

**Instructors Solutions Manual
for
David Poole's
LINEAR ALGEBRA
A Modern Introduction**

even- and odd-numbered exercises

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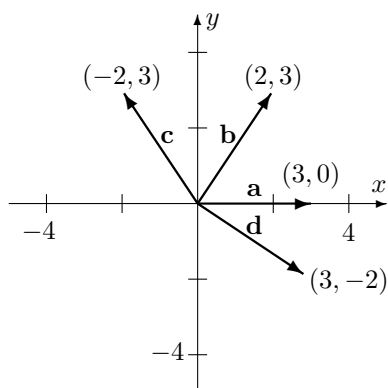
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Chapter 1

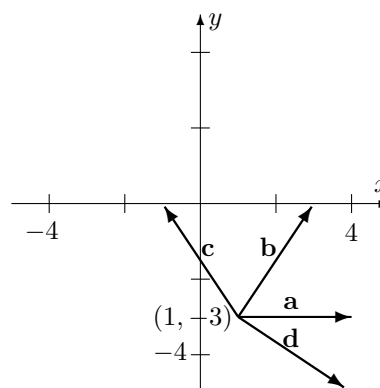
Vectors

1.1 The Geometry and Algebra of Vectors

1.



2.



3. See Figures 1.15 and 1.16.

4. (a) Similar to Example 1.1, we have the following:

If $[0, 2, 0]$ is translated to \overrightarrow{BC} where $B = (1, 2, 3)$,
 then we must have $C = (0 + 1, 2 + 2, 0 + 3) = (1, 4, 3)$ as the new tail.

In short, we added 1 to the first coordinate, 2 to the second coordinate,
 and 3 to the third coordinate. Why is this the correct thing to do?

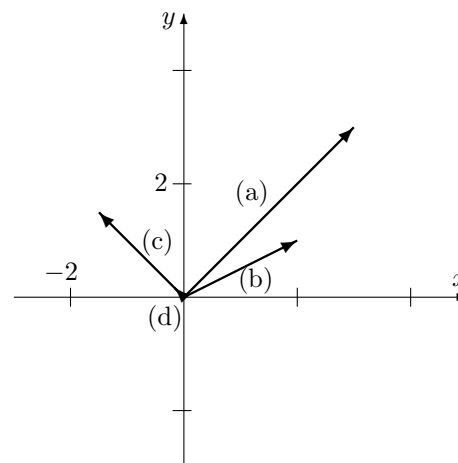
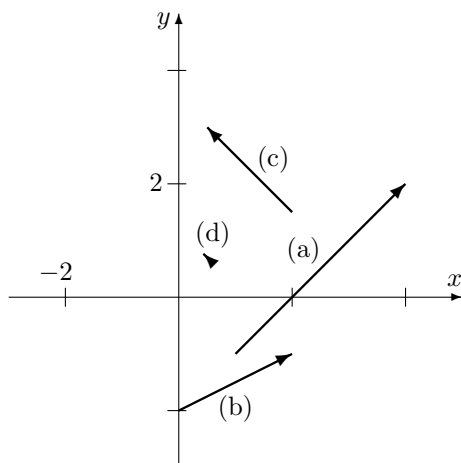
(b) Likewise, if $[3, 2, 1]$ is translated, $C = (3 + 1, 2 + 2, 1 + 3) = (4, 4, 4)$ is the new tail.

(c) If $[1, -2, 1]$ is translated, $C = (1 + 1, -2 + 2, 1 + 3) = (2, 0, 4)$ is the new tail.

(d) If $[-1, -1, -2]$ is translated, $C = (-1 + 1, -1 + 2, -2 + 3) = (0, 1, 1)$ is the new tail.

5. (a) $\overrightarrow{AB} = [4 - 1, 2 - (-1)] = [3, 3]$.

(b) $[2, 1]$ (c) $[-\frac{3}{2}, \frac{3}{2}]$ (d) $[-\frac{1}{6}, \frac{1}{6}]$.



6. Recall the notation that $[a, b]$ denotes a move of a units horizontally and b units vertically.

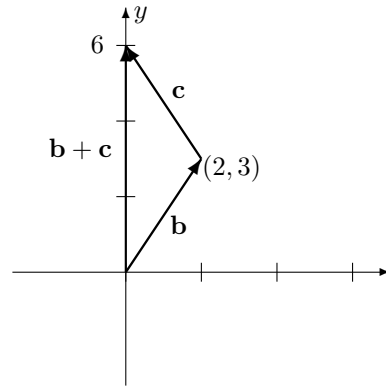
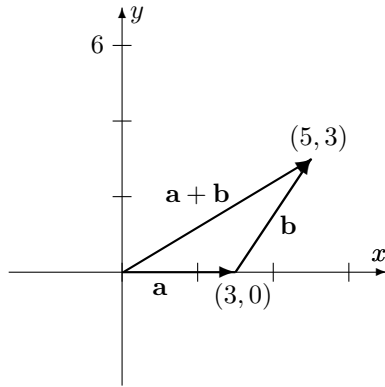
During the first part of the walk, the hiker walks 4 km north, so $\mathbf{a} = [0, 4]$.

During the second part of the walk, the hiker walks a distance of 5 km northeast.

From the components, we get $\mathbf{b} = [5 \cos 45^\circ, 5 \sin 45^\circ] = \left[\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right]$.

Thus, the net displacement vector is $\mathbf{c} = \mathbf{a} + \mathbf{b} = \left[\frac{5\sqrt{2}}{2}, 4 + \frac{5\sqrt{2}}{2}\right]$.

$$7. \mathbf{a} + \mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3+2 \\ 0+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \quad 8. \mathbf{b} + \mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



$$9. \mathbf{d} - \mathbf{c} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}. \quad 10. \mathbf{a} - \mathbf{d} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

$$11. 2\mathbf{a} + 3\mathbf{c} = 2[0, 2, 0] + 3[1, -2, 1] = [2 \cdot 0, 2 \cdot 2, 2 \cdot 0] + [3 \cdot 1, 3(-2), 3 \cdot 1] = [3, -2, 3].$$

$$12. 2\mathbf{c} - 3\mathbf{b} - \mathbf{d} = 2[1, -2, 1] - 3[3, 2, 1] - [-1, -1, -2] = [-6, -9, 1].$$

$$13. \mathbf{u} = [\cos 60^\circ, \sin 60^\circ] = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right], \mathbf{v} = [\cos 210^\circ, \sin 210^\circ] = \left[-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right] \Rightarrow (\text{implies})$$

$$\mathbf{u} + \mathbf{v} = \left[\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} - \frac{1}{2}\right], \mathbf{u} - \mathbf{v} = \left[\frac{1}{2} + \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} + \frac{1}{2}\right].$$

$$14. (a) \overrightarrow{AB} = \mathbf{b} - \mathbf{a}.$$

$$(b) \overrightarrow{BC} = \overrightarrow{OC} - \mathbf{b} = (\mathbf{b} - \mathbf{a}) - \mathbf{b} = -\mathbf{a}.$$

$$(c) \overrightarrow{AD} = -2\mathbf{a}.$$

$$(d) \overrightarrow{CF} = \overrightarrow{CB} + \overrightarrow{BA} + \overrightarrow{AF} = -\overrightarrow{BC} - \overrightarrow{AB} + (-\overrightarrow{AB} - \mathbf{a}) - \overrightarrow{AB} = 2(\mathbf{a} - \mathbf{b}).$$

$$(e) \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = (\mathbf{b} - \mathbf{a}) + (-\mathbf{a}) = \mathbf{b} - 2\mathbf{a}.$$

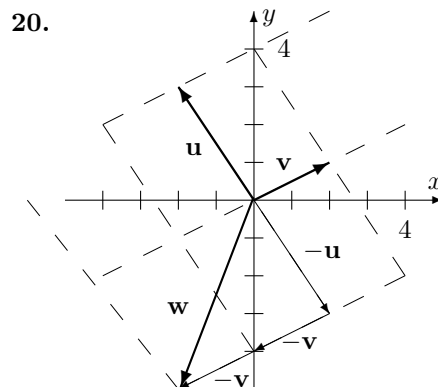
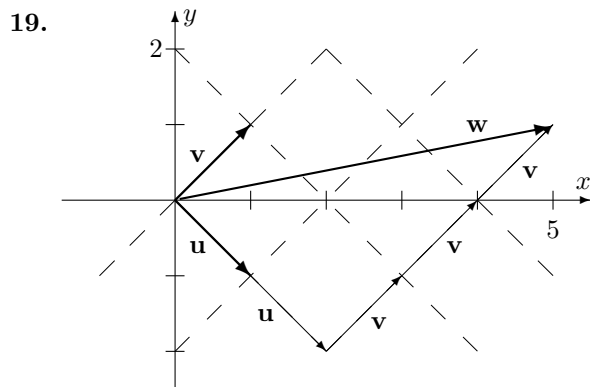
$$(f) \overrightarrow{BC} + \overrightarrow{DE} + \overrightarrow{FA} = -\mathbf{a} + (-\overrightarrow{AB}) + (\overrightarrow{AB} + \mathbf{a}) = \mathbf{0}.$$

$$15. 2(\mathbf{a} - 3\mathbf{b}) + 3(2\mathbf{b} + \mathbf{a}) \stackrel{\text{property e. distributivity}}{=} (2\mathbf{a} - 6\mathbf{b}) + (6\mathbf{b} + 3\mathbf{a}) \stackrel{\text{property b. associativity}}{=} (2\mathbf{a} + 3\mathbf{a}) + (-6\mathbf{b} + 6\mathbf{b}) = 5\mathbf{a}.$$

$$\begin{aligned}
 16. \quad -3(\mathbf{a} - \mathbf{c}) + 2(\mathbf{a} + 2\mathbf{b}) + 3(\mathbf{c} - \mathbf{b}) & \stackrel{\text{property e. distributivity}}{=} (-3\mathbf{a} + 3\mathbf{c}) + (2\mathbf{a} + 4\mathbf{b}) + (3\mathbf{c} - 3\mathbf{b}) \stackrel{\text{property b. associativity}}{=} \\
 & (-3\mathbf{a} + 2\mathbf{a}) + (4\mathbf{b} - 3\mathbf{b}) + (3\mathbf{c} + 3\mathbf{c}) = -\mathbf{a} + \mathbf{b} + 6\mathbf{c}.
 \end{aligned}$$

$$17. \quad \mathbf{x} - \mathbf{a} = 2(\mathbf{x} - 2\mathbf{a}) = 2\mathbf{x} - 4\mathbf{a} \Rightarrow \mathbf{x} - 2\mathbf{x} = \mathbf{a} - 4\mathbf{a} \Rightarrow -\mathbf{x} = -3\mathbf{a} \Rightarrow \mathbf{x} = 3\mathbf{a}.$$

$$18. \quad \mathbf{x} + 2\mathbf{a} - \mathbf{b} = 3(\mathbf{x} + \mathbf{a}) - 2(2\mathbf{a} - \mathbf{b}) = 3\mathbf{x} + 3\mathbf{a} - 4\mathbf{a} + 2\mathbf{b} \Rightarrow \mathbf{x} - 3\mathbf{x} = -\mathbf{a} - 2\mathbf{a} + 2\mathbf{b} + \mathbf{b} \Rightarrow -2\mathbf{x} = -3\mathbf{a} + 3\mathbf{b} \Rightarrow \mathbf{x} = \frac{3}{2}\mathbf{a} - \frac{3}{2}\mathbf{b}.$$

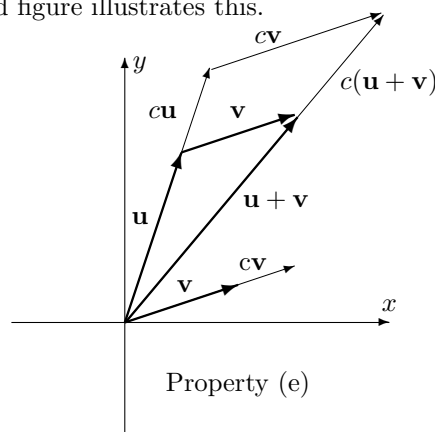
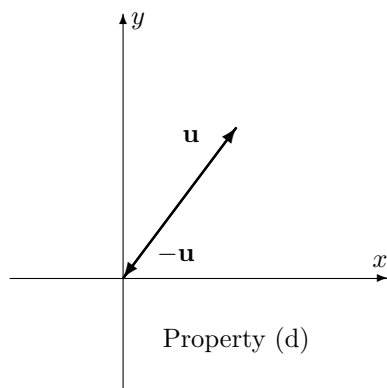


21. See Exercise 19.

22. See Exercise 20.

23. Property (d) states that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. The first diagram below shows \mathbf{u} along with $-\mathbf{u}$. Then, as the diagonal of the parallelogram, the resultant vector is $\mathbf{0}$.

Property (e) states $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$. The second figure illustrates this.



24. Let $\mathbf{u} = [u_1, u_2, \dots, u_n]$, $\mathbf{v} = [v_1, v_2, \dots, v_n]$, and let c and d be scalars in \mathbb{R} .

Property (d):

$$\begin{aligned}\mathbf{u} + (-\mathbf{u}) &= [u_1, u_2, \dots, u_n] + (-1[u_1, u_2, \dots, u_n]) = [u_1, u_2, \dots, u_n] + [-u_1, -u_2, \dots, -u_n] \\ &= [u_1 + (-u_1), u_2 + (-u_2), \dots, u_n + (-u_n)] = [0, 0, \dots, 0] = \mathbf{0}\end{aligned}$$

Property (e):

$$\begin{aligned}c(\mathbf{u} + \mathbf{v}) &= c([u_1, u_2, \dots, u_n] + [v_1, v_2, \dots, v_n]) = c([u_1 + v_1, u_2 + v_2, \dots, u_n + v_n]) \\ &= [c(u_1 + v_1), c(u_2 + v_2), \dots, c(u_n + v_n)] = [cu_1 + cv_1, cu_2 + cv_2, \dots, cu_n + cv_n] \\ &= [cu_1, cu_2, \dots, cu_n] + [cv_1, cv_2, \dots, cv_n] = c[u_1, u_2, \dots, u_n] + c[v_1, v_2, \dots, v_n] = c\mathbf{u} + c\mathbf{v}\end{aligned}$$

Property (f):

$$\begin{aligned}(c + d)\mathbf{u} &= (c + d)[u_1, u_2, \dots, u_n] = [(c + d)u_1, (c + d)u_2, \dots, (c + d)u_n] \\ &= [cu_1 + du_1, cu_2 + du_2, \dots, cu_n + du_n] = [cu_1, cu_2, \dots, cu_n] + [du_1, du_2, \dots, du_n] \\ &= c[u_1, u_2, \dots, u_n] + d[u_1, u_2, \dots, u_n] = c\mathbf{u} + d\mathbf{u}\end{aligned}$$

Property (g):

$$\begin{aligned}c(d\mathbf{u}) &= c(d[u_1, u_2, \dots, u_n]) = c[du_1, du_2, \dots, du_n] = [cd u_1, cd u_2, \dots, cd u_n] \\ &= [(cd)u_1, (cd)u_2, \dots, (cd)u_n] = (cd)[u_1, u_2, \dots, u_n] = (cd)\mathbf{u}\end{aligned}$$

25. $\mathbf{u} + \mathbf{v} = [0, 1] + [1, 1] = [1, 0]$, $\mathbf{u} \cdot \mathbf{v} = 0 + 1 = 1$.

26. $\mathbf{u} + \mathbf{v} = [1, 1, 0] + [1, 1, 1] = [0, 0, 1]$, $\mathbf{u} \cdot \mathbf{v} = 1 + 1 + 0 = 0$.

27. $\mathbf{u} + \mathbf{v} = [1, 0, 1, 1] + [1, 1, 1, 1] = [0, 1, 0, 0]$, $\mathbf{u} \cdot \mathbf{v} = 1 + 0 + 1 + 1 = 1$.

28. $\mathbf{u} + \mathbf{v} = [1, 1, 0, 1, 0] + [0, 1, 1, 1, 0] = [1, 0, 1, 0, 0]$, $\mathbf{u} \cdot \mathbf{v} = 0 + 1 + 0 + 1 + 0 = 0$.

29.

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| · | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 0 | 2 |
| 3 | 0 | 3 | 2 | 1 |

30.

| + | 0 | 1 | 2 | 3 | 3 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 |

| · | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | 0 | 2 | 4 | 1 | 3 |
| 3 | 0 | 3 | 1 | 4 | 2 |
| 4 | 0 | 4 | 3 | 2 | 1 |

31. $2 + 2 + 2 = 6 = 0$ in \mathbb{Z}_3 .

32. $2 \cdot 2 \cdot 2 = 3 \cdot 2 + 2 = 2$ in \mathbb{Z}_3 .

- 33.** $2(2 + 1 + 2) = 2(2) = 3 \cdot 1 + 1 = 1$ in \mathbb{Z}_3 . **34.** $3 + 1 + 2 + 3 = 4 \cdot 2 + 1 = 1$ in \mathbb{Z}_4 .
- 35.** $2 \cdot 3 \cdot 2 = 4 \cdot 3 + 0 = 0$ in \mathbb{Z}_4 . **36.** $3(3 + 3 + 2) = 4 \cdot 6 + 0 = 0$ in \mathbb{Z}_4 .
- 37.** $2 + 1 + 2 + 2 + 1 = 2$ in \mathbb{Z}_3 , $2 + 1 + 2 + 2 + 1 = 0$ in \mathbb{Z}_4 , $2 + 1 + 2 + 2 + 1 = 3$ in \mathbb{Z}_5 .
- 38.** $(3 + 4)(3 + 2 + 4 + 2) = 2(1) = 2$ in \mathbb{Z}_5 .
- 39.** $8(6 + 4 + 3) = 8(4) = 5$ in \mathbb{Z}_9 . **40.** $2^{100} = (2^{10})^{10} = (1024)^{10} = 1^{10} = 1$ in \mathbb{Z}_{11} .
- 41.** $[2, 1, 2] + [2, 0, 1] = [1, 1, 0]$ in \mathbb{Z}_3^3 . **42.** $[2, 1, 2] \cdot [2, 2, 1] = 1 + 2 + 2 = 2$ in \mathbb{Z}_3^3 .
- 43.** $[2, 0, 3, 2] \cdot ([3, 1, 1, 2] + [3, 3, 2, 1]) = [2, 0, 3, 2] \cdot [2, 0, 3, 3] = 0 + 0 + 2 + 1 = 3$ in \mathbb{Z}_4^4 .
 $[2, 0, 3, 2] \cdot ([3, 1, 1, 2] + [3, 3, 2, 1]) = [2, 0, 3, 2] \cdot [1, 4, 3, 3] = 2 + 0 + 4 + 1 = 2$ in \mathbb{Z}_5^4 .
- 44.** $x = 2 + (-3) = 2 + 2 = 4$ in \mathbb{Z}_5 . **45.** $x = 1 + (-5) = 1 + 1 = 2$ in \mathbb{Z}_6 .
- 46.** $x = (2)^{-1} = 2$ in \mathbb{Z}_3 . **47.** No solution. Why? Consider: $\frac{4}{2} = 2$.
- 48.** $x = (2)^{-1} = 3$ in \mathbb{Z}_5 . **49.** $x = (3)^{-1} 4 = (2) 4 = 3$ in \mathbb{Z}_5 .
- 50.** No solution. Why? Consider: $\frac{6}{3} = 2$. **51.** No solution. Why? $\frac{6}{2} = 3$, $\frac{8}{2} = 4$.
- 52.** $x = (8)^{-1} 9 = (7) 4 = 6$ in \mathbb{Z}_{11} . **53.** $x = (2)^{-1} (2 + (-3)) = (3) (2 + 2) = 2$ in \mathbb{Z}_5 .
- 54.** $x = (4)^{-1} (2 + (-5)) = (4)^{-1} (2 + 1)$, but $(4)^{-1}$ does not exist in \mathbb{Z}_6 .
Therefore there is no solution.
- 55.** Add 5 to both sides $\Rightarrow 6x = 6$, so $x = 1, 5$ (because $5 \cdot 8 = 40 = 8 \cdot 4 + 6$).
- 56.** (a) All values. (b) All values. (c) All values.
- 57.** (a) All $a \neq 0$ in \mathbb{Z}_5 have a solution because 5 is a prime number.
(b) $a = 1, 5$ because they have no common factors with 6 other than 1.
(c) a and m can have no common factors other than 1,
that is, the greatest common divisor (gcd) of a and m is 1.

1.2 Length and Angle: The Dot Product

1. Following Example 1.15, $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (-1) \cdot 3 + 2 \cdot 1 = -3 + 2 = -1.$

2. Following Example 1.15, $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 6 \end{bmatrix} = 2 \cdot 9 + (-3) \cdot 6 = 18 - 18 = 0.$

3. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 = 2 + 6 + 3 = 11.$

4. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1.5 \\ 0.4 \\ -2.1 \end{bmatrix} \cdot \begin{bmatrix} 3.0 \\ 5.2 \\ -0.6 \end{bmatrix} = (1.5) \cdot (3.0) + (0.4) \cdot (5.2) + (-2.1) \cdot (-0.6) = 7.84.$

5. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -\sqrt{2} \\ 0 \\ -5 \end{bmatrix} = 1 \cdot 4 + (\sqrt{2}) \cdot (-\sqrt{2}) + \sqrt{3} \cdot 0 + 0 \cdot (-5) = 4 - 2 = 2.$

6. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1.12 \\ -3.25 \\ 2.07 \\ -1.83 \end{bmatrix} \cdot \begin{bmatrix} -2.29 \\ 1.72 \\ 4.33 \\ -1.54 \end{bmatrix} = -1.12 \cdot 2.29 + 1.72 \cdot 4.33 + 1.83 \cdot 1.54 = 3.6265.$

7. In the remarks prior to Example 1.19, we note that finding a unit vector \mathbf{v} in the same direction as a given vector \mathbf{u} is called **normalizing** the vector \mathbf{u} .

Therefore, we proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}, \text{ so a unit vector } \mathbf{v} \text{ in the same direction as } \mathbf{u} \text{ is}$$

$$\mathbf{v} = (1/\|\mathbf{u}\|)\mathbf{u} = (1/\sqrt{5}) \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

8. Following Example 1.19, we have:

$$\|\mathbf{u}\| = \sqrt{2^2 + (-3)^2} = \sqrt{13}, \text{ so a unit vector } \mathbf{v} \text{ in the same direction as } \mathbf{u} \text{ is}$$

$$\mathbf{v} = (1/\|\mathbf{u}\|)\mathbf{u} = (1/\sqrt{13}) \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{13} \\ -3/\sqrt{13} \end{bmatrix}.$$

9. Following Example 1.19, we have:

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \text{ so a unit vector } \mathbf{v} \text{ in the same direction as } \mathbf{u} \text{ is}$$

$$\mathbf{v} = (1/\|\mathbf{u}\|)\mathbf{u} = (1/\sqrt{14}) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}.$$

10. Following Example 1.19, we have:

$$\|\mathbf{u}\| = \sqrt{(1.5)^2 + (0.4)^2 + (-2.1)^2} = \sqrt{6.82}, \text{ so unit vector } \mathbf{v} \text{ in the same direction as } \mathbf{u}$$

$$\text{is } \mathbf{v} = (1/\|\mathbf{u}\|)\mathbf{u} = (1/\sqrt{6.82}) \begin{bmatrix} 1.5 \\ 0.4 \\ -2.1 \end{bmatrix} = \begin{bmatrix} 1.5/\sqrt{6.82} \\ 0.4/\sqrt{6.82} \\ -2.1/\sqrt{6.82} \end{bmatrix} \approx \begin{bmatrix} 0.57 \\ 0.15 \\ -0.80 \end{bmatrix}.$$

11. $\|\mathbf{u}\| = \sqrt{1^2 + (\sqrt{2})^2 + (\sqrt{3})^2 + 0^2} = \sqrt{6}$, so a unit vector in the direction of \mathbf{u} is

$$\mathbf{v} = (1/\|\mathbf{u}\|)\mathbf{u} = (1/\sqrt{6}) \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ \sqrt{2}/\sqrt{6} \\ \sqrt{3}/\sqrt{6} \\ 0/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ \sqrt{3}/3 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}.$$

12. $\|\mathbf{u}\| = \sqrt{(1.12)^2 + (-3.25)^2 + (2.07)^2 + (-1.83)^2} = \sqrt{19.4507}$, so the unit vector \mathbf{v} is

$$\mathbf{v} = (1/\|\mathbf{u}\|)\mathbf{u} = (1/\sqrt{19.4507}) \begin{bmatrix} 1.12 \\ -3.25 \\ 2.07 \\ -1.83 \end{bmatrix} = \begin{bmatrix} 1.12/\sqrt{19.4507} \\ -3.25/\sqrt{19.4507} \\ 2.07/\sqrt{19.4507} \\ -1.83/\sqrt{19.4507} \end{bmatrix} \approx \begin{bmatrix} 0.2540 \\ -0.7369 \\ 0.4694 \\ -0.4149 \end{bmatrix}.$$

13. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-4)^2 + 1^2} = \sqrt{17}.$$

14. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-7)^2 + (-9)^2} = \sqrt{130}.$$

15. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1)^2 + (-1)^2 + 2^2} = \sqrt{6}.$$

16. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1.5 \\ 0.4 \\ -2.1 \end{bmatrix} - \begin{bmatrix} 3.0 \\ 5.2 \\ -0.6 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -4.8 \\ -1.5 \end{bmatrix}$, so

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1.5)^2 + (-4.8)^2 + (-1.5)^2} = \sqrt{27.54} \approx 5.25.$$

17. (a) $\mathbf{u} \cdot \mathbf{v}$ is a real number, so $\|\mathbf{u} \cdot \mathbf{v}\|$ is the norm of a number, which is not defined.
 (b) $\mathbf{u} \cdot \mathbf{v}$ is a scalar, while \mathbf{w} is a vector.
 Thus, $\mathbf{u} \cdot \mathbf{v} + \mathbf{w}$ adds a scalar to a vector, which is not a defined operation.
 (c) \mathbf{u} is a vector, while $\mathbf{v} \cdot \mathbf{w}$ is a scalar.
 Thus, $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$ is the dot product of a vector and a scalar, which is not defined.
 (d) $c \cdot (\mathbf{u} + \mathbf{v})$ is the dot product of a scalar and a vector, which is not defined.

18. From trigonometry, we have:

$$\cos \theta > 0 \Rightarrow \theta \text{ is acute, } \cos \theta < 0 \Rightarrow \theta \text{ is obtuse, and } \cos \theta = 0 \Rightarrow \theta \text{ is right.}$$

$$\text{From } \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}, \text{ we see } \mathbf{u} \cdot \mathbf{v} \text{ determines the sign of } \cos \theta. \text{ Why?}$$

Therefore, as in Example 1.23, we calculate:

$$\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + 1 \cdot 1(-3) = -1 < 0 \Rightarrow \cos \theta < 0 \Rightarrow \theta \text{ is obtuse.}$$

19. From trigonometry, we have:

$$\cos \theta > 0 \Rightarrow \theta \text{ is acute, } \cos \theta < 0 \Rightarrow \theta \text{ is obtuse, and } \cos \theta = 0 \Rightarrow \theta \text{ is right.}$$

$$\text{From } \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}, \text{ we see } \mathbf{u} \cdot \mathbf{v} \text{ determines the sign of } \cos \theta. \text{ Why?}$$

Therefore, as in Example 1.23, we calculate:

$$\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + (-1) \cdot (-2) + 1 \cdot (-1) = 4 > 0 \Rightarrow \cos \theta > 0 \Rightarrow \theta \text{ is acute.}$$

20. Following the first step in Example 1.23, we calculate:

$$\mathbf{u} \cdot \mathbf{v} = 5 \cdot 1 + 4 \cdot (-2) + (-3) \cdot (-3) = 5 - 8 + 3 = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta \text{ is right.}$$

21. Following the first step in Example 1.23, we calculate:

$$\mathbf{u} \cdot \mathbf{v} = (0.9) \cdot (-4.5) + (2.1) \cdot (2.6) + (1.2) \cdot (-0.8) = 0.45 \Rightarrow \cos \theta > 0 \Rightarrow \theta \text{ is acute.}$$

22. $\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-3) + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot (-2) = -3 \Rightarrow \cos \theta < 0 \Rightarrow \theta \text{ is obtuse.}$

23. Since $\mathbf{u} \cdot \mathbf{v}$ is obviously > 0 , we have $\cos \theta > 0$ which implies θ is acute.

Note: $\mathbf{u} \cdot \mathbf{v}$ is > 0 because the components of both \mathbf{u} and \mathbf{v} are positive.

24. As in Example 1.21, we begin by calculating $\mathbf{u} \cdot \mathbf{v}$ (if $\mathbf{u} \cdot \mathbf{v} = 0$, we're done. Why?):

$$\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + 1 \cdot (-3) = -1, \|\mathbf{u}\| = \sqrt{2^2 + 1^2} = \sqrt{5}, \|\mathbf{v}\| = \sqrt{1^2 + (-3)^2} = \sqrt{10}.$$

$$\text{So, } \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-1}{\sqrt{5}\sqrt{10}} = -\frac{1}{5 \cdot \sqrt{2}} \text{ and } \theta = \cos^{-1}\left(\frac{1}{5 \cdot \sqrt{2}}\right) = 1.7 \text{ radians or } 98^\circ.$$

25. As in Example 1.21, we begin by calculating $\mathbf{u} \cdot \mathbf{v}$ (because if $\mathbf{u} \cdot \mathbf{v} = 0$ we're done. Why?):

$$\mathbf{u} \cdot \mathbf{v} = 2 \cdot 1 + (-1) \cdot (-2) + 1 \cdot (-1) = 2 + 2 - 1 = 3,$$

$$\|\mathbf{u}\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}, \text{ and } \|\mathbf{v}\| = \sqrt{1^2 + (-2)^2 + (-1)^2} = \sqrt{6}.$$

$$\text{Therefore, } \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}, \text{ so } \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \text{ radians or } 60^\circ.$$

26. As in Example 1.21, we begin by calculating $\mathbf{u} \cdot \mathbf{v}$:

$$\mathbf{u} \cdot \mathbf{v} = 5 \cdot 1 + 4 \cdot (-2) + (-3) \cdot (-3) = 5 - 8 + 3 = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta \text{ is right.}$$

If we wished to be more explicit, we could continue following Example 1.15:

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{0}{\|\mathbf{u}\| \|\mathbf{v}\|} = 0, \text{ so } \theta = \cos^{-1}(0) = \frac{\pi}{2} \text{ radians or } 90^\circ.$$

27. Following Example 1.21, we calculate:

$$\mathbf{u} \cdot \mathbf{v} = (0.9) \cdot (-4.5) + (2.1) \cdot (2.6) + (1.2) \cdot (-0.8) = 0.45,$$

$$\|\mathbf{u}\| = \sqrt{(0.9)^2 + (2.1)^2 + (1.2)^2} = \sqrt{6.66}, \text{ and}$$

$$\|\mathbf{v}\| = \sqrt{(-4.5)^2 + (2.6)^2 + (-0.8)^2} = \sqrt{27.65}.$$

$$\text{Therefore, } \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{0.45}{\sqrt{6.66}\sqrt{27.65}} = \frac{0.45}{\sqrt{182.817}},$$

$$\text{so } \theta = \cos^{-1}\left(\frac{0.45}{\sqrt{182.817}}\right) \approx 1.5375 \text{ radians or } 88.09^\circ.$$

Note: To minimize error, we do not approximate until the last step.

$$\text{Since } \frac{0.45}{\sqrt{182.817}} \approx 0.0332816 \text{ is a positive number close to zero,}$$

we should expect θ to be close to but less than 90° . Why?

28. Following Example 1.21, we calculate:

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot (-3) + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot (-2) = -3,$$

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}, \text{ and}$$

$$\|\mathbf{v}\| = \sqrt{(-3)^2 + 1^2 + 2^2 + (-2)^2} = \sqrt{18}.$$

$$\text{Therefore, } \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-3}{\sqrt{30}\sqrt{18}} = -\frac{1}{6\sqrt{15}},$$

$$\text{so } \theta = \cos^{-1}\left(-\frac{1}{6\sqrt{15}}\right) \approx 1.70 \text{ radians or } 97.42^\circ.$$

Note: To minimize error, we do not approximate until the last step.

$$\text{Since } -\frac{1}{6\sqrt{15}} \approx -0.13 \text{ is a negative number close to zero,}$$

we should expect θ to be close to but greater than 90° . Why?

29. Following Example 1.21, we calculate:

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 5 + 2 \cdot 6 + 3 \cdot (-3) + 3 \cdot 7 + 4 \cdot 8 = 70,$$

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}, \text{ and}$$

$$\|\mathbf{v}\| = \sqrt{5^2 + 6^2 + 7^2 + 8^2} = \sqrt{174}.$$

$$\text{Therefore, } \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{70}{\sqrt{30}\sqrt{174}} = \frac{35}{3\sqrt{145}},$$

$$\text{so } \theta = \cos^{-1} \left(\frac{35}{3\sqrt{145}} \right) \approx 0.2502 \text{ radians or } 14.34^\circ.$$

Note: To minimize error, we do not approximate until the last step.

$$\text{Since } \frac{35}{3\sqrt{145}} \approx 0.9688639 \text{ is a positive number close to } 1,$$

we should expect θ to be close to but greater than 0° .

30. To show $\triangle ABC$ is right, we need only show one pair of its sides meet at a right angle.

So, we let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{AC}$, then by the definition of *orthogonal* given prior to Example 1.23, we need only show $\mathbf{u} \cdot \mathbf{v}$, or $\mathbf{u} \cdot \mathbf{w}$, or $\mathbf{v} \cdot \mathbf{w} = 0$.

Following Example 1.1 of Section 1.1, we calculate the sides of $\triangle ABC$:

$$\mathbf{u} = \overrightarrow{AB} = [1 - (-3), 0 - 2] = [4, -2], \mathbf{v} = \overrightarrow{BC} = [4 - 1, 6 - 0] = [3, 6],$$

$$\mathbf{w} = \overrightarrow{AC} = [4 - (-3), 6 - 2] = [7, 4], \text{ so } \mathbf{u} \cdot \mathbf{v} = 4 \cdot 3 + (-2) \cdot 6 = 12 - 12 = 0 \Rightarrow$$

The angle between $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{BC}$ is $90^\circ \Rightarrow \triangle ABC$ is a right triangle.

Note: It is obvious that \mathbf{v} is not orthogonal to \mathbf{w} . Why?

31. To show $\triangle ABC$ is right, we need only show one pair of its sides meet at a right angle.

So, we let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{AC}$, then by the definition of *orthogonal* given prior to Example 1.23, we need only show $\mathbf{u} \cdot \mathbf{v}$, or $\mathbf{u} \cdot \mathbf{w}$, or $\mathbf{v} \cdot \mathbf{w} = 0$.

Following Example 1.1 of Section 1.1, we calculate the sides of $\triangle ABC$:

$$\mathbf{u} = \overrightarrow{AB} = [-3 - 1, 2 - 1, (-2) - (-1)] = [-4, 1, -1],$$

$$\mathbf{v} = \overrightarrow{BC} = [2 - (-3), 2 - 2, (-4) - (-2)] = [5, 0, -2],$$

$$\mathbf{w} = \overrightarrow{AC} = [2 - 1, 2 - 1, (-4) - (-1)] = [1, 1, -3].$$

$$\text{Then } \mathbf{u} \cdot \mathbf{w} = (-4) \cdot 1 + 1 \cdot 1 + (-1) \cdot (-3) = -4 + 1 + 3 = 0 \Rightarrow$$

The angle between $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{AC}$ is $90^\circ \Rightarrow \triangle ABC$ is a right triangle.

- 32.** Following Example 1.22, we make a similar argument:

The dimensions do not matter, so we consider a cube with sides of length 1.

Also, the cube is symmetric, so we need only consider one diagonal and adjacent edge.

Orient the cube relative to the coordinate axes in \mathbb{R}^3 , as shown in Figure 1.34.

Take the diagonal to be $[1, 1, 1]$ and take the adjacent edge to be $[1, 0, 0]$.

Then the angle θ between these two vectors satisfies:

$$\cos \theta = \frac{1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0}{\sqrt{3}\sqrt{1}} = \frac{1}{\sqrt{3}}, \text{ so } \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74^\circ.$$

So, the diagonal and adjacent edge meet at 54.74° .

- 33.** Following Example 1.15, we make a similar argument:

The dimensions do not matter, so we consider a cube with sides of length 1.

Also, the cube is symmetric, so we need only consider one pair of diagonals.

Orient the cube relative to the coordinate axes in \mathbb{R}^3 , as shown in Figure 1.34.

Take the diagonals to be $[1, 1, 1]$ and $\mathbf{v} = [1, 1, -1]$ (from $(1, 1, 0)$ to $(0, 0, 1)$).

Then dot product between these two vectors satisfies:

$$\mathbf{u} \cdot \mathbf{v} = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot (-1) = 1 + 1 - 1 = 1 \neq 0 \Rightarrow$$

The diagonals of a cube are not perpendicular. How might we generalize this result?

- 34.** To show the parallelogram is a rhombus, we need only show the diagonals are perpendicular.

So, all we have to show is that the dot product of the diagonals is zero.

$$\mathbf{d}_1 \cdot \mathbf{d}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = 2 \cdot 1 + 2 \cdot (-1) + 0 \cdot 3 = 0.$$

To find the length of the side, we use the Pythagorean Theorem.

$$\text{Note that } \|\mathbf{d}_1\|^2 = 2^2 + 2^2 + 0^2 = 8 \text{ and } \|\mathbf{d}_2\|^2 = 1^2 + (-1)^2 + 3^2 = 11.$$

$$\text{So, the length of the side is } \frac{1}{2}\sqrt{\|\mathbf{d}_1\|^2 + \|\mathbf{d}_2\|^2} = \frac{1}{2}\sqrt{8 + 11} = \frac{1}{2}\sqrt{19} \approx 2.18.$$

- 35.** To resultant velocity of the airplane is the sum of the velocity of the airplane and the velocity of the wind.

$$\text{So, the resultant velocity } \mathbf{r} = \mathbf{p} + \mathbf{w} = \begin{bmatrix} 200 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -40 \end{bmatrix} = \begin{bmatrix} 200 \\ -40 \end{bmatrix}.$$

- 36.** Since $ABCD$ is a rectangle, opposite sides BA and CD are both parallel and congruent. So, we can use the method of Example 1.1 in Section 1.1 to find the coordinates of vertex D .

Specifically, we compute $\overrightarrow{BA} = [1 - 3, 2 - 6, 3 - (-2)] = [-2, -4, 5]$.

If \overrightarrow{BA} is then translated to \overrightarrow{CD} , where $C = (0, 5, -4)$, then we must have

$$D = (0 + (-2), 5 + (-4), -4 + 5) = (-2, 1, 1).$$

- 37.** Since $ABCD$ is a rectangle, opposite sides BA and CD are both parallel and congruent. So, we can use the method of Example 1.1 in Section 1.1 to find the coordinates of vertex D .

Specifically, we compute $\overrightarrow{BA} = [1 - 3, 2 - 6, 3 - (-2)] = [-2, -4, 5]$.

If \overrightarrow{BA} is then translated to \overrightarrow{CD} , where $C = (0, 5, -4)$, then we must have

$$D = (0 + (-2), 5 + (-4), -4 + 5) = (-2, 1, 1).$$

- 38.** Since $\mathbf{v}t = \mathbf{d}$, use the given information to find \mathbf{v} , then solve for t and complete \mathbf{d} .

Since the speed of the boat is 20 km/h and the speed of the flow is -5 km/h, $\mathbf{v} = \begin{bmatrix} 20 \\ -5 \end{bmatrix}$.

The width of the river is 2 km and the distance downstream is unknown. So, $\mathbf{d} = \begin{bmatrix} 2 \\ y \end{bmatrix}$.

That gives us $\mathbf{v}t = \begin{bmatrix} 20 \\ -5 \end{bmatrix} t = \begin{bmatrix} 2 \\ y \end{bmatrix}$.

So, $20t = 2$ which implies $t = 0.1$ and $y = -5(0.1) = -0.5$. Therefore:

- (a) Ann lands 0.5 or half a kilometer downstream.
- (b) It takes Ann 0.1 hours or 6 minutes to cross the river.

The river flow does not increase the amount of time it takes to cross. Why not?

39. Find the angle between Bert's resultant vector, \mathbf{r} , and his velocity vector upstream, \mathbf{v} .

Bert's velocity vector across the river is unknown, \mathbf{u} . So, $\mathbf{u} = \begin{bmatrix} x \\ 0 \end{bmatrix}$.

Bert's velocity vector upstream compensates for the downstream flow. So, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

So, his resultant vector is $\mathbf{r} = \mathbf{u} + \mathbf{v} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 1 \end{bmatrix}$.

Since Bert's speed is 2 mph, $\|\mathbf{r}\| = 2$. Why?

So, we have, $\|\mathbf{r}\|^2 = x^2 + 1 = 4$ which implies $x = \sqrt{3}$.

Now we follow Example 1.21 to find the angle between \mathbf{r} and \mathbf{v} .

$$\cos \theta = \frac{\mathbf{r} \cdot \mathbf{v}}{\|\mathbf{r}\| \|\mathbf{v}\|} = \frac{\sqrt{3}}{2}. \text{ So, } \theta = \cos^{-1} \left(\frac{\sqrt{3}}{2} \right) = 60^\circ.$$

So, Bert must swim at a 60° angle to the bank to swim directly across the river.

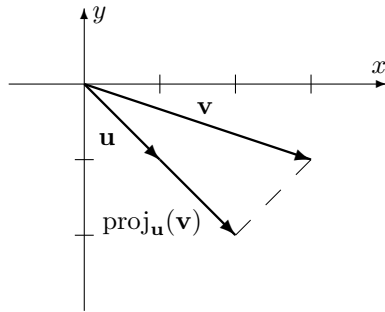
Could you use Pythagoras' Theorem to solve this problem? If so, how would you do it?

40. Following Example 1.17, we compute:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 4 \text{ and}$$

$$\mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2, \text{ so}$$

$$\begin{aligned} \text{proj}_{\mathbf{u}}(\mathbf{v}) &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \frac{4}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\mathbf{u}. \end{aligned}$$

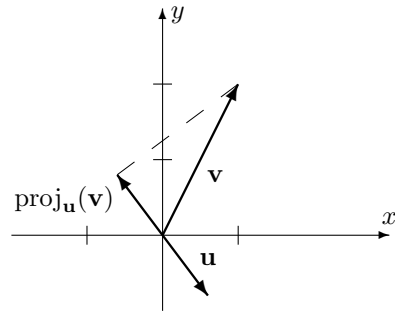


41. Following Example 1.17, we compute:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -1 \text{ and}$$

$$\mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} = 1, \text{ so}$$

$$\begin{aligned} \text{proj}_{\mathbf{u}}(\mathbf{v}) &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \frac{-1}{1} \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} \\ &= \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = -\mathbf{u}. \end{aligned}$$



42. Following Example 1.17, we compute:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2 \text{ and} \\ \mathbf{u} \cdot \mathbf{u} &= \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix} = 1, \text{ so} \\ \text{proj}_{\mathbf{u}}(\mathbf{v}) &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \frac{2}{1} \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix} \\ &= \begin{bmatrix} 4/3 \\ -4/3 \\ -2/3 \end{bmatrix} = 2\mathbf{u}.\end{aligned}$$

43. Following Example 1.17, we compute:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ -1 \\ -2 \end{bmatrix} = 6 \text{ and} \\ \mathbf{u} \cdot \mathbf{u} &= \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 4, \text{ so} \\ \text{proj}_{\mathbf{u}}(\mathbf{v}) &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \frac{6}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 3/2 \\ -3/2 \\ 3/2 \\ -3/2 \end{bmatrix} = \frac{3}{2}\mathbf{u}.\end{aligned}$$

44. Following Example 1.17, we compute:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \cdot \begin{bmatrix} 2.1 \\ 1.2 \end{bmatrix} = 2.85 \text{ and} \\ \mathbf{u} \cdot \mathbf{u} &= \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \cdot \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} = 2.5, \text{ so} \\ \text{proj}_{\mathbf{u}}(\mathbf{v}) &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \frac{2.85}{2.5} \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.57 \\ 1.71 \end{bmatrix} = 1.14\mathbf{u}.\end{aligned}$$

45. Following Example 1.17, we compute:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \begin{bmatrix} 3.01 \\ -0.33 \\ 2.52 \end{bmatrix} \cdot \begin{bmatrix} 1.34 \\ 4.25 \\ -1.66 \end{bmatrix} = -1.5523 \text{ and} \\ \mathbf{u} \cdot \mathbf{u} &= \begin{bmatrix} 3.01 \\ -0.33 \\ 2.52 \end{bmatrix} \cdot \begin{bmatrix} 3.01 \\ -0.33 \\ 2.52 \end{bmatrix} = 15.5194, \text{ so} \\ \text{proj}_{\mathbf{u}}(\mathbf{v}) &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \frac{-1.5523}{15.5194} \begin{bmatrix} 3.01 \\ -0.33 \\ 2.52 \end{bmatrix} \\ &\approx \begin{bmatrix} -0.301 \\ 0.033 \\ -0.252 \end{bmatrix} \approx -\frac{1}{10}\mathbf{u}.\end{aligned}$$

46. Let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix} 2-1 \\ 2-(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix} 4-1 \\ 0-(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(a) We compute the necessary values ...

(b) We compute the necessary values ...

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 6,$$

$$\mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 10 \quad (\|\mathbf{u}\| = \sqrt{10}),$$

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \begin{bmatrix} 3/5 \\ 9/5 \end{bmatrix} \Rightarrow$$

$$\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{bmatrix} 12/5 \\ -4/5 \end{bmatrix} \Rightarrow$$

$$\|\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})\| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \frac{4\sqrt{10}}{5}$$

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 6,$$

$$\|\mathbf{u}\| = \sqrt{1^2 + 3^2} = \sqrt{10},$$

$$\|\mathbf{v}\| = \sqrt{3^2 + 1^2} = \sqrt{10} \Rightarrow$$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{6}{\sqrt{10}\sqrt{10}} = \frac{3}{5} \Rightarrow$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

... then substitute into the formula for \mathcal{A} :

$$\mathcal{A} = \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})\|$$

$$= \frac{1}{2} \sqrt{10} \frac{4\sqrt{10}}{5} = 4.$$

... then substitute into the formula for \mathcal{A} :

$$\mathcal{A} = \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

$$= \frac{1}{2} \sqrt{10} \sqrt{10} \frac{4}{5} = 4.$$

47. Let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix} 4-3 \\ -2-(-1) \\ 6-4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix} 5-3 \\ 0-(-1) \\ 2-4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

(a) We compute the necessary values ...

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = -3,$$

$$\mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 6 \quad (\|\mathbf{u}\| = \sqrt{6}),$$

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1 \end{bmatrix} \Rightarrow$$

$$\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{bmatrix} 5/2 \\ 1/2 \\ -1 \end{bmatrix} \Rightarrow$$

$$\begin{aligned} \|\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})\| &= \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2} \\ &= \frac{\sqrt{30}}{2} \end{aligned}$$

... then substitute into the formula for \mathcal{A} :

$$\begin{aligned} \mathcal{A} &= \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})\| \\ &= \frac{1}{2} (\sqrt{6}) \left(\frac{\sqrt{30}}{2} \right) = \frac{3\sqrt{5}}{2}. \end{aligned}$$

(b) We compute the necessary values ...

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = -3,$$

$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6},$$

$$\|\mathbf{v}\| = \sqrt{2^2 + 1^2 + (-2)^2} = 3 \Rightarrow$$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{-3}{3\sqrt{6}} = -\frac{\sqrt{6}}{6} \Rightarrow$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \left(-\frac{\sqrt{6}}{6}\right)^2} = \frac{\sqrt{30}}{6}$$

... then substitute into the formula for \mathcal{A} :

$$\begin{aligned} \mathcal{A} &= \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta \\ &= \frac{1}{2} (\sqrt{6}) (3) \left(\frac{\sqrt{30}}{6} \right) = \frac{3\sqrt{5}}{2}. \end{aligned}$$

48. Two vectors \mathbf{u} and \mathbf{v} are orthogonal **if and only if** $[\Leftrightarrow]$ their dot product is zero. That is $\mathbf{u} \cdot \mathbf{v} = 0$. So, we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for k :

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} k+1 \\ k-1 \end{bmatrix} = 0 \Rightarrow 2(k+1) + 3(k-1) = 0 \Rightarrow 5k - 1 = 0 \Rightarrow k = \frac{1}{5}.$$

Substituting k back into the expression for \mathbf{v} we get: $\mathbf{v} = \begin{bmatrix} \frac{1}{5} + 1 \\ \frac{1}{5} - 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \end{bmatrix}$.

We check our answer by computing $\mathbf{u} \cdot \mathbf{v}$ (it should be zero):

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \end{bmatrix} = \frac{12}{5} - \frac{12}{5} = 0 \text{ as required.}$$

49. Two vectors \mathbf{u} and \mathbf{v} are orthogonal **if and only if** $[\Leftrightarrow]$ their dot product is zero. That is $\mathbf{u} \cdot \mathbf{v} = 0$. So, we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for k :

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix} = 0 \Rightarrow k^2 - k - 6 = (k+2)(k-3) = 0 \Rightarrow k = -2, 3.$$

Substituting k back into the expression for \mathbf{v} we get:

$$\text{When } k = -2, \mathbf{v}_1 = \begin{bmatrix} (-2)^2 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}. \text{ When } k = 3, \mathbf{v}_2 = \begin{bmatrix} 3^2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix}.$$

We check by computing $\mathbf{u} \cdot \mathbf{v}_1$ and $\mathbf{u} \cdot \mathbf{v}_2$ (they should both be zero):

$$\mathbf{u} \cdot \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix} = 4 + 2 - 6 = 0 \text{ and } \mathbf{u} \cdot \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix} = 9 - 3 - 6 = 0.$$

50. Two vectors \mathbf{u} and \mathbf{v} are orthogonal **if and only if** $[\Leftrightarrow]$ their dot product is zero. That is $\mathbf{u} \cdot \mathbf{v} = 0$. So, we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for y in terms of x :

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow 3x + y = 0 \Rightarrow y = -3x.$$

Substituting $y = -3x$ back into the expression for \mathbf{v} we get: $\mathbf{v} = \begin{bmatrix} x \\ -3x \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Conclusion: Any vector orthogonal to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ must be a multiple of $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Check: $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ -3x \end{bmatrix} = 3x - 3x = 0$ for all values of x .

Note: We could also have solved for x in terms of y yielding $\mathbf{v} = \begin{bmatrix} -\frac{1}{3}y \\ y \end{bmatrix} = y \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$.

51. As noted in the remarks just prior to Example 1.16:

The zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow a = b = 0$ is orthogonal to all vectors in \mathbb{R}^2 .

Having covered this case, we will now assume that at least one of a or $b \neq 0$.

Two vectors \mathbf{u} and \mathbf{v} are orthogonal *if and only if* $[\Leftrightarrow]$ their dot product is zero. That is $\mathbf{u} \cdot \mathbf{v} = 0$. So, we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for y in terms of x .

$$\text{Case 1: } b \neq 0: \mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow ax + by = 0 \Rightarrow y = -\frac{a}{b}x.$$

$$\text{Substituting } y = -\frac{a}{b}x \text{ back into the expression for } \mathbf{v} \text{ we get: } \mathbf{v} = \begin{bmatrix} x \\ -\frac{a}{b}x \end{bmatrix} = x \begin{bmatrix} 1 \\ -\frac{a}{b} \end{bmatrix}.$$

$$\text{If we let } x = b \text{ we find } b \begin{bmatrix} 1 \\ -\frac{a}{b} \end{bmatrix} = \begin{bmatrix} b \\ -a \end{bmatrix}, \text{ which clarifies the relationship between } \mathbf{u} \text{ and } \mathbf{v}.$$

$$\text{Case 2: } b = 0 (\Rightarrow a \neq 0): \mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow ax + by = 0 \Rightarrow x = -\frac{b}{a}y.$$

$$\text{Substituting } x = -\frac{b}{a}y \text{ back into the expression for } \mathbf{v} \text{ we get: } \mathbf{v} = \begin{bmatrix} -\frac{b}{a}y \\ y \end{bmatrix} = y \begin{bmatrix} -\frac{b}{a} \\ 1 \end{bmatrix}.$$

$$\text{If we let } y = -a \text{ we find } -a \begin{bmatrix} -\frac{b}{a} \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ -a \end{bmatrix} \text{ exactly as in Case 1.}$$

Conclusion: Any vector orthogonal to $\begin{bmatrix} a \\ b \end{bmatrix}$ ($\neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$) must be a multiple of $\begin{bmatrix} b \\ -a \end{bmatrix}$.

$$\text{Check: } \mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} bx \\ -ax \end{bmatrix} = abx - bax = 0 \text{ for all values of } x.$$

52. (a) The geometry of the vectors in Figure 1.26 suggests the following assertion:

For $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$, \mathbf{u} and \mathbf{v} must point in the same direction.

So, the angle θ between \mathbf{u} and \mathbf{v} must be $0 \Rightarrow \cos \theta = \cos 0 = 1$.

$$\text{So, we have } \cos \theta = \cos 0 = 1 = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \Rightarrow \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\|.$$

Emulating the statement of Theorem 1.6 (Pythagoras' Theorem), we state:

For all vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^2 and \mathbb{R}^3 , $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$

if and only if \mathbf{u} and \mathbf{v} point in the same direction.

Again, we must note that pointing in same direction is equivalent to $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\|$.

Figuring out when to apply this condition will be the key to a successful proof.

PROOF: Proceeding as in the proof of Theorem 1.5 (The Triangle Inequality), we note:

Since both sides of the equality are nonnegative,
showing that the *square* of the left-hand side, $\|\mathbf{u} + \mathbf{v}\|^2$,
is equal to the *square* of the right hand side, $(\|\mathbf{u}\| + \|\mathbf{v}\|)^2$,
is equivalent to proving the condition.

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| + \|\mathbf{v}\|^2 \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2. \end{aligned}$$

By Example 1.9

By $\mathbf{w} \cdot \mathbf{w} = \|\mathbf{w}\|^2$ for any vector \mathbf{w}

By $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\|$ (**key condition**)

By $x^2 + 2xy + y^2 = (x + y)^2$

- (b) Since the left-hand side, $\|\mathbf{u} + \mathbf{v}\|$, is always non-negative, we have to impose an initial condition of $\|\mathbf{u}\| \geq \|\mathbf{v}\|$.

The geometry of the vectors in Figures 1.26 and 1.30 suggests the following assertion:

For $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$ \mathbf{u} and \mathbf{v} must point in the opposite directions.

So, the angle θ between \mathbf{u} and \mathbf{v} must be $\pi \Rightarrow \cos \theta = \cos \pi = -1$.

$$\text{So, we have } \cos \theta = \cos \pi = -1 = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \Rightarrow \mathbf{u} \cdot \mathbf{v} = -\|\mathbf{u}\| \|\mathbf{v}\|.$$

Emulating the statement of Theorem 1.6 (Pythagoras' Theorem), we state:

For all vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^2 and \mathbb{R}^3 , $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$
if and only if \mathbf{u} and \mathbf{v} point in the opposite directions.

Again, we must note that pointing in same direction is equivalent to $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\|$.
Figuring out when to apply this condition will be the key to a successful proof.

PROOF: Proceeding as in the proof of Theorem 1.5 (The Triangle Inequality), we note:

Since both sides of the equality are nonnegative,
showing that the *square* of the left-hand side, $\|\mathbf{u} + \mathbf{v}\|^2$,
is equal to the *square* of the right hand side, $(\|\mathbf{u}\| - \|\mathbf{v}\|)^2$,
is equivalent to proving the condition.

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 \\ &= \|\mathbf{u}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| + \|\mathbf{v}\|^2 \\ &= (\|\mathbf{u}\| - \|\mathbf{v}\|)^2. \end{aligned}$$

By Example 1.9

By $\mathbf{w} \cdot \mathbf{w} = \|\mathbf{w}\|^2$ for any vector \mathbf{w}

By $\mathbf{u} \cdot \mathbf{v} = -\|\mathbf{u}\| \|\mathbf{v}\|$ (**key condition**)

By $x^2 - 2xy + y^2 = (x - y)^2$

53. We prove Theorem 1.2(b) by applying the definition of the dot product.

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \cdots + u_n(v_n + w_n) \\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \cdots + u_nv_n + u_nw_n \\ &= (u_1v_1 + u_2v_2 + \cdots + u_nv_n) + (u_1w_1 + u_2w_2 + \cdots + u_nw_n) \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.\end{aligned}$$

54. We prove the three parts of Theorem 1.2(d) by applying the definition of the dot product and key properties of real numbers.

Part 1: For any vector \mathbf{u} , we need to show $\mathbf{u} \cdot \mathbf{u} \geq 0$.

We begin by noting that for any real number x , we have $x^2 \geq 0$.

So, $\mathbf{u} \cdot \mathbf{u} = u_1u_1 + u_2u_2 + \cdots + u_nu_n = u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$.

Note: $u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$ because the u_i are real numbers.

Part 2: We need to show if $\mathbf{u} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{u} = 0$.

We begin by noting that if $\mathbf{u} = \mathbf{0}$, then $u_i = 0$ for all i .

If $\mathbf{u} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{u} = \mathbf{0} \cdot \mathbf{0} = u_1^2 + u_2^2 + \cdots + u_n^2 = 0^2 + 0^2 + \cdots + 0^2 = 0$.

Part 3: We need to show if $\mathbf{u} \cdot \mathbf{u} = 0$, then $\mathbf{u} = \mathbf{0}$.

We begin by noting that for any real number x , if $x^2 = 0$ then $x = 0$.

If $\mathbf{u} \cdot \mathbf{u} = u_1u_1 + u_2u_2 + \cdots + u_nu_n = u_1^2 + u_2^2 + \cdots + u_n^2 = 0$

then $u_i^2 = 0$ for all i which implies $u_i = 0$ because the u_i are real numbers.

Therefore, since $u_i = 0$ for all i , by definition $\mathbf{u} = \mathbf{0}$.

55. We need to show $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|\mathbf{v} - \mathbf{u}\| = d(\mathbf{v}, \mathbf{u})$.

If we let $c = -1$ in Theorem 1.3(b), then $\| -\mathbf{w} \| = \|\mathbf{w}\|$. We use this **key fact** below.

| | |
|---------------|---|
| <i>PROOF:</i> | $\begin{aligned}d(\mathbf{u}, \mathbf{v}) &= \ \mathbf{u} - \mathbf{v}\ && \text{By definition} \\ &= \ -(\mathbf{v} - \mathbf{u})\ && \text{By the fact that } (x - y) = -(y - x) \\ &= \ \mathbf{v} - \mathbf{u}\ && \text{By } \ -\mathbf{w}\ = \ \mathbf{w}\ \text{ (key fact)} \\ &= d(\mathbf{v}, \mathbf{u}). && \text{By definition}\end{aligned}$ |
|---------------|---|

56. We need to show $d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$. That is, $\|\mathbf{u} - \mathbf{w}\| \leq \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\|$.

This follows immediately from Theorem 1.5:

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \quad \text{with } \mathbf{x} = \mathbf{u} - \mathbf{v} \text{ and } \mathbf{y} = \mathbf{v} - \mathbf{w}.$$

57. We need to show $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = 0$ if and only if $\mathbf{u} = \mathbf{v}$.

This follows immediately from Theorem 1.3(a): $\|\mathbf{w}\| = 0$ if and only if $\mathbf{w} = \mathbf{0}$, with $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

58. We will show $\mathbf{u} \cdot c\mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ by applying the definitions.

$$\begin{aligned}\mathbf{u} \cdot c\mathbf{v} &= [u_1, u_2, \dots, u_n] \cdot [cv_1, cv_2, \dots, cv_n] = u_1cv_1 + u_2cv_2 + \cdots + u_ncv_n \\ &= cu_1v_1 + cu_2v_2 + \cdots + cu_nv_n = c(u_1v_1 + u_2v_2 + \cdots + u_nv_n) = c(\mathbf{u} \cdot \mathbf{v})\end{aligned}$$

59. We need to show $\|\mathbf{u} - \mathbf{v}\| \geq \|\mathbf{u}\| - \|\mathbf{v}\|$. That is, $\|\mathbf{u}\| \leq \|\mathbf{u} - \mathbf{v}\| + \|\mathbf{v}\|$.

This follows immediately from Theorem 1.5, $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$, with $\mathbf{x} = \mathbf{u} - \mathbf{v}$ and $\mathbf{y} = \mathbf{v}$.

60. If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ it does *not* follow that $\mathbf{v} = \mathbf{w}$.

An instructive counterexample is suggested by the remarks just prior to Example 1.16. Since $\mathbf{0} \cdot \mathbf{v} = 0$ for every vector \mathbf{v} in \mathbb{R}^n , the zero vector is orthogonal to every vector. So, if $\mathbf{u} = \mathbf{0}$, we know nothing about \mathbf{v} and \mathbf{w} except that they are vectors in \mathbb{R}^n .

However, we note that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ implies $\mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$.

So, if $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, it *does* follow that $\mathbf{v} - \mathbf{w}$ is orthogonal to \mathbf{u} .

61. We need to show $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for all vectors in \mathbb{R}^n .

Recall, by the definitions of the dot product and the norm, $\mathbf{w} \cdot \mathbf{w} = \|\mathbf{w}\|^2$.

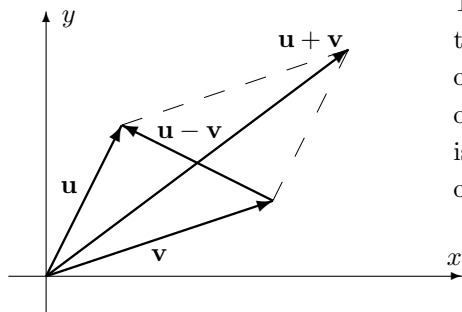
We apply Theorem 1.2(b) and this **key fact** to complete our *PROOF*:

$$\begin{aligned} (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} && \text{By Theorem 1.2(b)} \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} && \text{By the fact that } -xy + yx = 0 \\ &= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2. && \text{By the fact that } \mathbf{w} \cdot \mathbf{w} = \|\mathbf{w}\|^2 \text{ (key fact)} \end{aligned}$$

62. (a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \\ &= (\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + 2\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v}) \\ &= (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) + 2\mathbf{u} \cdot \mathbf{v} + (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2) - 2\mathbf{u} \cdot \mathbf{v} = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2. \end{aligned}$$

(b)



The proof in part (a) tells us that the sum of the squares of the lengths of the diagonals of a parallelogram is twice the sum of the squares of the lengths of its sides.

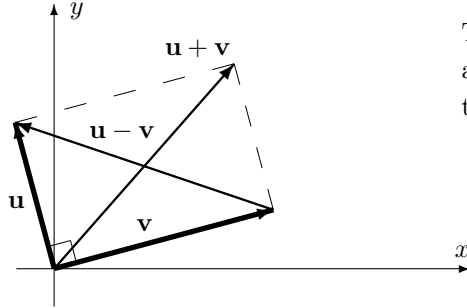
63. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and consider $\frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$. By definition, we have:

$$\begin{aligned} \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2 &= \frac{1}{4}[(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})] \\ &= \frac{1}{4}[(\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + 2\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v})] \\ &= \frac{1}{4}[(\|\mathbf{u}\|^2 - \|\mathbf{u}\|^2) + (\|\mathbf{v}\|^2 - \|\mathbf{v}\|^2) + 4\mathbf{u} \cdot \mathbf{v}] = \mathbf{u} \cdot \mathbf{v}. \end{aligned}$$

64. (a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\| &\Leftrightarrow \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2 \\ &\Leftrightarrow \|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 0 \\ &\Leftrightarrow \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2 = 0 \\ &\Leftrightarrow \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.} \end{aligned}$$

(b)



The proof in part (a) tells us that a parallelogram is a rectangle if and only if the lengths of its diagonals are equal.

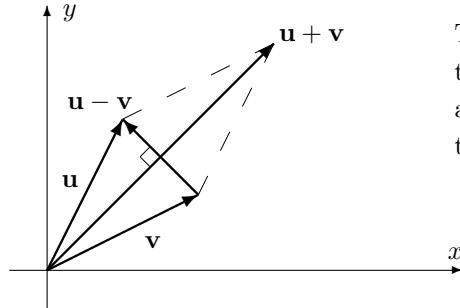
65. (a) We need to show $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = 0$ in \mathbb{R}^n if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

By Exercise 55, $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$.

Therefore, $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$ if and only if $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$.

It follows immediately that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal in \mathbb{R}^n if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

(b)



The proof in part (a) tells us that the diagonals of a parallelogram are perpendicular if and only if the lengths of its sides are equal.

66. From Example 1.9 and the fact that $\mathbf{w} \cdot \mathbf{w} = \|\mathbf{w}\|^2$, we have $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2$.

Taking the square root of both sides yields $\|\mathbf{u} + \mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2}$.

Substituting in the given values of $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = \sqrt{3}$, and $\mathbf{u} \cdot \mathbf{v} = 1$

gives us $\|\mathbf{u} + \mathbf{v}\| = \sqrt{2^2 + 2(1) + (\sqrt{3})^2} = \sqrt{4 + 2 + 3} = \sqrt{9} = 3$.

67. We need to show $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 2$ imply $\mathbf{u} \cdot \mathbf{v} \neq 3$.

From Theorem 1.4 (the Cauchy-Schwarz Inequality), we have $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$.

Substituting in the given values of $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 2$ shows $|\mathbf{u} \cdot \mathbf{v}| \leq 2$.

Therefore, $-2 \leq \mathbf{u} \cdot \mathbf{v} \leq 2$. It follows immediately that $\mathbf{u} \cdot \mathbf{v} \neq 3$.

68. (a) Assume that \mathbf{u} is orthogonal to both \mathbf{v} and \mathbf{w} , so $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} = 0$.

Then $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = 0 + 0 = 0$, so \mathbf{u} is orthogonal to $\mathbf{v} + \mathbf{w}$.

(b) Assume that \mathbf{u} is orthogonal to both \mathbf{v} and \mathbf{w} , so $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} = 0$.

Then $\mathbf{u} \cdot (s\mathbf{v} + t\mathbf{w}) = \mathbf{u} \cdot (s\mathbf{v}) + \mathbf{u} \cdot (t\mathbf{w}) = s(\mathbf{u} \cdot \mathbf{v}) + t(\mathbf{u} \cdot \mathbf{w}) = s(0) + t(0) = 0 + 0 = 0$,
so \mathbf{u} is orthogonal to $s\mathbf{v} + t\mathbf{w}$.

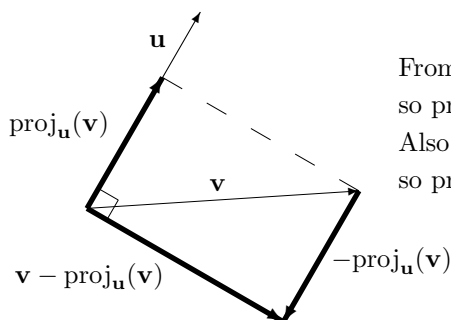
69. Two vectors (\mathbf{u} and \mathbf{v}) are orthogonal if their dot product equals zero. So we evaluate:

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})) &= \mathbf{u} \cdot \left(\mathbf{v} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} \right) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} \\ &= \mathbf{u} \cdot \mathbf{v} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) (\mathbf{u} \cdot \mathbf{u}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} = 0. \end{aligned}$$

70. (a) $\text{proj}_{\mathbf{u}}(\text{proj}_{\mathbf{u}}(\mathbf{v})) = \text{proj}_{\mathbf{u}}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}\right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \text{proj}_{\mathbf{u}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} = \text{proj}_{\mathbf{u}}(\mathbf{v})$.

$$\begin{aligned} \text{(b) } \text{proj}_{\mathbf{u}}(\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})) &= \text{proj}_{\mathbf{u}}\left(\mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}\right) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \left(\frac{\mathbf{u} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} = 0. \end{aligned}$$

(c)



From the diagram, we see that $\text{proj}_{\mathbf{u}}(\mathbf{v}) \parallel \mathbf{u}$,
so $\text{proj}_{\mathbf{u}}(\text{proj}_{\mathbf{u}}(\mathbf{v})) = \text{proj}_{\mathbf{u}}(\mathbf{v})$.

Also, $(\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})) \perp \mathbf{u}$,

so $\text{proj}_{\mathbf{u}}(\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})) = 0$.

71. (a) The Cauchy-Schwarz Inequality tells us $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$.

Squaring both sides, we get $|\mathbf{u} \cdot \mathbf{v}|^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2$.

In \mathbb{R}^2 with $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, this becomes $(u_1 v_1 + u_2 v_2)^2 \leq (u_1^2 + u_2^2)(v_1^2 + v_2^2) \Leftrightarrow$

$$0 \leq (u_1^2 + u_2^2)(v_1^2 + v_2^2) - (u_1 v_1 + u_2 v_2)^2 \Leftrightarrow 0 \leq u_1^2 v_2^2 + u_2^2 v_1^2 - 2u_1 u_2 v_1 v_2 \Leftrightarrow$$

$$0 \leq \frac{1}{2}(u_1 v_2 - u_2 v_1)^2 + \frac{1}{2}(u_2 v_1 - u_1 v_2)^2.$$

Since the final statement is true, all the statements are true.

- (b) Let \mathbf{u} and \mathbf{v} be elements of \mathbb{R}^3 . Then $|\mathbf{u} \cdot \mathbf{v}|^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \Leftrightarrow$

$$(u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \leq (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) \Leftrightarrow$$

$$0 \leq (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \Leftrightarrow$$

$$0 \leq u_1^2 v_2^2 + u_1^2 v_3^2 + u_2^2 v_1^2 + u_2^2 v_3^2 + u_3^2 v_1^2 + u_3^2 v_2^2 - 2u_1 v_1 u_2 v_2 - 2u_1 v_1 u_3 v_3 - 2u_2 v_2 u_3 v_3 \Leftrightarrow$$

$$0 \leq \frac{1}{2}(u_1 v_2 - u_2 v_1)^2 + \frac{1}{2}(u_2 v_1 - u_1 v_2)^2 + \frac{1}{2}(u_1 v_3 - u_3 v_1)^2 \\ + \frac{1}{2}(u_3 v_1 - u_1 v_3)^2 + \frac{1}{2}(u_2 v_3 - u_3 v_2)^2 + \frac{1}{2}(u_3 v_2 - u_2 v_3)^2.$$

Since the final statement is true, all the statements are true.

72. $\|\text{proj}_{\mathbf{u}}(\mathbf{v})\| = \left\| \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} \right\| = \left\| \frac{\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta}{\|\mathbf{u}\| \|\mathbf{u}\|} \right\| \|\mathbf{u}\| = \frac{\|\mathbf{u}\| \|\mathbf{v}\|}{\|\mathbf{u}\| \|\mathbf{u}\|} \|\mathbf{u}\| \cos \theta$
 $= \|\mathbf{v}\| |\cos \theta| \leq \|\mathbf{v}\| \text{ (since } |\cos \theta| \leq 1 \text{)}.$

73. We have $\text{proj}_{\mathbf{u}}(\mathbf{v}) = c\mathbf{u}$. From the figure, we see that $\cos \theta = \frac{c\|\mathbf{u}\|}{\|\mathbf{v}\|}$, so $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \frac{c\|\mathbf{u}\|}{\|\mathbf{v}\|}$
 which we solve for c to get $c = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}$. Thus, $\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u}$ since $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u}$.

74. Proof by Induction

See Appendix B for further discussion and examples of *Mathematical Induction*. Also see Chapter 3 of this Study Guide.

Using induction, we will prove the generalized Triangle Inequality:

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_n\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \dots + \|\mathbf{v}_n\| \text{ for all } n \geq 1.$$

$$1: \|\mathbf{v}_1\| \leq \|\mathbf{v}_1\|$$

This is obvious, so there is nothing to show.

$$k: \|\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \dots + \|\mathbf{v}_k\|$$

This is the induction hypothesis, so there is nothing to show.

$$k+1: \|\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k + \mathbf{v}_{k+1}\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \dots + \|\mathbf{v}_k\| + \|\mathbf{v}_{k+1}\|$$

This is the statement we must prove using the induction hypothesis.

The Triangle inequality, $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$, will also be key.

Let $\mathbf{v}_k + \mathbf{v}_{k+1}$ by the k th vector in the induction hypothesis. Then:

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k + \mathbf{v}_{k+1}\| \stackrel{\text{by induction}}{\leq} \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \dots + \|\mathbf{v}_k + \mathbf{v}_{k+1}\|$$

Applying The Triangle Inequality, we have $\|\mathbf{v}_k + \mathbf{v}_{k+1}\| \leq \|\mathbf{v}_k\| + \|\mathbf{v}_{k+1}\|$. So:

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_k + \mathbf{v}_{k+1}\| \stackrel{\text{by Tri Inq}}{\leq} \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \dots + \|\mathbf{v}_k\| + \|\mathbf{v}_{k+1}\|$$

We have proven (by induction) that

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \dots + \mathbf{v}_n\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \dots + \|\mathbf{v}_n\| \text{ for all } n \geq 1.$$

Induction can seem a little bit like *magic* at first glance.

Pay close attention to how the induction hypothesis is used to in the proof.

Exploration: Vectors and Geometry

Since Explorations are self-contained, only solutions will be provided.

1. Like Example 1.18: $\mathbf{m} - \mathbf{a} = \overrightarrow{AM} = \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}(\mathbf{b} - \mathbf{a}) \Rightarrow \mathbf{m} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{1}{3}(2\mathbf{a} + \mathbf{b})$.

In general, $\mathbf{m} - \mathbf{a} = \overrightarrow{AM} = \frac{1}{n}\overrightarrow{AB} = \frac{1}{n}(\mathbf{b} - \mathbf{a}) \Rightarrow \mathbf{m} = \mathbf{a} + \frac{1}{n}(\mathbf{b} - \mathbf{a}) = \frac{1}{n}((n-1)\mathbf{a} + \mathbf{b})$.

Note as $n \rightarrow \infty$, $\mathbf{m} \rightarrow \mathbf{a}$.

2. We use the notation that the vector beginning at the origin and ending at the point X is \mathbf{x} .

Therefore, from Exercise 1, we have: $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$ and $\mathbf{q} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) \Rightarrow$

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\overrightarrow{AB}.$$

3. Draw in \overrightarrow{AC} . Then from Exercise 2, we have: $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB} = \overrightarrow{SR}$.

Draw in \overrightarrow{BD} . Then from Exercise 2, we have: $\overrightarrow{PS} = \frac{1}{2}\overrightarrow{BD} = \overrightarrow{QR} \Rightarrow$

$PQRS$ is a parallelogram (opposite sides are parallel and congruent).

4. Following the hint, we find \mathbf{m} , the point that is two-thirds of the distance from A to P .

From Exercise 1, we have: $\mathbf{p} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) \Rightarrow \mathbf{m} = \frac{1}{3}(2\mathbf{p} + \mathbf{a}) = \frac{1}{3}(2 \cdot \frac{1}{2}(\mathbf{b} + \mathbf{c}) + \mathbf{a}) = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

Next, we find \mathbf{m}' , the point that is two-thirds of the distance from B to Q .

From Exercise 1, we have: $\mathbf{q} = \frac{1}{2}(\mathbf{a} + \mathbf{c}) \Rightarrow \mathbf{m}' = \frac{1}{3}(2\mathbf{q} + \mathbf{b}) = \frac{1}{3}(2 \cdot \frac{1}{2}(\mathbf{a} + \mathbf{c}) + \mathbf{b}) = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

Finally, we find \mathbf{m}'' , the point that is two-thirds of the distance from C to R .

From Exercise 1, we have: $\mathbf{r} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) \Rightarrow \mathbf{m}'' = \frac{1}{3}(2\mathbf{r} + \mathbf{c}) = \frac{1}{3}(2 \cdot \frac{1}{2}(\mathbf{a} + \mathbf{b}) + \mathbf{c}) = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

We have shown $\mathbf{m} = \mathbf{m}' = \mathbf{m}''$. That is, the medians intersect at the centroid, G .

5. We are given \overrightarrow{AH} is orthogonal to \overrightarrow{BC} , that is $\overrightarrow{AH} \cdot \overrightarrow{BC} = 0$ and

\overrightarrow{BH} is orthogonal to \overrightarrow{AC} , that is $\overrightarrow{BH} \cdot \overrightarrow{AC} = 0$.

We need to show \overrightarrow{CH} is orthogonal to \overrightarrow{AB} , that is $\overrightarrow{CH} \cdot \overrightarrow{AB} = 0$.

$$\begin{aligned} \overrightarrow{AH} \cdot \overrightarrow{BC} = 0 &\Rightarrow (\mathbf{h} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{c}) = 0 \Rightarrow \mathbf{h} \cdot \mathbf{b} - \mathbf{h} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 0 \\ \overrightarrow{BH} \cdot \overrightarrow{AC} = 0 &\Rightarrow (\mathbf{h} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{a}) = 0 \Rightarrow \mathbf{h} \cdot \mathbf{c} - \mathbf{h} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \end{aligned}$$

$$0 = \mathbf{h} \cdot \mathbf{b} - \mathbf{h} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = (\mathbf{h} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = \overrightarrow{CH} \cdot \overrightarrow{AB} = 0 \Rightarrow$$

\overrightarrow{CH} is orthogonal to \overrightarrow{AB} , so all the altitudes intersect at the orthocenter, H .

6. We are given \overrightarrow{QK} is orthogonal to \overrightarrow{AC} , that is $\overrightarrow{QK} \cdot \overrightarrow{AC} = 0$ and \overrightarrow{PK} is orthogonal to \overrightarrow{CB} , that is $\overrightarrow{PK} \cdot \overrightarrow{CB} = 0$.

We need to show \overrightarrow{RK} is orthogonal to \overrightarrow{AB} , that is $\overrightarrow{RK} \cdot \overrightarrow{AB} = 0$.

By Exercise 1, we have $\mathbf{q} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$ and $\mathbf{p} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$. So:

$$\begin{aligned} \overrightarrow{QK} \cdot \overrightarrow{AC} = 0 &\Rightarrow (\mathbf{k} - \mathbf{q}) \cdot (\mathbf{c} - \mathbf{a}) = 0 \Rightarrow (\mathbf{k} - \frac{1}{2}(\mathbf{a} + \mathbf{c})) \cdot (\mathbf{c} - \mathbf{a}) = 0 \\ \overrightarrow{PK} \cdot \overrightarrow{CB} = 0 &\Rightarrow (\mathbf{k} - \mathbf{p}) \cdot (\mathbf{b} - \mathbf{c}) = 0 \Rightarrow (\mathbf{k} - \frac{1}{2}(\mathbf{b} + \mathbf{c})) \cdot (\mathbf{b} - \mathbf{c}) = 0 \end{aligned} \Rightarrow$$

$$\begin{aligned} \mathbf{k} \cdot \mathbf{c} - \mathbf{k} \cdot \mathbf{a} - \frac{1}{2} \mathbf{a} \cdot \mathbf{c} + \frac{1}{2} \mathbf{a} \cdot \mathbf{a} - \frac{1}{2} \mathbf{c} \cdot \mathbf{c} + \frac{1}{2} \mathbf{a} \cdot \mathbf{c} &= 0 \\ \mathbf{k} \cdot \mathbf{b} - \mathbf{k} \cdot \mathbf{c} - \frac{1}{2} \mathbf{b} \cdot \mathbf{b} + \frac{1}{2} \mathbf{b} \cdot \mathbf{c} - \frac{1}{2} \mathbf{c} \cdot \mathbf{b} + \frac{1}{2} \mathbf{c} \cdot \mathbf{c} &= 0 \end{aligned} \Rightarrow$$

$$0 = \mathbf{k} \cdot \mathbf{b} - \mathbf{k} \cdot \mathbf{a} - \frac{1}{2} \mathbf{b} \cdot \mathbf{b} + \frac{1}{2} \mathbf{a} \cdot \mathbf{a} = (\mathbf{k} - \frac{1}{2}(\mathbf{b} + \mathbf{a})) \cdot (\mathbf{b} - \mathbf{a}) = (\mathbf{k} - \mathbf{r}) \cdot (\mathbf{b} - \mathbf{a}) = \overrightarrow{RK} \cdot \overrightarrow{AB} = 0 \Rightarrow \overrightarrow{RK} \text{ is orthogonal to } \overrightarrow{AB}, \text{ so all perpendicular bisectors intersect at the circumcenter, } K.$$

7. Let O be the origin, then we have: $\mathbf{b} = -\mathbf{a}$ and $\|\mathbf{a}\|^2 = \|\mathbf{c}\|^2 = r^2$, $r = \text{radius of the circle}$.

We need to show \overrightarrow{AC} is orthogonal to \overrightarrow{BC} , that is $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0 \Rightarrow$

$$(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{b}) = (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} + \mathbf{a}) = \|\mathbf{c}\|^2 + \mathbf{c} \cdot \mathbf{a} - \|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{c} = (\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{c}) + (r^2 - r^2) = 0 \Rightarrow \overrightarrow{AC} \text{ is orthogonal to } \overrightarrow{BC}, \text{ so } \angle ACB \text{ is a right angle.}$$

8. As in Exercise 5, we find \mathbf{m} , the point that is one-half of the distance from P to R .

From Exercise 1, we have: $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\mathbf{r} = \frac{1}{2}(\mathbf{c} + \mathbf{d}) \Rightarrow$

$$\mathbf{m} = \frac{1}{2}(\mathbf{p} + \mathbf{r}) = \frac{1}{2}(\frac{1}{2}(\mathbf{a} + \mathbf{b}) + \frac{1}{2}(\mathbf{c} + \mathbf{d})) = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}).$$

Next, we find \mathbf{m}' , the point that is one-half of the distance from Q to S .

From Exercise 1, we have: $\mathbf{q} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$ and $\mathbf{s} = \frac{1}{2}(\mathbf{a} + \mathbf{d}) \Rightarrow$

$$\mathbf{m}' = \frac{1}{2}(\mathbf{q} + \mathbf{s}) = \frac{1}{2}(\frac{1}{2}(\mathbf{b} + \mathbf{c}) + \frac{1}{2}(\mathbf{a} + \mathbf{d})) = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}).$$

We have shown $\mathbf{m} = \mathbf{m}'$.

That is, PR and QS bisect each other because they intersect at their mutual midpoint.

1.3 Lines and Planes

1. Following Example 1.27, we will:

- (a) find the normal form by substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ and
- (b) find the general form by computing those dot products.

(a) $\mathbf{n} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ The normal form is $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(b) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 3x + 2y$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \Rightarrow$ The general form is $3x + 2y = 0$.

2. Following Example 1.27, we will:

- (a) find the normal form by substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ and
- (b) find the general form by computing those dot products.

(a) $\mathbf{n} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow$ Normal form is $\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 3x - 4y$ and $\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \Rightarrow$ The general form is $3x - 4y = 2$.

3. Following Example 1.28, we will:

- (a) find the vector form by substituting into $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and
- (b) find the parametric form by equating components.

(a) $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \Rightarrow$ The vector form is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(b) The vector form in (a) implies the parametric form is $\begin{matrix} x = 1 - t \\ y = 3t \end{matrix}$.

4. Following Example 1.28, we will:

- (a) find the vector form by substituting into $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and
- (b) find the parametric form by equating components.

(a) $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow$ The vector form is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(b) The vector form in (a) implies the parametric form is $\begin{matrix} x = 3 - t \\ y = -3 + t \end{matrix}$.

5. Following Example 1.28, we will:

- (a) find the vector form by substituting into $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and
- (b) find the parametric form by equating components.

$$(a) \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \mathbf{d} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \text{The vector form is } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}.$$

$$(b) \text{ The vector form in (a) implies the parametric form is } \begin{array}{l} x = t \\ y = -t \\ z = 4t \end{array}.$$

6. Following Example 1.28, we will:

- (a) find the vector form by substituting into $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and
- (b) find the parametric form by equating components.

$$(a) \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{p} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \text{ and } \mathbf{d} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \Rightarrow \text{The vector form is } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}.$$

$$(b) \text{ The vector form in (a) implies the parametric form is } \begin{array}{l} x = -3 + t \\ y = 1 \\ z = 2 + 5t \end{array}.$$

7. Following Example 1.30, we will:

- (a) find the normal form by substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ and
- (b) find the general form by computing those dot products.

$$(a) \mathbf{n} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \text{The normal form is } \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$(b) \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3x + 2y + z \text{ and } \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2 \Rightarrow \text{The general form is } 3x + 2y + z = 2.$$

8. Following Example 1.30, we will:

- (a) find the normal form by substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ and
- (b) find the general form by computing those dot products.

$$(a) \mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{p} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \text{Normal form } \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}.$$

$$(b) \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x + 5z, \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = 7 \Rightarrow \text{The general form is } x + 5z = 7.$$

9. Following Example 1.31, we will:

- (a) find the vector form by substituting into $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$ and
 (b) find the parametric form by equating components.

$$(a) \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow$$

$$\text{The vector form is } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

$$(b) \text{ The vector form in (a) implies the parametric form is } \begin{aligned} x &= 2s - 3t \\ y &= s + 2t \\ z &= 2s + t \end{aligned}$$

10. Following Example 1.31, we will:

- (a) find the vector form by substituting into $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$ and
 (b) find the parametric form by equating components.

$$(a) \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{p} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow$$

$$\text{The vector form is } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$$(b) \text{ The vector form in (a) implies the parametric form is } \begin{aligned} x &= 4 + s - t \\ y &= -1 + s + t \\ z &= 3 + t \end{aligned}$$

11. Following Example 1.31, we realize we may choose any point on ℓ , so we will use P (Q would also be fine).

A convenient direction vector is $\mathbf{d} = \overrightarrow{PQ} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (or any scalar multiple of this).

$$\begin{aligned} \text{Thus we obtain: } \mathbf{x} &= \mathbf{p} + t\mathbf{d} \\ &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{aligned}$$

12. Following Example 1.31, we realize we may choose any point on ℓ , so we will use P (Q would also be fine).

A convenient direction vector is $\mathbf{d} = \overrightarrow{PQ} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$ (or any scalar multiple of this).

$$\begin{aligned} \text{Thus we obtain: } \mathbf{x} &= \mathbf{p} + t\mathbf{d} \\ &= \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}. \end{aligned}$$

13. Following Example 1.31, we realize we need to find two direction vectors, \mathbf{u} and \mathbf{v} . Since $P = (1, 1, 1)$, $Q = (4, 0, 2)$, and $R = (0, 1, -1)$ lie in plane \mathcal{P} , we compute:

$$\mathbf{u} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \overrightarrow{PR} = \mathbf{r} - \mathbf{p} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}.$$

Since \mathbf{u} and \mathbf{v} are not scalar multiples of each other, they will serve as direction vectors. If \mathbf{u} and \mathbf{v} were scalar multiples of each other, we would not have a plane but simply a line.

Therefore, we have the vector equation of \mathcal{P} :
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}.$$

14. Following Example 1.31, we realize we need to find two direction vectors, \mathbf{u} and \mathbf{v} . Since $P = (1, 0, 0)$, $Q = (0, 1, 0)$, and $R = (0, 0, 1)$ lie in plane \mathcal{P} , we compute:

$$\mathbf{u} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v} = \overrightarrow{PR} = \mathbf{r} - \mathbf{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Since \mathbf{u} and \mathbf{v} are not scalar multiples of each other, they will serve as direction vectors. If \mathbf{u} and \mathbf{v} were scalar multiples of each other, we would not have a plane but simply a line.

Therefore, we have the vector equation of \mathcal{P} :
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

15. The parametric equations and associated vector forms $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ found below are **not** unique.

- (a) As in the remarks prior to Example 1.20, we begin by letting $x = t$.

When we substitute $x = t$ into $y = 3x - 1$, we get $y = 3(t) - 1$. So, we have the following:

Parametric equations $\begin{matrix} x = & t \\ y = -1 + 3t \end{matrix}$ and vector form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$

- (b) In this case since the coefficient of y is 2, we begin by letting $x = 2t$.

When we substitute $x = 2t$ into $3x + 2y = 5$, we get $3(2t) + 2y = 5$.

Solving for y yields $y = -3t + 2.5$. So, we have the following:

Parametric equations: $\begin{matrix} x = & 2t \\ y = 2.5 - 3t \end{matrix}$ and vector form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$

We discover the following pattern: if line ℓ has equation $ax + by = c$, then $\mathbf{d} = \begin{bmatrix} b \\ -a \end{bmatrix}.$

16. By convention, \mathbf{u} is the vector with its tail at the origin and its head at the point U . So, when $\mathbf{x} = \mathbf{u}$, the line described by $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ passes through the point U .

We note $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ where $\mathbf{d} = \mathbf{q} - \mathbf{p}$ is the line that passes through P and Q , since it passes through P (when $t = 0$) in the direction of $\overrightarrow{PQ} (= \mathbf{q} - \mathbf{p})$.

- (a) To show \mathbf{x} describes the line segment \overline{PQ} as t varies from 0 to 1, we need to show:
 when $t = 0$, $\mathbf{x} = \mathbf{p}$ which implies the line described by \mathbf{x} passes through the point P and
 when $t = 1$, $\mathbf{x} = \mathbf{q}$ which implies the line described by \mathbf{x} passes through the point Q .
 When $t = 0$, $\mathbf{x} = \mathbf{p} + 0(\mathbf{q} - \mathbf{p}) = \mathbf{p}$.
 When $t = 1$, $\mathbf{x} = \mathbf{p} + 1(\mathbf{q} - \mathbf{p}) = \mathbf{q}$.
 Therefore, $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$ describes the line segment \overline{PQ} as t varies from 0 to 1.
- (b) As shown in *Exploration: Vectors and Geometry* to find the midpoint of \overline{PQ} , we start at P and travel half the length of \overline{PQ} in the direction of the vector $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$. In the language of vectors, we add $\frac{1}{2}\overrightarrow{PQ} = \frac{1}{2}(\mathbf{q} - \mathbf{p})$ to the vector \mathbf{p} . So, the vector whose head is the midpoint of \overline{PQ} is $\mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p})$. Equating this to our expression for \mathbf{x} yields: $\mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p}) = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$. It follows immediately that $t = \frac{1}{2}$ and $\mathbf{x} = \mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p}) = \frac{1}{2}(\mathbf{p} + \mathbf{q})$.
- (c) Given $\mathbf{p} = [2, -3]$, $\mathbf{q} = [0, 1]$, and $\mathbf{x} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$, we have:
 $\mathbf{x} = \frac{1}{2}([0, 1] + [2, -3]) = [1, -1]$. So, the midpoint of \overline{PQ} is $(1, -1)$.
- (d) Given $\mathbf{p} = [1, 0, 1]$, $\mathbf{q} = [4, 1, -2]$, and $\mathbf{x} = \frac{1}{2}(\mathbf{p} + \mathbf{q})$, we have:
 $\mathbf{x} = \frac{1}{2}([1, 0, 1] + [4, 1, -2]) = [\frac{5}{2}, \frac{1}{2}, -\frac{1}{2}]$. So, the midpoint of \overline{PQ} is $(\frac{5}{2}, \frac{1}{2}, -\frac{1}{2})$.
- (e) We want two points (parameterized by t_1, t_2) to split \overline{PQ} into three equal segments. As in (c), the vectors whose heads are one and two-thirds of \overline{PQ} from P are
 $\mathbf{p} + \frac{1}{3}(\mathbf{q} - \mathbf{p})$ and $\mathbf{p} + \frac{2}{3}(\mathbf{q} - \mathbf{p})$.
 Equating these to our expression for \mathbf{x} yields:
 $\mathbf{p} + \frac{1}{3}(\mathbf{q} - \mathbf{p}) = \mathbf{p} + t_1(\mathbf{q} - \mathbf{p})$ and $\mathbf{p} + \frac{2}{3}(\mathbf{q} - \mathbf{p}) = \mathbf{p} + t_2(\mathbf{q} - \mathbf{p})$.
 It follows immediately that $t_1 = \frac{1}{3}$ and $t_2 = \frac{2}{3}$, so
 $\mathbf{x}_1 = \mathbf{p} + \frac{1}{3}(\mathbf{q} - \mathbf{p}) = \frac{1}{3}(2\mathbf{p} + \mathbf{q})$ and $\mathbf{x}_2 = \mathbf{p} + \frac{2}{3}(\mathbf{q} - \mathbf{p}) = \frac{1}{3}(\mathbf{p} + 2\mathbf{q})$.
 Given $\mathbf{p} = [2, -3]$, $\mathbf{q} = [0, 1]$, $\mathbf{x}_1 = \frac{1}{3}(2\mathbf{p} + \mathbf{q})$, and $\mathbf{x}_2 = \frac{1}{3}(\mathbf{p} + 2\mathbf{q})$, we have:
 $\mathbf{x}_1 = \frac{1}{3}(2[2, -3] + [0, 1]) = [\frac{4}{3}, -\frac{5}{3}]$ and $\mathbf{x}_2 = \frac{1}{3}([2, -3] + 2[0, 1]) = [\frac{2}{3}, -\frac{1}{3}]$.
 So, the two points that divide \overline{PQ} into three equal parts are $(\frac{4}{3}, -\frac{5}{3})$ and $(\frac{2}{3}, -\frac{1}{3})$.
- (f) Likewise $\mathbf{p} = [1, 0, -1]$, $\mathbf{q} = [4, 1, -2]$, $\mathbf{x}_1 = \frac{1}{3}(2\mathbf{p} + \mathbf{q})$, and $\mathbf{x}_2 = \frac{1}{3}(\mathbf{p} + 2\mathbf{q})$, yields:
 $\mathbf{x}_1 = \frac{1}{3}(2[1, 0, -1] + [4, 1, -2]) = [2, \frac{1}{3}, -\frac{4}{3}]$, $\mathbf{x}_2 = \frac{1}{3}([1, 0, -1] + 2[4, 1, -2]) = [3, \frac{2}{3}, -\frac{1}{3}]$.
 So, the two points that divide \overline{PQ} into three equal parts are $(2, \frac{1}{3}, -\frac{4}{3})$ and $(3, \frac{2}{3}, -\frac{1}{3})$.

17. Need to show ℓ_1 with slope m_1 is perpendicular to ℓ_2 with slope m_2 if and only if $m_1 m_2 = -1$.

By definition, one possible form of the general equation for ℓ_1 with slope m_1 is $-m_1 x + y = b_1$.

So, the normal vector for ℓ_1 is $\mathbf{n}_1 = \begin{bmatrix} -m_1 \\ 1 \end{bmatrix}$ and the normal vector for ℓ_2 is $\mathbf{n}_2 = \begin{bmatrix} -m_2 \\ 1 \end{bmatrix}$.

Now we note ℓ_1 is perpendicular to line ℓ_2 if and only if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$, so we have:

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \begin{bmatrix} -m_1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -m_2 \\ 1 \end{bmatrix} = m_1 m_2 + 1 = 0 \text{ which implies } m_1 m_2 = -1 \text{ as we were to show.}$$

18. Given \mathbf{d} is the direction vector of line ℓ and \mathbf{n} is the normal vector to the plane \mathcal{P} , we have:
 If \mathbf{d} and \mathbf{n} are orthogonal which implies $\mathbf{d} \cdot \mathbf{n} = 0$, then line ℓ is parallel to plane \mathcal{P} .
 If \mathbf{d} and \mathbf{n} are parallel which implies $\mathbf{d} = c\mathbf{n}$ (scalar multiples), then ℓ is perpendicular to \mathcal{P} .

(a) Since the general form of \mathcal{P} is $2x + 3y - z = 1$, its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \mathbf{d}$.

Since $\mathbf{d} = 1\mathbf{n}$, ℓ is perpendicular to \mathcal{P} .

(b) Since the general form of \mathcal{P} is $4x - y + 5z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$.

Since $\mathbf{d} \cdot \mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-1) + (-1) \cdot 5 = 0$, ℓ is parallel to \mathcal{P} .

(c) Since the general form of \mathcal{P} is $x - y - z = 3$, its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

Since $\mathbf{d} \cdot \mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 2 \cdot 1 + 3 \cdot (-1) + (-1) \cdot (-1) = 0$, ℓ is parallel to \mathcal{P} .

(d) Since the general form of \mathcal{P} is $4x + 6y - 2z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$.

Since $\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = \frac{1}{2}\mathbf{n}$, ℓ is perpendicular to \mathcal{P} .

19. Given \mathbf{n}_1 is the normal vector of \mathcal{P}_1 and \mathbf{n} is the normal vector of \mathcal{P} , we have:
 If \mathbf{n}_1 and \mathbf{n} are orthogonal which implies $\mathbf{n}_1 \cdot \mathbf{n} = 0$, then \mathcal{P}_1 is perpendicular to \mathcal{P} .
 If \mathbf{n}_1 and \mathbf{n} are parallel which implies $\mathbf{n}_1 = c\mathbf{n}$ (scalar multiples), then \mathcal{P}_1 is parallel to \mathcal{P} .

(a) Since the general form of \mathcal{P} is $2x + 3y - z = 1$, its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

Since $\mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = 4 \cdot 2 + (-1) \cdot 3 + 5 \cdot (-1) = 0$, \mathcal{P}_1 is perpendicular to \mathcal{P} .

(b) Since the general form of \mathcal{P} is $4x - y + 5z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$.

Since $\mathbf{n}_1 = 1\mathbf{n}$, \mathcal{P}_1 is parallel to \mathcal{P} .

(c) Since the general form of \mathcal{P} is $x - y - z = 3$, its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

Since $\mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 0$, \mathcal{P}_1 is perpendicular to \mathcal{P} .

(d) Since the general form of \mathcal{P} is $4x + 6y - 2z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$.

Since $\mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = 0$, \mathcal{P}_1 is perpendicular to \mathcal{P} .

20. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine \mathbf{p} and \mathbf{d} .

The general equation of the given line is $2x - 3y = 1$, so its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Our line is perpendicular to the given line, so it has direction vector $\mathbf{d} = \mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Furthermore, since our line passes through the point $P = (2, -1)$, we have $\mathbf{p} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

So, the vector form of the line perpendicular to $2x - 3y = 1$ through the point $P = (2, -1)$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

- 21.** Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine \mathbf{p} and \mathbf{d} .

The general equation of the given line is $2x - 3y = 1$, so its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Our line is parallel to the given line, so it has direction vector $\mathbf{d} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

This comes from the solution of Exercise 45 in Section 1.2: $\mathbf{n} \cdot \mathbf{d} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} b \\ -a \end{bmatrix} = ab - ab = 0$.

Continuing, since our line passes through the point $P = (2, -1)$, we have $\mathbf{p} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

So, the vector form of the line parallel to $2x - 3y = 1$ through the point $P = (2, -1)$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

- 22.** Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine \mathbf{p} and \mathbf{d} .

A line is perpendicular to a plane if its direction vector $\mathbf{d} = \mathbf{n}$ the normal vector of the plane.

The general equation of the given plane is $x - 3y + 2z = 5$, so its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$.

Therefore, the direction vector of our line is $\mathbf{d} = \mathbf{n} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$.

Furthermore, since our line passes through the point $P = (-1, 0, 3)$, we have $\mathbf{p} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$.

So, the vector form of the line perpendicular to $x - 3y + 2z = 5$ through $P = (-1, 0, 3)$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}.$$

- 23.** Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine \mathbf{p} and \mathbf{d} .

A line with parametric equations $\begin{matrix} x = a + et \\ y = b + ft \\ z = c + gt \end{matrix}$ has vector form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + t \begin{bmatrix} e \\ f \\ g \end{bmatrix}$.

Therefore, its direction vector is $\mathbf{d} = \begin{bmatrix} e \\ f \\ g \end{bmatrix}$. We use this key observation below.

Since the given line has parametric equations $\begin{matrix} x = 1 - t \\ y = 2 + 3t \\ z = -2 - t \end{matrix}$,

it has vector form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$. So, its direction vector is $\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$.

Since our line is parallel to the given line, its direction vector is also $\mathbf{d} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$.

Furthermore, since our line passes through the point $P = (-1, 0, 3)$, we have $\mathbf{p} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$.

So, the vector form of the line parallel to the given line through $P = (-1, 0, 3)$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

- 24.** Since the normal form is $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$, we use the given information to determine \mathbf{n} and \mathbf{p} .

A plane is parallel to a given plane if their normal vectors \mathbf{n} are equal.

Since the general form of the given plane is $6x - y + 2z = 3$, its normal vector is $\mathbf{n} = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$.

Since our plane is parallel to the given plane, its normal vector is also $\mathbf{n} = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$.

Furthermore, since our plane passes through the point $P = (0, -2, 5)$, we have $\mathbf{p} = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$.

So, the normal form of the plane parallel to $6x - y + 2z = 3$ through $P = (0, -2, 5)$ is

$$\begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 12.$$

25. Following Example 1.23, we will determine the general equations in two simple steps:
 First, we will use Figure 1.31 in Section 1.2 to find a normal vector \mathbf{n} and a point vector \mathbf{p} .
 Then we will substitute into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ and compute the dot products to find the equations.

(a) We start with \mathcal{P}_1 determined by the face of the cube in the yz -plane.

It is clear that a normal vector for \mathcal{P}_1 is $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or any vector parallel to the x -axis.

Also we see that \mathcal{P}_1 passes through the origin $P = (0, 0, 0)$, so we set $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ yields $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or $1 \cdot x + 0 \cdot y + 0 \cdot z = 0$.

So, the general equation for \mathcal{P}_1 determined by the face in the yz -plane is $x = 0$.

Likewise, the general equation for \mathcal{P}_2 determined by the face in the xz -plane is $y = 0$

and the general equation for \mathcal{P}_3 determined by the face in the xy -plane is $z = 0$.

We have found equations for the planes that pass through the origin.

We will use this information to find equations for the planes that pass through $(1, 1, 1)$.

We begin with \mathcal{P}_4 passing through the face parallel to the face in the yz -plane.

Since \mathcal{P}_4 is parallel to the face in the yz -plane, its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

As previously noted \mathcal{P}_4 passes through the point $P = (1, 1, 1)$, so we set $\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Substituting into $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ yields $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $1 \cdot x + 0 \cdot y + 0 \cdot z = 1$.

So, the general equation for \mathcal{P}_4 is $x = 1$.

Likewise, the general equations for \mathcal{P}_5 and \mathcal{P}_6 are $y = 1$ and $z = 1$ respectively.

- (b) We will use the given information to determine \mathbf{n} and \mathbf{p} , then compute $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$.

We begin by observing the two key facts that will enable us to find \mathbf{n} and \mathbf{p} :

Two planes \mathcal{P}_1 , \mathcal{P} are perpendicular if their normal vectors are orthogonal, so $\mathbf{n}_1 \cdot \mathbf{n} = 0$. Every vector \mathbf{u} in the plane \mathcal{P}_1 is orthogonal to its normal vector \mathbf{n}_1 , so $\mathbf{n}_1 \cdot \mathbf{u} = 0$.

Condition 1: Our plane must be perpendicular to the xy -plane, so $\mathbf{n}_1 \cdot \mathbf{n} = 0$. From (a),

$$\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ so } \mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \Rightarrow z = 0. \text{ So, } \mathbf{n}_1 \text{ is of the form } \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

Condition 2: \mathbf{n}_1 must be perpendicular to the vector \mathbf{u} from the origin to $(1, 1, 1)$.

$$\text{Since } \mathbf{u} = \begin{bmatrix} 1-0 \\ 1-0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ we have } \mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow x + y = 0 \Rightarrow y = -x.$$

$$\text{So, } \mathcal{P}_1 \text{ must be of the form } \mathbf{n}_1 = \begin{bmatrix} x \\ -x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}. \text{ Letting } x = 1 \text{ yields } \mathbf{n}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

As previously noted \mathcal{P}_1 passes through the origin $P = (0, 0, 0)$, so we set $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\text{Now } \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p} \text{ yields } \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } 1 \cdot x + (-1) \cdot y + 0 \cdot z = 0.$$

Therefore, the general equation for the plane perpendicular to the xy -plane and containing the diagonal from the origin to $(1, 1, 1)$ is $x - y = 0$.

- (c) As above, use $\mathbf{u} = [0, 1, 1]$ and $\mathbf{v} = [1, 0, 1]$ from Example 1.15 of Section 1.2 to find \mathbf{n} .

$$\text{From } \mathbf{n} \cdot \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \Rightarrow y + z = 0 \Rightarrow y = -z.$$

$$\text{From } \mathbf{n} \cdot \mathbf{v} = \begin{bmatrix} x \\ -z \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \Rightarrow x + z = 0 \Rightarrow x = -z.$$

$$\text{So, the normal vector } \mathbf{n} = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}. \text{ When } z = -1, \text{ we have } \mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

It is obvious the side diagonals pass through the origin $P = (0, 0, 0)$, so we set $\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\text{Now } \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p} \text{ yields } \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } 1 \cdot x + 1 \cdot y + (-1) \cdot z = 0.$$

The general equation for the plane containing the side diagonals is $x + y - z = 0$.

- 26.** Finding the distance between points A and B is equivalent to finding $d(\mathbf{a}, \mathbf{b})$.

Given $\mathbf{x} = [x, y, z]$, $\mathbf{p} = [1, 0, -2]$, and $\mathbf{q} = [5, 2, 4]$, we have the condition $d(\mathbf{x}, \mathbf{p}) = d(\mathbf{x}, \mathbf{q})$.

We simplify that equation to find the condition all points $X = (x, y, z)$ must satisfy.

$$d(\mathbf{x}, \mathbf{p}) = \sqrt{(x-1)^2 + (y-0)^2 + (z+2)^2} = \sqrt{(x-5)^2 + (y-2)^2 + (z-4)^2} = d(\mathbf{x}, \mathbf{q}).$$

Squaring both sides, we have: $(x-1)^2 + (y-0)^2 + (z+2)^2 = (x-5)^2 + (y-2)^2 + (z-4)^2 \Rightarrow$
 $(x^2 - 2x + 1) + y^2 + (z^2 + 4z + 4) = (x^2 - 10x + 25) + (y^2 - 4y + 4) + (z^2 - 8z + 16).$

Noting the squares cancel and combining the other like terms, we have: $8x + 4y + 12z = 40$.

Dividing both sides by 4, we see all points $X = (x, y, z)$ lie in the plane $2x + y + 3z = 10$.

- 27.** We will first follow Example 1.25, then use $d(Q, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$ and compare results.

Comparing $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we see ℓ has $P = (-1, 2)$ and $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

As suggested by Figure 1.63, we need to calculate the length of \overrightarrow{RQ} , where R is the point on ℓ at the foot of the perpendicular from Q .

Now if we let $\mathbf{v} = \overrightarrow{PQ}$, then $\overrightarrow{PR} = \text{proj}_{\mathbf{d}}(\mathbf{v})$ and $\overrightarrow{RQ} = \mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})$.

Step 1. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$

Step 2. $\text{proj}_{\mathbf{d}}(\mathbf{v}) = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}} \right) \mathbf{d} = \left(\frac{1 \cdot 3 + (-1) \cdot 0}{1 \cdot 1 + (-1) \cdot (-1)} \right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix}.$

Step 3. The vector we want is $\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v}) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}.$

Step 4. The distance $d(Q, \ell)$ from Q to ℓ is $\|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \left\| \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix} \right\|.$

$$\text{So Theorem 1.3(b) implies } \|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \frac{3}{2} \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \frac{3}{2} \sqrt{1+1} = \frac{3\sqrt{2}}{2}.$$

Now in order to calculate $d(Q, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$ we need to put ℓ into general form.

If $\mathbf{d} = \begin{bmatrix} a \\ b \end{bmatrix}$, then $\mathbf{n} = \begin{bmatrix} b \\ -a \end{bmatrix}$ because $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} b \\ -a \end{bmatrix} = 0$. For ℓ , $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

From $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ we have $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ so $x + y = 1$ and $a = b = c = 1$.

Furthermore, since $Q = (2, 2) = (x_0, y_0)$ we have $x_0 = y_0 = 2$.

$$\text{So } d(Q, \ell) = \frac{|2 + 2 - 1|}{\sqrt{1^2 + 1^2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \text{ exactly as we found by following Example 1.25.}$$

28. We will follow Example 1.25, then use $d(Q, \ell) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ and compare results.

Even though the formula $d(Q, \ell) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ was developed for planes, it can work for lines in \mathbb{R}^3 with the proper choice of $\mathbf{n} = [a, b, c]$.

Comparing $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$ to $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we see $P = (1, 1, 1)$ and $\mathbf{d} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$.

As suggested by Figure 1.63, we need to calculate the length of \overrightarrow{RQ} , where R is the point on ℓ at the foot of the perpendicular from Q .

Now if we let $\mathbf{v} = \overrightarrow{PQ}$, then $\overrightarrow{PR} = \text{proj}_{\mathbf{d}}(\mathbf{v})$ and $\overrightarrow{RQ} = \mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})$.

Step 1. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$.

Step 2. $\text{proj}_{\mathbf{d}}(\mathbf{v}) = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}} \right) \mathbf{d} = \left(\frac{(-2) \cdot (-1) + (3) \cdot (-1)}{(-2) \cdot (-2) + 3 \cdot 3} \right) \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/13 \\ 0 \\ -3/13 \end{bmatrix}$.

Step 3. The vector we want is $\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v}) = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 2/13 \\ 0 \\ -3/13 \end{bmatrix} = \begin{bmatrix} -15/13 \\ 0 \\ -10/13 \end{bmatrix}$.

Step 4. The distance $d(Q, \ell)$ from Q to ℓ is $\|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \left\| \begin{bmatrix} -15/13 \\ 0 \\ -10/13 \end{bmatrix} \right\|$.

So Theorem 1.3(b) implies $\|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \frac{5}{13} \left\| \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \right\| = \frac{5}{13} \sqrt{9+4} = \frac{5\sqrt{13}}{13}$.

Now in order to calculate $d(Q, \ell) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ we need to put ℓ into general form.

The appropriate choice of \mathbf{n} mentioned at the top follows from the following observation:

Vector $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$ found using Theorem 1.3(b) in Step 4 is orthogonal to \mathbf{d} , so let $\mathbf{n} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

From $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$, $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so $3x + 2z = 5$ and $a = 3, b = 0, c = 2, d = 5$.

Furthermore, since $Q = (0, 1, 0) = (x_0, y_0, z_0)$ we have $x_0 = 0, y_0 = 1$, and $z_0 = 0$.

So $d(Q, \ell) = \frac{|0 + 0 + 0 - 5|}{\sqrt{3^2 + 2^2}} = \frac{5}{\sqrt{13}} = \frac{5\sqrt{13}}{13}$ exactly as we found by following Example 1.25.

29. We will follow Example 1.26, then use $d(Q, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ and compare results.

By definition $ax + by + cz = d$ implies $\mathbf{n} = [a, b, c]$, so $x + y - z = 0$ implies $\mathbf{n} = [1, 1, -1]$.

As suggested by Figure 1.64, we need to calculate the length of $\overrightarrow{RQ} = \text{proj}_{\mathbf{n}}(\mathbf{v})$, where $\mathbf{v} = \overrightarrow{PQ}$.

Step 1. By trial and error, we find $P = (1, 0, 1)$ satisfies $x + y - z = 0$.

$$\text{Step 2. } \mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$$\text{Step 3. } \text{proj}_{\mathbf{n}}(\mathbf{v}) = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{n}} \right) \mathbf{n} = \left(\frac{1 \cdot 1 + 1 \cdot 2 - 1 \cdot 1}{1^2 + 1^2 + (-1)^2} \right) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}.$$

$$\text{Step 4. The distance from } Q \text{ to } \mathcal{P} \text{ is } \|\text{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix} \right\| = \frac{2}{3} \left\| \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\| = \frac{2\sqrt{3}}{3}.$$

Now for $d(Q, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ we need identify a, b, c, d , and x_0, y_0, z_0 .

Since $x + y - z = 0$, $a = 1, b = 1, c = -1, d = 0$. From $Q = (2, 2, 2)$, $x_0 = y_0 = z_0 = 2$.

$$\text{So } d(Q, \mathcal{P}) = \frac{|2 + 2 - 2 + 0|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \text{ as we found by following Example 1.26.}$$

30. We will follow Example 1.26, then use $d(Q, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ and compare results.

By definition $ax + by + cz = d$ implies $\mathbf{n} = [a, b, c]$, so $x - 2y + 2z = 1$ implies $\mathbf{n} = [1, -2, 2]$.

As suggested by Figure 1.64, we need to calculate the length of $\overrightarrow{RQ} = \text{proj}_{\mathbf{n}}(\mathbf{v})$, where $\mathbf{v} = \overrightarrow{PQ}$.

Step 1. By trial and error, we find $P = (1, 0, 0)$ satisfies $x - 2y + 2z = 1$.

$$\text{Step 2. } \mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\text{Step 3. } \text{proj}_{\mathbf{n}}(\mathbf{v}) = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{n}} \right) \mathbf{n} = \left(\frac{-1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0}{1^2 + (-2)^2 + 2^2} \right) \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = -\frac{1}{9} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 2/9 \\ -2/9 \end{bmatrix}.$$

$$\text{Step 4. The distance from } Q \text{ to } \mathcal{P} \text{ is } \|\text{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \begin{bmatrix} -1/9 \\ 2/9 \\ -2/9 \end{bmatrix} \right\| = \frac{1}{9} \left\| \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right\| = \frac{1}{3}.$$

Now for $d(Q, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ we need identify a, b, c, d , and x_0, y_0, z_0 .

Since $x - 2y + 2z = 1$, $a = 1, b = -2, c = 2, d = 1$. From $Q = (0, 0, 0)$, $x_0 = y_0 = z_0 = 0$.

$$\text{So } d(Q, \mathcal{P}) = \frac{|0 - 0 + 0 - 1|}{\sqrt{1^2 + (-2)^2 + 2^2}} = \frac{1}{\sqrt{9}} = \frac{1}{3} \text{ as we found by following Example 1.26.}$$

31. Similar to Example 1.25, Figure 1.63 suggests we let $\mathbf{v} = \overrightarrow{PQ}$, then $\mathbf{w} = \overrightarrow{PR} = \text{proj}_{\mathbf{d}}(\mathbf{v})$.

Comparing $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we see ℓ has $P = (-1, 2)$ and $\mathbf{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\text{Step 1. } \mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

$$\text{Step 2. } \mathbf{w} = \text{proj}_{\mathbf{d}}(\mathbf{v}) = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}} \right) \mathbf{d} = \left(\frac{1 \cdot 3 + (-1) \cdot 0}{1 \cdot 1 + (-1) \cdot (-1)} \right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix}.$$

$$\text{Step 3. So, } \mathbf{r} = \mathbf{p} + \overrightarrow{PR} = \mathbf{p} + \text{proj}_{\mathbf{d}}(\mathbf{v}) = \mathbf{p} + \mathbf{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

Therefore, the point R on ℓ that is closest to Q is $(\frac{1}{2}, \frac{1}{2})$.

32. Similar to Example 1.25, Figure 1.63 suggests we let $\mathbf{v} = \overrightarrow{PQ}$, then $\mathbf{w} = \overrightarrow{PR} = \text{proj}_{\mathbf{d}}(\mathbf{v})$.

Comparing $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$ to $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we see ℓ has $P = (1, 1, 1)$ and $\mathbf{d} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$.

$$\text{Step 1. } \mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

$$\text{Step 2. } \mathbf{w} = \text{proj}_{\mathbf{d}}(\mathbf{v}) = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}} \right) \mathbf{d} = \left(\frac{(-2) \cdot (-1) + 3 \cdot (-1)}{(-2)^2 + 3^2} \right) \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/13 \\ 0 \\ -3/13 \end{bmatrix}.$$

$$\text{Step 3. So, } \mathbf{r} = \mathbf{p} + \overrightarrow{PR} = \mathbf{p} + \text{proj}_{\mathbf{d}}(\mathbf{v}) = \mathbf{p} + \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2/13 \\ 0 \\ -3/13 \end{bmatrix} = \begin{bmatrix} 15/13 \\ 1 \\ 10/13 \end{bmatrix}.$$

Therefore, the point R on ℓ that is closest to Q is $(\frac{15}{13}, 1, \frac{10}{13})$.

33. Similar to Example 1.26, Figure 1.64 suggests we let $\mathbf{v} = \overrightarrow{PQ}$, then $\mathbf{w} = \overrightarrow{QR} = -\text{proj}_{\mathbf{n}}(\mathbf{v})$.

By definition $ax + by + cz = d$ implies $\mathbf{n} = [a, b, c]$, so $x + y - z = 0$ implies $\mathbf{n} = [1, 1, -1]$.

Step 1. By trial and error, we find $P = (1, 0, 1)$ satisfies $x + y - z = 0$.

$$\text{Step 2. } \mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

$$\text{Step 3. } \mathbf{w} = \text{proj}_{\mathbf{n}}(\mathbf{v}) = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} = \left(\frac{1 \cdot 1 + 1 \cdot 2 + (-1) \cdot 1}{1^2 + 1^2 + (-1)^2} \right) \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}.$$

$$\text{Step 4. So, } \mathbf{r} = \mathbf{p} + \overrightarrow{PQ} + \overrightarrow{QR} = \mathbf{p} + \mathbf{v} - \text{proj}_{\mathbf{n}}(\mathbf{v}) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4/3 \\ 8/3 \end{bmatrix}.$$

Therefore, the point R in \mathcal{P} that is closest to Q is $(\frac{4}{3}, \frac{4}{3}, \frac{8}{3})$.

34. Similar to Example 1.26, Figure 1.64 suggests we let $\mathbf{v} = \overrightarrow{PQ}$, then $\mathbf{w} = \overrightarrow{QR} = \text{proj}_{\mathbf{n}}(\mathbf{v})$.

By definition $ax + by + cz = d$ implies $\mathbf{n} = [a, b, c]$, so $x - 2y + 2z = 1$ implies $\mathbf{n} = [1, -2, 2]$.

Step 1. By trial and error, we find $P = (1, 0, 0)$ satisfies $x - 2y + 2z = 1$.

$$\text{Step 2. } \mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\text{Step 3. } \mathbf{w} = \text{proj}_{\mathbf{n}}(\mathbf{v}) = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} = \left(\frac{1 \cdot (-1)}{1^2 + (-2)^2 + 2^2} \right) \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 2/9 \\ -2/9 \end{bmatrix}.$$

$$\text{Step 4. So, } \mathbf{r} = \mathbf{p} + \overrightarrow{PQ} + \overrightarrow{QR} = \mathbf{p} + \mathbf{v} + \text{proj}_{\mathbf{n}}(\mathbf{v}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/9 \\ 2/9 \\ -2/9 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 2/9 \\ -2/9 \end{bmatrix}.$$

Therefore, the point R in \mathcal{P} that is closest to Q is $(-\frac{1}{9}, \frac{2}{9}, -\frac{2}{9})$.

- 35.** Since the given lines ℓ_1 and ℓ_2 are parallel, we can simply choose Q on ℓ_1 , P on ℓ_2 . Following Example 1.25, we have:

From ℓ_1 , $Q = (1, 1)$. From ℓ_2 , we have $P = (5, 4)$, $\mathbf{d} = [-2, 3]$, and $\mathbf{n} = [3, 2] = [a, b]$.

Step 1. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = [1, 1] - [5, 4] = [-4, -3]$.

Step 2. $\text{proj}_{\mathbf{d}}(\mathbf{v}) = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}} \right) \mathbf{d} = \left(\frac{(-2) \cdot (-4) + 3 \cdot (-3)}{(-2)^2 + 3^2} \right) \begin{bmatrix} -2 \\ 3 \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/13 \\ -3/13 \end{bmatrix}$.

Step 3. The vector we want is $\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v}) = \begin{bmatrix} -4 \\ -3 \end{bmatrix} - \begin{bmatrix} 2/13 \\ -3/13 \end{bmatrix} = \begin{bmatrix} -54/13 \\ -36/13 \end{bmatrix}$.

Step 4. The distance $d(Q, \ell_2)$ from ℓ_1 to ℓ_2 is $\|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \left\| \begin{bmatrix} -54/13 \\ -36/13 \end{bmatrix} \right\|$.

$$\text{So Theorem 1.3(b) implies } \|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \frac{18}{13} \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\| = \frac{18}{13} \sqrt{4+9} = \frac{18\sqrt{13}}{13}.$$

From $\mathbf{n} \cdot \mathbf{p} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \end{bmatrix} = 23$, $c = 23$. Since $Q = (1, 1) = (x_0, y_0)$, we have $x_0 = y_0 = 1$.

$$\text{Now compare: } d(\ell_1, \ell_2) = d(Q, \ell_2) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}} = \frac{|3 + 2 - 23|}{\sqrt{3^2 + 2^2}} = \frac{18}{\sqrt{13}} = \frac{18\sqrt{13}}{13}.$$

- 36.** Since the given lines ℓ_1 and ℓ_2 are parallel, we can simply choose Q on ℓ_1 , P on ℓ_2 . Following Example 1.25, we have:

From $[x, y, z] = [1, 0, -1] + t[1, 1, 1]$, we see ℓ_1 gives us $Q = (1, 0, -1)$.

From $[x, y, z] = [0, 1, 1] + t[1, 1, 1]$, we see ℓ_2 has $P = (0, 1, 1)$, $\mathbf{d} = [1, 1, 1]$, and $\mathbf{n} = [1, -2, 1]$.

Step 1. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = [1, 0, -1] - [0, 1, 1] = [1, -1, -2]$

Step 2. $\text{proj}_{\mathbf{d}}(\mathbf{v}) = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}} \right) \mathbf{d} = \left(\frac{1 \cdot 1 - 1 \cdot 1 - 1 \cdot 2}{1^2 + 1^2 + 1^2} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -\frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/3 \\ -2/3 \end{bmatrix}$.

Step 3. The vector we want is $\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v}) = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} -2/3 \\ -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ -1/3 \\ -4/3 \end{bmatrix}$.

Step 4. The distance $d(Q, \ell_2)$ from ℓ_1 to ℓ_2 is $\|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \left\| \begin{bmatrix} 5/3 \\ -1/3 \\ -4/3 \end{bmatrix} \right\|$.

$$\text{So Theorem 1.3(b) implies } \|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \frac{1}{3} \left\| \begin{bmatrix} 5 \\ -1 \\ -4 \end{bmatrix} \right\| = \frac{1}{3} \sqrt{25 + 1 + 16} = \frac{\sqrt{42}}{3}.$$

Set $\mathbf{n} = [5, -1, -4] = [a, b, c]$ we found in Step 4 because it is orthogonal to \mathbf{d} .

From $\mathbf{n} \cdot \mathbf{p} = d = [5, -1, -4] \cdot [0, 1, 1] = -5$. Since $Q = (1, 0, -1)$, $x_0 = 1$, $y_0 = 0$, $z_0 = -1$.

$$\text{Now compare: } d(Q, \ell_2) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|5 + 4 + 5|}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{14}{\sqrt{42}} = \frac{\sqrt{42}}{3}.$$

- 37.** Since the given planes \mathcal{P}_1 and \mathcal{P}_2 are parallel, we can simply choose Q in \mathcal{P}_1 , P in \mathcal{P}_2 . Following Example 1.26, we have:

Step 1. Since $2x + y - 2z = 0$, $Q = (0, 0, 0)$ is on \mathcal{P}_1 .

Since $2x + y - 2z = 5$, $P = (0, 5, 0)$ is on \mathcal{P}_2 and $\mathbf{n} = [2, 1, -2] = [a, b, c]$.

Step 2. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = [0, 0, 0] - [0, 5, 0] = [0, -5, 0]$

Step 3. $\text{proj}_{\mathbf{n}}(\mathbf{v}) = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} = \left(\frac{2 \cdot 0 - 1 \cdot 5 - 2 \cdot 0}{2^2 + 1^2 + (-2)^2} \right) \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = -\frac{5}{9} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -10/9 \\ -5/9 \\ 10/9 \end{bmatrix}.$

Step 4. The distance $d(Q, \mathcal{P}_2)$ from \mathcal{P}_1 to \mathcal{P}_2 is $\|\mathbf{v} - \text{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \begin{bmatrix} -10/9 \\ -5/9 \\ 10/9 \end{bmatrix} \right\|.$

$$\text{So Theorem 1.3(b) implies } \|\text{proj}_{\mathbf{n}}(\mathbf{v})\| = \frac{5}{9} \left\| \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\| = \frac{5}{9} \sqrt{4 + 1 + 4} = \frac{5}{3}.$$

From $\mathbf{n} \cdot \mathbf{p} = d = [2, 1, -2] \cdot [0, 5, 0] = 5$. Since $Q = (0, 0, 0)$, $x_0 = y_0 = z_0 = 0$.

$$\text{Now compare: } d(Q, \mathcal{P}_2) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|0 + 0 + 0 - 5|}{\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{5}{\sqrt{9}} = \frac{5}{3}.$$

- 38.** Since the given planes \mathcal{P}_1 and \mathcal{P}_2 are parallel, we can simply choose Q in \mathcal{P}_1 , P in \mathcal{P}_2 . Following Example 1.26, we have:

Step 1. Since $x + y + z = 1$, $Q = (1, 0, 0)$ is on \mathcal{P}_1 .

Since $x + y + z = 3$, $P = (3, 0, 0)$ is on \mathcal{P}_2 and $\mathbf{n} = [1, 1, 1] = [a, b, c]$.

Step 2. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = [1, 0, 0] - [3, 0, 0] = [-2, 0, 0]$

Step 3. $\text{proj}_{\mathbf{n}}(\mathbf{v}) = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} = \left(\frac{1 \cdot (-2) + 1 \cdot 0 + 1 \cdot 0}{1^2 + 1^2 + 1^2} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -\frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/3 \\ -2/3 \end{bmatrix}.$

Step 4. The distance $d(Q, \mathcal{P}_2)$ from \mathcal{P}_1 to \mathcal{P}_2 is $\|\mathbf{v} - \text{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \begin{bmatrix} -2/3 \\ -2/3 \\ -2/3 \end{bmatrix} \right\|.$

$$\text{So Theorem 1.3(b) implies } \|\text{proj}_{\mathbf{n}}(\mathbf{v})\| = \frac{2}{3} \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \frac{2}{3} \sqrt{1 + 1 + 1} = \frac{2\sqrt{3}}{3}.$$

From $\mathbf{n} \cdot \mathbf{p} = d = [1, 1, 1] \cdot [3, 0, 0] = 3$. Since $Q = (1, 0, 0)$, $x_0 = 1$, $y_0 = 0$, $z_0 = 0$.

$$\text{Now compare: } d(Q, \mathcal{P}_2) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|1 + 0 + 0 - 3|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

39. Will show $d(B, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$, where $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\mathbf{n} \cdot \mathbf{a} = c$, and $B = (x_0, y_0)$.

Step 1. From Figure 1.61, we see $d(B, \ell) = \|\text{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} \right\| = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|}$.

Step 2. Since $\mathbf{v} = \mathbf{b} - \mathbf{a}$, $\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - c = ax_0 + by_0 - c$.

Step 3. So, $d(B, \ell) = \|\text{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} \right\| = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$.

40. Will show $d(B, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$, where $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\mathbf{n} \cdot \mathbf{a} = d$, and $B = (x_0, y_0, z_0)$.

Step 1. From Figure 1.62, we see $d(B, \mathcal{P}) = \|\text{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} \right\| = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|}$.

Step 2. Since $\mathbf{v} = \mathbf{b} - \mathbf{a}$, $\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - d = ax_0 + by_0 + cz_0 - d$.

Step 3. So, $d(B, \mathcal{P}) = \|\text{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} \right\| = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

41. We will apply the formula from Exercise 39, $d(B, \ell) = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|}$.

Step 1. We select $B = (x_0, y_0)$ on ℓ_1 so that $\mathbf{n} \cdot \mathbf{b} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = ax_0 + by_0 = c_1$.

Step 2. We select A on ℓ_2 so that $\mathbf{n} \cdot \mathbf{a} = c_2$.

Step 3. Set $\mathbf{v} = \mathbf{b} - \mathbf{a}$, then $d(B, \mathcal{P}) = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a}|}{\|\mathbf{n}\|} = \frac{|c_1 - c_2|}{\|\mathbf{n}\|}$.

42. We will apply the formula from Exercise 40, $d(B, \mathcal{P}) = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|}$.

Step 1. We select $B = (x_0, y_0, z_0)$ on \mathcal{P}_1 so that $\mathbf{n} \cdot \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = ax_0 + by_0 + cz_0 = d_1$.

Step 2. We select A on \mathcal{P}_2 so that $\mathbf{n} \cdot \mathbf{a} = d_2$.

Step 3. Set $\mathbf{v} = \mathbf{b} - \mathbf{a}$, then $d(B, \mathcal{P}) = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a}|}{\|\mathbf{n}\|} = \frac{|d_1 - d_2|}{\|\mathbf{n}\|}$.

43. As in Example 1.14 of Section 1.2, we note that $\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

So, given two planes \mathcal{P}_1 with \mathbf{n}_1 and \mathcal{P}_2 with \mathbf{n}_2 , we have $\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|}$.

Step 1. Since \mathcal{P}_1 has equation $x + y + z = 0$, $\mathbf{n}_1 = [1, 1, 1]$.

Since \mathcal{P}_2 has equation $2x + y - 2z = 0$, $\mathbf{n}_2 = [2, 1, -2]$.

Step 2. Therefore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = [1, 1, 1] \cdot [2, 1, -2] = 1 \cdot 2 + 1 \cdot 1 - 1 \cdot 2 = 1$,

$\|\mathbf{n}_1\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, and $\|\mathbf{n}_2\| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$.

Step 3. So $\cos \theta = \frac{1}{3\sqrt{3}}$ and $\theta = \cos^{-1} \left(\frac{1}{3\sqrt{3}} \right) \approx 78.9^\circ$.

44. As in Example 1.14 of Section 1.2, we note that $\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

So, given two planes \mathcal{P}_1 with \mathbf{n}_1 and \mathcal{P}_2 with \mathbf{n}_2 , we have $\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|}$.

Step 1. Since \mathcal{P}_1 has equation $3x - y + 2z = 5$, $\mathbf{n}_1 = [3, -1, 2]$.

Since \mathcal{P}_2 has equation $x + 4y - z = 2$, $\mathbf{n}_2 = [1, 4, -1]$.

Step 2. Therefore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = [3, -1, 2] \cdot [1, 4, -1] = 3 \cdot 1 - 1 \cdot 4 - 2 \cdot 1 = -3$,

$\|\mathbf{n}_1\| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$, and $\|\mathbf{n}_2\| = \sqrt{1^2 + 4^2 + (-1)^2} = 3\sqrt{2}$.

Step 3. So $\cos \theta = -\frac{3}{3\sqrt{28}} = -\frac{\sqrt{7}}{14}$ and $\theta = \cos^{-1} \left(-\frac{\sqrt{7}}{14} \right) \approx 100.9^\circ$.

Since we need an acute angle, we take $180^\circ - 100.9^\circ = 79.1^\circ$.

45. As in Example 1.14 of Section 1.2, we note that $\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

So, given \mathcal{P} with \mathbf{n} and ℓ with \mathbf{d} , we have $\cos \theta = \frac{|\mathbf{n} \cdot \mathbf{d}|}{\|\mathbf{n}\| \|\mathbf{d}\|}$.

Step 1. To show \mathcal{P} and ℓ intersect, we note:

$$x + y + 2z = (2 + t) + (1 - 2t) + 2(3 + t) = 9 + t = 0 \text{ implies } t = -9.$$

So, \mathcal{P} and ℓ intersect at the point $[2 + (-9), 1 - 2(-9), 3 + (-9)] = [-7, 19, -6]$.

Step 2. Since \mathcal{P} has equation $x + y + 2z = 0$, $\mathbf{n} = [1, 1, 2]$.

$$\text{Given } \begin{matrix} x = 2 + t \\ y = 1 - 2t \\ z = 3 + t \end{matrix}, \ell \text{ satisfies } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \text{ So, } \mathbf{d} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Step 3. Therefore, $\mathbf{n} \cdot \mathbf{d} = [1, 1, 2] \cdot [1, -2, 1] = 1 \cdot 1 - 1 \cdot 2 + 2 \cdot 1 = 1$,

$$\|\mathbf{n}\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}, \text{ and } \|\mathbf{d}\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}.$$

Step 4. So $\cos \theta = \frac{1}{\sqrt{6}\sqrt{6}} = \frac{1}{6}$ and $\theta = \cos^{-1}\left(\frac{1}{6}\right) \approx 80.4^\circ$.

46. As in Example 1.14 of Section 1.2, we note that $\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{\|\mathbf{u}\| \|\mathbf{v}\|}$.

So, given \mathcal{P} with \mathbf{n} and ℓ with \mathbf{d} , we have $\cos \theta = \frac{|\mathbf{n} \cdot \mathbf{d}|}{\|\mathbf{n}\| \|\mathbf{d}\|}$.

Step 1. To show \mathcal{P} and ℓ intersect, we note:

$$4x - y - z = 4(t) - (1 + 2t) - (2 + 3t) = -3 - t = 6 \text{ implies } t = -9.$$

So, \mathcal{P} and ℓ intersect at the point $[(-9), 1 + 2(-9), 2 + 3(-9)] = [-9, -17, -25]$.

Step 2. Since \mathcal{P} has equation $4x - y - z = 0$, $\mathbf{n} = [4, -1, -1]$.

$$\text{Given } \begin{matrix} x = t \\ y = 1 + 2t \\ z = 2 + 3t \end{matrix}, \ell \text{ satisfies } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}. \text{ So, } \mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Step 3. Therefore, $\mathbf{n} \cdot \mathbf{d} = [4, -1, -1] \cdot [1, 2, 3] = 4 \cdot 1 - 1 \cdot 2 - 1 \cdot 3 = -1$,

$$\|\mathbf{n}\| = \sqrt{4^2 + (-1)^2 + (-1)^2} = 3\sqrt{2}, \text{ and } \|\mathbf{d}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$$

Step 4. So $\cos \theta = -\frac{1}{3\sqrt{28}} = -\frac{\sqrt{7}}{42}$ and $\theta = \cos^{-1}\left(-\frac{\sqrt{7}}{42}\right) \approx 93.6^\circ$.

Since we need an acute angle, we take $180^\circ - 93.6^\circ = 86.4^\circ$.

47. Will find an expression for \mathbf{p} in terms of \mathbf{v} and \mathbf{n} given \mathbf{n} is orthogonal to \mathbf{p} , that is $\mathbf{p} \cdot \mathbf{n} = 0$.

Step 1. We solve for c starting from the given equation $\mathbf{p} = \mathbf{v} - c\mathbf{n}$.

$$\begin{array}{ll}
 \mathbf{p} = \mathbf{v} - c\mathbf{n} & \Rightarrow \\
 c\mathbf{n} = \mathbf{v} - \mathbf{p} & \text{Given} \\
 (c\mathbf{n}) \cdot \mathbf{n} = (\mathbf{v} - \mathbf{p}) \cdot \mathbf{n} & \Rightarrow \text{By } x = y - z \text{ implies } z = y - x \\
 c(\mathbf{n} \cdot \mathbf{n}) = \mathbf{v} \cdot \mathbf{n} - \mathbf{p} \cdot \mathbf{n} & \text{Taking the dot product of } \mathbf{n} \text{ with both sides.} \\
 c(\mathbf{n} \cdot \mathbf{n}) = \mathbf{n} \cdot \mathbf{v} & \text{By properties of the dot product} \\
 & \text{By } \mathbf{v} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{v} \text{ and } \mathbf{p} \cdot \mathbf{n} = 0 \\
 c = \frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} & \text{Dividing both sides by } \mathbf{n} \cdot \mathbf{n} \text{ (a scalar)}
 \end{array}$$

Note: Figure 1.66 also shows $c\mathbf{n} = \text{proj}_{\mathbf{n}}(\mathbf{v}) = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n}$ which implies $c = \frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}$.

Step 2. Letting $c = \frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}$ in $\mathbf{p} = \mathbf{v} - c\mathbf{n}$, we have $\mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n}$.

48. We will find $\mathbf{p} = \text{proj}_{\mathcal{P}}(\mathbf{v})$ using $\mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n}$ and $\mathbf{v} = [1, 0, -2]$.

- (a) Step 1. Since $x + y + z = 0$, $\mathbf{n} = [1, 1, 1]$.

$$\text{So } \mathbf{n} \cdot \mathbf{v} = [1, 1, 1] \cdot [1, 0, -2] = 1 \cdot 1 + 1 \cdot 0 - 1 \cdot 2 = -1,$$

$$\mathbf{n} \cdot \mathbf{n} = [1, 1, 1] \cdot [1, 1, 1] = 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 3, \text{ and } c = \frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} = -\frac{1}{3}.$$

$$\text{Step 2. So } \mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -7/3 \end{bmatrix}.$$

- (b) Step 1. Since $3x - y + z = 0$, $\mathbf{n} = [3, -1, 1]$.

$$\text{So } \mathbf{n} \cdot \mathbf{v} = [3, -1, 1] \cdot [1, 0, -2] = 3 \cdot 1 - 1 \cdot 0 - 1 \cdot 2 = 1,$$

$$\mathbf{n} \cdot \mathbf{n} = [3, -1, 1] \cdot [3, -1, 1] = 3 \cdot 3 + 1 \cdot 1 + 1 \cdot 1 = 11, \text{ and } c = \frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} = \frac{1}{11}.$$

$$\text{Step 2. So } \mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{11} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 3/11 \\ -1/11 \\ 1/11 \end{bmatrix} = \begin{bmatrix} 10/11 \\ -1/11 \\ -23/11 \end{bmatrix}.$$

- (c) Step 1. Since $3x - 2z = 0$, $\mathbf{n} = [3, 0, -2]$.

$$\text{So } \mathbf{n} \cdot \mathbf{v} = [3, 0, -2] \cdot [1, 0, -2] = 3 \cdot 1 - 0 \cdot 0 + 2 \cdot 2 = 7,$$

$$\mathbf{n} \cdot \mathbf{n} = [3, 0, -2] \cdot [3, 0, -2] = 3 \cdot 3 + 0 \cdot 0 + 2 \cdot 2 = 13, \text{ and } c = \frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} = \frac{7}{13}.$$

$$\text{Step 2. So } \mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{7}{13} \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 21/13 \\ 0 \\ 14/13 \end{bmatrix} = \begin{bmatrix} 2/13 \\ -14/13 \\ -30/13 \end{bmatrix}.$$

- (d) Step 1. Since $2x - 3y + z = 0$, $\mathbf{n} = [2, -3, 1]$.

$$\text{So } \mathbf{n} \cdot \mathbf{v} = [2, -3, 1] \cdot [1, 0, -2] = 2 \cdot 1 - 3 \cdot 0 - 1 \cdot 2 = 0.$$

$$\text{Step 2. So } \mathbf{p} = \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

Note: The projection is the vector itself because the vector is parallel to the plane. That is equivalent to the vector being orthogonal to the normal of the plane.

Exploration: The Cross Product

Since Explorations are self-contained, only solutions will be provided.

$$1. \quad (a) \quad \mathbf{u} \times \mathbf{v} = \begin{bmatrix} 1(2) - 1(-1) \\ 1(3) - 0(2) \\ 0(-1) - 1(3) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}. \quad (b) \quad \mathbf{u} \times \mathbf{v} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}.$$

$$(c) \quad \mathbf{u} \times \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (d) \quad \mathbf{u} \times \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

$$2. \quad \mathbf{e}_1 \times \mathbf{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 - 0(1) \\ 0 - 0 \\ 1(1) - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{e}_3,$$

$$\mathbf{e}_2 \times \mathbf{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{e}_1, \text{ and } \mathbf{e}_3 \times \mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{e}_2.$$

3. Two vectors are orthogonal if their dot product equals zero. Check:

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} &= \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\ &= (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 \\ &= (u_2v_3u_1 - u_1v_3u_2) + (u_3v_1u_2 - u_2v_1u_3) + (u_1v_2u_3 - u_3v_2u_1) = 0. \end{aligned}$$

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} &= \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &= (u_2v_3 - u_3v_2)v_1 + (u_3v_1 - u_1v_3)v_2 + (u_1v_2 - u_2v_1)v_3 \\ &= (u_2v_3v_1 - u_2v_1v_3) + (u_3v_1v_2 - u_3v_1v_2) + (u_1v_2v_3 - u_1v_3v_2) = 0. \end{aligned}$$

4. (a) A vector normal to the plane is $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$.

Thus, the normal form for the equation of this plane is $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p} \Leftrightarrow$

$$\begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Leftrightarrow 3x + 3y - 3z = 9 \Leftrightarrow x + y - z = 3.$$

(b) Two vectors on the plane are $\mathbf{u} = \overrightarrow{PQ} = [2, 1, 1]$ and $\mathbf{v} = \overrightarrow{PR} = [1, 3, -2]$,

so a vector normal to the plane is $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ 5 \end{bmatrix}$.

Thus, the normal form for the equation of this plane is $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p} \Leftrightarrow$

$$\begin{bmatrix} -5 \\ 5 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Leftrightarrow -5x + 5y + 5z = 0 \Leftrightarrow -x + y + z = 0.$$

5. (a) $\mathbf{v} \times \mathbf{u} = \begin{bmatrix} v_2u_3 - v_3u_2 \\ v_3u_1 - v_1u_3 \\ v_1u_2 - v_2u_1 \end{bmatrix} = - \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = -(\mathbf{u} \times \mathbf{v}).$

(b) $\mathbf{u} \times \mathbf{0} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_2(0) - u_3(0) \\ u_3(0) - u_1(0) \\ u_1(0) - u_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}.$

(c) $\mathbf{u} \times \mathbf{u} = \begin{bmatrix} u_2u_3 - u_3u_2 \\ u_3u_1 - u_1u_3 \\ u_1u_2 - u_2u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}.$

(d) $\mathbf{u} \times k\mathbf{v} = \begin{bmatrix} u_2kv_3 - u_3kv_2 \\ u_3kv_1 - u_1kv_3 \\ u_1kv_2 - u_2kv_1 \end{bmatrix} = k \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = k(\mathbf{u} \times \mathbf{v}).$

(e) $\mathbf{u} \times k\mathbf{u} = k(\mathbf{u} \times \mathbf{u}) = k(\mathbf{0}) = \mathbf{0}.$

(f) We compute the cross product as follows:

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \begin{bmatrix} u_2(v_3 + w_3) - u_3(v_2 + w_2) \\ u_3(v_1 + w_1) - u_1(v_3 + w_3) \\ u_1(v_2 + w_2) - u_2(v_1 + w_1) \end{bmatrix} \\ &= \begin{bmatrix} (u_2v_3 - u_3v_2) + (u_2w_3 - u_3w_2) \\ (u_3v_1 - u_1v_3) + (u_3w_1 - u_1w_3) \\ (u_1v_2 - u_2v_1) + (u_1w_2 - u_2w_1) \end{bmatrix} \\ &= \begin{bmatrix} (u_2v_3 - u_3v_2) \\ (u_3v_1 - u_1v_3) \\ (u_1v_2 - u_2v_1) \end{bmatrix} + \begin{bmatrix} (u_2w_3 - u_3w_2) \\ (u_3w_1 - u_1w_3) \\ (u_1w_2 - u_2w_1) \end{bmatrix} = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}. \end{aligned}$$

6. (a) We compute the cross product as follows:

$$\begin{aligned}
 \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \\
 &= u_1 v_2 w_3 - u_1 v_3 w_2 + u_2 v_3 w_1 - u_2 v_1 w_3 + u_3 v_1 w_2 - u_3 v_2 w_1 \\
 &= (u_2 v_3 - u_3 v_2) w_1 + (u_3 v_1 - u_1 v_3) w_2 + (u_1 v_2 - u_2 v_1) w_3 \\
 &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.
 \end{aligned}$$

- (b) We compute the cross product as follows:

$$\begin{aligned}
 \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} = \begin{bmatrix} u_2 (v_1 w_2 - v_2 w_1) - u_3 (v_3 w_1 - v_1 w_3) \\ u_3 (v_2 w_3 - v_3 w_2) - u_1 (v_1 w_2 - v_2 w_1) \\ u_1 (v_3 w_1 - v_1 w_3) - u_2 (v_2 w_3 - v_3 w_2) \end{bmatrix} \\
 &= \begin{bmatrix} (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (u_1 v_1 + u_2 v_2 + u_3 v_3) w_1 \\ (u_1 w_1 + u_2 w_2 + u_3 w_3) v_2 - (u_1 v_1 + u_2 v_2 + u_3 v_3) w_2 \\ (u_1 w_1 + u_2 w_2 + u_3 w_3) v_3 - (u_1 v_1 + u_2 v_2 + u_3 v_3) w_3 \end{bmatrix} \\
 &= (u_1 w_1 + u_2 w_2 + u_3 w_3) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - (u_1 v_1 + u_2 v_2 + u_3 v_3) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\
 &= (\mathbf{u} \cdot \mathbf{w}) \mathbf{v} - (\mathbf{u} \cdot \mathbf{v}) \mathbf{w}.
 \end{aligned}$$

- (c) We compute the cross product as follows:

$$\begin{aligned}
 \|\mathbf{u} \times \mathbf{v}\|^2 &= \left\| \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \right\|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_1)^2 \\
 &= (u_1^2 + u_2^2 + u_3^2) (v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\
 &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2.
 \end{aligned}$$

7. 2: $\mathbf{e}_1 \times (\mathbf{e}_2 \times \mathbf{e}_3) = (\mathbf{e}_1 \cdot \mathbf{e}_3) \mathbf{e}_2 - (\mathbf{e}_1 \cdot \mathbf{e}_2) \mathbf{e}_3$ [by 6(b)] $= 0$ since $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$.

Thus, since the \mathbf{e}_i have length 1, we must have $\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3$, by 6(c).

Show $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$ and $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ by cyclically permuting the indices.

3: $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{u}) \cdot \mathbf{v}$ [by 10(a)] $= \mathbf{0} \cdot \mathbf{v}$ [by 9(c)] $= \mathbf{0}$, so \mathbf{u} is orthogonal to $\mathbf{u} \times \mathbf{v}$.

Similarly, $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (-1) (\mathbf{v} \times \mathbf{u}) = (\mathbf{v} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{0}$. So \mathbf{v} is orthogonal to $\mathbf{u} \times \mathbf{v}$.

8. (a) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2 \theta$
 $= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 (1 - \cos^2 \theta) = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2 \theta \Rightarrow$
 $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta.$

- (b) From Figure 1, and recalling the area of a triangle is $A = \frac{1}{2}(\text{base})(\text{height})$, we have:
 $A = \frac{1}{2} (\|\mathbf{u}\|) (\|\mathbf{v}\| \sin \theta)$. But $\|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$, so $A = \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\|$.

- (c) Let $\mathbf{u} = \overrightarrow{AB} = [1, -1, -1]$ and $\mathbf{v} = \overrightarrow{AC} = [4, -3, 2]$. Then (b) \Rightarrow
The area of triangle ABC is $A = \frac{1}{2} \|[1, -1, -1] \times [4, -3, 2]\| = \frac{1}{2} \sqrt{62}$.

1.4 Applications

1. Following Example 1.34, we find the resultant force, \mathbf{r} , using \mathbf{f}_1 and \mathbf{f}_2 .

The magnitude of \mathbf{r} is $\|\mathbf{r}\| = \sqrt{\|\mathbf{f}_1\|^2 + \|\mathbf{f}_2\|^2} = \sqrt{12^2 + 5^2} = 13$.

The angle θ between \mathbf{r} and east (the direction of \mathbf{f}_2) is $\theta = \tan^{-1}\left(\frac{12}{5}\right) \approx 67.4^\circ$.

The direction of the resultant is closer to north than east. Why does this make sense?

2. Following Example 1.34, we find the resultant force, \mathbf{r} , using \mathbf{f}_1 and \mathbf{f}_2 .

The magnitude of \mathbf{r} is $\|\mathbf{r}\| = \sqrt{\|\mathbf{f}_1\|^2 + \|\mathbf{f}_2\|^2} = \sqrt{15^2 + 20^2} = 25$.

The angle θ between \mathbf{r} and west (the direction of \mathbf{f}_1) is $\theta = \tan^{-1}\left(\frac{20}{15}\right) \approx 53.1^\circ$.

The direction of the resultant is closer to south than west. Why does this make sense?

3. Similar to Example 1.34, we find the resultant force, \mathbf{r} , using \mathbf{f}_1 and \mathbf{f}_2 .

If we let $\mathbf{f}_1 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$, then $\mathbf{f}_2 = \begin{bmatrix} 8 \cos(60^\circ) \\ 8 \sin(60^\circ) \end{bmatrix} = \begin{bmatrix} 4 \\ 4\sqrt{3} \end{bmatrix}$.

So, the resultant force is $\mathbf{r} = \mathbf{f}_1 + \mathbf{f}_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 4\sqrt{3} \end{bmatrix} = \begin{bmatrix} 12 \\ 4\sqrt{3} \end{bmatrix}$.

The magnitude of \mathbf{r} is $\|\mathbf{r}\| = \sqrt{12^2 + (4\sqrt{3})^2} = 8\sqrt{3}$.

The angle formed by \mathbf{r} and \mathbf{f}_1 is $\theta = \tan^{-1}\left(\frac{4\sqrt{3}}{12}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$.

The resultant force also forms a 30° with \mathbf{f}_2 . Why does this make sense?

4. Similar to Example 1.34, we find the resultant force, \mathbf{r} , using \mathbf{f}_1 and \mathbf{f}_2 .

If we let $\mathbf{f}_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, then $\mathbf{f}_2 = \begin{bmatrix} 6 \cos(135^\circ) \\ 6 \sin(135^\circ) \end{bmatrix} = \begin{bmatrix} -3\sqrt{2} \\ 3\sqrt{2} \end{bmatrix}$.

So, the resultant force is $\mathbf{r} = \mathbf{f}_1 + \mathbf{f}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3\sqrt{2} \\ 3\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 - 3\sqrt{2} \\ 3\sqrt{2} \end{bmatrix}$.

The magnitude of \mathbf{r} is $\|\mathbf{r}\| = \sqrt{(4 - 3\sqrt{2})^2 + (3\sqrt{2})^2} = \sqrt{52 - 24\sqrt{2}} \approx \sqrt{18} \approx 4.24$.

The angle formed by \mathbf{r} and \mathbf{f}_1 is $\theta = \tan^{-1} \left(\frac{3\sqrt{2}}{4 - 3\sqrt{2}} \right) \approx 93.3^\circ$.

More than one angle has a tangent of $\frac{3\sqrt{2}}{4 - 3\sqrt{2}}$. How do you determine the correct one?

5. Similar to Example 1.34, we find the resultant force, \mathbf{r} , using \mathbf{f}_1 , \mathbf{f}_2 , and \mathbf{f}_3 .

If we let $\mathbf{f}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ then $\mathbf{f}_2 = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$, and $\mathbf{f}_3 = \begin{bmatrix} 4 \cos(60^\circ) \\ 4 \sin(60^\circ) \end{bmatrix} = \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix}$.

So, the resultant force is $\mathbf{r} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -6 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2\sqrt{3} \end{bmatrix} = \begin{bmatrix} -2 \\ 2\sqrt{3} \end{bmatrix}$.

The magnitude of \mathbf{r} is $\|\mathbf{r}\| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$.

The angle formed by \mathbf{r} and \mathbf{f}_1 is $\theta = \tan^{-1} \left(\frac{2\sqrt{3}}{-2} \right) = 120^\circ$.

More than one angle has a tangent of $\frac{2\sqrt{3}}{-2}$. How do you determine the correct one?

6. Similar to Example 1.34, we find the resultant force, \mathbf{r} , using \mathbf{f}_1 , \mathbf{f}_2 , \mathbf{f}_3 , and \mathbf{f}_4 .

If we let $\mathbf{f}_1 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$, then $\mathbf{f}_2 = \begin{bmatrix} 0 \\ 13 \end{bmatrix}$, $\mathbf{f}_3 = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$, and $\mathbf{f}_4 = \begin{bmatrix} 0 \\ -8 \end{bmatrix}$.

So, the resultant force is $\mathbf{r} = \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \mathbf{f}_4 = \begin{bmatrix} 10 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 13 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -8 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

The magnitude of \mathbf{r} is $\|\mathbf{r}\| = \sqrt{5^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$.

The angle formed by \mathbf{r} and \mathbf{f}_1 is $\theta = \tan^{-1} \left(\frac{5}{5} \right) = 45^\circ$.

7. Following Example 1.35, we resolve \mathbf{f} into \mathbf{f}_x and \mathbf{f}_y .

Specifically, if we let $\mathbf{f}_x = \begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\mathbf{f}_y = \begin{bmatrix} 0 \\ y \end{bmatrix}$, then $\mathbf{f} = \mathbf{f}_x + \mathbf{f}_y = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Let the angle formed by \mathbf{f} and \mathbf{f}_x be 60° .

Then since $\|\mathbf{f}\| = 10 = \sqrt{x^2 + y^2}$, we have $\cos^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) = \cos^{-1}\left(\frac{x}{10}\right)$.

That implies $\frac{x}{10} = \cos 60^\circ = \frac{1}{2}$. So, $x = \frac{1}{2} \cdot 10 = 5$ and $y = \sqrt{10^2 - 5^2} = \sqrt{75} = 5\sqrt{3}$.

Therefore, \mathbf{f} resolves into $\mathbf{f}_x = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $\mathbf{f}_y = \begin{bmatrix} 0 \\ 5\sqrt{3} \end{bmatrix}$.

Why can we choose $\mathbf{f}_x = \begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\mathbf{f}_y = \begin{bmatrix} 0 \\ y \end{bmatrix}$?

Hint: We are told \mathbf{f} is resolved into forces that are perpendicular to each other.

8. Similar to Example 1.35, we resolve \mathbf{f} into \mathbf{f}_r and \mathbf{f}_p where \mathbf{f}_r is parallel to the ramp and \mathbf{f}_p is perpendicular to it.

Specifically, if we let $\mathbf{f}_r = \begin{bmatrix} r \\ 0 \end{bmatrix}$ and $\mathbf{f}_p = \begin{bmatrix} 0 \\ p \end{bmatrix}$, then $\mathbf{f} = \mathbf{f}_r + \mathbf{f}_p = \begin{bmatrix} r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ p \end{bmatrix} = \begin{bmatrix} r \\ p \end{bmatrix}$.

Note that the angle formed by \mathbf{f} and \mathbf{f}_r is 60° . Why?

Then since $\|\mathbf{f}\| = 10 = \sqrt{r^2 + p^2}$, we have $\cos^{-1}\left(\frac{r}{\sqrt{r^2 + p^2}}\right) = \cos^{-1}\left(\frac{r}{10}\right)$.

That implies $\frac{r}{10} = \cos 60^\circ = \frac{1}{2}$. So, $r = \frac{1}{2} \cdot 10 = 5$.

So, the force parallel to the ramp that must be applied to keep the block from sliding down the ramp is 5 N.

We can choose $\mathbf{f}_r = \begin{bmatrix} r \\ 0 \end{bmatrix}$ and $\mathbf{f}_p = \begin{bmatrix} 0 \\ p \end{bmatrix}$. Why? Hint: \mathbf{f}_r and \mathbf{f}_p are perpendicular to each other.

9. Similar to Example 1.35, we resolve \mathbf{f} into \mathbf{f}_x and \mathbf{f}_y where \mathbf{f}_x is parallel to the ground and \mathbf{f}_y is the lifting force perpendicular to it.

Specifically, if we let $\mathbf{f}_x = \begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\mathbf{f}_y = \begin{bmatrix} 0 \\ y \end{bmatrix}$, then $\mathbf{f} = \mathbf{f}_x + \mathbf{f}_y = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Note that the angle formed by \mathbf{f} and \mathbf{f}_y is 45° . Why?

Then since $\|\mathbf{f}\| = 1500 = \sqrt{x^2 + y^2}$, we have $\sin^{-1}\left(\frac{y}{\sqrt{x^2 + y^2}}\right) = \cos^{-1}\left(\frac{y}{1500}\right)$.

That implies $\frac{y}{1500} = \sin 45^\circ = \frac{\sqrt{2}}{2}$. So, $y = \frac{\sqrt{2}}{2} \cdot 1500 = 750\sqrt{2}$.

So, the vertical force that tends to lift the vehicle off the ground is $750\sqrt{2}$ N.

We seem to have ignored the mass of the car and the acceleration due to gravity. Did we?

10. Similar to Example 1.35, we resolve \mathbf{f} into \mathbf{f}_x and \mathbf{f}_y where \mathbf{f}_x is parallel to the ground and \mathbf{f}_y is perpendicular to it.

Specifically, if we let $\mathbf{f}_x = \begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\mathbf{f}_y = \begin{bmatrix} 0 \\ y \end{bmatrix}$, then $\mathbf{f} = \mathbf{f}_x + \mathbf{f}_y = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Note that the angle formed by \mathbf{f} and \mathbf{f}_x is 45° . Why?

Then since $\|\mathbf{f}\| = 100 = \sqrt{x^2 + y^2}$, we have $\sin^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) = \cos^{-1}\left(\frac{x}{100}\right)$.

That implies $\frac{x}{100} = \cos 45^\circ = \frac{\sqrt{2}}{2}$. So, $x = \frac{\sqrt{2}}{2} \cdot 100 = 50\sqrt{2}$.

So, the force that is causing the lawn mower to move forward is $50\sqrt{2}$ N.

We seem to have ignored the mass of the mower and the acceleration due to gravity. Did we?

11. Similar to Example 1.36, we find \mathbf{t} , the tension in the wire, using $\|\mathbf{f}_y\|$, the weight of the sign. Since the mass of the sign is 50 kg, $\|\mathbf{f}_y\| = 50 \cdot 9.8 = 490$ N.

The angle formed by \mathbf{t} and \mathbf{f}_y is 60° , so $\|\mathbf{f}_y\| = 490 = \|\mathbf{t}\| \cos 60^\circ = \frac{1}{2} \cdot \|\mathbf{t}\|$.

That implies the tension in the cable is $\|\mathbf{t}\| = 2 \cdot 490 = 980$ N.

We seem to have ignored the horizontal component of the weight of the sign. Did we?

12. Similar to Example 1.36, we find \mathbf{s} , the tension in each string, using $\|\mathbf{w}\|$, the weight of the sign. Since the mass of the sign is 1 kg, $\|\mathbf{w}\| = 1 \cdot 9.8 = 9.8$ N.

Furthermore, since the angle formed by \mathbf{s} and \mathbf{w} is 45° , the tension in each string is equal to exactly half of the weight of the sign.

$$\text{So, } \frac{1}{2} \cdot \|\mathbf{w}\| = 4.9 = \|\mathbf{s}\| \cos 45^\circ = \frac{1}{\sqrt{2}} \cdot \|\mathbf{s}\|.$$

That implies the tension in each string is $\|\mathbf{s}\| = 4.9\sqrt{2} \approx 6.9$ N.

13. Find the sine of the angles the wires make with the ceiling, then follow Example 1.36.

The triangle formed by the wires and the ceiling is right. Why? Hint: What are the lengths of the sides?

So, the angle the 15-cm wire makes with the ceiling has $\sin \theta_1 = \frac{20}{25} = 0.8$.

Likewise, the angle the 20-cm wire makes with the ceiling has $\sin \theta_2 = \frac{15}{25} = 0.6$.

Following Example 1.36, we have $\frac{\|\mathbf{f}_1\|}{\sin \theta_1} = \frac{\|\mathbf{f}_2\|}{\sin \theta_2} = \frac{\|\mathbf{r}\|}{\sin 90^\circ}$ where $\|\mathbf{r}\| = 15 \cdot 9.8 = 147$ N.

$$\text{So, } \mathbf{f}_1 = \frac{\|\mathbf{r}\| \sin \theta_1}{\sin 90^\circ} = \frac{147(0.8)}{1} = 117.6 \text{ N and } \mathbf{f}_2 = \frac{\|\mathbf{r}\| \sin \theta_2}{\sin 90^\circ} = \frac{147(0.6)}{1} = 88.2 \text{ N.}$$

14. Following Example 1.36, we have $\frac{\|\mathbf{f}_1\|}{\sin 30^\circ} = \frac{\|\mathbf{f}_2\|}{\sin 45^\circ} = \frac{\|\mathbf{r}\|}{\sin 105^\circ}$ where $\|\mathbf{r}\| = 20 \cdot 9.8 = 196$ N.

$$\text{So, } \mathbf{f}_1 = \frac{\|\mathbf{r}\| \sin 30^\circ}{\sin 105^\circ} = \frac{196(0.5)}{0.9659} \approx 101.46 \text{ N and } \mathbf{f}_2 = \frac{\|\mathbf{r}\| \sin 45^\circ}{\sin 105^\circ} = \frac{196(0.7071)}{0.9659} \approx 143.48 \text{ N.}$$

15. We require that $[1, 0, 1, 1, d] \cdot [1, 1, 1, 1, 1] = 0 \Rightarrow 1 + 1 + 1 + d = 0 \Rightarrow d = 1$.
So the associated parity check code vector is $\mathbf{v} = [1, 0, 1, 1, 1]$.
16. We require that $[1, 1, 0, 1, 1, d] \cdot [1, 1, 1, 1, 1, 1] = 0 \Rightarrow 1 + 1 + 1 + 1 + d = 0 \Rightarrow d = 0$.
So the associated parity check code vector is $\mathbf{v} = [1, 1, 0, 1, 1, 0]$.
17. We check if $\mathbf{v} \cdot \mathbf{1} = \mathbf{0}$: $[1, 0, 1, 0] \cdot [1, 1, 1, 1] = 1 + 0 + 1 + 0 = 0$.
So a single error could not have occurred.
18. We check if $\mathbf{v} \cdot \mathbf{1} = \mathbf{0}$: $[1, 1, 1, 0, 1, 1] \cdot [1, 1, 1, 1, 1, 1] = 1 + 1 + 1 + 1 + 1 + 1 = 1$.
So a single error could have occurred.
19. We check if $\mathbf{v} \cdot \mathbf{1} = \mathbf{0}$: $[0, 1, 0, 1, 1, 1] \cdot [1, 1, 1, 1, 1, 1] = 1 + 1 + 1 + 1 = 0$.
So a single error could not have occurred.
20. We check if $\mathbf{v} \cdot \mathbf{1} = \mathbf{0}$: $[1, 1, 0, 1, 0, 1, 1, 1] \cdot [1, 1, 1, 1, 1, 1, 1, 1] = 1 + 1 + 1 + 1 + 1 + 1 + 1 = 0$.
So a single error could not have occurred.
21. We require that $[1, 2, 2, 2, d] \cdot [1, 1, 1, 1, 1] = 0$ in \mathbb{Z}_3 . Thus $1 + 2 + 2 + 2 + d = 0 \Rightarrow d = 2$.
22. We require that $[3, 4, 2, 3, d] \cdot [1, 1, 1, 1, 1] = 0$ in \mathbb{Z}_5 . Thus $3 + 4 + 2 + 3 + d = 0 \Rightarrow d = 3$.

23. We require that $[1, 5, 6, 4, 5, d] \cdot [1, 1, 1, 1, 1, 1] = 0$ in \mathbb{Z}_7 .
Thus, $1 + 5 + 6 + 4 + 5 + d = 0 \Rightarrow d = 0$.

24. We require that $[3, 0, 7, 5, 6, 8, d] \cdot [1, 1, 1, 1, 1, 1, 1] = 0$ in \mathbb{Z}_9 .
Thus $3 + 7 + 5 + 6 + 8 + d = 0 \Rightarrow d = 7$.

25. Let \mathbf{u} and \mathbf{v} be arbitrary vectors in \mathbb{Z}_m^n that differ in exactly one component.
So: $\mathbf{u} = [u_1, u_2, \dots, u_i, \dots, u_n], \mathbf{v} = [v_1, v_2, \dots, v_j, \dots, v_n]$, where $u_i = v_i, i \neq j$:

$$\begin{aligned} \mathbf{c} \cdot (\mathbf{u} - \mathbf{v}) &= [1, 1, \dots, 1] \cdot ([u_1, u_2, \dots, u_i, \dots, u_n] - [v_1, v_2, \dots, v_j, \dots, v_n]) \\ &= [1, 1, \dots, 1] \cdot ([0, 0, \dots, u_i, \dots, 0] - [0, 0, \dots, v_j, \dots, 0]) = u_i - v_j \neq 0 \end{aligned}$$

So, $\mathbf{c} \cdot \mathbf{u} \neq \mathbf{c} \cdot \mathbf{v}$.

26. $\mathbf{c} \cdot \mathbf{u} = 0 \Rightarrow 3(0 + 9 + 6 + 7 + 0 + 7) + (5 + 4 + 4 + 0 + 2) + d = 0 \Rightarrow 7 + 5 + d = 0 \Rightarrow d = 8$.

27. $\mathbf{c} \cdot \mathbf{u} = 0 \Rightarrow 3(0 + 4 + 1 + 1 + 4 + 2) + (1 + 0 + 4 + 8 + 1) + d = 0 \Rightarrow 6 + 4 + d = 0 \Rightarrow d = 0$.

28. Let $\mathbf{u} = [0, 4, 6, 9, 5, 6, 1, 8, 2, 0, 1, 5]$ be a UPC vector.

(a) We check if $\mathbf{c} \cdot \mathbf{u} = 0$:

$$\begin{aligned} \mathbf{c} \cdot \mathbf{u} &= [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1] \cdot [0, 4, 6, 9, 5, 6, 1, 8, 2, 0, 1, 5] \\ &= 3(0 + 6 + 5 + 1 + 2 + 1) + (4 + 9 + 6 + 8 + 0) + 5 = 5 + 7 + 5 = 7 \neq 0 \end{aligned}$$

so, the UPC cannot be correct.

(b) Now $\mathbf{u} = [0, 4, u_3, 9, 5, 6, 1, 8, 2, 0, 1, 5]$, and we require that $\mathbf{c} \cdot \mathbf{u} = 0$:

$$\begin{aligned} 3(0 + u_3 + 5 + 1 + 2 + 1) + (4 + 9 + 6 + 8 + 0) + 5 &= 0 \Leftrightarrow \\ 3(u_3 + 9) + 2 &= 0 \Leftrightarrow 3u_3 + 7 = 8 \Leftrightarrow 3u_3 = 1 \Leftrightarrow u_3 = 7. \end{aligned}$$

So, the correct UPC is $[0, 4, 7, 9, 5, 6, 1, 8, 2, 0, 1, 5]$.

29. Let $\mathbf{u} = [u_1, u_2, \dots, u_{12}]$ be a correct UPC.

Let $\mathbf{u}' = [u_1, \dots, u'_i, \dots, u_{12}]$ be a UPC with an error in the i th component.

Now we check if \mathbf{u}' is correct, first noting that $\mathbf{u}' = \mathbf{u} - u_i \mathbf{e}_i + u'_i \mathbf{e}_i$.

We have $\mathbf{c} \cdot \mathbf{u}' = \mathbf{c} \cdot (\mathbf{u} - u_i \mathbf{e}_i + u'_i \mathbf{e}_i) = \mathbf{c} \cdot \mathbf{u} - u_i \mathbf{c} \cdot \mathbf{e}_i + u'_i \mathbf{c} \cdot \mathbf{e}_i = \mathbf{c} \cdot \mathbf{e}_i (u'_i - u_i)$.

But $\mathbf{c} \cdot \mathbf{e}_i = 1$ or $3 \neq 0$ and $u'_i \neq u_i$ since the i th component is wrong.

So $\mathbf{c} \cdot \mathbf{u}' \neq 0$, and the UPC will detect the single error.

30. (a) We check if $\mathbf{c} \cdot \mathbf{u}' = 0$:

$$\begin{aligned}\mathbf{c} \cdot \mathbf{u}' &= [3, 1, 3, 1, 3, 1, 3, 1, 3, 1] \cdot [0, 4, 7, 9, 2, 7, 0, 2, 0, 9, 4, 6] \\ &= 3(0 + 7 + 2 + 0 + 0 + 4) + (4 + 9 + 7 + 2 + 9) + 6 = 9 + 1 + 6 = 6 \neq 0\end{aligned}$$

So, the error will be detected.

- (b) Transposing the third and fourth components of \mathbf{u} gives $\mathbf{u}' = [0, 7, 9, 4, 2, 7, 0, 2, 0, 9, 4, 6]$, while $\mathbf{c} \cdot \mathbf{u}' = 0$ in \mathbb{Z}_{10} , so the error will not be detected.
- (c) Assume there is a transposition error between the i^{th} and $(i+1)^{th}$ components and that the new UPC $\mathbf{u}' = \mathbf{u} + \mathbf{e}_i(u_{i+1} - u_i) + \mathbf{e}_{i+1}(u_i - u_{i+1})$ satisfies $\mathbf{c} \cdot \mathbf{u}' = 0$. So:

$$\begin{aligned}0 &= \mathbf{c} \cdot (\mathbf{u} + \mathbf{e}_i(u_{i+1} - u_i) + \mathbf{e}_{i+1}(u_i - u_{i+1})) \\ &= \mathbf{c} \cdot \mathbf{u} + \mathbf{c} \cdot \mathbf{e}_i(u_{i+1} - u_i) + \mathbf{c} \cdot \mathbf{e}_{i+1}(u_i - u_{i+1}) \\ &= c_i(u_{i+1} - u_i) + c_{i+1}(u_i - u_{i+1}) = u_i(c_{i+1} - c_i) + u_{i+1}(c_i - c_{i+1})\end{aligned}$$

Case 1: If i is even then $c_i = 1$, $c_{i+1} = 3$, and the constraint becomes $0 = 2u_i + 8u_{i+1}$.

Case 2: If i is odd then $c_i = 3$, $c_{i+1} = 1$, and the constraint becomes $0 = 8u_i + 2u_{i+1}$.

So, if u_i and u_{i+1} are transposed and $8u_i + 2u_{i+1} = 0$, the error will not be detected.

NOTE: The trivial solution is $u_i = u_{i+1}$, that is the adjacent components are the same.

31. $\mathbf{c} \cdot \mathbf{u} = 0 \Rightarrow [10, 9, 8, 7, 6, 5, 4, 3, 2, 1] \cdot [0, 3, 8, 7, 9, 7, 9, 9, 3, d] = 1 + d \Rightarrow d = 10$.

32. $\mathbf{c} \cdot \mathbf{u} = 0 \Rightarrow [10, 9, 8, 7, 6, 5, 4, 3, 2, 1] \cdot [0, 3, 9, 4, 7, 5, 6, 8, 2, d] = 4 + d \Rightarrow d = 7$.

33. Let $\mathbf{u} = [0, 4, 4, 9, 5, 0, 8, 3, 5, 6]$ be an ISBN vector.

- (a) $\mathbf{c} \cdot \mathbf{u} = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1] \cdot [0, 4, 4, 9, 5, 0, 8, 3, 5, 6] = 218 = 9 \neq 0$ in \mathbb{Z}_{11} .
Thus the ISBN cannot be correct (because $\mathbf{c} \cdot \mathbf{u} \neq 0$).

- (b) Now $\mathbf{u} = [0, 4, 4, 9, u_5, 0, 8, 3, 5, 6]$, and we require that $\mathbf{c} \cdot \mathbf{u} = 0$: $[10, 9, 8, 7, 6, 5, 4, 3, 2, 1] \cdot [0, 4, 4, 9, u_5, 0, 8, 3, 5, 6] = 0 \Leftrightarrow 1 + 6u_5 = 0 \Leftrightarrow 6u_5 = 10 \Leftrightarrow u_5 = 9$.

34. (a) $\mathbf{u} = [0, 6, 7, 9, 7, 6, 2, 9, 0, 6]$ is a correct ISBN.

Consider a transposition between the fourth and fifth components.

Then $\mathbf{u}' = [0, 6, 7, 7, 9, 6, 2, 9, 0, 6]$.

Now check if \mathbf{u}' is correct: $\mathbf{c} \cdot \mathbf{u}' = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1] \cdot [0, 6, 7, 7, 9, 6, 2, 9, 0, 6] = 9 \neq 0$.

So, the error would be detected.

- (b) In general, an ISBN vector \mathbf{u} satisfies the check constraint $\sum_{i=1}^{10} (11-i)(u_i) = 0$,

but in \mathbb{Z}_{11} $(11-i) = -i$, so the constraint becomes $\sum_{i=1}^{10} (-i)(u_i) = 0$.

Consider a transposition of any two adjacent elements (say the u_j and u_{j+1} elements).

This gives a new vector $\mathbf{u}' = \mathbf{u} + \mathbf{e}_j(u_{j+1} - u_j) + \mathbf{e}_{j+1}(u_j - u_{j+1})$.

The check constraint on \mathbf{u}' becomes $\sum_{i=1}^{10} (-i)(u'_i) = 0$,

but $u'_i = u_i + (\mathbf{e}_j)_i(u_{j+1} - u_j) + (\mathbf{e}_{j+1})_i(u_j - u_{j+1})$, so this becomes

$$\begin{aligned} \sum_{i=1}^{10} (-i)(u'_i) &= \sum_{i=1}^{10} (-i)(u_i) + \sum_{i=1}^{10} (-i)(\mathbf{e}_j)_i(u_{j+1} - u_j) + \sum_{i=1}^{10} (-i)(\mathbf{e}_{j+1})_i(u_j - u_{j+1}) \\ &= \sum_{i=1}^{10} (-i)(\mathbf{e}_j)_i(u_{j+1} - u_j) + \sum_{i=1}^{10} (-i)(\mathbf{e}_{j+1})_i(u_j - u_{j+1}) \\ &= (-j)(u_{j+1} - u_j) + (-(j+1))(u_j - u_{j+1}) \\ &= (-j)(u_{j+1} - u_j) + (-j+10)(u_j - u_{j+1}) = 10(u_j - u_{j+1}) \end{aligned}$$

So the constraint $\sum_{i=1}^{10} (-i)(u'_i) = 0$ can only be met if $u_j = u_{j+1}$,

in which case the transposed vector is the same as the original.

So the constraint $\sum_{i=1}^{10} (-i)(u_i) = 0$ cannot be broken by transposing adjacent components.

An error made in any two adjacent components of the ISBN will be detected.

- (c) See the proof in part (b).

35. Consider the ISBN $\mathbf{u} = [0, 8, 3, 7, 0, 9, 9, 0, 2, 6]$.

- (a) $\mathbf{c} \cdot \mathbf{u} = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1] \cdot [0, 8, 3, 7, 0, 9, 9, 0, 2, 6] = 5$ in \mathbb{Z}_{11} . $\mathbf{c} \cdot \mathbf{u} \neq 0$, so this ISBN cannot be correct (because $\mathbf{c} \cdot \mathbf{u} \neq 0$).

- (b) Consider the ISBN \mathbf{u} with the first and second components transposed. Then:

$\mathbf{u} = [8, 0, 3, 7, 0, 9, 9, 0, 2, 6]$, $\mathbf{c} \cdot \mathbf{u} = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1] \cdot [8, 0, 3, 7, 0, 9, 9, 0, 2, 6] = 0$.

So, the correct ISBN is $\mathbf{u} = [8, 0, 3, 7, 0, 9, 9, 0, 2, 6]$.

- (c) Consider the ISBN $\mathbf{u} = [0, 3, 8, 7, 0, 9, 9, 0, 2, 6]$. If a transposition error occurs between the second and third components we get the vector $\mathbf{u}' = [0, 8, 3, 7, 0, 9, 9, 0, 2, 6]$, where $\mathbf{c} \cdot \mathbf{u}' = 5 \neq 0$, so the error would be detected.

However, this error cannot be corrected because transpositions to both

$\mathbf{u} = [0, 3, 8, 7, 0, 9, 9, 0, 2, 6]$ and $\mathbf{u}'' = [8, 0, 3, 7, 0, 9, 9, 0, 2, 6]$ give proper ISBNs.

Chapter 1 Review

1. We will explain and give counter examples to justify our answers below.

(a) **True**. Follows from the properties of \mathbb{R}^n listed in Theorem 1.1 in Section 1.1:

| | |
|---|--|
| $\mathbf{u} = \mathbf{u} + \mathbf{0}$ | Zero Property, Property (c) |
| $= \mathbf{u} + (\mathbf{w} + (-\mathbf{w}))$ | Additive Inverse Property, Property (d) |
| $= (\mathbf{u} + \mathbf{w}) + (-\mathbf{w})$ | Distributive Property, Property (b) |
| $= (\mathbf{v} + \mathbf{w}) + (-\mathbf{w})$ | By the given condition $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$ |
| $= \mathbf{v} + (\mathbf{w} + (-\mathbf{w}))$ | Distributive Property, Property (b) |
| $= \mathbf{v} + \mathbf{0}$ | Additive Inverse Property, Property (d) |
| $= \mathbf{v}$ | Zero Property, Property (c) |

(b) **False**. See Example 1.16 and Exercise 54 in Section 1.2. Two key counter examples:
 Since $\mathbf{0} \cdot \mathbf{v} = 0$ for every vector \mathbf{v} in \mathbb{R}^3 , $\mathbf{0}$ is orthogonal to every vector.

That is, if $\mathbf{u} = \mathbf{0}$, we know nothing about \mathbf{v} and \mathbf{w} .

Let \mathbf{u} and \mathbf{v} be orthogonal to \mathbf{w} then $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} = 0$.

E.g., consider $\mathbf{u} = [a, 0, 0]$, $\mathbf{v} = [0, b, 0]$, and $\mathbf{w} = [0, 0, c]$.

(c) **False**. Note this property is **not** listed in Theorem 1.2 in Section 1.2.

Let $\mathbf{v} = \mathbf{0}$ then $\mathbf{u} \cdot \mathbf{0} = 0$ and $\mathbf{0} \cdot \mathbf{w} = 0$, but there is no restriction on \mathbf{u} and \mathbf{w} .

Let $\mathbf{u} = \mathbf{w}$ then $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{u} = 0$, but $\mathbf{u} \cdot \mathbf{u} \neq 0$ unless $\mathbf{u} = \mathbf{0}$.

E.g., consider $\mathbf{u} = [a, 0, 0]$, $\mathbf{v} = [0, b, 0]$, and $\mathbf{w} = [c, 0, 0]$, then $\mathbf{u} \cdot \mathbf{w} = ac$.

(d) **False**. When a line is parallel to plane then $\mathbf{d} \cdot \mathbf{n} = 0$, that is \mathbf{d} is **orthogonal** to \mathbf{n} .
 See Figure 1.57 in Section 1.3.

(e) **True**. Every line in plane \mathcal{P} and parallel to \mathcal{P} is **orthogonal** to its normal vector \mathbf{n} .
 See Figure 1.62 in Section 1.3.

(f) **True**. See the remarks following Example 1.24 in Section 1.3.

(g) **False**. In \mathbb{R}^3 many non-parallel lines are **skew** (non-intersecting lines with $\mathbf{d}_1 \neq \mathbf{d}_2$).

For example, ℓ_1 with $\mathbf{x} = t\mathbf{d}$ with $\mathbf{d} = [1, 0, 0]$ (the x -axis) and

ℓ_2 with $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ with $\mathbf{p} = [0, 0, 1]$ and $\mathbf{d} = [0, 1, 0]$

(the line parallel to the y -axis through $[0, 0, 1]$).

(h) **False**. See Examples 1.27 to 1.29 in Section 1.4.

For example, $[1, 0, 1] \cdot [1, 0, 1] = 1 + 0 + 1 = 0$ in \mathbb{Z}_2 .

In general, the dot product of any binary vector with an even number of 1s is 0.

(i) **True**. See Example 1.37 in Section 1.4.

We have $\mathbf{c} \cdot \mathbf{u} = 3(0 + 1 + 7 + 5 + 7 + 8) + (4 + 7 + 1 + 2 + 0 + 2) = 100 = 0$ in \mathbb{Z}_{10} .

(j) **False**. See Example 1.38 in Section 1.4.

$\mathbf{c} \cdot \mathbf{u} = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1] \cdot [0, 5, 3, 2, 3, 4, 1, 7, 4, 8] = 162 \neq 0$ in \mathbb{Z}_{11} .

2. See Examples 1.1 and 1.5 in Section 1.1.

$$\text{Let } \mathbf{w} = \begin{bmatrix} 10 \\ -10 \end{bmatrix} \text{ then } \mathbf{z} = 4\mathbf{u} + \mathbf{v} + \mathbf{w} = 4 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}.$$

So the coordinates of the point at the head of $4\mathbf{u} + \mathbf{v}$ are (9, 12).

3. See Example 1.5 in Section 1.1. **Note:** We should do the vector arithmetic first.

$$2\mathbf{x} + \mathbf{u} = 3(\mathbf{x} - \mathbf{v}) \text{ implies } \mathbf{x} = \mathbf{u} + 3\mathbf{v} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}.$$

4. As in **Exploration: Vectors and Geometry**, we have $\overrightarrow{BA} = \mathbf{b} - (-\mathbf{a}) = \mathbf{b} + \mathbf{a}$.

5. We proceed as in Example 1.14 of Section 1.2.

We have $\mathbf{u} \cdot \mathbf{v} = -1 \cdot 2 + 1 \cdot 1 - 2 \cdot 1 = -3$, $\|\mathbf{u}\| = \sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{6}$, and $\|\mathbf{v}\| = \sqrt{6}$.

Therefore, $\cos \theta = -\frac{3}{\sqrt{6}\sqrt{6}} = -\frac{1}{2}$, so $\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ radians or 120° .

6. We proceed as in Example 1.17 of Section 1.2.

The length of \mathbf{v} and the angle it makes with \mathbf{u} determines how much of \mathbf{u} it covers.

This is basically what $\text{proj}_{\mathbf{u}}(\mathbf{v}) = c\mathbf{u}$ says. The projection of \mathbf{v} shadows a **fraction** of \mathbf{u} .

$$\text{So } \mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1, \mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 9, \text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\right) \mathbf{u} = \begin{bmatrix} 1/9 \\ -2/9 \\ 2/9 \end{bmatrix}.$$

In this case, the projection of \mathbf{v} shadows $\frac{1}{9}$ of \mathbf{u} .

7. We use the given conditions to find a unit vector in the xy -plane orthogonal to $\mathbf{v} = [1, 2, 3]$.

Step 1. Figure 1.15 in Section 1.1 implies any vector in the xy -plane has a z -component of 0.

So, the vector \mathbf{u} we are looking for must be of the form $\mathbf{u} = [a, b, 0]$.

Step 2. Like Exercise 42 in Section 1.2, since \mathbf{u} is orthogonal to \mathbf{v} , we have

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a + 2b = 0. \text{ So } a = -2b \text{ and } \mathbf{u} = \begin{bmatrix} -2b \\ b \\ 0 \end{bmatrix} = b \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

Step 3. As in Example 1.12 of Section 1.2, we **normalize** \mathbf{u} to create \mathbf{w} , the unit vector.

$$\text{Letting } b = 1 \text{ above gives us } \mathbf{u} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \|\mathbf{u}\| = \sqrt{(-2)^2 + 1^2 + 0^2} = \sqrt{5}.$$

$$\text{So } \mathbf{one} \text{ vector that works is } \mathbf{w} = \left(\frac{1}{\|\mathbf{u}\|}\right) \mathbf{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}.$$

Note: The fact that we got to choose a value for b implies there are infinitely many solutions.

8. We begin by noting $\begin{matrix} x = 2 - t \\ y = 3 + 2t \\ z = -1 + t \end{matrix}$ implies $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, so $\mathbf{d} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \mathbf{n}$.

From Example 1.23 of Section 1.3, we have $\mathbf{n} \cdot \mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 = 2$.

So the normal equation $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ becomes the general equation $-x + 2y + z = 2$.

9. Planes that are parallel have parallel normals, so since our plane is parallel to $2x + 3y - z = 0$,

a normal to given plane and therefore our plane is $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

As in Example 1.23 of Section 1.3, we find the plane through $P = (3, 2, 5)$ with $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

With $\mathbf{p} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we have $\mathbf{n} \cdot \mathbf{p} = 2 \cdot 3 + 3 \cdot 2 - 1 \cdot 5 = 7$.

So the normal equation $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ becomes the general equation $2x + 3y - z = 7$.

10. As in Exercise 9, we use the given conditions to find the normal of the plane, \mathbf{n} .

Step 1. The normal of the plane \mathbf{n} must be normal to every line in the plane.

So, we find $\mathbf{d}_1 = \overrightarrow{AB}$, $\mathbf{d}_2 = \overrightarrow{BC}$ and compute $\mathbf{n} \cdot \mathbf{d}_1 = 0$, $\mathbf{n} \cdot \mathbf{d}_2 = 0$.

$$\mathbf{d}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{d}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\mathbf{n} \cdot \mathbf{d}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = -b + c = 0 \Rightarrow c = b. \text{ So } \mathbf{n} = \begin{bmatrix} a \\ b \\ b \end{bmatrix}.$$

$$\mathbf{n} \cdot \mathbf{d}_2 = \begin{bmatrix} a \\ b \\ b \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = -a + b + b = 0 \Rightarrow a = 2b. \text{ So } \mathbf{n} = \begin{bmatrix} 2b \\ b \\ b \end{bmatrix} = b \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Step 2. Take $P = A = (1, 1, 0)$ and compute $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ to find the general equation.

Since $\mathbf{n} \cdot \mathbf{p} = 2 \cdot 1 + 1 \cdot 1 + 1 \cdot 0 = 3$, we have the general equation $2x + y + z = 3$.

Note: We could also have taken $\mathbf{p} = \mathbf{b}$ or \mathbf{c} since $\mathbf{n} \cdot \mathbf{b} = \mathbf{n} \cdot \mathbf{c} = 3$. Why?

11. We proceed as in Exercise 41 of Section 1.2. See the notes prior to Exercise 41.

$$\text{Let } \mathbf{u} = \overrightarrow{AB} = \begin{bmatrix} 1-1 \\ 0-1 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \overrightarrow{AC} = \begin{bmatrix} 0-1 \\ 1-1 \\ 2-0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

(a) We compute the necessary values ...

(b) We compute the necessary values ...

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = 2,$$

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = 2,$$

$$\mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 2 \quad (\|\mathbf{u}\| = \sqrt{2}),$$

$$\|\mathbf{u}\| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2},$$

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \Rightarrow$$

$$\|\mathbf{v}\| = \sqrt{(-1)^2 + 0^2 + 2^2} = \sqrt{5} \Rightarrow$$

$$\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow$$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5} \Rightarrow$$

$$\begin{aligned} \|\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})\| &= \sqrt{(-1)^2 + 1^2 + 1^2} \\ &= \sqrt{3} \end{aligned}$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \left(\frac{\sqrt{10}}{5}\right)^2} = \frac{\sqrt{15}}{5}$$

... then substitute into the formula for \mathcal{A} :

... then substitute into the formula for \mathcal{A} :

$$\begin{aligned} \mathcal{A} &= \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})\| \\ &= \frac{1}{2} \sqrt{2} \sqrt{3} = \frac{\sqrt{6}}{2}. \end{aligned}$$

$$\begin{aligned} \mathcal{A} &= \frac{1}{2} \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta \\ &= \frac{1}{2} \sqrt{2} \sqrt{5} \frac{\sqrt{15}}{5} = \frac{\sqrt{6}}{2}. \end{aligned}$$

12. From Example 1 in **Exploration: Vectors and Geometry**, we have $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

$$\text{So } \mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b}) = \frac{1}{2} \left(\begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ -7 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}.$$

Therefore, the midpoint of the line segment between A and B is $(4, -3, -1)$.

13. We proceed as in Exercise 61 from Section 1.2.

We need to show $\|\mathbf{u}\| = 2$ and $\|\mathbf{v}\| = 3$ imply $\mathbf{u} \cdot \mathbf{v} \neq -7$.

From Theorem 1.4 (the Cauchy-Schwarz Inequality), we have $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$.

Substituting in the given values of $\|\mathbf{u}\| = 2$ and $\|\mathbf{v}\| = 3$ shows $|\mathbf{u} \cdot \mathbf{v}| \leq 6$.

Therefore, $-6 \leq \mathbf{u} \cdot \mathbf{v} \leq 6$. It follows immediately that $\mathbf{u} \cdot \mathbf{v} \neq -7$.

14. We will follow Example 1.26 in Section 1.3, then use $d(Q, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

By definition $ax + by + cz = d$ implies $\mathbf{n} = [a, b, c]$, so $2x + 3y - z = 0$ implies $\mathbf{n} = [2, 3, -1]$.

By Figure 1.62 in Section 1.3, we calculate the length of $\overrightarrow{RQ} = \text{proj}_{\mathbf{n}}(\mathbf{v})$, where $\mathbf{v} = \overrightarrow{PQ}$.

Step 1. By trial and error, we find $P = (0, 0, 0)$ satisfies $2x + 3y - z = 0$.

Step 2. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = [3, 2, 5] - [0, 0, 0] = [3, 2, 5]$.

Step 3. $\text{proj}_{\mathbf{n}}(\mathbf{v}) = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{n}} \right) \mathbf{n} = \left(\frac{2 \cdot 3 + 3 \cdot 2 - 1 \cdot 5}{2^2 + 3^2 + (-1)^2} \right) \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ -1/2 \end{bmatrix}$.

Step 4. The distance from Q to \mathcal{P} is $\|\text{proj}_{\mathbf{n}}(\mathbf{v})\| = \left\| \begin{bmatrix} 1 \\ 3/2 \\ -1/2 \end{bmatrix} \right\| = \frac{1}{2} \left\| \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\| = \frac{\sqrt{14}}{2}$.

Now for $d(Q, \mathcal{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$ we need identify a, b, c, d , and x_0, y_0, z_0 .

Since $2x + 3y - z = 0$, $a = 1, b = 3, c = -1, d = 0$. From $Q = (3, 2, 5)$, $x_0 = 3, y_0 = 2, z_0 = 5$.

So $d(Q, \mathcal{P}) = \frac{|2 \cdot 3 + 3 \cdot 2 - 1 \cdot 5 - 0|}{\sqrt{2^2 + 3^2 + (-1)^2}} = \frac{7}{\sqrt{14}} = \frac{\sqrt{14}}{2}$ as in Example 1.26 in Section 1.3.

15. We follow Example 1.27 in Section 1.3, then use $d(Q, \ell) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

Comparing $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we see ℓ has $P = (0, 1, 2)$ and $\mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Now if we let $\mathbf{v} = \overrightarrow{PQ}$, then $\overrightarrow{PR} = \text{proj}_{\mathbf{d}}(\mathbf{v})$ and $\overrightarrow{RQ} = \mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})$.

Step 1. $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = [3, 2, 5] - [0, 1, 2] = [3, 1, 3]$.

Step 2. $\text{proj}_{\mathbf{d}}(\mathbf{v}) = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}} \right) \mathbf{d} = \left(\frac{1 \cdot 3 + 1 \cdot 1 + 1 \cdot 3}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{7}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 7/3 \\ 7/3 \end{bmatrix}$.

Step 3. The vector we want is $\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v}) = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 7/3 \\ 7/3 \\ 7/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -4/3 \\ 2/3 \end{bmatrix} = -\frac{2}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Step 4. The distance $d(Q, \ell)$ is $\|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \left\| \begin{bmatrix} 2/3 \\ -4/3 \\ 2/3 \end{bmatrix} \right\| = \frac{2}{3} \left\| \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\| = \frac{2\sqrt{6}}{3}$.

$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ so $x - 2y + z = 0$ and $a = 1, b = -2, c = 1, d = 0$.

So $d(Q, \ell) = \frac{|3 - 4 + 5 - 0|}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$ as in Example 1.27 of Section 1.3.

16. We follow Example 1.34 in Section 1.4.

$$3 - (2 + 4)^3(4 + 3)^2 = 3 - (1)^3(2)^2 = 3 - (1)(4) = -1 = 4.$$

Note: $-1 = 4$ in \mathbb{Z}_5 since $1 + 4 = 5 = 0 = 1 + (-1)$.

17. We begin by noting that $5 \cdot 3 = 15 = 1$ in \mathbb{Z}_7 so $3^{-1} = 5$ in \mathbb{Z}_7 since $5 \cdot 3 = 1 = 3 \cdot 3^{-1}$.

$$3(x + 2) = 5 \Rightarrow$$

$$5 \cdot 3(x + 2) = 5 \cdot 5 \Rightarrow$$

$$x + 2 = 25 = 4 \Rightarrow$$

$$x = 2.$$

We multiply both sides by 5

because $3^{-1} = 5$ in \mathbb{Z}_7 .

$$25 = 21 + 4 = 7 \cdot 3 + 4 = 4 \text{ in } \mathbb{Z}_7.$$

Simply subtract 2 from both sides.

Note: We should check our answer: $3(2 + 2) = 12 = 7 \cdot 1 + 5 = 5$ in \mathbb{Z}_7 .

18. It is **impossible** to solve $3(x + 2) = 5$ in \mathbb{Z}_9 since 3 does **not** have a multiplicative inverse.

Note: Any number that shares a common factor with the **base**,
like in this case 3 and 9, will **not** have a multiplicative inverse. Why?

19. Similar to Example 1.38, we compute the products in the dot product in \mathbb{Z}_5 .

$$[2, 1, 3, 3] \cdot [3, 4, 4, 2] = 2 \cdot 3 + 1 \cdot 4 + 3 \cdot 4 + 3 \cdot 2 = 1 + 4 + 2 + 1 = 3.$$

20. Similar to Example 1.39, we compute the dot product to find the conditions in \mathbb{Z}_2^4 .

$$[1, 1, 1, 0] \cdot [d_1, d_2, d_3, d_4] = 1 \cdot d_1 + 1 \cdot d_2 + 1 \cdot d_3 = d_1 + d_2 + d_3 = 2.$$

So, either $d_1 = d_2 = 1$ and $d_3 = 0$ or $d_1 = d_3 = 1$ and $d_2 = 0$ or $d_2 = d_3 = 1$ and $d_1 = 0$.
That is, there are 3 possibilities.

However, since d_4 can be 0 or 1 in each case, there are six binary vectors that satisfy $\mathbf{u} \cdot \mathbf{v} = 0$.