Chapter 1
Problem Solutions

Concept Problems

1. What is the difference between the transient response and the sinusoidal steady state response of a digital filter.

Solution

The transient response of a filter is generally taken to be the response to a unit impulse. If the filter is stable the response goes to zero in finite time. The sinusoidal steady state response is the response to sinusoids of unit amplitude at some set of frequencies. For a linear system the response is always sinusoidal and is usually expressed as a gain and a phase response. Any transient startup of a sinusoidal signal is ignored.

2. Give at least three applications in which linear phase is a requirement.

Solution

- •Any data transmission in which phase is used to encode information.
- • Any signal which is to be passed through a filter which must not distort the wave shape
- • Interpolators or decimators in which a linear phase FIR filter can be made more efficient than a comparable IIR filter.
- • Anywhere signals are filtered and added.

3. In the introduction to this chapter a digital filter was designed that approximated an analog filter consisting of a resistor and a capacitor. Since it is clear that the digital filter was much more complex in terms of hardware what kind of advantages might be gained by creating a digital filter to replace an analog filter. Consider flexibility of the design, the design effort, cost of the hardware, long term stability, and quality of the filter.

Solution

• *Flexibility* – While digital filters can be implemented in hardware using all hardware components, the more typical implementation uses mostly software. Software makes a filter much more flexible in terms of changes, error correction, scaling, or duplication.

• *Design effort* – Much of filter design, both analog and digital, comes from existing designs that are modified to meet needs. If we were starting from scratch, the analog filter which does not require an A/D or D/A converter is probably easier. In terms of the filter itself analog and digital filters are roughly the same amount of design effort.

• *Cost* – Digital filters become much more cost effective as the order increases as compared to analog filters. In terms of component cost for example, a second or third order analog filter can be much cheaper than a standalone digital counterpart. But as the order gets higher, analog filters become more expensive since parts precision and temperature compensation becomes important.

• *Long term stability* – This is decidedly in favor of the digital filter which does not suffer from aging or temperature drift to the degree that an analog counterpart does.

• *Quality of the filter* – At higher orders, digital filters are decidedly higher in quality than analog counterparts. At lower orders the two can be competitive depending on what one is willing to spend on hardware.

4. Explain why one would expect an IIR filter to be computationally more efficient for a given algorithm than an FIR filter.

Solution

An FIR filter has all of its poles at the origin. In terms of frequency space, it's as if the filter has a fixed gain and we locate zeros are specific locations where we want to drive the gain toward zero. The IIR filter, on the other hand, has poles located anywhere inside the unit circle. The designer can use poles and zeros together to adjust the gain up or down as needed. The poles located inside the unit circle for the IIR filter provide feedback terms in the difference equation. Thus, the output is fed back and can be used to further alter the gain.

The ability to locate poles anywhere inside the unit circle which provides feedback terms gives the IIR filter a slight advantage in altering the gain curve over an FIR filter of the same order.

5. FIR filters do not have a feedback path. What are the implications of this for system stability? Is oscillation possible? Why or why not?

Solution

FIR systems are always stable. A system is stable if its impulse response falls to zero in a finite amount of time. For an FIR system, once an impulse has passed through the system, the output must be exactly zero so all FIR systems are inherently stable.

Likewise, oscillation is not possible with an FIR filter. An oscillator has an output when there is not input. For an FIR filter the output must eventually drop to zero once the input goes to zero. Oscillation is not possible.

6. IIR filters have a feedback path and FIR filters do not. What would you expect to be the result of small errors in the formulation of the coefficients for these two systems? Would the impact of such an error be greater or less for an IIR filter? Explain.

Solution

For and IIR system a small error can be fed back to the input to reverberate through the system. Some frequencies caused by errors may be amplified. In general, IIR systems are more sensitive to coefficient quantization errors than are FIR systems which have no feedback.

7. Trapezoidal integration was used to approximate an integral and convert a differential equation to a difference equation. What would be the consequences of using an approximation based on a second order equation such as a parabola rather than the straight line fit of the trapezoidal method?

Solution

A second order approximation will likely give a better approximation to an integral but it will also double the order of the system. Higher order approximations are not used because of the increased computational complexity of the result.

8. Since it is often difficult in practice to approximate an impulse, describe another method of finding the impulse response indirectly from say, a step function which is readily available on most signal generators.

Solution
Since the impulse is the derivative of the step the impulse response is the derivative of the step response for a linear system. In practice, we can measure the step response, fit a curve to the response so that we have it in closed form, and take the derivative of the fitted curve to get the impulse response.

9. When a computer is used to evaluate a difference equation, the coefficients are represented as binary numbers where the number of bits used is limited. This causes the coefficient to have a small error called the *quantization error*. The same kind of error occurs in the A/D converter. What other errors are likely in the evaluation of a difference equation?
Solution
The evaluation of a difference equation is the summing of the products of the coefficients and input or output terms. When two N-bit numbers are multiplied a 2N-bit product is produced which must be rounded or truncated to N-bits. This is another source of error that can be mitigated by using extra bits for products called *guard bits*.

Overflow is another source of error. Filters can have overflow in a single stage at a particular frequency and must be properly scaled to avoid this error.

Aliasing is another source of error. Aliasing most often come into play by way of high frequency noise which comes with the input signal.

Finally, there can be error in the D/A and reconstruction process as the signal is converted from digital back to analog.

10. Suppose that in the evaluation of a difference equation the coefficients of the input variable and the delayed versions of the input variable are multiplied by a constant. What is the consequence of this multiplication in the frequency domain?
Solution
The input variable and its delayed versions appear in the numerator of the transfer function. If these are multiplied by a constant the effect will be go increase the gain of the filter by that constant.
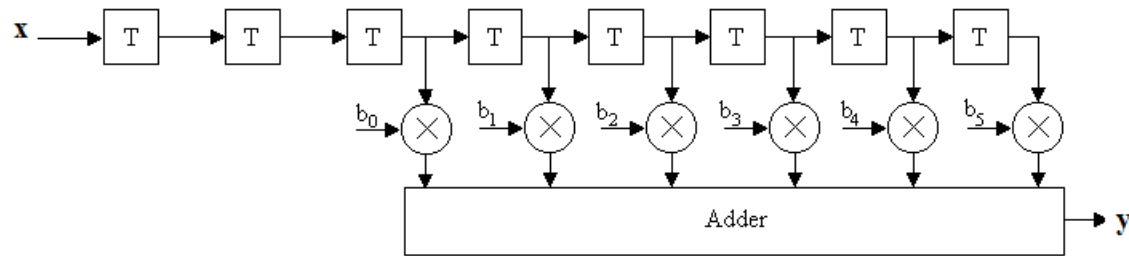
11. In problem 1.7 the impulse response of the FIR filters is given by
$$h(nT) = \{b_0, b_1, b_2, b_3, b_4, b_5\}$$
Redraw the filter diagram so that the impulse response is given by
$$h(nT) = \{0,0,0, b_0, b_1, b_2, b_3, b_4, b_5\}$$
What are the consequences of adding these zeros to the impulse response on the magnitude vs frequency plot? What are the consequences on the phase vs frequency plot?

Solution

The magnitude plot will be unchanged but the phase plot will be the same shape but shifted by three time periods across all frequencies.

12. Suppose the impulse response function for an FIR difference equation is symmetric. For example the response might be given by

$h(nT) = \{b_0, b_1, b_2, b_3, b_4, b_5, b_4, b_3, b_2, b_1, b_0\}$.

How can the difference equation be written to use this symmetry to reduce the number of multiplications necessary for its evaluation?

Solution

The difference equation can be written as:

$y(n) = b_0 x(n) + b_1 x(n-1) + ... + b_4 x(n-4) + b_5 x(n-5) + b_4 x(n-6) + ... + b_0 x(n-10)$

Because of the symmetry we can factor out some of the coefficients to write:

$y(n) = b_0 [x(n) + x(n-10)] + b_1 [x(1) + x(n-9)] + ... + b_4 [x(n-4) + x(n-6)] + b_5 x(n-5)$

Thus, instead of doing 11 multiplications we can do the additions first and do only 6 multiplications.

13. When the coefficients of a difference equation are implemented in real time we would like to represent the coefficients with as many bits as possible since this reduces the quantization error. Give three reasons why increasing the number of bits has negative consequences on the implementation of a difference equation.

Solution

• The coefficients are stored in registers or memory and increasing the number of bits forces the use of longer registers and more memory.

• The data path (busses) inside the DSP processor unit need to be as wide as the coefficients or at least some integer multiple of the coefficient size. Increasing the number of bits makes the data path wider and increases the hardware cost and complexity.

• Multiplication time will be longer since multiplication time is usually done on a bit by bit basis. For cases where multiplication is done in parallel hardware, the cost and complexity of this hardware is increased.

14. In general a sinusoid can be represented by the equation $y = A\sin(2\pi f t)$ where A is the amplitude and f is the frequency. This equation has two unknowns so that at least two values for y at two different time samples are needed in order to determine A and f. What other information do I need in order to *uniquely* determine values for A and f?

Solution

You need to know that the two samples are both within a single cycle of the sinusoid.

15. A given low pass filter has a cutoff frequency f_c and a sample frequency f_s. The sample period is $T = 1/f_s$. In implementing the filter it is essential that the evaluation of the difference equation be completed in less than T seconds. What are the consequences with regard to f_c if this is not the case. For example, suppose it takes 1.2T to do the evaluation of the difference equation.

Solution

This effectively decreases the sample frequency and there will be a corresponding decrease in the cut off frequency of a low pass filter. In addition, if the sample frequency is slower than the design called for, aliasing could be a problem.

Analysis and Design Problems

1.1 Using the numerical integration by trapezoids outlined in this chapter find the difference equation for a digital filter to approximate the analog filter below.
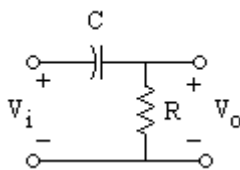
Figure P1.1
An analog filter.

Solution

The loop equation for the circuit is

$$v_i = v_c + v_0$$

But $v_0 = iR$ and

$$v_c = (1/C)\int i\,dt = (1/RC)\int v_0\,dt$$

The loop equation can be written as

$$(1/RC)\int v_0\,dt = v_i - v_o$$

Using (1.8) for trapezoidal integration we get

$$v_i(n) - v_o(n) = v_i(n-1) - v_o(n-1) + (T/2RC)[v_o(n) + v_o(n-1)]$$

Which simplifies to

$$v_0(n) = K_1[v_i(n) - v_i(n-1)] + K_2 v_o(n-1)$$

Where $K_1 = \dfrac{2RC}{2RC + T}$ and $K_2 = \dfrac{2RC - T}{2RC + T}$

1.2 A typical computer program for a digital filter was given as:

```
Initialize Variables
DO Forever
   Call AtoD(Vi)          ;Get a sample from the A to D
   Vo = K1*Vi + K2*Vo1
   Call DtoA(Vo)          ;Output Vo to D to A
   Vo1 = Vo               ;Reset the value of the old variable.
   Wait for T seconds to pass
Loop
```

```
End
```

Answer the following questions about the program:

    A) What is the purpose of the statement "Wait for T seconds to pass." What are the consequences of removing this statement.

    B) What variable(s) need to be initialized by the "Initialize Variables" statement.

    C) If the A to D converter takes 10 microseconds to complete a conversion, the computer requires 8 microseconds for each multiply and .5 microseconds for each add and the D to A converter requires .1 microsecond, approximately what is the maximum sample frequency for this filter with this difference equation.

Solution

A) This statement forces the loop to complete every T seconds. If it is removed, the sampling frequency will change.

B) K1 and Vo1

C) There are 2 multiplies and 1 add = 16.5 µsec plus the A/D = 8 µsec plus the D/A = 0.1 µsec comes to 24.6 µsec. Maximum sample frequency s $1/24.6 \mu sec = 40.65$ KHz.

1.3 For the difference equation given by:

$$v_o(nT) = 0.3333v_i(nT) + 0.3333v_i(n-1)T + 0.3333v_o(n-1)T$$

The sample frequency is 1 KHz.

    A) Find and tabulate the impulse response.

    B) Find and tabulate the step response.

    C) Express the frequency response of the output/input in terms of sines and cosines.

    D) Plot the magnitude and phase of the frequency response for the frequency range 0 to fs/2.

    E) Does this filter represent a FIR or an IIR filter? Explain.

Solution

A)

n	v_i	$v_i/3$	$v_i(n-1)/3$	$v_o(n-1)/3$	v_o
0	1	1/3	0	0	1/3
1	0	0	1/3	1/9	4/9
2	0	0	0	4/27	4/27
3	0	0	0	4/81	4/81
4	0	0	0	4/243	4/243
…	…	…	…	…	…

B)

n	v_i	$v_i/3$	$v_i(n-1)/3$	$v_o(n-1)/3$	v_o
0	1	1/3	0	0	1/3
1	1	1/3	1/3	1/9	7/9
2	1	1/3	1/3	7/27	25/27
3	1	1/3	1/3	25/81	79/81
4	1	1/3	1/3	79/243	241/243
…	…	…	…	…	…

C) The difference equation is

$$v_o(nT) = 0.3333v_i(nT) + 0.3333v_i(n-1)T + 0.3333v_o(n-1)T$$

Let $v_i(nT) = e^{j\omega nT}$, $v_i(n-1)T = e^{j\omega(n-1)T}$, $v_o(nT) = Ae^{j\omega nT}$ and $v_o(n-1)T = Ae^{j\omega(n-1)T}$

This gives

$$Ae^{j\omega nT} = (1/3)e^{j\omega nT} + (1/3)e^{j\omega(n-1)T} + (1/3)Ae^{j\omega(n-1)T}$$

Divide both sides by $e^{j\omega nT}$

$$A = (1/3) + (1/3)e^{-j\omega T} + (1/3)Ae^{-j\omega T}$$

Solve this for A to get

$$A = \frac{(1/3)(1+e^{-j\omega T})}{1-e^{-j\omega T}}$$

$$\frac{v_o(e^{j\omega nT})}{v_i(e^{j\omega nT})} = \frac{Ae^{j\omega nT}}{e^{j\omega nT}} = A$$

Applying Euler's identity we get

$$\frac{v_o(e^{j\omega nT})}{v_i(e^{j\omega nT})} = \frac{(1/3)[\cos(\omega T)+1-j\sin(\omega T)]}{1-\cos(\omega T)+j\sin(\omega T)}$$

$$\left|\frac{v_o e^{j\omega nT}}{v_i e^{j\omega nT}}\right| = (1/3)\frac{\sqrt{(\cos(\omega T)+1)^2+\sin^2(\omega T)}}{\sqrt{(1-\cos(\omega T))^2+\sin^2(\omega T)}}$$

$$\theta(\omega T) = \tan^{-1}\left(\frac{\sin(\omega T)}{\cos(\omega T)+1}\right) - \tan^{-1}\left(\frac{\sin(\omega T)}{1-\cos(\omega T)}\right)$$

D) Using MATLAB®

```
fs = 1000;T = 1/fs;
f = 0:fs/2;
w = 2*pi*f;
A = (1/3)*(1+exp(-j*w*T))./(1-(1/3)*exp(-j*w*T));
figure(1);clf;
subplot(2, 1, 1);
plot(f, abs(A));
title('Magnitude of A');
xlabel('frequency in Hz');
ylabel('Gain');
subplot(2, 1, 2);
plot(f, angle(A)*180/pi);
title('Phase of A');
xlabel('frequency in Hz');
ylabel('Angle in degrees');
```
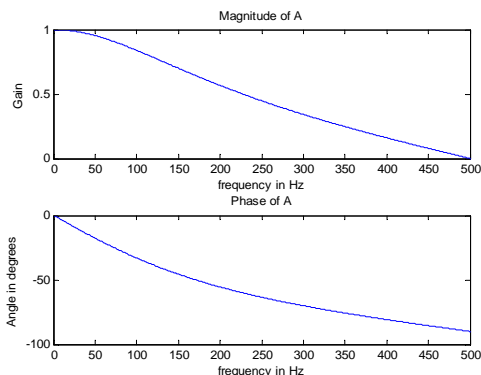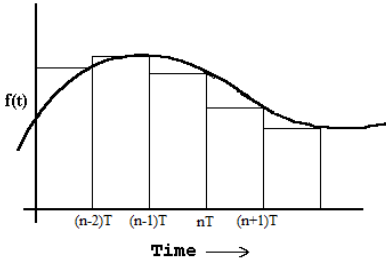

E) This represents an IIR filter since it has feedback.

1.4 The algorithm used in this chapter relied on a discrete approximation to the integral using trapezoids.
    A) Find the difference equation to approximate the integral if rectangles are used in place of trapezoids.
    B) Plot the frequency response of a trapezoidal and a rectangular integrator.

Solution

A)

If we take the variable y to represent the integral we can write:

$$y(nT) = \int_{0}^{nT} f(t) \cdot dt$$

This equation can be broken into two separate parts and rewritten as:

$$y(nT) = \int_{0}^{(n-1)T} f(t) \cdot dt + \int_{(n-1)T}^{nT} f(t) \cdot dt$$

which can be written as

$$y(nT) = y([n-1]T) + \int_{(n-1)T}^{nT} f(t) \cdot dt$$

If we approximate the integral with the area under the rectangle this equation becomes
$$y(nT) = y([n-1]T) + Tf(nT)$$

B) To find the frequency response we let $f(nT) = e^{j\omega nT}$ and we take $y(nT) = Ae^{j\omega nT}$
Putting these values into the difference equation from part A gives:
$$Ae^{j\omega nT} = Ae^{j\omega(n-1)T} + Te^{j\omega nT}$$
Solve this equation for A to get
$$A(1 - e^{-j\omega T}) = T \text{ or}$$

$$A = \frac{T}{1 - e^{-j\omega T}}$$ Rectangular integration sinusoidal gain equation.

For the trapezoidal integrator (1.8) gives the difference equation:
$$y(nT) = y([n-1]T) + (T/2)[f(nT) + f([n-1]T)]$$

Let $f(nT) = e^{j\omega nT}$ and we take $y(nT) = Ae^{j\omega nT}$ to get
$$Ae^{j\omega nT} = Ae^{j\omega(n-1)T} + (T/2)[e^{j\omega nT} + e^{j\omega(n-1)T}]$$
Solve for A to get
$$A = T/2 \frac{1 + e^{-j\omega T}}{1 - e^{-j\omega T}}$$ Trapezoidal integration sinusoidal gain equation

Use MATLAB® to plot the gain of the rectangular and trapezoidal gain equations.

```
fs = 1;T = 1/fs;
f = 0:.001:fs/2;
w = 2*pi*f;
Ar = T./(1-exp(-j*w*T));
At = (T/2)*(1+exp(-j*w*T))./(1-exp(-j*w*T));
figure(1);clf;
plot(f, abs(Ar));
title('Magnitude of rectangular and trapezoidal integrators');
xlabel('frequency in Hz');
ylabel('Gain');
hold on;
plot(f, abs(At), 'r');
axis([0 fs/2 0 10]);
```
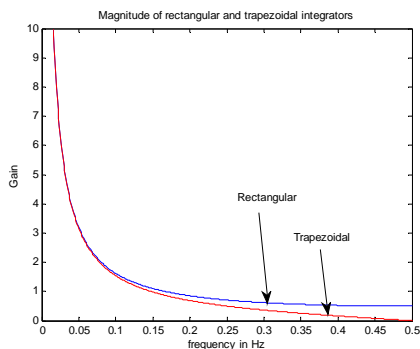

1.5 Describe in detail how you could use a sinusoidal signal generator and an oscilloscope to determine the gain vs. frequency plot for a digital filter.
Solution
• Set up the function generator for sinusoidal outputs with a 1 volt amplitude.
• Apply the function generator to the digital filter and using a dual trace oscilloscope observe both the input and the output.
• The gain is the amplitude of the output divided by the amplitude of the input and the phase is the phase shift with the input as the reference. Measure the gain and phase for frequencies from near zero to fs/2.

1.6 Show that the difference equation for the second order IIR filter Figure P1.6 is given by
$$y_k = b_0 x_k + b_1 x_{k-1} + b_2 x_{k-2} - a_1 y_{k-1} - a_2 y_{k-2}$$
where y_k is the output variable at time k and u_k is the input variable at time k.
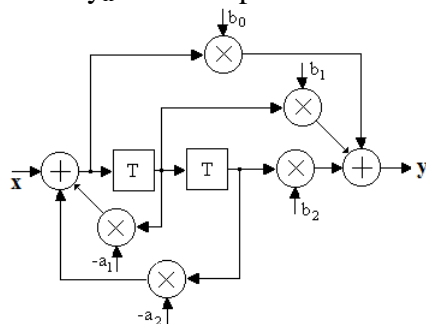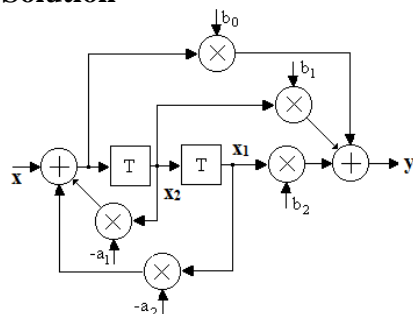

Figure P1.6

An IIR filter.
Solution

$$y(k) = b_2 x_1(k) + b_1 x_2(k) + b_0 x_2(k+1)$$

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = -a_2 x_1(k) - a_1 x_2(k)$$

| k	x(k)	x_2(k+1)	x_2(k)	x_1(k)	y(k)
0	1	1	0	0	b_0
1	0	$-a_1$	1	0	$-b_0 a_1 + b_1$
2	0	$-a_2 + a_1^2$	$-a_1$	1	$b_2 - b_1 a_1 + b_0(-a_2 + a_1^2)$
3	0	$a_1 a_2 + a_1 a_2 - a_1^3$	$-a_2 + a_1^2$	$-a_1$	$-b_2 a_1 + b_1(-a_2 + a_1^2) + b_0(a_1 a_2 + a_1 a_2 - a_1^3)$

We can also get the impulse response from the difference equation:

$$y_k = b_0 x_k + b_1 x_{k-1} + b_2 x_{k-2} - a_1 y_{k-1} - a_2 y_{k-2}$$

k	x(k)	x(k-1)	x(k-2)	y(k)	y(k-1)	y(k-2)
0	1	0	0	b_0	0	0
1	0	1	0	$-b_0 a_1 + b_1$	b_0	0
2	0	0	1	$b_2 - b_1 a_1 + b_0(-a_2 + a_1^2)$	$-b_0 a_1 + b_1$	b_0
3	0	0	0	$-b_2 a_1 + b_1(-a_2 + a_1^2) + b_0(a_1 a_2 + a_1 a_2 - a_1^3)$	$b_2 - b_1 a_1 + b_0(-a_2 + a_1^2)$	$-b_0 a_1 + b_1$

Since the difference equation has the same response to an impulse as does the evaluation of the state diagram the two forms must represent the same system.

1.7 Find the general expression for the impulse response for the FIR filter shown in Figure P1.7. Show that this response is always finite in length and has a length no greater than n+1 where n is the subscript of the final coefficient.
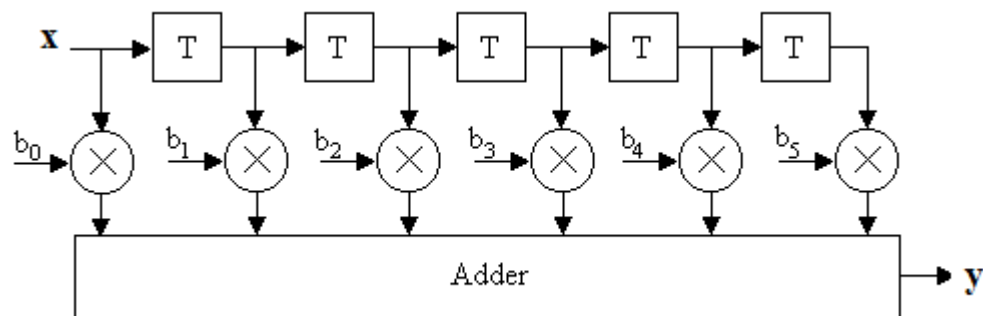
Figure P1.7
An FIR filter.

Solution

$h(nT) = \{b_0, b_1, b_2, b_3, b_4, b_5, 0, 0, 0, \ldots\}$

1.8 Answer the questions below for the difference equation given by:

$v_o(kT) = 0.5v_i(kT) - .035v_i(k-1)T - 0.825v_0(k-1)T$

    A) Does this equation represent an IIR or an FIR system? Explain.

    B) Write the first 5 terms of the impulse response for this system.

    C) Write the expression for the frequency response for this system if the sample frequency is
       1,000 Hz.

Solution

A) This is an IIR system since it has feedback.

B)

k	$v_i(k)$	$v_i(k-1)$	$v_o(k)$	$v_o(k-1)$
0	1	0	0.5	0
1	0	1	-0.4475	0.5
2	0	0	0.3692	-0.4475
3	0	0	-0.3046	0.3692
4	0	0	0.2513	-0.3046

C) The difference equation is:

$v_o(nT) = 0.5v_i(nT) - .035v_i(n-1)T - 0.825v_0(n-1)T$

Let $v_i(nT) = e^{j\omega nT}$, $v_i(n-1)T = e^{j\omega(n-1)T}$, $v_o(nT) = Ae^{j\omega nT}$ and $v_o(n-1)T = Ae^{j\omega(n-1)T}$

This gives

$Ae^{j\omega nT} = 0.5e^{j\omega nT} - 0.035e^{j\omega(n-1)T} - 0.825Ae^{j\omega(n-1)T}$

Divide both sides by $e^{j\omega nT}$

$A = 0.5 - 0.035e^{-j\omega T} - 0.825Ae^{-j\omega T}$

Solve this for A to get

$A = \dfrac{(0.5 - 0.035e^{-j\omega T})}{1 - 0.825e^{-j\omega T}}$

$\dfrac{v_o(e^{j\omega nT})}{v_i(e^{j\omega nT})} = \dfrac{Ae^{j\omega nT}}{e^{j\omega nT}} = A$

Applying Euler's identity we get

$\dfrac{v_o(e^{j\omega nT})}{v_i(e^{j\omega nT})} = \dfrac{[0.5 - 0.035\cos(\omega T) + j0.035\sin(\omega T)]}{1 - 0.825\cos(\omega T) + j0.825\sin(\omega T)}$

The magnitude response is:

$\left| \dfrac{v_o e^{j\omega nT}}{v_i e^{j\omega nT}} \right| = (1/3)\dfrac{\sqrt{(0.5 - 0.035\cos(\omega T))^2 + 0.035^2\sin^2(\omega T)}}{\sqrt{(1 - 0.825\cos(\omega T))^2 + 0.825^2\sin^2(\omega T)}}$

The phase response is:

$\theta(\omega T) = \tan^{-1}\left(\dfrac{0.035\sin(\omega T)}{0.5 - 0.035\cos(\omega T)}\right) - \tan^{-1}\left(\dfrac{0.825\sin(\omega T)}{1 - 0.825\cos(\omega T)}\right)$

$T = 0.001$.

1.9 Consider the difference equation given by
$$y(nT) = x(nT) + Ky(n-1)T$$
    A) Find the expression for the frequency response for y/x in terms of K. Take T = 1.
    B) Use MATLAB® to plot the frequency response for two cases: K = 0.8 and K = 1/0.8 = 1.25. What is the same and what is different about these two responses?
    C) Use MATLAB® to plot the impulse response for two cases: K = 0.8 and K = 1/0.8 = 1.25. Plot at least 10 terms.
    D) What can you conclude from the impulse response of these two difference equations that is not evident from the frequency response?

Solution

A) The difference equation is:
$$y(nT) = x(nT) + Ky(n-1)T$$
Let $x(nT) = e^{j\omega nT}$, $y(nT) = Ae^{j\omega nT}$ and $y(n-1)T = Ae^{j\omega(n-1)T}$

This gives
$$Ae^{j\omega nT} = e^{j\omega nT} + KAe^{j\omega(n-1)T}$$
Divide both sides by $e^{j\omega nT}$
$$A = 1 + KAe^{-j\omega T}$$
Solve this for A to get
$$A = \frac{1}{1 - Ke^{-j\omega T}}$$
$$\frac{y(e^{j\omega nT})}{x(e^{j\omega nT})} = \frac{Ae^{j\omega nT}}{e^{j\omega nT}} = A$$
Applying Euler's identity we get
$$\frac{y(e^{j\omega nT})}{x(e^{j\omega nT})} = \frac{1}{1 - K\cos(\omega T) + jK\sin(\omega T)}$$

The magnitude response is:
$$\left|\frac{ye^{j\omega nT}}{xe^{j\omega nT}}\right| = \frac{1}{\sqrt{(1 - K\cos(\omega T))^2 + K^2\sin^2(\omega T)}}$$

The phase response is:
$$\theta(\omega T) = 0 - \tan^{-1}\left(\frac{K\sin(\omega T)}{1 - K\cos(\omega T)}\right)$$

B) The magnitude plots have the same shape but when K = 0.8 the magnitude is overall slightly higher. If both plots were normalized to a gain of unity at 0 Hz they would be the same. The phase plots are very different. For K = 0.8 the phase curve begins and ends at 0°. It reaches a maximum of about -50° at about 0.1 Hz. When K = 1.25 the phase curve begins at -180° and ends at 0°. The total phase changes is about 180°.

```
fs = 1;T = 1/fs;
f = 0:.001:fs/2;
w = 2*pi*f;
K = 0.8;
A = 1./(1-K*exp(-j*w*T));
figure(1);clf;
subplot(2, 1, 1);
plot(f, abs(A));
title('Magnitude of A, K = 0.8');
xlabel('frequency in Hz');
ylabel('Gain');
subplot(2, 1, 2);
plot(f, angle(A)*180/pi);
title('Phase of A, K = 0.8');
xlabel('frequency in Hz');
ylabel('Angle in degrees');
```
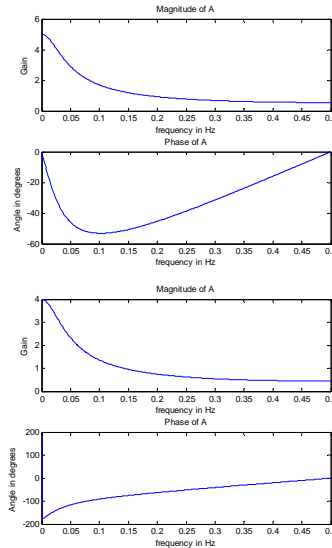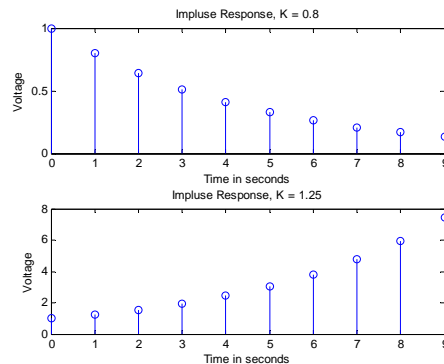

C)
```
x = [1 zeros(1, 9)]; %impulse in
y = zeros(1, 10);
y(1) = x(1);
K = 0.8;
for i = 2:length(y)
    y(i) = x(i) + K*y(i-1);
end
figure(4);clf;
fs = 1;T = 1/fs;
nT = (0:9)*T;
subplot(2, 1, 1);
stem(nT, y);
title('Impluse Response, K = 0.8');
xlabel('Time in seconds');
ylabel('Voltage');
```


D) When K = 1.25 the system is unstable.

1.10 A certain difference equation has an impulse response given by
$h(nT) = \{1.0, 0.5, 0.25, 0, 0, 0, \dots\}$
The step input function can be viewed as a series of impulses beginning at $nT = 0$. The step output must then be a series of impulse responses since we are dealing with linear systems. Use this information to find the first 5 terms of the step response of this system.

Solution

We view the step as a sequence of impulses each shifted one place further to the right from the last. The step response is the sum of the shifted impulse responses.

					n				
	0	1	2	3	4	5	6	7	8
0	1	0.5	0.25	0	0	0	0	0	0
1	0	1	0.5	0.25	0	0	0	0	0
2	0	0	1	0.5	0.25	0	0	0	0

-13-

3	0	0	0	1	0.5	0.25	0	0	0
4	0	0	0	0	1	0.5	0.25	0	0
5	0	0	0	0	0	1	0.5	0.25	0
sum	1	1.5	1.75	1.75	1.75	1.75	1.75	1.75	1.75

1.11 When a step input is applied to a certain difference equation, the output, or step response, is given by

$s(nT) = \{0.25, 0.5, 1.0, 1.0, 1.0, …\}$

A step function in discrete time can be viewed as a sequence of shifted impulses to the step response must be a sequence of shifted impulse responses. Use this information to derive the impulse response for this system.

Solution

From the previous problem we see that the next term of the step response is always the previous term plus the present term of the impulse response. We can write (from the solution of Problem 10)

$stp(n) = stp(n-1) + h(n)$

From this we get

$h(n) = stp(n) – stp(n-1)$

If the step response is

$s(nT) = \{0.25, 0.5, 1.0, 1.0, 1.0, …\}$

The impulse response is:

$h(nT) = \{0.25, 0.25, 0.5, 0, 0, 0, …\}$

1.12 Consider the difference equation given by

$y(nT) = 0.075x(nT) + 0.075x(n-1)T + 1.5y(n-1)T – 0.65y(n-2)T$

The frequency response for y/x for this equation is given in Figure P1.12. The sample frequency is $f_s = 1000$. The exploded view of the magnitude and phase plots show that at an input frequency of 100 Hz the gain is approximately 0.54 and the phase is approximately -111°. Using MATLAB$^{\circledR}$ allow x to be a sinusoid at 100 Hz with an amplitude of 1.0. Apply x to the difference equation and calculate the sequence for y(nT). Verify the gain and phase shift that were approximated from the frequency response curves. Note that you will have to produce several cycles of y to get an accurate reading since the first few cycles will contain the transient start up terms as well as the steady state response.
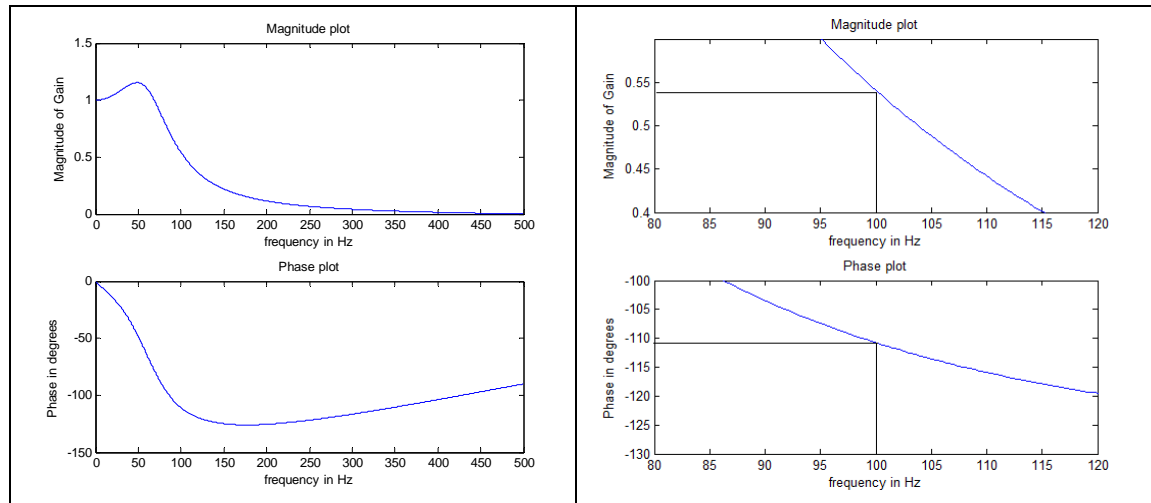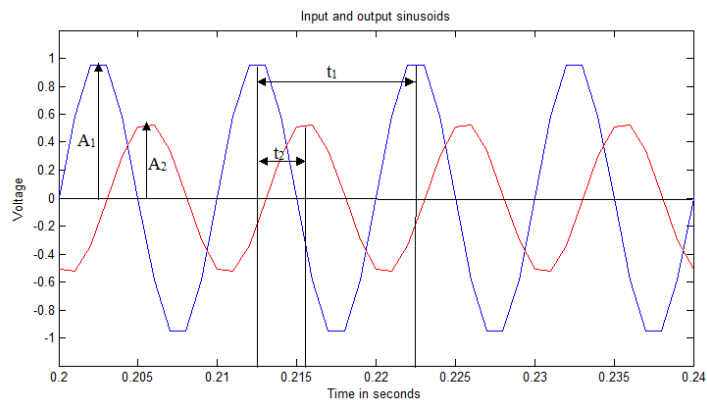
Figure P1.12
The frequency response curve left and a blow up right for a difference equation.

Solution
```
fs = 1000;T = 1/fs;
t = 0:T:1;
fsig = 100;
x = sin(2*pi*fsig*t);
y = zeros(1, length(x));
y(1) = .075*x(1);
y(2) = .075*x(2) + .075*x(1) + 1.5*y(1);
for i = 3:length(y)
    y(i) = .075*x(i) + .075*x(i-1) + 1.5*y(i-1) - .65*y(i-2);
end
figure(1);clf;
plot(t, x);
hold on;
plot(t, y, 'r');
axis([.2 .24 -1.2 1.2]);
title('Input and output sinusoids');
xlabel('Time in seconds');
ylabel('Voltage');
```


From the figure: $A_1 / A_2 \approx 0.54$ and $(t_2 / t_1)360 \approx 111^o$

-15-

1.13 In the difference equation below K is a real number greater than 0.
$$y(nT) = x(nT) + x(n-1)T + Ky(n-1)T$$
    A) Find the range of values of K that makes the impulse response grow without bound.
    B) Find the range of values of K that makes the impulse response fall to zero.

Solution

$$y(nT) = x(nT) + x(n-1)T + Ky(n-1)T$$

Tabulate the impulse response.

n	x(n)	x(n-1)	y(n)	y(n-1)
0	1	0	1	0
1	0	1	1+K	1
2	0	0	$K(1+K)$	1+K
3	0	0	$K^2(1+K)$	$K(1+K)$
4	0	0	$K^3(1+K)$	$K^2(1+K)$
…	…	…	…	…
N	0	0	$K^{(N-1)}(1+K)$	$K^{(N-2)}(1+K)$

A) If $K^{(N-1)} + K^N$ is to grow without bound then $|K| > 1$
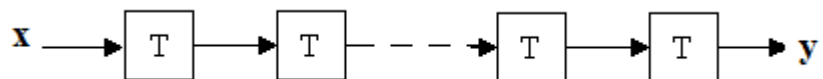
B) In order for $K^{(N-1)} + K^N$ to approach zero we need $|K| < 1$

1.14 An N-stage delay line is shown in Figure P1.15.
    A) Write the difference equation for the output.
    B) Write the frequency response for y/x.
    C) Show that the magnitude of the frequency response is always 1 and the phase shift is
        linearly related to the frequency.

Figure P1.14
A delay line

Solution

A) $y(n) = x(n-N)$

B) $x(nT) = e^{j\omega nT}$ and $y(nT) = Ae^{j\omega nT}$

This gives

$$Ae^{j\omega nT} = e^{j\omega(n-N)T}$$

Divide both sides by $e^{j\omega nT}$

$$A = e^{-j\omega NT}$$

C) Magnitude of A

$$|A| = 1$$

Phase of A

$$\angle A = -\omega NT$$