Chapter 1

Mathematical Preliminaries

5. To that $\overline{X} \cap \overline{Y} = \overline{(X \cup Y)}$ requires establishing both of the inclusions $\overline{X} \cap \overline{Y} \subseteq \overline{(X \cup Y)}$ and $\overline{(X \cup Y)} \subseteq \overline{X} \cap \overline{Y}$.

   i) Let $a \in \overline{X} \cap \overline{Y}$. By the definition of intersection, $a \in \overline{X}$ and $a \in \overline{Y}$. Thus $a \notin X$ and $a \notin Y$ or, in terms of union, $a \notin (X \cup Y)$. It follows that $a \in \overline{(X \cup Y)}$.

   ii) Assume $a \in \overline{(X \cup Y)}$. Then $a \notin X \cup Y$. This implies that $a \notin X$ and $a \notin Y$. Consequently, $a \in \overline{X} \cap \overline{Y}$.

   Part (i) shows that $\overline{X} \cap \overline{Y}$ is a subset of $\overline{(X \cup Y)}$, while (ii) establishes the inclusion $\overline{(X \cup Y)} \subseteq \overline{X} \cap \overline{Y}$.

   A completely analogous argument can be used to establish the equality of the sets $\overline{(X \cap Y)}$ and $\overline{X} \cup \overline{Y}$.

6. a) The function $f(n) = 2n$ is total and one-to-one. However, it is not onto since the range is the set of even numbers.

   b) The function
   $$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ n-1 & \text{otherwise} \end{cases}$$
   is total and onto. It is not one-to-one since $f(0) = f(1) = 0$.

   c) The function
   $$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n = 1 \\ n & \text{otherwise} \end{cases}$$
   is total, one-to-one, and onto but not the identity.

   d) The function
   $$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ \uparrow & \text{otherwise} \end{cases}$$
   maps the even natural numbers onto the entire set of natural numbers. It is not total, since it is not defined for odd natural numbers.

13. To prove that $\equiv_p$ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive. This is shown using the same argument given in Example 1.3.1, which explicitly considers the case when $p = 2$.

   i) *Reflexivity:* For every natural number n, $n \bmod p = n \bmod p$.

ii) *Symmetry:* If $n \bmod p = m \bmod p$, then $m \bmod p = n \bmod p$.

iii) *Transitivity:* If $n \bmod p = m \bmod p$ and $m \bmod p = k \bmod p$, then $n \bmod p = k \bmod p$.

The equivalence classes of $\equiv_p$ are the sets consisting of natural numbers that are equal mod p:

$$\begin{aligned}
[0]_{\equiv_p} &= \{0, p, 2p, 3p, \ldots\} \\
[1]_{\equiv_p} &= \{1, p+1, 2p+1, 3p+1, \ldots\} \\
[2]_{\equiv_p} &= \{2, p+2, 2p+2, 3p+2, \ldots\} \\
&\vdots \\
[p-1]_{\equiv_p} &= \{p-1, 2p-1, 3p-1, 4p-1, \ldots\}.
\end{aligned}$$

15. i) *Reflexivity:* To demonstrate reflexivity, we must show that every ordered pair $[m, n]$ is related to itself. The requirement for $[m, n]$ to be related to $[m, n]$ by $\equiv$ is $m+n = n+m$, which follows from the commutativity of addition.

ii) *Symmetry:* If $[m, n] \equiv [j, k]$, then $m+k = n+j$. Again, by commutativity, $j+n = k+m$ and $[j, k] \equiv [m, n]$.

iii) *Transitivity:* $[m, n] \equiv [j, k]$ and $[j, k] \equiv [s, t]$ imply $m+k = n+j$ and $j+t = k+s$. Adding the second equality to the first, we obtain

$$m + k + j + t = n + j + k + s.$$

Subtracting $j + k$ from each side yields $m + t = n + s$, showing that $[m, n] \equiv [s, t]$ as desired.

18. The set of non-negative rational numbers is defined by

$$\{n/m \mid n \in \mathbf{N},\ m \in \mathbf{N} - \{0\}\}$$

A rational number n/m can be represented by the ordered pair $[n, m]$. This representation defines a one-to-one correspondence between the rational numbers and the set $\mathbf{N} \times (\mathbf{N} - \{0\})$. The latter set is known to be countable by Theorem 1.4.4.

22. Diagonalization is used to prove that there are an uncountable number of monotone increasing functions. Assume that the set of monotone increasing functions is countable. Then these functions can be listed in a sequence $f_0, f_1, f_2, \ldots, f_n, \ldots$. Define a function f as follows:

$$\begin{aligned}
f(0) &= f_0(0) + 1 \\
f(i) &= f_i(i) + f(i-1)
\end{aligned}$$

for $i > 0$. Since $f_i(i) > 0$, it follows that $f(i) > f(i-1)$ for all i.

Clearly $f(i) \neq f_i(i)$ for any i, contradicting the assumption that $f_0, f_1, \ldots, f_n, \ldots$ exhaustively enumerates the monotone increasing functions. Consequently, the set is uncountable.

25. We first observe that every real number in $(0, 1]$ can be expressed by an infinite decimal $.x_0x_1x_2 \ldots x_n \ldots$. With such a representation, the number $\frac{1}{2}$ is represented by both $.50000\ldots$ and $.49999\ldots$. To obtain a unique representation, we consider only decimal expansions that do not end with an infinite sequence of zeros.

Assume the set of real numbers in $(0, 1]$ is countable. This implies that there is a sequence

$$r_0, r_1, r_2, \ldots, r_n, \ldots$$

that contains all of the real numbers in the interval $(0, 1]$. Let the decimal expansion of r_n be denoted $.x_{n0}x_{n1}x_{n2}\ldots$. The enumeration given above is used to construct an infinite two-dimensional array, the i^{th} row of which consists of the expansion of r_i.

$$\begin{array}{rllll} r_0 = & x_{00} & x_{01} & x_{02} & \cdots \\ r_1 = & x_{10} & x_{11} & x_{12} & \cdots \\ r_2 = & x_{20} & x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

With the unique decimal representation, two numbers are distinct if they differ at any position in the decimal expansion. A real number $r = x_0 x_1 \ldots$ is defined using the diagonal elements in the array formed by the x_{ii}'s as follows:

$$x_i = \begin{cases} 2 & \text{if } x_{ii} = 1 \\ 1 & \text{otherwise.} \end{cases}$$

Clearly $r \neq r_i$ for any i since the i^{th} position of r, x_i, is not identical to the i^{th} position of r_i. Therefore the assumption that the enumeration contains all real numbers in $(0, 1]$ fails, and we conclude that the set is uncountable.

28. Before proving the Schröder-Bernstein Theorem, we consider the special case where $card(\mathrm{B}) \leq card(\mathrm{A})$, $card(\mathrm{A}) \leq card(\mathrm{B})$, and $\mathrm{B} \subseteq \mathrm{A}$.

    The relationship $card(\mathrm{B}) \leq card(\mathrm{A})$ follows immediately from the inclusion $\mathrm{B} \subseteq \mathrm{A}$ since the identity function $id : \mathrm{B} \to \mathrm{A}$ is a one-to-one function from B into A.

    By definition, $card(\mathrm{A}) \leq card(\mathrm{B})$ means there is a one-to-one function f from A into B. We will use f to construct a one-to-one function $h : \mathrm{A} \to \mathrm{B}$ from A onto B. The function h demonstrates that A and B have the same cardinality.

    The diagram

    INSERT FIGURE Chapter 1: exercise 28 HERE

    illustrates the mapping f. The function f is defined for all elements in A and the values of f, indicated by the heads of the arrows, must all be in B.

    For each $x \in \mathrm{A} - \mathrm{B}$, we define the set

$$ch(x) = \{x, f(x), f(f(x))), \ldots, f^i(x), \ldots\}.$$

    Every element in $ch(x)$ is in B, except for x itself which is in $\mathrm{A} - \mathrm{B}$. Let

$$\mathrm{C} = \bigcup_{x \in \mathrm{A} - \mathrm{B}} ch(x).$$

    Now define the function $h : \mathrm{A} \to \mathrm{B}$ as follows:

$$h(z) = \begin{cases} f(z), & \text{if } z \in \mathrm{C}; \\ z, & \text{otherwise.} \end{cases}$$

    To show that h is a one-to-one, we must prove that $h(x) = h(y)$ implies that $x = y$. There are four cases to consider.

    Case 1: $x, y \notin \mathrm{C}$. Then $x = h(x) = h(y) = y$.

Case 2: $x \in$ C and $y \notin$ C. Since $x \in$ C, $h(x) = f(x)$ is in C. But $h(x) = h(y) = y$. This implies that $y \in$ C, which is a contradiction. Thus $h(x)$ cannot equal $h(y)$ in this case.

Case 3: $x \notin$ C and $y \in$ C. Same argument as case 2.

Case 4: $x, y \in$ C. Let f^i denote the composition of f with itself i times, and f^0 denote the identity function. The proof uses the fact that the composition of one-to-one functions is one-to-one. Although you will be familiar with functional composition from previous mathematical studies, a description and formal definition are given in Section 9.4 of the text if you need a reminder.

Since x and y are both in C, $x = f^m(s)$ and $y = f^n(t)$ for some $s, t \in$ A $-$ B. Then

$$h(x) = f(f^m(s)) = h(y) = f(f^n(t)).$$

If $m = n$, then $s = t$ and $x = y$ and we are done. Assume that $m > n$. Then $f^{m-n}(s) = t$. Applying the function f^n to both sides we get $f^m(s) = f^n(t)$, or equivalently, $x = y$. A similar argument shows $x = y$ when $m < n$.

We now show that h maps A onto B. For each $x \in$ B but not in C, $h(x) = x$ and x is covered by the mapping. If $x \in$ B and $x \in$ C, then $x = f(t)$ for some $t \in$ C because each element in C is either in A $-$ B or obtained by the result of applying f to an element in C. Consequently, each element of B is 'hit' by the mapping h. Because h is a one-to-one function from A to B, we conclude that $card$(A) = $card$(B).

To prove the Schröder-Bernstein Theorem in its generality, we must show that $card$(X) $\leq$ $card$(Y) and $card$(Y) $\leq$ $card$(X) implies $card$(X) = $card$(Y) for arbitrary sets X and Y. By the assumption, there are one-to-one functions f : X $\to$ Y and g : Y $\to$ X. Let

$$Im(\mathrm{Y}) = \{x \in \mathrm{X} \mid x = g(y) \text{ for some } y \text{ in } \mathrm{Y}\}$$

be the image of Y in X under g. Now

- Im(Y) is a subset of X,
- Im(Y) has the same cardinality as Y (g is one-to-one and onto), and
- the composition $f \circ g$ is a one-to-one mapping from X into Im(Y).

By the preceding result, $card$(X) = $card$(Im(Y)). It follows that $card$(X) = $card$(Y) by Exercise 27.

31. Let L be the set of the points in $\mathbf{N} \times \mathbf{N}$ on the line defined by $n = 3 \cdot m$. L can be defined recursively by

    Basis: $[0, 0] \in$ L.

    Recursive step: If $[m, n] \in$ L, then $[s(m), s(s(s(n)))] \in$ L.

    Closure: $[m, n] \in$ L only if it can be obtained from $[0, 0]$ using finitely many applications of the recursive step.

33. The product of two natural numbers can be defined recursively using addition and the successor operator s.

    Basis: if $n = 0$ then $m \cdot n = 0$

    Recursive step: $m \cdot s(n) = m + (m \cdot n)$

    Closure: $m \cdot n = k$ only if this equality can be obtained from $m \cdot 0 = 0$ using finitely many applications of the recursive step.

37. The set $\mathcal{F}$ of finite subsets of the natural numbers can be defined recursively as follows:

Basis: $\emptyset, \{0\} \in \mathcal{F}$

Recursive step: If $\{n\} \in \mathcal{F}$, then $\{s(n)\} \in \mathcal{F}$.
If X, Y $\in \mathcal{F}$, then X $\cup$ Y $\in \mathcal{F}$.

Closure: A set X is in $\mathcal{F}$ only if it can be obtained from the basis elements by a finite number of applications of the recursive step.

The first rule in the recursive step generates all sets containing a single natural number. The second rule combines previously generated sets to obtain sets of larger cardinality.

39. We prove, by induction on n, that

$$\sum_{i=0}^{n} 2^i \;=\; 2^{n+1} - 1$$

Basis: The basis is $n = 0$. We explicitly show that the equality holds for this case.

$$\sum_{i=0}^{0} 2^i \;=\; 2^0 \;=\; 1 \;=\; 2^1 - 1$$

Inductive hypothesis: Assume, for all values $k = 1, 2, \ldots, n$, that

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

Inductive step: We need to show that

$$\sum_{i=0}^{n+1} 2^i \;=\; 2^{(n+1)+1} - 1$$

To utilize the inductive hypothesis, the summation is decomposed into the sum of the first n powers of 2 and 2^{n+1}.

$$\begin{aligned}\sum_{i=0}^{n+1} 2^i &= \sum_{i=0}^{n} 2^i + 2^{n+1} \\ &= 2^{n+1} - 1 + 2^{n+1} \qquad \text{(inductive hypothesis)} \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{(n+1)+1} - 1\end{aligned}$$

43. The set R of nodes reachable from a given node x in a directed graph is defined recursively using the adjacency relation A.

Basis: $x \in$ R.

Recursive step: If $y \in$ R and $[y, z] \in$ A, then $z \in$ R.

Closure: $y \in$ R only if y can be obtained from x by finitely many applications of the recursive step.

46. a) The depth of the tree is 4.

b) The set of ancestors of x_{11} is $\{x_{11}, x_7, x_2, x_1\}$. Recall that by our definition is node is an ancestor of itself, which is certainly not the case in family trees.

c) The minimal common ancestor of x_{14} and x_{11} is x_2; of x_{15} and x_{11} is x_1.

d) The subtree generated by x_2 is comprised of the arcs $[x_2, x_5]$, $[x_2, x_6]$, $[x_2, x_7]$, $[x_5, x_{10}]$, $[x_7, x_{11}]$, and $[x_{10}, x_{14}]$.

e) The frontier is the set $\{x_{14}, x_6, x_{11}, x_3, x_8, x_{12}, x_{15}, x_{16}\}$.

48. Induction on the depth of the tree is used to prove that a complete binary tree T of depth n has $2^{n+1} - 1$ nodes. Let $nodes(\mathrm{T})$ and $leaves(\mathrm{T})$ denote the number of nodes and leaves in a tree T.

Basis: The basis consists of trees of depth zero; that is, trees consisting solely of the root. For any such tree T, $nodes(\mathrm{T}) = 1 = 2^1 - 1$.

Inductive hypothesis: Assume that every complete binary tree T of depth k, $k = 0, \ldots, n$, satisfies $nodes(\mathrm{T}) = 2^{k+1} - 1$.

Inductive step: Let T be a complete binary tree of depth $n + 1$, where $n \geq 0$. We need to show that $nodes(\mathrm{T}) = 2^{(n+1)+1} - 1$. T is obtained by adding two children to each leaf of a complete binary tree T′ of depth n. Since T′ is complete binary, it is also strictly binary and

$$leaves(\mathrm{T}') = (nodes(\mathrm{T}') + 1)/2$$

by Exercise 47. Thus

$$\begin{aligned}
nodes(\mathrm{T}) &= nodes(\mathrm{T}') + 2 \cdot leaves(\mathrm{T}') \\
&= nodes(\mathrm{T}') + 2 \cdot [(nodes(\mathrm{T}') + 1)/2] && \text{(Exercise 47)} \\
&= 2 \cdot nodes(\mathrm{T}') + 1 \\
&= 2 \cdot (2^{n+1} - 1) + 1 && \text{(inductive hypothesis)} \\
&= 2^{(n+1)+1} - 1
\end{aligned}$$