that the slopes of the tangent lines to $f(x) = \sqrt[3]{x}$ and $g(x) = |x|$ are not defined at $x = 0$. Note that f has a tangent line (which is vertical), but g does not (it has a cusp). The absolute value function can be explored graphically using TEC.

Workshop/Discussion

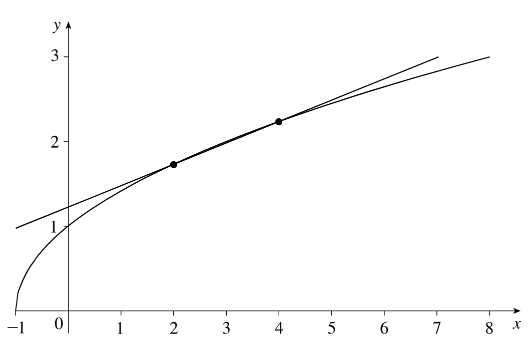
- Estimate slopes from discrete data, as in Exercises 2 and 7.
- Estimate the slope of $y = \dfrac{3}{1+x^2}$ at the point $\left(1, \frac{3}{2}\right)$ using the graph, and then numerically. Draw the tangent line to this curve at the indicated point. Do the same for the points $(0, 3)$ and $\left(2, \frac{3}{5}\right)$.
  ANSWER -1.5, 0, -0.48
- Draw tangent lines to the curve $y = \sin\left(\dfrac{1}{x}\right)$ at $x = \dfrac{1}{2\pi}$ and $x = \dfrac{1}{\pi/2}$. Notice the difference in the quality of the tangent line approximations.

Group Work 1: What's the Pattern?

The students will not be able to do Problem 3 from the graph alone, although some will try. After a majority of them are working on Problem 3, announce that they can do this numerically.
If they are unable to get Problem 6, have them repeat Problem 4 for $x = 15$, and again for $x = 0$.

ANSWERS

1, 2.



3. $2 - \sqrt{3} \approx 0.268$, $\sqrt{5} - 2 \approx 0.236$,
$\sqrt{4.5} - \sqrt{3.5} \approx 0.250$, $\dfrac{\sqrt{4.2} - \sqrt{3.8}}{0.4} \approx 0.250$

4. $\frac{1}{4}$ is a good estimate.

5. $\frac{1}{6}$ is a good estimate.

6. $\dfrac{1}{2\sqrt{a+1}}$

Group Work 2: Slope Patterns

When introducing this activity, it may be best to fill out the first line of the table with your students, or to estimate the slope at $x = -1$. If a group finishes early, have them try to justify the observations made in the last part of Problem 2.

ANSWERS

1. (a) 0, 0.2, 0.4, 0.6 (b) 11.5

2. (a) Estimating from the graph gives that the function is increasing for $x < -3.2$, decreasing for $-3.2 < x < 3.2$, and increasing for $x > 3.2$.

(b) The slope of the tangent line is positive when the function is increasing, and the slope of the tangent line is negative when the function is decreasing.

(c) The slope of the tangent line is zero somewhere between $x = -3.2$ and -3.1, and somewhere between $x = 3.1$ and 3.2. The graph has a local maximum at the first point and a local minimum at the second.

(d) The tangent line approximates the curve worst at the maximum and the minimum. It approximates best at $x = 0$, where the curve is "straightest," that is, at the point of inflection.

Homework Problems

CORE EXERCISES 1, 5

SAMPLE ASSIGNMENT 1, 3, 5, 7

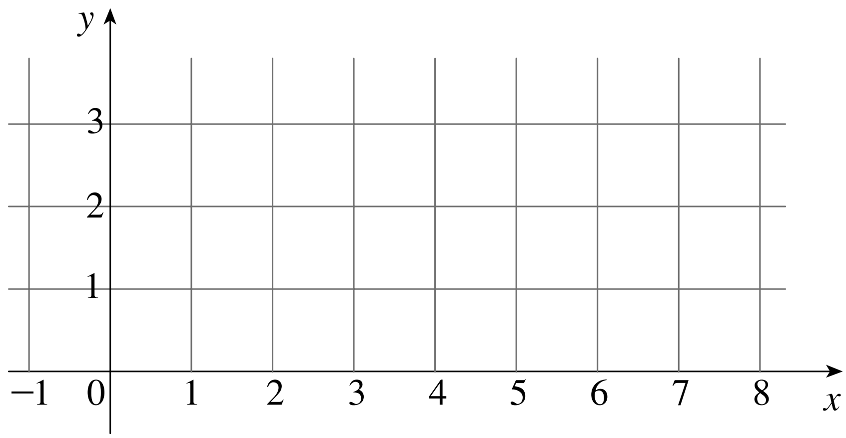
EXERCISE	D	A	N	G
1		×	×	×
3		×		×
5		×		
7		×	×	×

GROUP WORK 1, SECTION 1.4

What's the Pattern?

Consider the function $f(x) = \sqrt{1+x}$.

1. Carefully sketch a graph of this function on the grid below.



2. Sketch the secant line to f between the points with x-coordinates $x = 2$ and $x = 4$.

3. Sketch the secant lines to f between the pairs of points with the following x-coordinates, and compute their slopes:

(a) $x = 2$ and $x = 3$ (b) $x = 3$ and $x = 4$ (c) $x = 2.5$ and $x = 3.5$ (d) $x = 2.8$ and $x = 3.2$

4. Using the slopes you've found so far, estimate the slope of the tangent line at $x = 3$.

5. Repeat Problem 4 for $x = 8$.

6. Based on Problems 4 and 5, guess the slope of the tangent line at any point $x = a$, for $a > -1$.

GROUP WORK 2, SECTION 1.4

Slope Patterns

1. (a) Estimate the slope of the line tangent to the curve $y = 0.1x^2$, where $x = 0, 1, 2, 3$. Use your information to fill in the following table:

x	slope of tangent line
0	
1	
2	
3	

(b) You should notice a pattern in the above table. Using this pattern, estimate the slope of the line tangent to $y = 0.1x^2$ at the point $x = 57.5$.

2. Consider the function $f(x) = 0.1x^3 - 3x$.

(a) On what intervals is this function increasing? On what intervals is it decreasing?

(b) On what interval or intervals is the slope of the tangent line positive? On what interval or intervals is the slope of the tangent line negative? What is the connection between these questions and part (a)?

(c) Where does the slope of the tangent line appear to be zero? What properties of the graph occur at these points?

(d) Where does the tangent line appear to approximate the curve the best? The worst? What properties of the graph seem to make it so?

1.5 The Limit of a Function

Suggested Time and Emphasis

1 class Essential material

Points to Stress

1. The various meanings of "limit" (descriptive, numeric, graphic), both finite and infinite. Note that algebraic manipulations are not yet emphasized.
2. The geometric and limit definitions of vertical asymptotes.
3. The advantages and disadvantages of using a calculator to compute a limit.

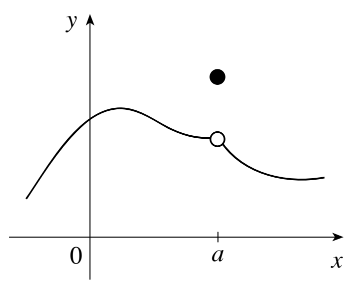
Quiz Questions

- **TEXT QUESTION** What is the difference between the statements "$f(a) = L$" and "$\lim_{x \to a} f(x) = L$"?

  ANSWER The first is a statement about the value of f at the point $x = a$, the second is a statement about the values of f at points near, but not equal to, $x = a$.

- **DRILL QUESTION** The graph of a function f is shown below. Are the following statements about f true or false? Why?

  (a) $x = a$ is in the domain of f (b) $\lim_{x \to a} f(x)$ exists (c) $\lim_{x \to a^+} f(x)$ is equal to $\lim_{x \to a^-} f(x)$



  ANSWER

  (a) True, because f is defined at $x = a$.

  (b) True, because as x gets close to a, $f(x)$ approaches a value.

  (c) True, because the same value is approached from both directions.

Materials for Lecture

- Present the "motivational definition of limit": *We say that* $\lim_{x \to a} f(x) = L$ *if as x gets close to a*, $f(x)$ *gets close to* L, and then lead into the definition in the text. (Note that limits will be defined more precisely in Section 1.7.)
- Describe asymptotes verbally, and then give graphic and limit definitions. If foreshadowing horizontal asymptotes, note that a function *can* cross a horizontal asymptote. Perhaps foreshadow the notion of slant asymptotes, which are covered later in the text.
- Discuss how we can rephrase the last section's concept as "the slope of the tangent line is the limit of the slopes of the secant lines as $\Delta x \to 0$".

- Stress that if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then we still don't know anything about $\lim_{x \to a} \left[f(x) - g(x) \right]$.

Workshop/Discussion

- Explore the greatest integer function $f(x) = [\![x]\!]$ on the interval $[-1, 2]$ in terms of left-hand and right-hand limits.

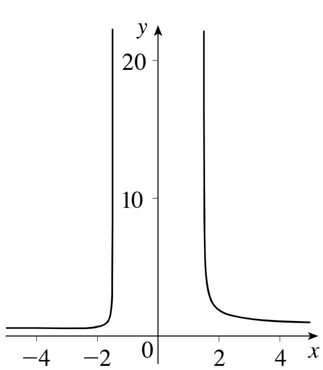
- Discuss a limit such as $\lim_{x \to 3} \frac{x-3}{4x+2}$. When can you "just plug in the numbers"?

Group Work 1: Infinite Limits

After the students are finished, Problem 2 can be used to initiate a discussion of left and right hand limits, and of the precise definition of a vertical asymptote, as presented in the text. In addition, Section 3.4 can be foreshadowed by asking the students to explore the behavior of $\frac{3x^2 + 4x + 5}{\sqrt{16x^4 - 81}}$ for large positive and large negative values of x, both on the graph and numerically. If there is time, the students can be asked to analyze the asymptotes of $f(x) = \sec x$ and the other trigonometric functions.

ANSWERS

1. Answers will vary. The main thing to check is that there are vertical asymptotes at $\pm\frac{\pi}{2}$ and at $\pm\frac{3\pi}{2}$.

2. 

There are vertical asymptotes at $x = \pm 1.5$.

Group Work 2: The Shape of Things to Come

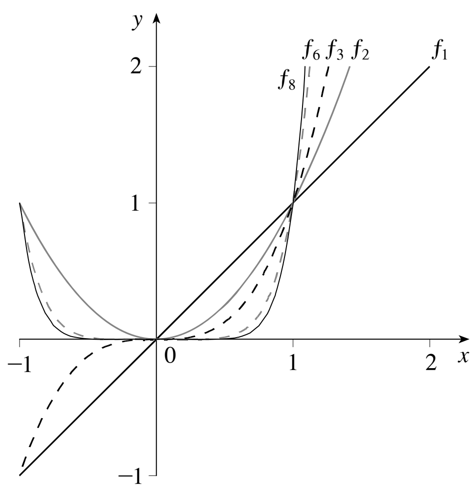
This exercise foreshadows concepts that will be discussed later, but can be introduced now. The idea is to show the students that the concept of "limit" can get fairly subtle, and that care is needed. The second page anticipates Section 3.4, and the third page anticipates Section 1.7. Pages 2 and 3 are independent of each other; either or both can be used. Problem 4 on page 3 is a little tricky and can be omitted if desired.

ANSWERS

PAGE 1

1. $2^1 = 2$, $(0.6)^2 = 0.36$, $(0.8)^3 = 0.512$, $0^4 = 0$, $(1.01)^8 = 1.0829$

2.



3. $f_n(1) = 1$ for all n.

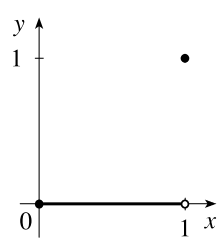
4. The curves all go through the origin.

PAGE 2

1. (a) 0 (b) 1 (c) 1

2. (a) 0 (b) 0 (c) 0 (d) 1

3. The function $g(x)$ is important in real analysis. Its graph looks like this:



4. (a) 1 (b) 0

PAGE 3

1. $\frac{1}{10}$ (or any positive number less than $\frac{1}{10}$)

2. Estimates will vary.

3. $\frac{\sqrt{101}}{10} - 1 \approx 0.0049876$ (or any positive number less than 0.0049876)

4. Yes, the problems could have been done with any smaller positive number.

5. The students can be forgiven for not answering this question. It will be fully answered in Section 1.7. The short answer: Let a be the "small number you can name." Then we have shown that we can always find a small interval about x such that $|x^2 - 0| < a$. A similar argument can be made for the second part. The main idea here is to set up ideas that will be explored more fully in Section 1.7.

Group Work 3: Why Can't We Just Trust the Table?

This activity was inspired by the article "An Introduction to Limits" from *College Mathematics Journal*, January 1997, page 51, and extends Example 4.

Put the students into groups and give each group two different digits between 1 and 9, and then let them proceed with the problems in the handout.

ANSWERS

1. The answer, of course, depends on the starting digit:

x	$\sin \frac{\pi}{x}$
0.1	0
0.01	0
0.001	0
0.0001	0
0.00001	0
0.000001	0

x	$\sin \frac{\pi}{x}$
0.2	0
0.02	0
0.002	0
0.0002	0
0.00002	0
0.000002	0

x	$\sin \frac{\pi}{x}$
0.3	$-\frac{\sqrt{3}}{2}$
0.03	$-\frac{\sqrt{3}}{2}$
0.003	$-\frac{\sqrt{3}}{2}$
0.0003	$-\frac{\sqrt{3}}{2}$
0.00003	$-\frac{\sqrt{3}}{2}$
0.000003	$-\frac{\sqrt{3}}{2}$

x	$\sin \frac{\pi}{x}$
0.4	1
0.04	0
0.004	0
0.0004	0
0.00004	0
0.000004	0

x	$\sin \frac{\pi}{x}$
0.5	0
0.05	0
0.005	0
0.0005	0
0.00005	0
0.000005	0

x	$\sin \frac{\pi}{x}$
0.6	$-\frac{\sqrt{3}}{2}$
0.06	$\frac{\sqrt{3}}{2}$
0.006	$\frac{\sqrt{3}}{2}$
0.0006	$\frac{\sqrt{3}}{2}$
0.00006	$\frac{\sqrt{3}}{2}$
0.000006	$\frac{\sqrt{3}}{2}$

x	$\sin \frac{\pi}{x}$
0.7	−0.974927912
0.07	0.781831482
0.007	0.433883739
0.0007	0.974927912
0.00007	−0.781831482
0.000007	−0.433883739

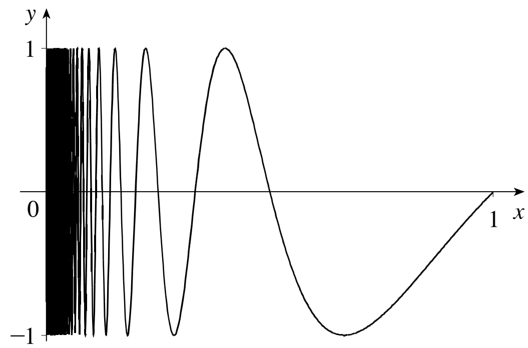
x	$\sin \frac{\pi}{x}$
0.8	$-\frac{\sqrt{2}}{2}$
0.08	1
0.008	0
0.0008	0
0.00008	0
0.000008	0

x	$\sin \frac{\pi}{x}$
0.9	−0.342020143
0.09	−0.342020143
0.009	−0.342020143
0.0009	−0.342020143
0.00009	−0.342020143
0.000009	−0.342020143

2. Answers will vary.

3. Answers will vary.

4. There is no limit.



Homework Problems

CORE EXERCISES 1, 3, 5, 11, 13, 19, 29, 31, 39

SAMPLE ASSIGNMENT 1, 3, 5, 7, 9, 11, 13, 17, 19, 29, 31, 32, 35, 39

EXERCISE	D	A	N	G
1	×			
3	×			
5	×			×
7	×			×
9	×			×
11				×
13	×			×

EXERCISE	D	A	N	G
17				×
19		×		
29		×		
31		×		
32		×		
35		×		
39		×		×

GROUP WORK 1, SECTION 1.5
Infinite Limits

1. Draw an odd function which has the lines $x = \frac{\pi}{2}$ and $x = -\frac{3\pi}{2}$ among its vertical asymptotes.

2. Analyze the vertical asymptotes of $\dfrac{3x^2 + 4x + 5}{\sqrt{16x^4 - 81}}$.

GROUP WORK 2, SECTION 1.5

The Shape of Things to Come

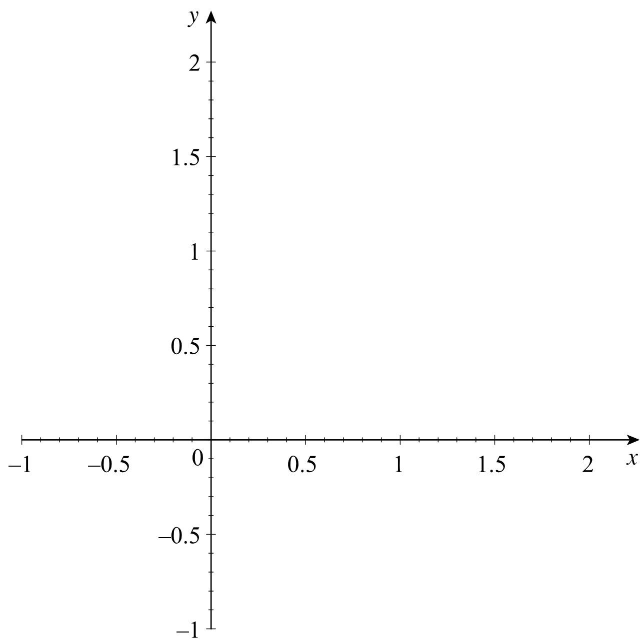
In this activity we are going to explore a set of functions:

$$\begin{aligned} f_1(x) &= x \\ f_2(x) &= x^2 \\ f_3(x) &= x^3 \\ &\vdots \\ f_n(x) &= x^n, \ n \text{ any positive integer} \end{aligned}$$

1. To start with, let's practice the new notation. Compute the following:

$f_1(2) =$ _____ $f_2(0.6) =$ _____ $f_3(0.8) =$ _____ $f_4(0) =$ _____ $f_8(1.01) =$ _____

2. Sketch the functions f_1, f_2, f_3, f_6, and f_8 on the set of axes below.



3. The number 0 plays a special role, since $f_n(0) = 0^n = 0$ for all positive integers n. Find another number $a > 0$ such that $f_n(a) = a$ for all positive integers n.

4. We know that $\lim_{x \to 0} f_n(x) = 0$ for all positive integers n. How is this fact reflected on your graphs above?

GROUP WORK 2, SECTION 1.5

The Shape of Things to Come: Approaching Infinity

1. Using what you know about limits, compute the following quantities:

(a) $\lim_{x \to 0} f_3(x)$

(b) $\lim_{x \to 1} f_4(x)$

(c) $\lim_{x \to 1} f_{15}(x)$

2. Using what you know about limits, compute the following quantities:

(a) $\lim_{n \to \infty} f_n\left(\frac{1}{2}\right)$

(b) $\lim_{n \to \infty} f_n(0.99)$

(c) $\lim_{n \to \infty} f_n(x)$, where $|x| < 1$

(d) $\lim_{n \to \infty} f_n(1)$

3. Let $g(x) = \lim_{n \to \infty} f_n(x)$ for $0 \le x \le 1$. Sketch $g(x)$, paying particular attention to $g(1)$ and values of x close to 1.

4. Are the following quantities defined? If so, what are they? If not, why not?

(a) $\lim_{n \to \infty} \left(\lim_{x \to 1^-} f_n(x) \right)$

(b) $\lim_{x \to 1^-} \left(\lim_{n \to \infty} f_n(x) \right)$

GROUP WORK 2, SECTION 1.5

The Shape of Things to Come: The Nitty-Gritty

By definition, "$\lim_{x \to 0} f_2(x) = 0$" means that by taking x very close to zero, we can make $\left|x^2 - 0\right|$ smaller than any small number you can name.

1. Find a number $\boldsymbol{\delta} > 0$ such that if $-\boldsymbol{\delta} < x < \boldsymbol{\delta}$, then $|f_2(x)| < \frac{1}{100}$.

2. Use a graph to find a number $\boldsymbol{\delta} > 0$ such that if $-\boldsymbol{\delta} < x - 1 < \boldsymbol{\delta}$, then $\left|x^2 - 1\right| < \frac{1}{100}$.

3. Now use algebra to find a number $\boldsymbol{\delta} > 0$ such that if $-\boldsymbol{\delta} < x - 1 < \boldsymbol{\delta}$, then $\left|x^2 - 1\right| < \frac{1}{100}$.

4. When constructing this problem, $\frac{1}{100}$ was used as an arbitrary smallish number. Could you have done the previous problems if we replaced $\frac{1}{100}$ by $\frac{1}{10,000}$? How about $\frac{1}{1,000,000}$?

5. Reread the first sentence on this page. How do your answers to Problems 1 and 4 show that $\lim_{x \to 0} f_2(x) = 0$? Do your answers to Problems 2, 3, and 4 show that $\lim_{x \to 1} \left(x^2 - 1\right) = 0$? Why?

GROUP WORK 3, SECTION 1.5

Why Can't We Just Trust the Table?

We are going to investigate $\lim_{x\to 0} \sin \frac{\pi}{x}$. We will take values of x closer and closer to zero, and see what value the function approaches.

1. Your teacher has given you a digit — let's call it d. Fill out the following table. If, for example, your digit is 3, then you would compute $\sin\left(\frac{\pi}{0.3}\right)$, $\sin\left(\frac{\pi}{0.03}\right)$, $\sin\left(\frac{\pi}{0.003}\right)$, $\sin\left(\frac{\pi}{0.0003}\right)$, etc.

x	$\sin \frac{\pi}{x}$
$0.d$	
$0.0d$	
$0.00d$	
$0.000d$	
$0.0000d$	
$0.00000d$	

2. What is $\lim_{x\to 0} \sin \frac{\pi}{x}$?

3. Now fill out the table with a different digit.

x	$\sin \frac{\pi}{x}$
$0.d$	
$0.0d$	
$0.00d$	
$0.000d$	
$0.0000d$	
$0.00000d$	

Do you get the same result?

4. What is $\lim_{x\to 0} \sin \frac{\pi}{x}$?

1.6 Calculating Limits Using the Limit Laws

Suggested Time and Emphasis

1 class   Essential material

Points to Stress

1. The algebraic computation of limits: manipulating algebraically, examining left- and right-hand limits, using the limit laws to break monstrous functions into pieces, and analyzing the pieces.
2. The evaluation of limits from graphical representations.
3. Examples where limits don't exist (using algebraic and graphical approaches).
4. The computation of limits when the limit laws do not apply, and the use of direct substitution property when they do.

Quiz Questions

- **TEXT QUESTION** In Example 4, why isn't $\lim_{x\to 1} g(x) = \pi$?

  ANSWER Because the limit isn't affected by the function when $x = 1$, only when x is near 1.

- **DRILL QUESTION** If $a \neq 0$, find $\lim_{x\to a} \dfrac{x^2 - 2ax + a^2}{x^2 - a^2}$.

  (A) $\dfrac{1}{2a}$ (B) $\dfrac{1}{2a^2}$ (C) $-\dfrac{1}{2a^2}$ (D) 0 (E) Does not exist

  ANSWER (D)

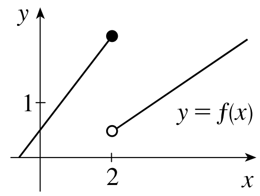
Materials for Lecture

- Discuss why $\lim_{x\to 0} [\![x]\!] \sin x = 0$ is not a straightforward application of the Product Law.
- Have the students determine the existence of $\lim_{x\to 0^+} \sqrt{x}$ and determine why we cannot compute $\lim_{x\to 0} \sqrt{x}$.
- Use the Squeeze Theorem to show that $\lim_{x\to 0} x^2 [\![x]\!] = 0$.

Workshop/Discussion

- Compute some limits of quotients, such as $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$, $\lim_{x\to 0} \dfrac{x^3-8}{x-2}$, $\lim_{x\to 3} \dfrac{x^3-8}{x-2}$, and $\lim_{x\to 2} \dfrac{x^3-8}{x-2}$, always attempting to plug values in first.
- Have the students check if $\lim_{x\to -5} \dfrac{x+5}{|x+5|}$ exists, and then compute left- and right-hand limits. Then check $\lim_{x\to -5} \dfrac{(x+5)^2}{|x+5|}$.
- Do some subtle product and quotients, such as $\lim_{x\to 0} \left(\dfrac{x}{|x|} \sin x\right)$ and $\lim_{x\to 1} \dfrac{\sqrt{x+3}-2}{x-1}$.

- Present some graphical examples, such as $\lim_{x \to 0} f(x)$ and $\lim_{x \to 2} f(x)$ in the graph below.



Group Work 1: Exploring Limits

Have the students work on this exercise in groups. Problem 2 is more conceptual than Problem 1, but makes an important point about the sums and products of limits.

ANSWERS

1. (a) (i) Does not exist (ii) Does not exist (iii) 4 (iv) Does not exist

(b) (i) Does not exist (ii) 1 (c) (i) 0 (ii) Does not exist

2. (a) Both quantities exist.

(b) Each quantity may or may not exist.

Group Work 2: Fixing a Hole

This exercise foreshadows concepts used later in the discussion of continuity, in addition to giving the students practice in taking limits. After the exercise, point out that mathematicians use the word "puncture" as well as "hole".

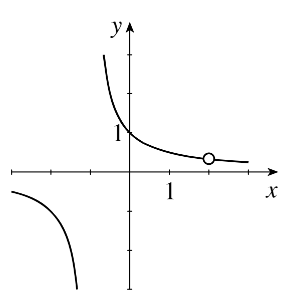
ANSWERS

1. No, yes, yes, no

2. $\{x \mid x \neq -1, 2\}$

3. Does not exist, $\frac{1}{3}$

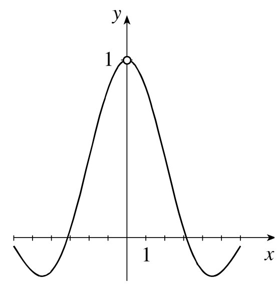
4.



One of the discontinuities can be "filled in" and the other cannot.

5. A "hole" is an x-value at which the function is not defined, yet the left- and right-hand limits exist.
Or: A "hole" is an x-value where the function is undefined, yet the function is defined near x.
Or: A "hole" is an x-value at which we can add a point to the function and thus make it continuous there.

6.



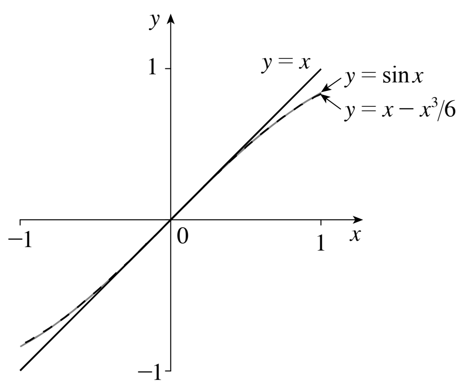
g has a hole at $x = 0$.

Group Work 3: The Squeeze Theorem

This exercise gives an informal graphical way to show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. A more careful geometric argument is given in Section 2.4.

ANSWERS

1, 2.



3. For $x > 0$, $\sin x < x \Rightarrow \frac{\sin x}{x} < 1$.

For $x < 0$, $\sin x > x \Rightarrow \frac{\sin x}{x} < 1$ (reversing the second inequality because $x < 0$).

4. $f(x) = 1 - \frac{x^2}{6}$

5. The Squeeze Theorem now gives $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Homework Problems

CORE EXERCISES 1, 7, 15, 17, 51

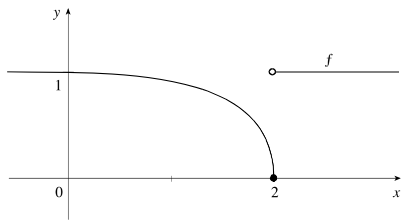
SAMPLE ASSIGNMENT 1, 7, 9, 11, 15, 17, 21, 27, 41, 51, 53

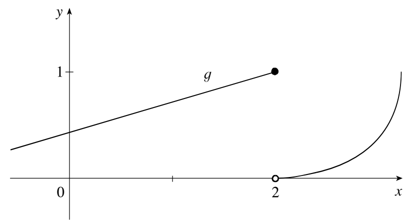
EXERCISE	D	A	N	G
1	×	×		
7		×		
9		×		
11		×		
15		×		
17		×		
21		×		
27		×		
41		×		
51		×		
53		×		

GROUP WORK 1, SECTION 1.6

Exploring Limits

1. Given the functions f and g (defined graphically below) and h and j (defined algebraically), compute each of the following limits, or state why they don't exist:





$$h(x) = \frac{x^2 - 4}{x - 2}$$

$$j(x) = \begin{cases} 1 & \text{if } x < 2 \\ 0 & \text{if } x \geq 2 \end{cases}$$

(a) (i) $\lim_{x\to 2} f(x)$ (ii) $\lim_{x\to 2} g(x)$ (iii) $\lim_{x\to 2} h(x)$ (iv) $\lim_{x\to 2} j(x)$

(b) (i) $\lim_{x\to 2} \left[g(x) + h(x)\right]$ (ii) $\lim_{x\to 2} \left[f(x) + j(x)\right]$

(c) (i) $\lim_{x\to 2} \left[f(x)\, g(x)\right]$ (ii) $\lim_{x\to 2} \left[f(x)\, j(x)\right]$

2. (a) In general, if $\lim_{x\to a} m(x)$ exists and $\lim_{x\to a} n(x)$ exists, is it true that $\lim_{x\to a} [m(x) + n(x)]$ exists? How about $\lim_{x\to a} [m(x)\, n(x)]$? Justify your answers.

(b) In general, if $\lim_{x\to a} m(x)$ does not exist and $\lim_{x\to a} n(x)$ does not exist, is it true that $\lim_{x\to a} [m(x) + n(x)]$ does not exist? How about $\lim_{x\to a} [m(x)\, n(x)]$? Compare these with your answers to part (a).

GROUP WORK 2, SECTION 1.6

Fixing a Hole

Consider $f(x) = \dfrac{x-2}{x^2-x-2}$.

1. Is $f(x)$ defined for $x = -1$? For $x = 0$? For $x = 1$? For $x = 2$?

2. What is the domain of f?

3. Compute $\lim\limits_{x\to -1} \dfrac{x-2}{x^2-x-2}$ and $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-x-2}$. Notice that one limit exists, and one does not.

4. Graph $y = \dfrac{x-2}{x^2-x-2}$. There are two x-values that are not in the domain of f. Later, we will call these "discontinuities". Geometrically, what is the difference between the two discontinuities?

5. We say that $f(x)$ has one *hole* in it. Where do you think that the hole is? Define "hole" in this context.

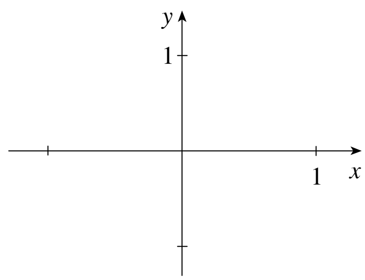
6. The function $g(x) = \dfrac{\sin x}{x}$ is not defined at $x = 0$. Sketch this function. Does it have a hole at $x = 0$?

GROUP WORK 3, SECTION 1.6

The Squeeze Theorem

In this exercise, we take a graphical approach to computing $\lim_{x\to 0} \frac{\sin x}{x}$.

1. Using a graphing calculator, show that if $0 \le x \le 1$, then $x - \frac{x^3}{6} \le \sin x \le x$. Give a rough sketch of the three functions over the interval [0, 1] on the graph below.



2. Again using a graphing calculator, show that if $-1 \le x \le 0$, then $x - \frac{x^3}{6} \ge \sin x \ge x$. If you have not done so already, add these portions of the three functions to your graph above.

3. Explain why $\frac{\sin x}{x} \le 1$ for $-1 \le x \le 1$, $x \ne 0$. Use the inequalities in parts 1 and 2 to help you.

4. Again using parts 1 and 2, can you find a function $f(x)$ with $f(x) \le \frac{\sin x}{x}$ on $-1 \le x \le 1$, $x \ne 0$, such that $\lim_{x\to 0} f(x) = 1$?

5. Using parts 3 and 4, compute $\lim_{x\to 0} \frac{\sin x}{x}$.

1.7 The Precise Definition of a Limit

Suggested Time and Emphasis

$1\text{–}1\frac{1}{2}$ classes Optional material

Points to Stress

1. The geometry of the ε-δ definition, what the notation means, and how it relates to the geometry.
2. The "narrow range" definition of a limit, as defined below.
3. Extending the precise definition to one-sided and infinite limits.

Quiz Questions

- **TEXT QUESTION** Example 1 finds a number δ such that $\left|\left(x^3 - 5x + 6\right) - 2\right| < 0.2$ whenever $|x - 1| < \delta$. Why does this *not* prove that $\lim_{x\to 1}\left(x^3 - 5x + 6\right) = 2$?

  ANSWER It is not a proof because we only dealt with $\varepsilon = 0.2$; a proof would hold for *all* ε.
- **DRILL QUESTION** Let $f(x) = 5x + 2$. Find δ such that $|f(x) - 12| < 0.01$ whenever $-\delta < x - 2 < \delta$.

  ANSWER $\delta = 0.002$ works, as does any smaller δ.

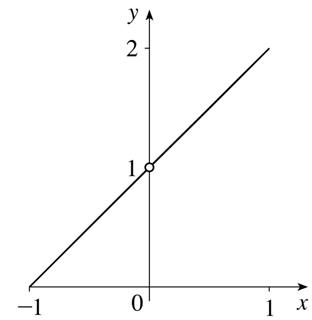
Materials for Lecture

- The "narrow range" definition of limit may be covered as a way of introducing the ε–δ definition to the students in a familiar numerical context. *We say that* $\lim_{x\to a} f(x) = L$ *if for any y-range centered at L there is an x-range centered at a such that the graph is "trapped" in the window* — that is, does not go off the top or the bottom of the window. The transition to the traditional definition can now be made easier by observing that the width of the y-range is 2ε and the width of the x-range is 2δ. If the students are familiar with graphing calculators, this definition can be illustrated with setting different viewing windows for a particular graph.
- Make sure the students understand that limit proofs, as described in the book, are two-step processes. The act of finding δ is separate from writing the proof that the students' choice of δ works in the limit definition. This fact is stated clearly in the text, but it is a novel enough idea that it should be reinforced.
- Discuss how close x needs to be to 4, first to ensure that $\dfrac{1}{(x-4)^2} > 1000$, and then so that $\dfrac{1}{(x-4)^2} > 20,000$. Then argue intuitively that $\lim_{x\to 4}\dfrac{1}{(x-4)^2} = \infty$.
- Using the formal definition of limit, show that neither 1 nor -1 is the limit of $h(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$ as x goes to 0. Emphasize that although this result is obvious from the graph, the idea is to see how the definition works using a function that is easy to work with.

Workshop/Discussion

- Estimate how close x must be to 0 to ensure that $(\sin x)/x$ is within 0.03 of 1. Then estimate how close x must be to 0 to ensure that $(\sin x)/x$ is within 0.001 of 1. Describe what you did in terms of the definition of a limit.
- Discuss why $f(x) = [\![x]\!]$ does not have a limit at $x = 0$, first using the "narrow range" definition of limit, and then possibly the ε-δ definition of limit.

  Then discuss why $\lim_{x\to 0} \frac{x^2 + x}{x} = 1$, using the "narrow range" definition of limit and a graph like the one at right.



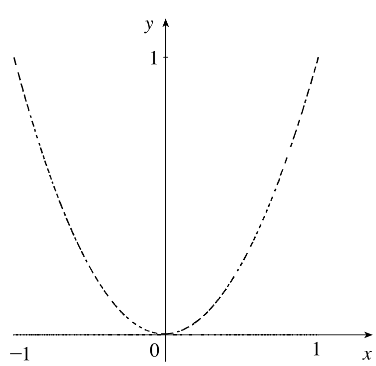
- Find, numerically or algebraically, a $\delta > 0$ such that if $0 \le |x - 0| \le \delta$, then $\left|x^3 - 0\right| < 10^{-3}$. Similarly, compute a $\delta > 0$ such that if $0 \le |x - 2| \le \delta$, then $\left|x^3 - 8\right| < 10^{-3}$.

Group Work 1: A Jittery Function

This exercise can be done several ways. After they have worked for a while, perhaps ask one group to try to solve it using the Squeeze Theorem, another to solve it using the "narrow range" definition of limit, and a third to solve it using the ε-δ definition of limit. They should show why their method works for Problem 2, and fails for Problem 3.

ANSWERS

1.



2. $\lim_{x\to 0} f(x) = 0$. Choose ε with $\varepsilon > 0$. Let $\delta = \sqrt{\varepsilon}$. Now if $-\delta < x < \delta$, then $x^2 < \varepsilon$, regardless of whether x is rational or irrational. This can also be shown using the Squeeze Theorem and the fact that $0 < f(x) < x^2$, and then using the Limit Laws to compute $\lim_{x\to 0} 0$ and $\lim_{x\to 0} x^2$.

3. It does not exist. Assume that $\lim_{x\to 1} f(x) = L$. Choose $\varepsilon = \frac{1}{10}$. Now, whatever your choice of δ, there are some x-values in the interval $(1 + \delta, 1 - \delta)$ with $f(x) = 0$, so L must be less than $\frac{1}{10}$. But there are also values of x in the interval with $f(x) > \frac{2}{10}$, so L must be greater than $\frac{1}{10}$. So L cannot exist. The "narrow range" definition of limit can also be used to solve this problem.

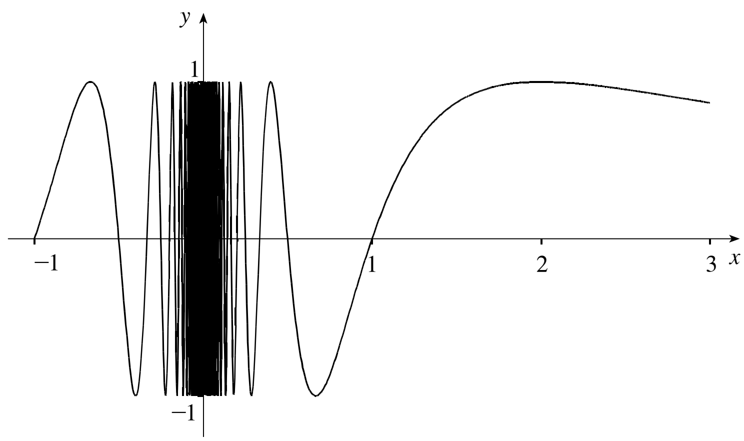
4. We can conjecture that the limit does not exist by applying the reasoning from Problem 3.

Group Work 2: The Dire Wolf Collects His Due

The students will not be able to do Problem 1 with any kind of accuracy. Let them discover for themselves how deceptively difficult it is, and then tell them that they should do the best that they can to show what is happening as x goes to zero. Ask them to compare their result with $\lim_{x\to 0} x \sin(\pi/x)$. If a group finishes early, pass out the supplementary problems.

ANSWERS

1.



2. (a) 1, 1, 1 (b) 1 (c) 0 (d) A function must approach only one number for the limit to exist.

ANSWERS TO SUPPLEMENTARY PROBLEMS

1. The length of the boundary is infinite. There are infinitely many wiggles, each adding at least 2 to the total perimeter length.

2. The area is finite. It is less than the area of the rectangle defined by $0 \le x \le 1$, $-2 \le y \le 1$.

3. Answers will vary.

Group Work 3: Infinity is Very Big

The precise definition of infinite limits is similar to the standard definition, but it is different enough that most students need a little practice before they can grasp it.

ANSWERS

1. $x < 0.001$

2. (a) Choose M. Now let $\delta = \frac{1}{\sqrt{M}}$. If $0 < x < \frac{1}{\sqrt{M}}$, then $\frac{1}{x^2} > M$. Values of δ less than $\frac{1}{\sqrt{M}}$ work, too.

(b) $\frac{1}{x}$ is large negative for small negative values of x, and large positive for small positive values of x.

Group Work 4: The Significance of the "For Every"

The purpose of this exercise is to allow the students to discover that rigor in mathematics is often necessary and useful. Problem 1 is designed to lead the students to make a false assumption about the third function, $h(x)$. Problem 2 should dispel that assumption.

This exercise is longer than it appears. Allow the students plenty of time to do the first three questions, which should help them to internalize and understand the formal definition of a limit. Closure is important to ensure that the "punchline" isn't lost in the algebra.

When the students are finishing up, it is *crucial* to pass out Problem 2. This part asks them to look at the functions a third time, with $\varepsilon = 0.01$. Make sure that the students remember to check values of $h(x)$ for $x > 0$ and for $x < 0$. Finish up by having them draw a graph of $h(x)$.

NOTE If time is limited, allow the students to find a δ that works from looking at graphs, as opposed to finding the largest possible δ algebraically.

ANSWERS

PART 1

1. (a) $\delta = \frac{1}{4}$ (b) $\delta = \frac{1}{2}$ (c) Any δ will work.

2. $\delta = \frac{1}{20}$, $\delta = \frac{1}{10}$, any δ will work. $\left|h(x) - \frac{1}{25}\right| = \begin{cases} 0.08 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$ which is always less than $\frac{1}{10}$.

3. Students may or may not see the wrinkle in $h(x)$ at this point.

PART 2

$\delta = \frac{1}{200}$, $\delta = \frac{1}{100}$, no δ will work. $\left|h(x) - \frac{1}{25}\right| = \begin{cases} 0.08 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$ which is always greater than 0.01.

Homework Problems

CORE EXERCISES 1, 7, 9, 11, 19

SAMPLE ASSIGNMENT 1, 3, 5, 7, 9, 11, 19, 21, 25, 29, 43

EXERCISE	D	A	N	G
1				×
3				×
5				×
7		×		
9		×		
11	×	×		
19		×		
21		×		
25		×		
29		×		
43		×		

GROUP WORK 1, SECTION 1.7

A Jittery Function

Not all functions that occur in mathematics are simple combinations of the "toolkit" functions usually seen in calculus. Consider this function:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$$

1. It is obvious that you can't graph this function in the same literal way that you would graph $y = \cos x$, but it is useful to have some idea of what this function looks like. Try to sketch the graph of $y = f(x)$.

2. Does $\lim_{x\to 0} f(x)$ exist? If so, what is its value? If not, why not? Make sure to justify your answer carefully.

3. Does $\lim_{x\to 1} f(x)$ exist? Carefully justify your answer.

4. What do you conjecture about $\lim_{x\to a} f(x)$ if $a \neq 0$?

GROUP WORK 2, SECTION 1.7

The Dire Wolf Collects his Due

In this activity we will explore a function that is particularly loved by mathematicians everywhere, $\sin(\pi/x)$.

1. Sketch the graph of $y = \sin(\pi/x)$ on the interval $[-1, 3]$.

2. It appears that this function is not defined at $x = 0$, does not have a limit at $x = 0$, and in fact, does not even have a right-hand limit.

(a) Evaluate $\sin(\pi/x)$ at $x = \frac{2}{1}, \frac{2}{5}$, and $\frac{2}{9}$.

(b) Evaluate $\sin(\pi/x)$ for $x = \frac{2}{4n+1}$, n a positive integer, using the pattern from part (a).

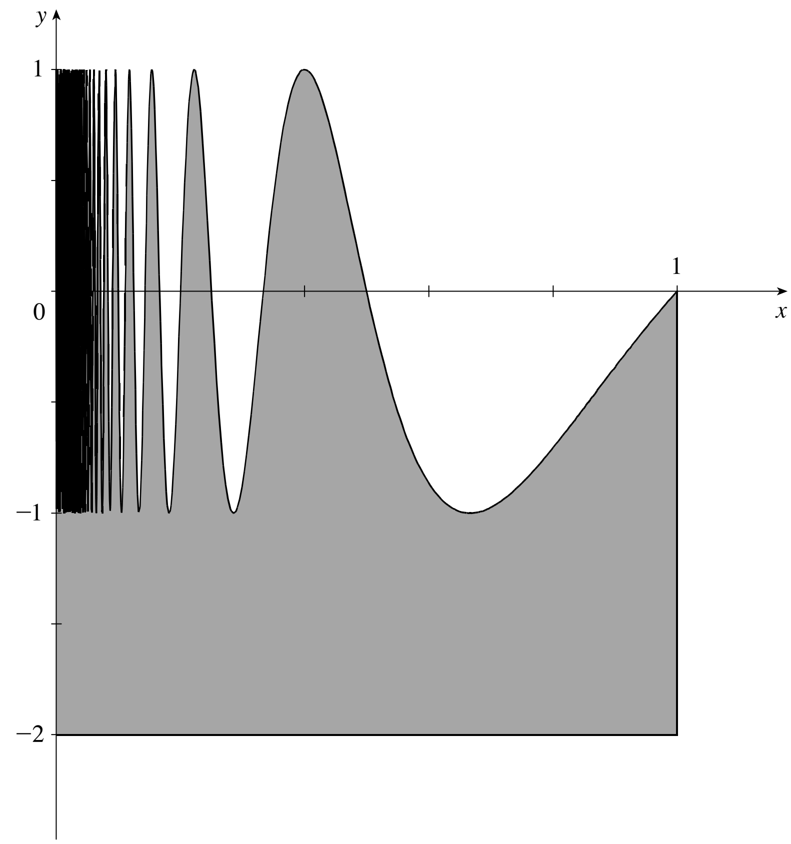
(c) Evaluate $\sin(\pi/x)$ for $x = \frac{1}{1}, \frac{1}{2}$, and $\frac{1}{3}$. Using this pattern, evaluate $\sin(\pi/x)$ for $x = \frac{1}{n}$, n a positive integer.

(d) Give an argument to show that $\lim_{x \to 0^+} \sin(\pi/x)$ does not exist.

GROUP WORK 2, SECTION 1.7

The Dire Wolf Collects his Due (Supplementary Problems)

Consider the region bounded on the bottom by the line $y = -2$, on the left by the line $x = 0$, on the right by the line $x = 1$, and on top by the graph of $y = \sin(\pi/x)$ as shown:



1. Is the length of the boundary of this region finite or infinite? Justify your answer.

2. Is the area of this region finite or infinite? Justify your answer.

3. Do you think this result is as interesting as we do? Why or why not?

GROUP WORK 3, SECTION 1.7

Infinity is Very Big

1. For what values of x near 0 is it true that $\frac{1}{x^2} > 1{,}000{,}000$?

2. The precise definition of $\lim_{x \to a} f(x) = \infty$ states that for every positive number M, no matter how large, there is a corresponding positive number δ such that $f(x) > M$ whenever $0 < |x - a| < \delta$.

(a) Use this definition to prove that $\lim_{x \to 0} \left(\frac{1}{x^2} \right) = \infty$.

(b) Why is it not true that $\lim_{x \to 0} \left(\frac{1}{x} \right) = \infty$? Give reasons for your answer.

GROUP WORK 4, SECTION 1.7

The Significance of the "For Every" (Part 1)

Consider the following functions:

$$f(x) = 2x + 3 \qquad g(x) = \frac{x^2 - 4}{x - 2} \qquad h(x) = \frac{|x|}{25x}$$

We want to try to prove the following statements:

$$\lim_{x \to 1} f(x) = 5 \qquad \lim_{x \to 2} g(x) = 4 \qquad \lim_{x \to 0} h(x) = \frac{1}{25}$$

Notice that these are not obvious statements, since $g(2)$ and $h(0)$ are both undefined.

1. We start with $\varepsilon = \frac{1}{2}$.

(a) Can you find a number δ with the property that, when $|x - 1| < \delta$, $|f(x) - 5| < \frac{1}{2}$? Illustrate your answer with a graph, and prove it algebraically.

(b) Can you find a number δ with the property that, when $|x - 2| < \delta$, $|g(x) - 4| < \frac{1}{2}$?

(c) Can you find a number δ with the property that, when $|x - 0| < \delta$, $\left|h(x) - \frac{1}{25}\right| < \frac{1}{2}$?

2. We now have some reason to believe that the above statements are true. But just having "some reason to believe" isn't enough for mathematicians. Repeat the previous problem for $\varepsilon = \frac{1}{10}$.

3. Now, what do you believe about these limits?

GROUP WORK 4, SECTION 1.7

The Significance of the "For Every" (Part 2)

Try the three limits again, this time for $\varepsilon = \frac{1}{100}$. Make sure that when you are trying to verify the condition $|x - x_0| < \delta$, you check values of $x_0 > x$ and $x_0 < x$. Do you wish to change your answer to Problem 3 from Part 1?

1.8 Continuity

Suggested Time and Emphasis

$1\text{–}1\frac{1}{2}$ classes   Essential material

Points to Stress

1. The graphical and mathematical definitions of continuity, and the basic principles.
2. Examples of discontinuity.
3. The Intermediate Value Theorem: mathematical statement, graphical examples, and applied examples.

Quiz Questions

- **TEXT QUESTION** The text says that $y = \tan x$ is discontinuous at $x = \frac{\pi}{2}$. This would seem to contradict Theorem 7. Does it? Why or why not?

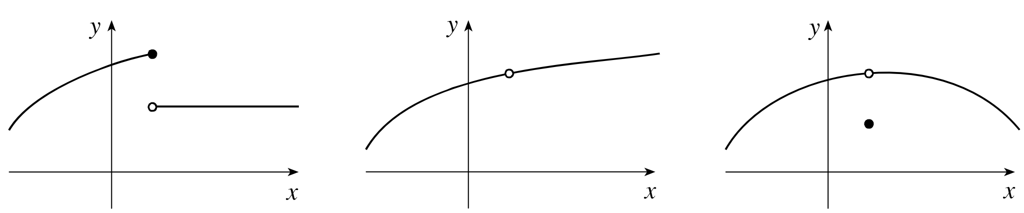
  ANSWER It does not; $\tan x$ is indeed continuous at every point in its domain, but $x = \frac{\pi}{2}$ is not in its domain.
- **DRILL QUESTION** Assume that $f(1) = -5$, and $f(3) = 5$. Does there have to be a value of x, between 1 and 3, such that $f(x) = 0$?

  ANSWER No, there does not. Only if the function is continuous does the IVT indicate that there must be such a value.

Materials for Lecture

- Discuss the idea of continuity at a point, continuity on an interval, and the basic types of discontinuities.
- Note that the statement "f is continuous at $x = a$" is implicitly saying three things:
  1. $f(a)$ exists.
  2. $\lim_{x \to a} f(x)$ exists.
  3. The two quantities are equal.

  To show that all three statements are important to continuity, have the students come up with examples where the first holds and the second does not, the second holds and the first does not, and where the first two hold and the third does not. Examples are sketched below.



- Some students tend to believe that all piecewise functions are discontinuous at the border points. Examine the function $f(x) = \begin{cases} \frac{2}{\pi}x & \text{if } x < \frac{\pi}{2} \\ \sin x & \text{if } \frac{\pi}{2} \le x \le 2\pi \\ 2\pi & \text{if } x \ge 2\pi \end{cases}$ at the points $x = \frac{\pi}{2}$ and $x = 2\pi$. This would be a good time to point out that the function $|x|$ is continuous everywhere, including at $x = 0$.
- Start by stating the basic idea of the Intermediate Value Theorem (IVT) in broad terms. (Given a function on an interval, the function hits every y-value between the starting and ending y-values.) Then attempt

to translate this statement into precise mathematical notation. Show that this process reveals some flaws in our original statement that have to be corrected (the interval must be closed; the function must be continuous.)

- To many students the IVT says something trivial to the point of uselessness. It is important to show examples where the IVT is used to do non-trivial things.

  Example: A graphing calculator uses the IVT when it graphs a function. A pixel represents a starting and ending y-value, and it is assumed that all the intermediate values are there. This is why graphing calculators are notoriously bad at graphing discontinuous functions.

  Example: Assume a circular wire is heated. Use the IVT to show that there exist two diametrically opposite points with the same temperature.

  ANSWER Let $f(x)$ be the difference between the temperature at a point x and the temperature at the point opposite x. f is a difference of continous functions, and is thus continuous itself. If $f(x) > 0$, then $f(-x) < 0$, so by the IVT there must exist a point at which $f = 0$.

  Example: Show that there exists a number whose cube is one more than the number itself. (This is Exercise 65.)

  ANSWER Let $f(x) = x^3 - x - 1$. f is continuous, and $f(0) < 0$ and $f(2) > 0$. So by the IVT, there exists an x with $f(x) = 0$.

- Have the students look at the function $f(x) = \begin{cases} 0 & x \text{ irrational} \\ \frac{1}{q} & x = \frac{p}{q}\text{, where } p \text{ and } q \text{ are integers, } q \text{ is positive, and the fraction is in lowest terms} \end{cases}$

  This function, discovered by Riemann, has the property that it is continuous where x is irrational, and not continuous where x is rational.

Workshop/Discussion

- If the group exercise "A Jittery Function" was assigned, revisit $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$ Ask the students to guess if this function is continuous at $x = 0$. Many will not believe that it is. Now look at it using the definition of continuity. They should agree that $f(0) = 0$. In the exercise it was shown that $\lim_{x \to 0} f(x)$ existed and was equal to 0. So, this function is continuous at $x = 0$. A sketch such as the one found in the answer to that group work may be helpful.
- Present the following scenario: two ice fishermen are fishing in the middle of a lake. One of them gets up at 6:00 P.M. and wanders back to camp along a scenic route, taking two and a half hours to get there. The second one leaves at 7:00 P.M., and walks to camp along a direct route, taking one hour to get there. Show that there was a time where they were equidistant from camp.
- Revisit Exercise 10 in Section 1.5, discussing why the function is discontinuous.

Group Work 1: Exploring Continuity

The first problem is appropriate for all classes. Problem 2 assumes the students have previously seen the activity "A Jittery Function". If they have not, skip it and go directly to Problem 3. Before handing out Problem 3, make sure that the students recall the definition of the greatest integer ("floor") function $y = [\![x]\!]$.

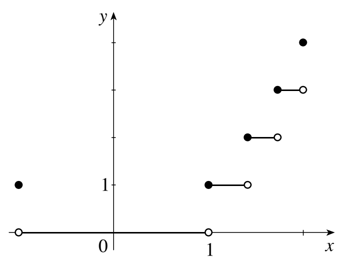
After this exercise, discuss the continuity of $[\![x]\!]$ at integer and at non-integer values. Problem 4 is intended for classes with a more theoretical bent.

ANSWERS

1. $c = 4, m = 5$

2. (b) 0 (c) 0 (d) It is continuous because $f(0) = \lim_{x\to 0} f(x)$.

3. (a)



(b) All values except $a = \pm 1, \sqrt{2}, \sqrt{3}, 2$

(c) $\lim_{x\to 0} [\![x^2]\!] = 0$; $\lim_{x\to 2} [\![x^2]\!]$ does not exist because the left- and right-hand limits are different.

4. (a) The fact that f is continuous implies that $\lim_{x\to a} f(x) = f(a)$ for all a. Then, by the Limit Laws,

$$\lim_{x\to a} h(x) = \lim_{x\to a} f(x)^2 = \left(\lim_{x\to a} f(x)\right)^2 = f(a)^2 = h(a).$$

(b) False. For example, let $f(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$

Group Work 2: The Area Function

This activity is designed to reinforce the notion of continuity by presenting it in an unfamiliar context. It will also ease the transition to area functions in Chapter 5. It is important that this activity be well set up. Do Problem 1 with the students, making sure to compute a few values of $A(r)$ and to sketch it. The students should try to answer Problems 2 and 3 using their intuition and the definition of continuity. It may be desirable to have the students restrict themselves to $r \geq 0$. Note that in this exercise, one can "prove" continuity by looking at the actual formulas for $A(r)$ and $B(r)$, but that the goal of the exercise is that the students understand intuitively why both area functions are continuous.

Students may disagree on the answer to Problem 3. If you are fortunate enough to have groups that have reached opposite conclusions, break up one or more of them, and have representatives go to other groups to try to convince them of the error of their ways.

ANSWER Yes to all three questions. For all r, $A(r)$, $B(r)$, and $C(r)$ exist; and $\lim_{x\to r} A(x) = A(r)$, $\lim_{x\to r} B(x) = B(r)$, and $\lim_{x\to r} C(x) = C(r)$. (The limits can be shown to exist by looking at the left- and right-hand limits.)

Group Work 3: The Twin Problem

When students see this problem, there is a good chance that they will disagree among themselves about the answer. Let them argue for a while. Ideally, they will come up with the idea of using the Intermediate Value Theorem to prove that Dr. Stewart was correct. If they don't, this may need to be given to them as a hint. Another hint they may need is that the Intermediate Value Theorem deals with a single continuous function, whereas the problem is talking about two functions, Stewart's temperature and Shasta's temperature. They will have to figure out a way to find a single function that they can use. Encourage them to write up a solution to the exact degree of rigor that will be expected of them on homework and exams; this is a good opportunity to convey the course's expectations to the students.

ANSWER Let $S(t)$ and $O(t)$ be Dr. Stewart's and Shasta's temperatures at time t. Now let $T(t) = S(t) - O(t)$. $T(t)$ is continuous (being a difference of continuous functions), $T(0) > 0$ (Dr. Stewart is warmer at first), and $T(f) < 0$ (where f represents the end of the vacation; Shasta is warmer at the end). Therefore, by the IVT, there exists a time a at which $T(a) = 0$ and hence $S(a) = O(a)$. Notice that most students who try to argue that the conclusion is false (using things such as stasis chambers and exceeding the speed of light) are really trying to construct a scenario where the continuity of the temperature function is violated.

Group Work 4: Swimming to the Shore

Emphasize to the students that they are not trying to *find* x, but simply trying to prove its existence. As in the Twin Problem, a first hint might be to use the IVT, and a second could be to find a single continuous function of x.

It is probably best to do this activity after the students have seen the solution to the ice fisherman problem above, or the Twin Problem.

ANSWER Let $D(x) = d(P_x, A) - d(P_x, B)$. $D(x)$ is continuous, $D(A) > 0$, and $D(B) < 0$. Therefore, by the IVT, there is a place where $D(x) = 0$.

Homework Problems

CORE EXERCISES 1, 3, 13, 15, 23, 27, 43, 51

SAMPLE ASSIGNMENT 1, 3, 7, 9, 13, 15, 19, 23, 27, 33, 35, 43, 51, 53

EXERCISE	D	A	N	G
1		×		×
3	×			×
7				×
9	×	×		×
13		×		
15		×		
19	×	×		×

EXERCISE	D	A	N	G
23	×	×		
27	×	×		
33		×		×
35		×		
43		×		×
51		×		
53		×		

GROUP WORK 1, SECTION 1.8

Exploring Continuity

1. Are there values of c and m that make $h(x) = \begin{cases} cx^2 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ -x^3 + mx & \text{if } x > 1 \end{cases}$ continuous at $x = 1$? Find c and m, or explain why they do not exist.

2. Recall the function $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$

(a) Do you believe that $f(x)$ is continuous at $x = 0$? Why or why not?

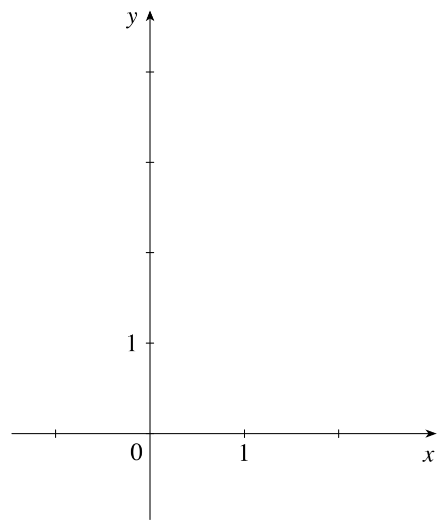
(b) What is $f(0)$?

(c) What is $\lim_{x \to 0} f(x)$?

(d) Use parts (b) and (c) either to revise your answer to part (a), or to prove that your answer is correct.

3. Consider the function $h(x) = [\![x^2]\!]$

(a) Sketch the graph of the function for $-1 \le x \le 2$.



(b) For what values of a, $-1 \le a \le 2$, is $\lim_{x \to a} h(x) = h(a)$?

(c) Compute $\lim_{x \to 0} [\![x^2]\!]$ and $\lim_{x \to 2} [\![x^2]\!]$, if they exist. Explain your answers.

4. We know that the function $g(x) = x^2$ is continuous everywhere.

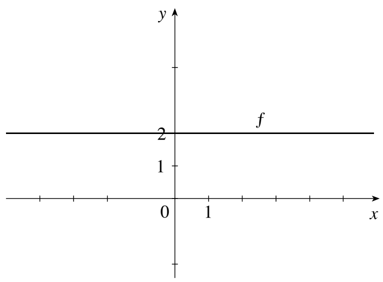
(a) Show that if f is continuous everywhere, then $h(x) = f(x)^2$ is continuous everywhere, using a limit argument.

(b) Is it true or false that if $h(x) = f(x)^2$ is continuous everywhere, then f is continuous everywhere? If it is true, prove it. If it is false, give a counterexample.

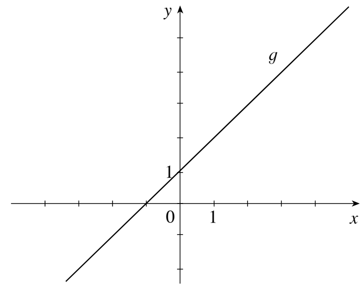
GROUP WORK 2, SECTION 1.8

The Area Function

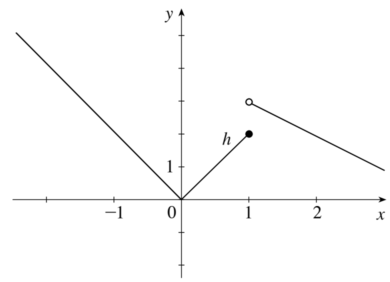
The following are graphs of $y = f(x)$, $y = g(x)$, and $y = h(x)$:



$f(x) = 2$



$g(x) = x + 1$



$$h(x) = \begin{cases} |2x| & \text{if } x \leq 1 \\ -x + 4 & \text{if } x > 1 \end{cases}$$

1. Let $A(r)$ be the area enclosed by the x-axis, the y-axis, the graph of the function f, and the line $x = r$. Would you conjecture that $A(r)$ is continuous at every point in the domain of f? Why or why not?

2. Let $B(r)$ be the area enclosed by the x-axis, the y-axis, the graph of the function g, and the line $x = r$. Would you conjecture that $B(r)$ is continuous at every point in the domain of g? Why or why not?

3. Let $C(r)$ be the area enclosed by the x-axis, the y-axis, the graph of the function h, and the line $x = r$. Would you conjecture that $C(r)$ is continuous at every point in the domain of h? Why or why not?

GROUP WORK 3, SECTION 1.8

The Twin Problem

There is a bit of trivia about the author of your textbook, Dr. James Stewart, that very few people know. He has an evil twin sister named Shasta. Although he loves his sister dearly, she dislikes him and tries to be different from him in all things.

Last winter, they both went on vacation. Dr. Stewart went to Hawaii. Shasta had planned on going to Aruba, but she decided against it. She hates her brother so much that she was afraid there would be a chance that they might be experiencing the same temperature at the same time, and that prospect was distasteful to her. So she decided to vacation in northern Alaska.

After a few days, Dr. Stewart received a call: "This is Shasta. I am very cold and uncomfortable here. That's good, since you are undoubtedly warm and comfortable, and I want us to be different. But I'm not sure why I should be the one in northern Alaska. I think we should switch places for the last half of our trip."

"It is only fair," he agreed.

So they each traveled again. Dr. Stewart took a trip from Hawaii to Alaska, while Shasta took a trip from Alaska to Hawaii. They each traveled their own different routes, perhaps stopping at different places along the way. Eventually, they had reversed locations. Dr. Stewart was shivering in Alaska; Shasta was in Hawaii, warm and happy. She received a call from her brother.

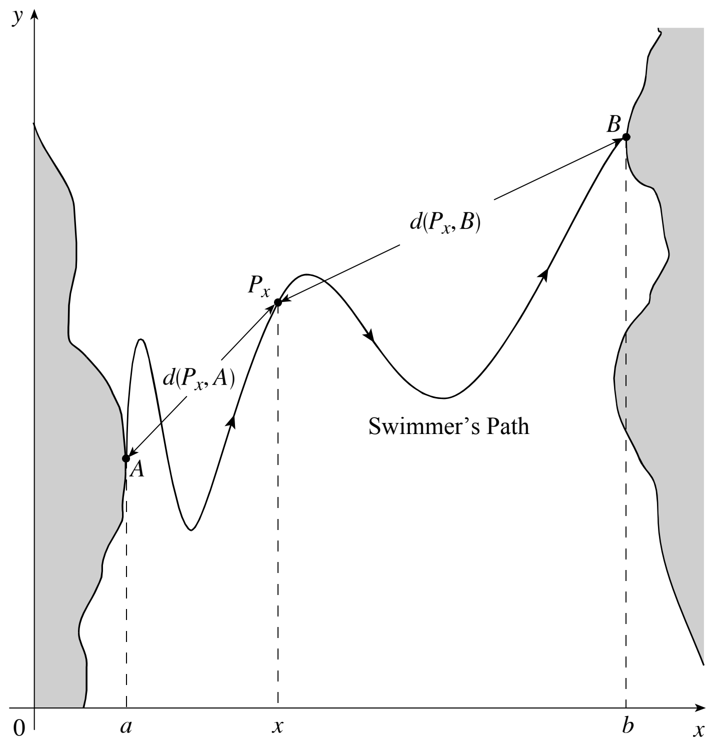
"Hi, Shasta. Guess what? At some time during our travels, we were experiencing exactly the same temperature at the same time. So Ha!"

Is Dr. Stewart right? Has Good triumphed over Evil? He would try to write out a proof of his statement, but his hands are too frozen to grasp his pen. Help him out. Either prove him right, or prove him wrong, using mathematics.

GROUP WORK 4, SECTION 1.8

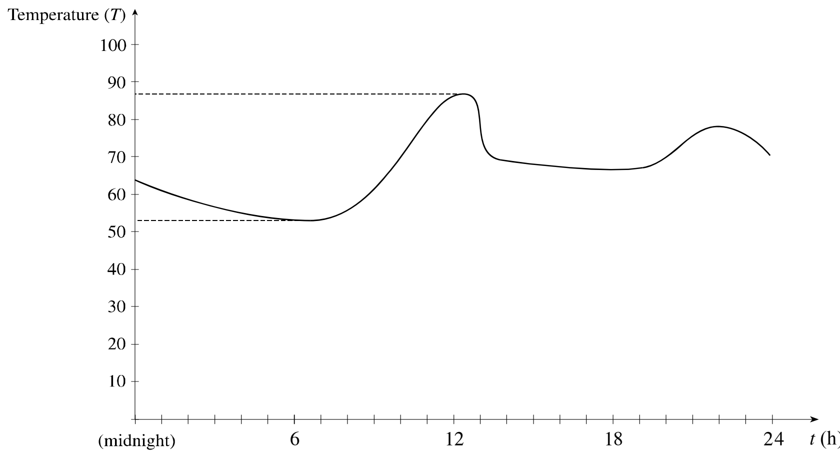
Swimming to the Shore

A swimmer crosses a river starting at point A and ending at point B, following the path shown below. Prove that for some value x, the swimmer's distance $d\,(P_x, A)$ from A is the same as the distance $d\,(P_x, B)$ from B.



Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. The graph below shows the temperature of a room during a summer day as a function of time, starting at midnight.



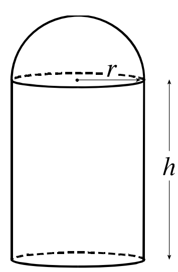
(a) Evaluate f (noon) and f (6 P.M.). State the range of f.

(b) Where is f increasing? Decreasing?

(c) Give a possible explanation for what happened at noon.

(d) Give a possible explanation why f attains its minimum value at 6 A.M.

2. A proposed new grain silo consists of a cylinder of height h and radius r, capped by a hemisphere.



Express its volume as a function of h and r.

3. The Slopps® trading card company has decided to put out its best line of trading cards ever: The "Famous Mathematicians" Series. Each pack of cards contains eight famous mathematicians, a mathematical puzzle, and a mathematics sticker. Naturally, you want a complete set, but you will have to buy a lot of cards because the really good ones (like the Galois, Sylvester, Hesse, Newton, and Leibniz cards) are very rare. Your local dealer will sell you an individual card, randomly selected, for 50 cents. Most people are interested in buying the packs of 8 for \$2.80. When you tell the dealer you want to buy a *lot* of them, he offers to sell you a box (containing 10 packs) for \$25, or a carton (containing 10 boxes) for \$230.

Let $c(x)$ be the (least) cost of buying x cards. Note that it is acceptable to buy more than x cards if it costs less than buying exactly x cards.

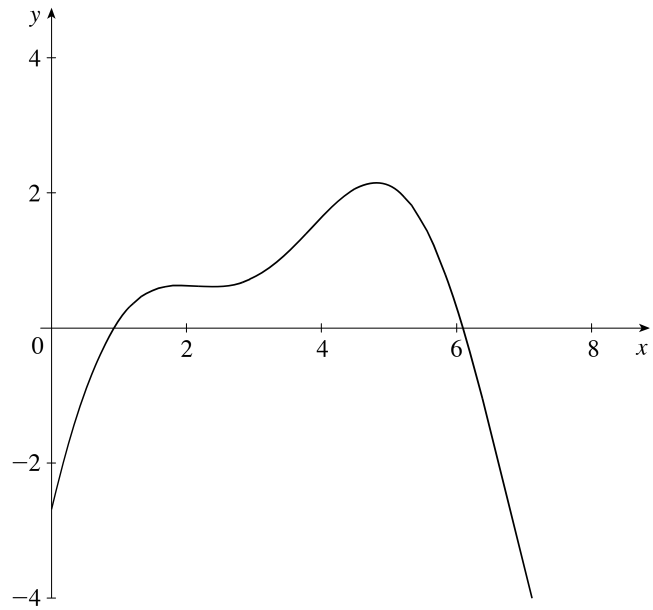
(a) Explain why the cheapest way to buy 6 cards is to buy a pack of 8. What is $c(6)$?

(b) Sketch a graph of $y = c(x)$ from $x = 0$ to $x = 24$.

(c) Find a formula for $c(x)$, valid for $x = 80$ to $x = 90$.

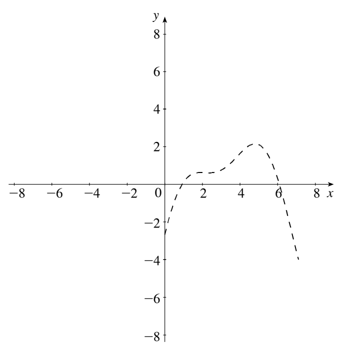
(d) If you wanted to buy 1005 cards, what is the least you would have to pay?

4. The following is a graph of $y = f(x)$.

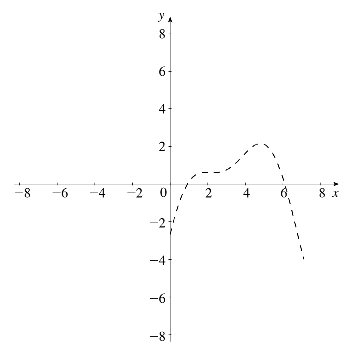


On the same axes, draw and label graphs of

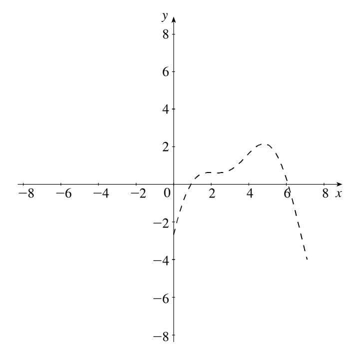
(a) $2f(x+2)$



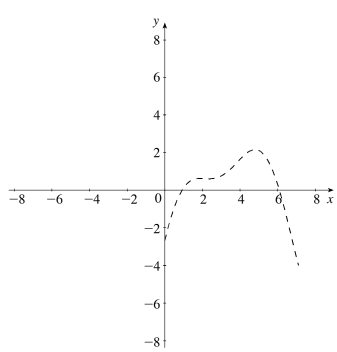
(b) $\frac{1}{2}f(-x)$



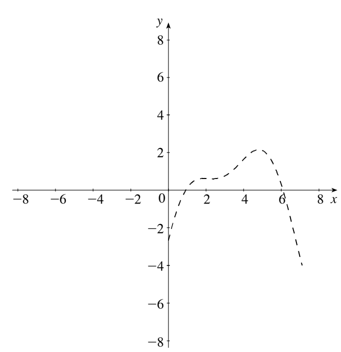
(c) $f(2x)$



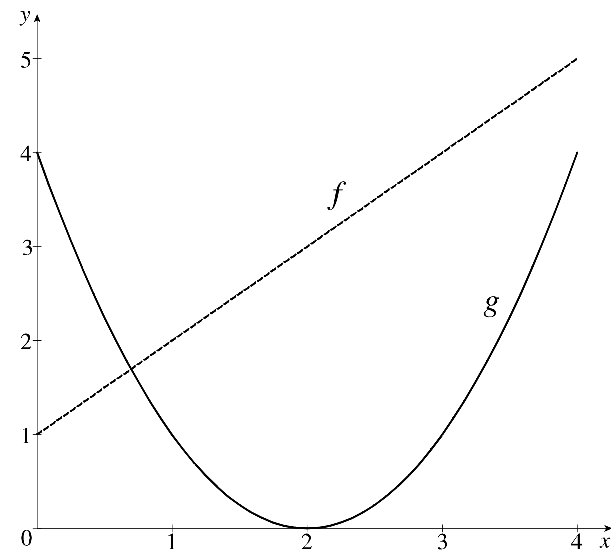
(d) $2f(2x-2)$



(e) $f(2x) - 2$



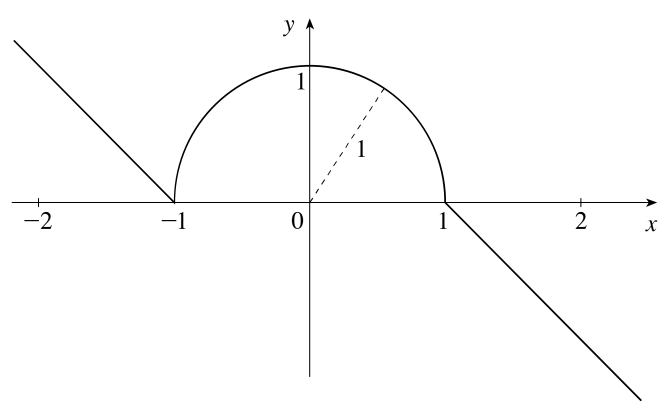
5. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.



(a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$ (c) $(f \circ f)(2)$

(d) $(g \circ g)(2)$ (e) $(f + g)(2)$ (f) $(f/g)(2)$

6. Find a formula that describes the following function.



7. Let $f(x) = 2.912345x^2 + 3.131579x - 0.099999$

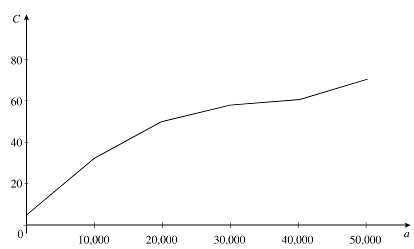
(a) To simplify approximation of f, write a quadratic function $g(x)$ with *integer* coefficients that closely models $f(x)$ for $-10 < x < 10$.

(b) Compute $g(4)$ and $f(4)$.

(c) Compute the error in using $g(4)$ to approximate $f(4)$ as a percentage of the correct answer $f(4)$.

(d) For larger values of x (say $x = 10$ or $x = 20$), would $g(x)$ be an overestimate or an underestimate of $f(x)$? Justify your answer without computing specific values of f and g.

8. A manufacturer hires a mathematician to come up with a function f that models the cost C of producing a alternative-music compact discs, where C is in thousands of dollars. The graph of f is given below.

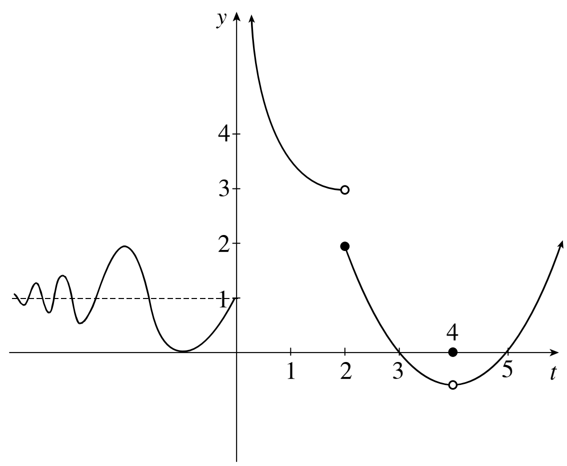


(a) What does $f(50{,}000) - f(49{,}999)$ represent?

(b) For what value or range of values is the cost per disc the least?

(c) Give a possible explanation for the sudden increase in the curve's slope at the end.

9. Consider the following graph of f.



(a) What is $\lim_{t \to 0^+} f(t)$? $\lim_{t \to 0^-} f(t)$? $\lim_{t \to 2^-} f(t)$?

(b) For what values of x does $\lim_{t \to x} f(t)$ exist?

(c) Does f have any vertical asymptotes? If so, where?

(d) For what values of x is f discontinuous?

10. Find values for a and b that will make f continuous everywhere, if

$$f(x) = \begin{cases} 3x + 1 & \text{if } x < 2 \\ ax + b & \text{if } 2 \le x < 5 \\ x^2 & \text{if } 5 \le x \end{cases}$$

11. Consider the function $f(x) = \begin{cases} 2x^2 & \text{if } x \ge -1 \\ x + 2 & \text{if } x < -1 \end{cases}$

(a) Let $L = \lim_{x \to 0} f(x)$. Find L.

(b) Find a number $\delta > 0$ so that if $0 < |x| < \delta$, then $|f(x) - L| < 0.01$.

(c) Show that f does not have a limit at -1.

(d) Explain what would go wrong if you tried to show that $\lim_{x \to -1} f(x) = 1$ using the ε-δ definition.

Hint: Try $\varepsilon = \frac{1}{2}$.

12. Let

$$f(x) = \begin{cases} \sqrt{3-x} & \text{if } x \le 1 \\ x^2 & \text{if } 1 < x < 3 \\ 27/x & \text{if } x \ge 3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim_{x \to 1^-} f(x)$ (ii) $\lim_{x \to 1^+} f(x)$ (iii) $\lim_{x \to 1} f(x)$ (iv) $\lim_{x \to 3^-} f(x)$

(v) $\lim_{x \to 3^+} f(x)$ (vi) $\lim_{x \to 3} f(x)$ (vii) $\lim_{x \to 9} f(x)$ (viii) $\lim_{x \to -6} f(x)$

(b) Where is f discontinuous?

1 SAMPLE EXAM SOLUTIONS

1. Approximate answers are acceptable for this problem.

(a) $f(\text{noon}) = 87^\circ$, $f(6 \text{ P.M.}) = 67^\circ$, range of f is $[53, 87]$.

(b) f is increasing on $(6, 12)$ and $(20, 22)$; f is decreasing on $(0, 6)$, $(12, 20)$ and $(22, 24)$.

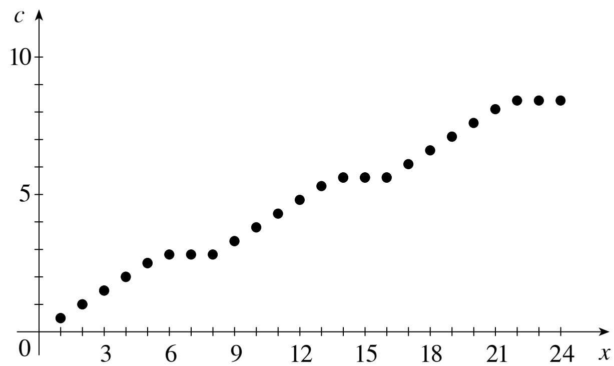
(c) Possible explanations for the drop in temperature at noon are a sudden thundershower, or an air conditioner being turned on.

(d) A possible explanation for f attaining its minimum value at 6 A.M. is that this is just before sunrise.

2. The total volume is the volume of a cylinder of height and radius r plus the volume of a hemisphere of radius r, that is, $V = \pi r^2 h + \frac{2}{3}\pi r^3$.

3. (a) If we buy 8 cards for \$2.80, then this costs less than buying 6 individual cards at \$0.50 apiece. Hence, $C(6) = \$2.80$.

(b)

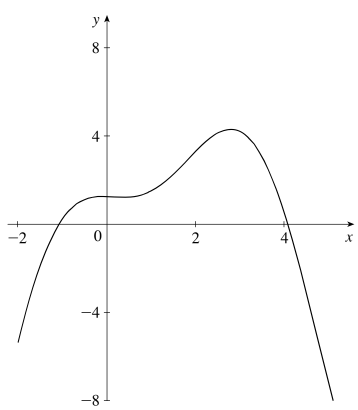


(c) $c(x) = \begin{cases} 25 + 0.5(x - 80) & \text{if } 80 \le x \le 85 \\ 27.8 & \text{if } 86 \le x \le 88 \\ 27.8 + 0.5(x - 88) & \text{if } 89 \le x \le 90 \end{cases}$

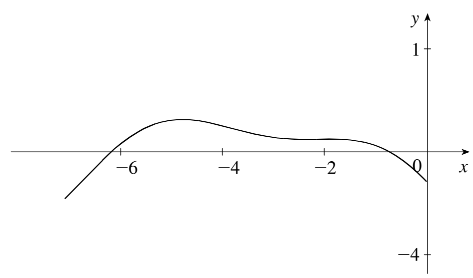
(d) To buy 1005 cards, the best deal is to buy one carton (800 cards), two boxes (160 cards), five packs (40 cards) and five individual cards. The total cost would be

$$230 + 2(25) + 5(2.80) + 5(0.50) = \$296.50$$

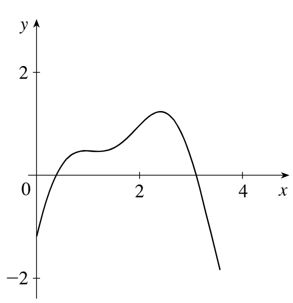
4. (a)



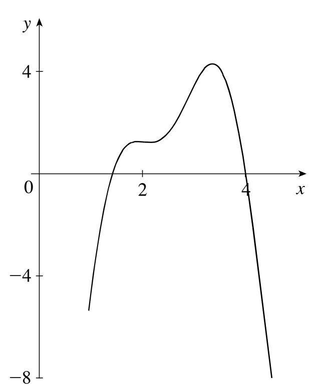
(b)



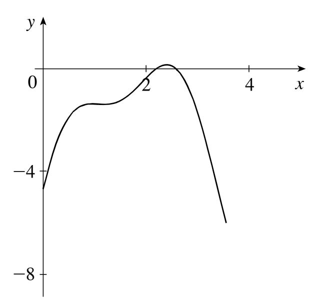
(c)



(d)



(e)



5. (a) $(f \circ g)(2) = f(0) = 1$

(b) $(g \circ f)(2) = g(3) = 1$

(c) $(f \circ f)(2) = f(3) = 4$

(d) $(g \circ g)(2) = g(0) = 4$

(e) $(f + g)(2) = f(2) + g(2) = 3 + 0 = 3$

(f) $\left(\dfrac{f}{g}\right)(2)$ is undefined because $g(2) = 0$.

6. $f(x) = \begin{cases} -x - 1 & \text{if } x < -1 \\ \sqrt{1 - x^2} & \text{if } -1 \leq x \leq 1 \\ -x + 1 & \text{if } x > 1 \end{cases}$

7. (a) $g(x) = 3x^2 + 3x$

(b) $g(4) = 60,\ f(4) = 59.023837$

(c) The percentage error in using $g(4)$ as an approximation for $f(4)$ is $100\left|\dfrac{f(4) - g(4)}{g(4)}\right| = 1.63\%$.

(d) For larger values of x, $g(x)$ is an overestimate of $f(x)$ because the coefficient of the dominant term (x^2) is larger.

8. (a) $f(50,000) - f(49,999)$ represents the cost of producing the 50,000th disc.

(b) The cost per disc is cheapest for $30,000 < a < 40,000$. This is where the slope of f is the smallest.

(c) One possible explanation for the sudden increase in the curve's slope is scarcity of materials.

9. (a) $\lim\limits_{t \to 0^+} f(t) = \infty,\ \lim\limits_{t \to 0^-} f(t) = 1,\ \lim\limits_{t \to 2^-} f(t) = 3$

(b) $\lim\limits_{t \to x} f(t)$ exists for all x except $x = 0$ and $x = 2$.

(c) There is a vertical asymptote at $x = 0$.

(d) f is discontinuous at $x = 0$, 2, and 4.

10. Solve $3(2) + 1 = 2a + b$ and $5^2 = 5a + b$ to get $a = 6,\ b = -5$.

11. (a) $L = 0$

(b) Let δ be any number greater than zero and less than $\sqrt{\frac{0.01}{2}}$. $\delta = 0.07$ works, for example.

(c) The left hand limit is 2, and the right hand limit is 1.

(d) Choose $\varepsilon = \frac{1}{2}$. We now need a δ such that $|f(x) - 1| < \frac{1}{2}$ for all x with $|x + 1| < \delta$. But if $x > -1$, as x approaches -1, $f(x)$ approaches 2, and $|f(x) - 1|$ approaches 1, which is greater than $\frac{1}{2}$.

12. (a) (i) $\sqrt{2}$ (ii) 1 (iii) Does not exist (iv) 9 (v) 9 (vi) 9 (vii) 3 (viii) 3

(b) f is discontinuous at $x = 1$.