INSTRUCTOR'S SOLUTIONS MANUAL

TO ACCOMPANY

INTRODUCTION TO DIGITAL COMMUNICATIONS

MICHAEL B. PURSLEY

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Errata for INTRODUCTION TO DIGITAL COMMUNICATIONS

January 8, 2005

Page 20, Problem 1.4(b): The word "they" refers to U and V. A better wording is: If, in addition, X and Y have the same mean, are U and V uncorrelated?

Page 97, six lines from the bottom: An integral sign is missing in the equation for $R_Y(t, s)$. It should be a double integral.

Page 324, first displayed equation: In the first equation for the average probability of error, each numerical subscript is off by one. The subscripts should range from 0 to 3, not from 1 to 4. The subscripts are correct in the second displayed equation.

Page 377, six lines below equation (6.73): "for some signal constellations the bit error depends on" should be "for some signal constellations the bit error probability depends on" (the word "probability" is missing)

Page 409, Problem 6.15: The first sentence should be "Consider the symbol error probabilities for regular QASK signal constellations with maximum-likelihood receivers."

Page 410, Problem 6.20(a): $3d$ should be $2d$

Page 650, first sentence of last paragraph: "locally symmetry" should be "local symmetry"

Chapter 1 Solutions

1.2 We assume, of course, that $n > 1$. For such n, **(a)** the probability that the code can correct the error pattern is

$$P_c = (1-p)^n + np(1-p)^{n-1} = [1+(n-1)p](1-p)^{n-1},$$

and **(b)** the probability that the code can detect the error pattern is

$$P_d = (1-p)^n + np(1-p)^{n-1} + \binom{n}{2} p^2(1-p)^{n-2} = \left[1+(n-2)p + \frac{(n-2)(n-1)p^2}{2}\right](1-p)^{n-2}.$$

1.4 **(a)** $E\{UV\} = E\{(X+Y)(X-Y)\} = E\{X^2\} - E\{Y^2\} = 0$, so U and V are orthogonal.

(b) Yes, U and V are uncorrelated. In part **(a)** we found that $E\{UV\} = 0$, so it suffices to show that $E\{U\}E\{V\} = 0$, which follows from $E\{V\} = E\{X-Y\} = E\{X\} - E\{Y\} = 0$.

1.6 **(a)** $E\{(X_3+X_9)^2\} = E\{X_3^2\} + E\{X_9^2\} + 2E\{X_3X_9\} = 24\beta$.

(b) $\beta = E\{X_i^2\} < \infty$ implies that $\beta < \infty$, $\beta = E\{X_i^2\} \geq 0$ implies that $\beta \geq 0$, and $P(X_i = 0) < 1$ implies that $\beta \neq 0$; therefore, $0 < \beta < \infty$.

1.8 The probability of error is given by

$$P_e = P(X_1 > X_2) = P(X_1 - X_2 > 0) = 1 - P(X_1 - X_2 \leq 0) = 1 - P(X \leq 0),$$

where $X = X_1 - X_2$. X has mean $\mu = \mu_1 - \mu_2$. Because X_1 and X_2 are uncorrelated, X has variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$. Thus,

$$P_e = 1 - \Phi[-\mu/\sigma] = \Phi[\mu/\sigma] = \Phi\left[(\mu_1-\mu_2)/\sqrt{\sigma_1^2+\sigma_2^2}\right] = Q\left[(\mu_2-\mu_1)/\sqrt{\sigma_1^2+\sigma_2^2}\right].$$

1.9

$$F_Y(v) = \Phi\left[(v-\mu_1-\mu_2-\mu_3)/\sqrt{\Lambda_{1,1}+\Lambda_{2,2}+\Lambda_{3,3}+2\Lambda_{1,2}+2\Lambda_{2,3}+2\Lambda_{1,3}}\right]$$

1.10 The answer can be given in terms of Φ or Q:

$$\begin{aligned} P(Z^2 < 2) &= \Phi[(\sqrt{2}-\mu)/\sigma] - \Phi[(-\sqrt{2}-\mu)/\sigma] = \Phi[(\mu+\sqrt{2})/\sigma] - \Phi[(\mu-\sqrt{2})/\sigma] \\ &= Q[(\mu-\sqrt{2})/\sigma] - Q[(\mu+\sqrt{2})/\sigma] \end{aligned}$$

1.11 $f_{X_1}(x) = f_{X_2}(x)$ is equivalent to $\exp\{-(x-\mu_1)^2/2\sigma^2\} = \exp\{-(x-\mu_2)^2/2\sigma^2\}$, which is equivalent to $(x-\mu_1)^2 = (x-\mu_2)^2$, which is true if and only if the distance from x to μ_1 is the same as the distance from x to μ_2. That is, x is located half way between μ_1 and μ_2, which is true if and only if $x = (\mu_1+\mu_2)/2$. The reason for the restriction $\mu_1 \neq \mu_2$ should be obvious to you. (If $\mu_1 = \mu_2$ then $f_{X_1}(x) = f_{X_2}(x)$ for all x: The densities are identical.)

1.12 $f_{X_1}(y) > f_{X_2}(y)$ is equivalent to $\exp\{-(y-\mu_1)^2/2\sigma^2\} > \exp\{-(y-\mu_2)^2/2\sigma^2\}$. Since $\exp\{-u\}$ is a decreasing function of u, it follows that $f_{X_1}(y) > f_{X_2}(y)$ is equivalent to $(y-\mu_1)^2 < (y-\mu_2)^2$, which is equivalent to $|y-\mu_1| < |y-\mu_2|$. Thus, $f_{X_1}(y) > f_{X_2}(y)$ if and only if y is closer to μ_1 than to μ_2.

1.13 The square is the set $S = \{(u, v) : 1 \le u \le 2, 1 \le v \le 2\}$. The probability that the point (X_1, X_2) is in the square is

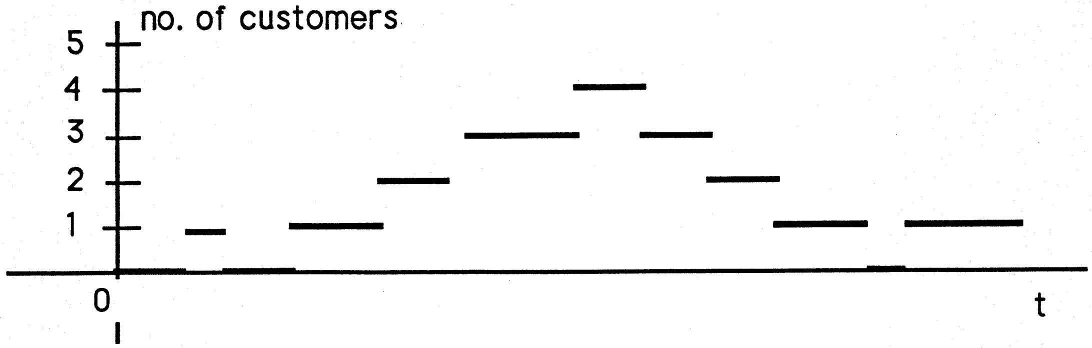
$$\begin{aligned} P[(X_1, X_2) \in S] &= P(1 \le X_1 \le 2, 1 \le X_2 \le 2) = P(1 \le X_1 \le 2)P(1 \le X_2 \le 2) \\ &= \{\Phi[(2 - \mu_1)/\sigma] - \Phi[(1 - \mu_1)/\sigma]\}\{\Phi[(2 - \mu_2)/\sigma] - \Phi[(1 - \mu_2)/\sigma]\}. \end{aligned}$$

Notice that

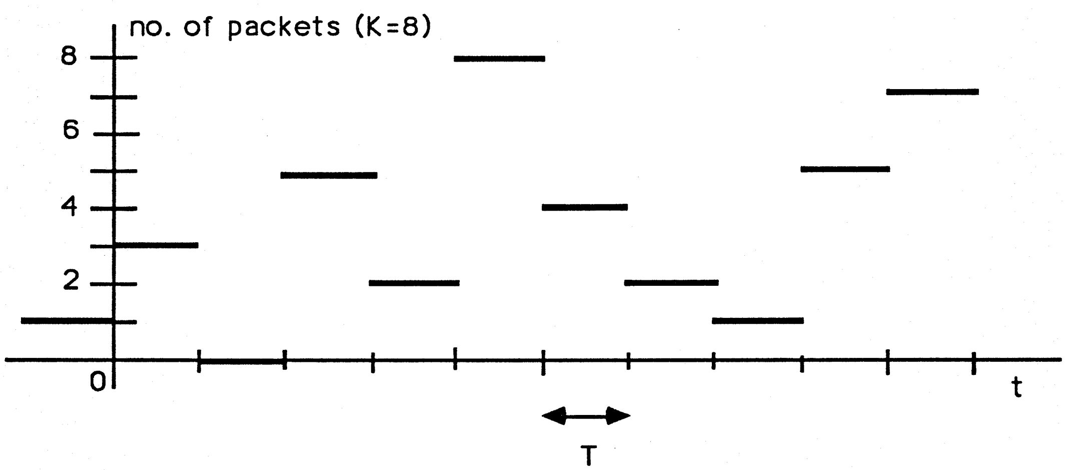
$$\Phi[(2 - \mu)/\sigma] - \Phi[(1 - \mu)/\sigma] = \int_{1-\mu}^{2-\mu} \frac{1}{\sqrt{2\pi}\,\sigma} \exp(-v^2/2\sigma^2)\, dv.$$

Because $\exp(-v^2/2\sigma^2)$ is a decreasing function of v^2, the integral is maximized when the interval of integration is centered about zero; that is, when $(2 - \mu) = -(1 - \mu)$, which requires that $\mu = 3/2$. Thus, $P[(X_1, X_2) \in S]$ is maximized for $\mu_1 = \mu_2 = 3/2$.

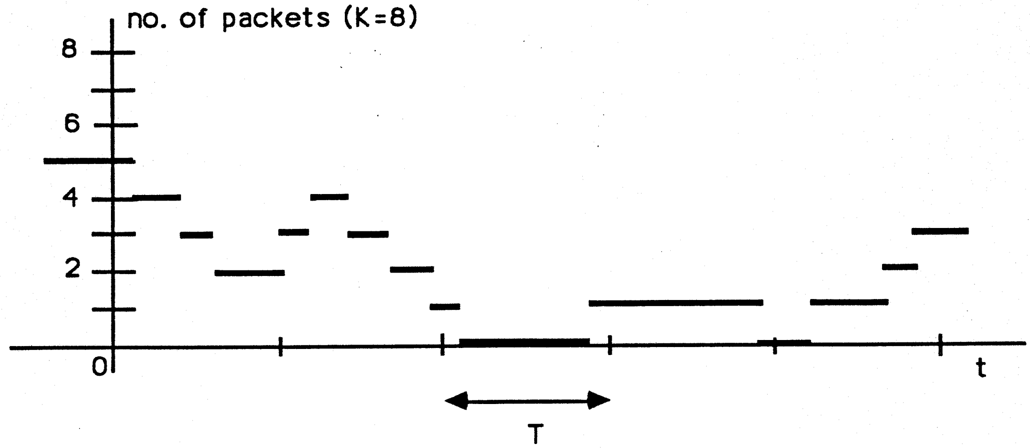
2.2 A typical sample function for the number of customers waiting in line is shown below. Notice that the sample functions for this random process may increase or decrease (the sample functions for the counting process of Example 2-3 are nondecreasing as illustrated in Figure 2-5). As in the counting process of Example 2-3, we asume that two customers do not arrive at precisely the same time. The sample function shown below satisfies N(t)=0 for t=0, but that is not necessary (consider a store that is open 24 hours per day).



2.3 (a) Regardless of when the packet is ready, the terminal must wait until the beginning of the next time slot in order to begin the transmission. In effect, the arrivals in a particular time slot are not "counted" until they are actually transmitted at the ***beginning*** of the ***next*** time slot. Therefore, the jump discontinuities in this random process can occur at T-second intervals only, the size of the jump at one of these times is arbitrary in magnitude, and the jump can be positive or negative. The only constraint on the size and magnitude of a jump is that the process N(t) can never be less than zero or greater than K (because K is the total number of terminals, it is the maximum number of packets that can be transmitted simultaneously). A typical sample function is shown below. In this figure, the intervals are shown as all starting at multiples of T, but that is not necessary; in general, the left-hand end points will be at times t_0+nT for each integer n.



2.3 (b) For asynchronous transmission without slotting, a terminal begins a transmission just as soon as the packet is ready. As with the counting process, we assume that no two packets are ready at precisely the same time. A typical sample function is shown below.



2.4 (1) The number of customers in a check-out line has no upper bound (except for the population of the world), whereas the number of packets being transmitted at time t can never exceed the number of transmitters in the network (which is given as K, presumably much smaller than the population of the world).

(2) Packet transmissions take a *fixed* amount of time (T seconds) for the system described in Problem 2.3, but the time required to check out a customer is *variable*. For instance, when N(t)=1 and there are no new arrivals, this feature results in fixed-length "flat segments" for the packet transmission sample paths (see the sample function in 2.3(b)), but not for the check-out line (see 2.2 solution).

2.5 (a) Continuous-time, discrete-amplitude random process
(b) Discrete-time, discrete-amplitude random process
(c) Discrete-time, continuous-amplitude random process
(d) Discrete-time, discrete-ampitude random process
(e) Continuous-time, continuous-amplitude random process

2.6 For $t \geq 2$, $f_{Y,1}(0,t) = P[Y(t)=0] = P[X(t)=X(t-1)=X(t-2)=0] = (1/2)^3 = 1/8$, and

$$f_{Y,1}(1,t) = P[Y(t)=1]$$
$$= P[X(t)=1,X(t-1)=X(t-2)=0]+P[X(t-1)=1,X(t)=X(t-2)=0]$$
$$+P[X(t-2)=1,X(t)=X(t-1)=0]$$
$$= 3(1/8) = 3/8.$$

Similarly, $f_{Y,1}(2,t) = 3/8$ and $f_{Y,1}(3,t) = 1/8$. Clearly $f_{Y,1}(k,t) = 0$ for other values of k.

For the second part of the problem, first observe that

$$\{Y(t)=3\}\cap\{Y(t+1)=2\} = \{X(t)=X(t-1)=X(t-2)=1,X(t+1)=0\}.$$

So, $f_{Y,2}(3,2;t,t+1) = P[Y(t)=3,Y(t+1)=2] = (1/2)^4 = 1/16$.

2.7 $F_{Y,1}(u;t) = P[Y(t)\le u] = P(e^{-Xt}\le u)$. Because $P(0\le X\le 1) = 1$, then $P(-t\le -Xt\le 0) = 1$ and $P(e^{-t}\le e^{-Xt}\le 1) = 1$. Therefore,

$$F_{Y,1}(u;t) = P(e^{-Xt}\le u) = 1 \text{ if } u\ge 1,$$

and

$$F_{Y,1}(u;t) = P(e^{-Xt}\le u) = 0 \text{ if } u\le e^{-t}.$$

So, we are left with the range $e^{-t}<u<1$. In this range,

$$F_{Y,1}(u;t) = P(e^{-Xt}\le u) = P([X\ge -t^{-1}\ln(u)] = 1 - P[X< -t^{-1}\ln(u)] = 1 - F_X(-t^{-1}\ln(u)). \qquad (2.7.1)$$

In the last step we have used the fact that X is a continuous random variable. Now, $e^{-t}<u<1$ implies $0< -t^{-1}\ln(u)<1$, which, together with the fact that X is uniformly distributed on [0,1] (i.e., $F_X(w) = w$ for $0<w<1$) implies

$$F_X(-t^{-1}\ln(u)) = -t^{-1}\ln(u) \text{ for } e^{-t}<u<1. \qquad (2.7.2)$$

combining (2.7.1) and (2.7.2), we have $F_{Y,1}(u;t) = 1 + t^{-1}\ln(u)$ for $e^{-t}<u<1$. Thus,

$$F_{Y,1}(u;t) = \begin{cases} 1, & u\ge 1 \\ 1 + t^{-1}\ln(u), & e^{-t}<u<1 \\ 0, & u\le e^{-t}. \end{cases}$$

2.8 First observe that $\tau > \tau_0 \Rightarrow |\tau| > \tau_0 \Rightarrow E\{X(t_0)X(t_0+\tau)\} = \mu_X(t_0)\mu_X(t_0+\tau)$. Also, the fact that X(t) is WSS implies $\mu_X(t_0)\equiv$ a constant (call it μ_X) and $E\{X(t_0)X(t_0+\tau)\} = R_X(\tau)$. Therefore, $\tau>\tau_0 \Rightarrow R_X(\tau) = \mu_X{}^2$. In particular, as $\tau\to\infty$ we eventually have $\tau>\tau_0$, so $\lim_{\tau\to\infty} R_X(\tau) = \mu_X{}^2$.

2.9 $E\{Y(t)\} = 0$ because $E\{X(u)\} = 0$ for all u. Because Y(t) is WSS (why?), $E\{[Y(t)]^2\} = E\{[Y(T)]^2\} = E\{[c_1X(T) + c_2X(0)]^2\} = c_1{}^2R_X(0) + c_2{}^2R_X(0) + 2c_1c_2R_X(T)$. Therefore,

$$\sigma_Y = [(c_1{}^2 + c_2{}^2)R_X(0) + 2c_1c_2R_X(T)]^{1/2}.$$

Next observe that

$$\begin{aligned} P[Y(t_0)>\gamma] &= 1 - P[Y(t_0)\le\gamma] \\ &= 1 - \Phi(\gamma/\sigma_Y) \quad \text{(because Y(t) has zero mean)} \\ &= 1 - \Phi(\gamma[(c_1{}^2 + c_2{}^2)R_X(0) + 2c_1c_2R_X(T)]^{-1/2}). \end{aligned}$$

2.10 Let $W(t) = X(t)/\sqrt{R_X(0)}$, so $Y(t) = \Phi(W(t))$. First observe that $P[Y(t)\le 0]=0$ and $P[Y(t)>1]=0$. From this it follows that

$$F_{Y,1}(v;t) = P[Y(t)\le v]=0 \text{ for all } v\le 0$$

$$F_{Y,1}(v;t) = P[Y(t)\le v]=1 \text{ for all } v\ge 1.$$

Next consider $0<v<1$. In this range,

$$F_{Y,1}(v;t) = P[Y(t)\le v] = P[\Phi(W(t))\le v] = P[W(t)\le \Phi^{-1}(v)].$$

Because W(t) is Gaussian with zero mean and unit variance (prove this),

$$F_{Y,1}(v;t) = P[W(t)\le\Phi^{-1}(v)] = \Phi(\Phi^{-1}(v)) = v,$$

for $0<v<1$. Summarizing, we have

$$F_{Y,1}(v;t) = \begin{cases} 1, & v\ge 1, \\ v, & 0<v<1, \\ 0, & v\le 0. \end{cases}$$

2.10 (cont'd) This is just the uniform distribution on the interval [0,1]. The corresponding density function is given by $f_{Y,1}(v;t) = 1$ for $0 < v < 1$ and $f_{Y,1}(v;t) = 0$ for other values of v.

2.11 Because the given function is a function of a single variable only, we can use (2.25)-(2.27) in place of (2.15)-(2.17). For each τ, $R(\tau)$ is either 0 or 1, so it satisfies $R(\tau) \geq 0$ for $-\infty < \tau < \infty$. The given function is symmetrical; that is, $R(\tau) = R(-\tau)$ for each τ. The maximum value of $R(\tau)$ is 1 and $R(0) = 1$, so $|R(\tau)| \leq R(0)$ for $-\infty < \tau < \infty$. Hence the given function satisfies (2.25) -(2.27).

Now let $t_1 = 0$, $t_2 = T$, $t_3 = 2T$, $\alpha_1 = \alpha_3 = 1$, and $\alpha_2 = -1$. This choice gives

$$\sum_{i=1}^{3} \sum_{k=1}^{3} \alpha_i \alpha_k^* R_X(t_i - t_k) = -1,$$

so the given function is not nonnegative definite (it does not satisfy (2.18)). Alternative: Note that for these values of t_1, t_2, and t_3, the given function has the same correlation matrix as the function of part (b) of Exercise 2-10. This correlation matrix has a negative determinant, so it cannot be a nonnegative definite matrix.

2.14 The autocorrelation function can be expressed as $R_X(t,s) = \cos(\omega_0(t-s))$, which is a function of t-s only. Since the mean is also constant, X(t) is WSS.

2.15 $\mu_X(t) = E\{A\}E\{\cos(2\pi\Lambda t+\Theta)\} = 10[E\{\cos\Theta\}E\{\cos 2\pi\Lambda t\} - E\{\sin\Theta\}E\{\cos 2\pi\Lambda t\}] = 0$, because $E\{\cos\Theta\} = E\{\sin\Theta\} = 0$. So, $\mu_X(t) = 0$ for all t.

$$R_X(t+\tau,t) = E\{A^2\}E\{\cos(2\pi\Lambda(t+\tau)+\Theta)\cos(2\pi\Lambda t+\Theta)\}$$
$$= (1/2)E\{A^2\}[E\{\cos 2\pi\Lambda\tau\} + E\{\cos(2\pi\Lambda(2t+\tau)+2\Theta)\}].$$

But $E\{\cos 2\Theta\} = E\{\sin 2\Theta\} = 0 \Rightarrow E\{\cos(2\pi\Lambda(2t+\tau)+2\Theta)\} = 0$. Also, $E\{A^2\} = 200$. So, the autocorrelation function is given by

$$R_X(t+\tau,t) = 100E\{\cos 2\pi\Lambda\tau\} = 100 \int_{-W}^{W} \cos(2\pi\lambda\tau)\,d\lambda/2W = 100\sin(2\pi W\tau)/2\pi W\tau.$$

Clearly, $R_X(t+\tau,t)$ depends on τ only, and, since the mean is identically zero, the process is WSS. Note that $R_X(\tau)$ can be written as $100\,\mathrm{sinc}(2W\tau)$, where the sinc function is defined as $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$.
(This is the original and most common definition for the sinc function.)

2.16 (a) $$R_Z(t,t+\tau) = c^2E\{X(t)Y(t)X(t+\tau)Y(t+\tau)\} + cd[E\{X(t)Y(t)\} + E\{X(t+\tau)Y(t+\tau)\} + d^2$$
$$= c^2R_X(\tau)R_Y(\tau) + d^2,$$

where we have used the fact that the processes X(t) and Y(t) are independent (hence, uncorrelated) and zero mean, so $E\{X(u)Y(v)\} = \mu_X(u)\mu_Y(v) = 0$ for all u and v. Because $R_Z(t,t+\tau)$ depends on τ only and $E\{Z(t)\} = d$, which is contant, Z(t) is WSS.

2.16 (b) Because X(t) and Y(t) are uncorrelated, zero-mean random processes,

$$R_Z(t,t+\tau) = R_X(\tau)\cos(\omega_0 t)\cos[\omega_0(t+\tau)] + R_Y(\tau)\sin(\omega_0 t)\sin[\omega_0(t+\tau)]$$
$$= (1/2)[R_X(\tau)+R_Y(\tau)]\cos(\omega_0\tau) + (1/2)[R_X(\tau)-R_Y(\tau)]\cos[\omega_0(2t+\tau)].$$

But this depends on t in general, so Z(t) is not WSS in general (it is WSS if and only if $R_X(\tau)=R_Y(\tau)$ for all τ).

2.17 (a) Yes, as is shown in Exercise 2-8.

(b) $R_{X,Y}(t,s) = \alpha_1\alpha_2 E\{\cos(2\pi f_0 t+\Theta)\sin(2\pi f_0 s+\Theta)\}$

$= (\alpha_1\alpha_2/2)\, E\{\sin[2\pi f_0(s-t) + \sin[2\pi f_0(s+t)+2\Theta]\}$

$= (\alpha_1\alpha_2/2)\, \sin[2\pi f_0(s-t)]$

so $R_{X,Y}(t,t+\tau) = (\alpha_1\alpha_2/2)\sin(2\pi f_0\tau)$, which depends on τ only.

(c) Yes, they are jointly WSS since each random process is WSS and the cross-correlation function depends on τ only.

(d) No. In fact, for $t_0 = (4f_0)^{-1}$, $R_{X,Y}(0,t_0) = \alpha_1\alpha_2/2 \neq 0 = \mu_X(0)\mu_Y(t_0)$.

(e) No, they are not independent. This follows from (d) or from the observation that if t_0 is defined as in (d), $Y(t) = (\alpha_2/\alpha_1)X(t-t_0)$; that is, Y(t) is just a constant multiple of a time-delayed version of X(t).

2.18 (a) $R_{X,Y}(t,s) = E\{V^3\}ts = 0$ for all t and s (use $f_V(u) = f_V(-u)$ for all u).

(b) No, they are not individually WSS (e.g., $R_X(t,t)=E\{V^2\}t^2$, which depends on t).

(c) Yes, $R_{X,Y}(t,s) = 0$ and $E\{X(t)\} = 0$, so $R_{X,Y}(t,s) = 0 = \mu_X(t)\mu_Y(s)$ for all t and s.

(d) Certainly not! For instance, $Y(1) = [X(1)]^2$; in fact, $Y(t) = t^{-1}[X(t)]^2$ if $t\neq 0$.

2.19 (a) $\beta = 0 \Rightarrow E\{[X(t)]^2\} = 0 \Rightarrow P(X(t)=0) = 1$ for all t.

(b) For a fixed value of t, X(t) is a Gaussian random variable with zero mean and variance $\sigma^2 = R_X(0) = \beta$, so it has distribution function $F_{X,1}(u;t) = \Phi(u/\sqrt{\beta})$.

(c) For $t_1 = T$ and $t_2 = 2T$, $E\{X(t_1)X(t_2)\} = R_X(T) = 0$. Because X(T) and X(2T) also have zero means, this implies they are uncorrelated; because they are jointly Gaussian, uncorrelatedness implies independence. Therefore,

$$f_{X,2}(u_1,u_2;T,2T) = f_{X,1}(u_1;T)\, f_{X,1}(u_2;2T) = \exp\{-u_1^2/2\beta\}\exp\{-u_2^2/2\beta\}/2\pi\beta$$
$$= \exp\{-(u_1^2+u_2^2)/2\beta\}/2\pi\beta.$$

(d) For $t_1 = T/2$ and $t_2 = T$, $E\{X(t_1)X(t_2)\} = R_X(T/2) = \beta/2$. The process X(t) has zero mean, so the correlation coefficient (see p.27) is $\rho(T/2,T) = (\beta/2)/\beta = 1/2$. The two-dimensional density $f_{X,2}(u_1,u_2;T/2,T)$ is the bivariate Gaussian density with zero means, common variance $\sigma^2 = R_X(0) = \beta$, and correlation coefficient 1/2, which reduces to

$$f_{X,2}(u_1,u_2;T/2,T) = \exp\{-2(u_1^2 - u_1u_2 + u_2^2)/3\beta\}/\sqrt{3}\pi\beta.$$

2.19 (e) The random variable $Z = X(0) + X(T/2) + X(T)$ is Gaussian (sum of jointly Gaussian random variables) with zero mean and variance $E\{Z^2\} = 3R_X(0) + 4R_X(T/2) + 2R_X(T) = 5\beta$. Therefore,

$$P(Z > 10) = 1 - P(Z \le 10) = 1 - \Phi(10/\sqrt{5\beta}).$$

2.20 (a) $E\{X^2(t)\} < \infty$ implies $c_1 < \infty$, $R_X(0) \ge 0$ implies $c_1 \ge 0$, and $R_X(\tau) \le |R_X(0)|$ implies $c_2 \ge 0$. Thus c_1 and c_2 must satisfy $0 \le c_1 < \infty$ and $c_2 \ge 0$. If $X(t)$ is to be a nontrivial random process, we must also have $E\{X^2(t)\} > 0$, which implies both $c_1 > 0$ and $c_2 < \infty$. Thus if $X(t)$ is a nontrivial WSS random process, c_1 and c_2 must satisfy $0 < c_1 < \infty$ and $0 \le c_2 < \infty$. There is no problem with $c_2 = 0$, because $c_2 = 0$ gives $R_X(\tau) = c_1 \cos(\omega_0 \tau)$, which is a valid autocorrelation function for a nontrivial random process if $0 < c_1 < \infty$.

(b) $E\{X^2(t)\} = R_X(0) = c_1$

(c) $E\{Y(t)\} = E\{3X(4t)\} = 3\mu_X$, which is a constant.

$E\{Y(t+\tau)Y(t)\} = 9E\{X(4(t+\tau))X(4t)\} = 9R_X(4\tau)$, which depends on τ only.

So the random process $Y(t)$ is WSS.

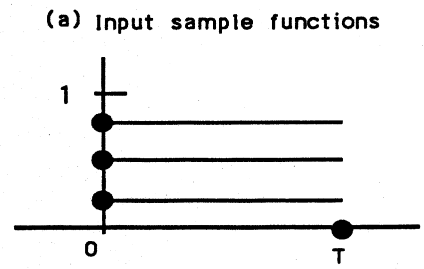
(d) $R_Y(\tau) = 9R_X(4\tau) = 9c_1 \exp(-4c_2|\tau|)\cos(4\omega_0\tau)$

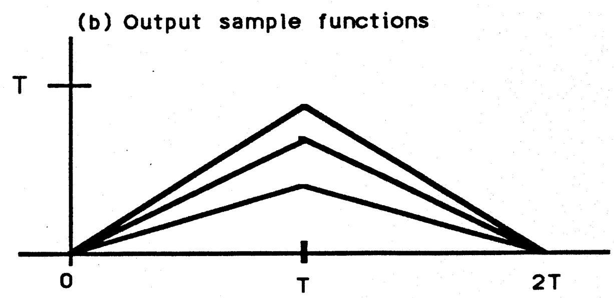
2.21 (a) $\mu_Y(t) = 0$ (b) $\mathrm{Var}\{Y(t)\} = 4kTRB = 4kRB(ct+b)$

(c) $f_{Y,1}(u;t) = \exp\{-u^2/[8kRB(ct+b)]\} / \sqrt{8\pi kRB(ct+b)}$

(d) $Y(t)$ has a constant mean, but it is not stationary; in fact, the variance is time varying, as shown in part (b).

3.1





(c) No, in fact the mean is not constant.

(d) $\mu_Y = \mu_Y * h = (1/2) p_T * p_{T_0}$, which gives $\mu_Y(t) = 0$ for $t < 0$ and $t \geq T_0 + T$, and

$$\mu_Y(t) = \begin{cases} t/2\,, & 0 \leq t < T, \\ T/2\,, & T \leq t < T_0, \\ (T_0+T-t)/2\,, & T_0 \leq t < T_0+T. \end{cases}$$

(e) $R_X(t,s) = E\{X(t)X(s)\} = E\{V^2 p_T(t) p_T(s)\} = 1/3$ for $0 \leq t < T$ and $0 \leq s < T$ and $R_X(t,s) = 0$, otherwise.

(f) $R_Y(t,t) = E\{[Y(t)]^2\} = (1/3)[\Lambda(t-T)]^2$, where Λ is the triangular function:

$$\Lambda(u) = T - |u| \text{ for } 0 \leq |u| < T \text{ and } \Lambda(u) = 0 \text{ otherwise.}$$

3.2 $E\{Z(0)+Z(1)+Z(2)\} = 0$ so $\sigma^2 = \mathrm{Var}\{Z(0)+Z(1)+Z(2)\} = E\{[Z(0)+Z(1)+Z(2)]^2\}$

$$= 3R_Z(0) + 4R_Z(1) + 2R_Z(2) = 3 + 4\exp(-1) + 2\exp(-4)$$

Because $Z(0)+Z(1)+Z(2)$ is Gaussian with zero mean and standard deviation σ, $P[Z(0)+Z(1)+Z(2) > \gamma] = 1 - \Phi(\gamma/\sigma) = 1 - \Phi(\gamma/\sqrt{3+4\exp(-1)+2\exp(-4)})$.

3.3 (a) $\mu_Y(n) = \mu_X(n) + \mu_X(n-1) + \mu_X(n-2)$

(b) $R_{X,Y}(i,j) = E\{X(i)Y(j)\} = E\{X(i)[X(j)+X(j-1)+X(j-2)]\} = R_X(i,j)+R_X(i,j-1)+R_X(i,j-2)$

$$= R_X(i-j)+R_X(i-j+1)+R_X(i-j+2) = \exp\{(i-j)^2\} + \exp\{(i-j+1)^2\} + \exp\{(i-j+2)^2\}$$

(c) $R_Y(i,j) = E\{Y(i)Y(j)\} = E\{[X(i)+X(i-1)+X(i-2)][X(j)+X(j-1)+X(j-2)]\}$

$$= 3R_X(i-j) + 2R_X(i-j+1) + 2R_X(i-j-1) + R_X(i-j+2) + R_X(i-j-2)$$

$$= 3\exp\{(i-j)^2\} + 2\exp\{(i-j+1)^2\} + 2\exp\{(i-j-1)^2\} + \exp\{(i-j+2)^2\} + \exp\{(i-j-2)^2\}$$

d) **Yes**. First, $\mu_X(n)$ does not depend on n and $R_X(n,k)$ depends on n-k only, so the input is WSS. Now, there are two acceptable statements to make to conclude the answer: (1) the linear filter is time invariant and a WSS input to a time-invariant linear system gives input and output processes that are jointly WSS ***or*** (2) the output mean is constant, the output autocorrelation $R_Y(i,j)$ depends on i-j only (from part (c)), and the crosscorrelation $R_{X,Y}(i,j)$ depends on i-j only (from part (b)). In either case, the conclusion is that the two proceses are jointly WSS.

3.4 (a) YES. The input is WSS and the linear filter is time invariant, so the input and output random processes are jointly WSS.

(b) All of the following integrals are from $-\infty$ to ∞:

$$R_{X,Y}(t,s) = E\{X(t)Y(s)\} = E\left\{X(t)\int X(u)h(s-u)\,du\right\} = \int E\{X(u)X(t)\}h(s-u)\,du$$

$$= \int R_X(u-t)h(s-u)\,du = \beta\int \delta(u-t)h(s-u)\,du = \beta\, h(s-t)$$

But $h(v)=1$ for $t_1 \le v < t_2$ and $h(v)=0$ otherwise, so

$R_{X,Y}(t,s)=\beta$ for $t_1 \le s-t < t_2$ and $R_{X,Y}(t,s)=0$ otherwise.

Alternative: Because the two random processes are jointly WSS, it is also valid to write $R_{X,Y}(\tau) = R_{X,Y}(s+\tau,s) = h(-\tau)$ and therefore

$R_{X,Y}(\tau)=\beta$ for $t_1 \le -\tau < t_2$ (i.e., $-t_2 < \tau \le -t_1$ and $R_{X,Y}(\tau)=0$ otherwise.

(c) The convolution $h*\tilde{h}$ is just the convolution of two rectangular functions of unit height, and each has duration t_2-t_1. Such a convolution gives a triangular function with base $2(t_2-t_1)$ and height t_2-t_1. Because $f = h*\tilde{h}$ is always symmetric, this triangular function is therefore centered at the origin. Formally, it is given by $f(t)=(t_2-t_1)-|t|$ for $|t|<(t_2-t_1)$ and $f(t)=0$ otherwise. Because Y(t) is WSS, $R_Y = f*R_X$. Because $R_X(\tau)=\beta\delta(\tau)$, this gives, $R_Y(\tau)=\beta f(\tau)$, so $R_Y(t,s)=\beta[(t_2-t_1)-|t-s|]$ for $|t-s|<(t_2-t_1)$ and $R_Y(t,s)=0$ otherwise.

It is also valid to write $R_Y(\tau)=\beta[(t_2-t_1)-|\tau|]$ for $|\tau|<(t_2-t_1)$ and $R_Y(\tau)=0$ otherwise.

3.5

$$R_Y(0) = \int_{-\infty}^{\infty} f(u)R_X(u)\,du = 2\int_0^{\infty} f(u)R_X(u)\,du = 2\int_0^{T}(T-u)\eta\exp(-\gamma u)\,du$$

$$= (2\eta/\gamma^2)[\exp(-\gamma T)-1] + (2\eta/\gamma)T$$

3.6 Because X(t) is WSS and the linear filter is time invariant, the random processes X(t) and Y(t) are automatically jointly WSS.

$$R_{X,Y}(\tau) = \int_{-\infty}^{\infty} h(u)R_X(\tau+u)\,du = \int_0^{\infty} b\exp(-bu)R_X(\tau+u)\,du.$$

For $\tau\ge 0$:

$$R_{X,Y}(\tau) = \int_0^{\infty} b\exp(-bu)\,\eta\exp\{-\gamma(\tau+u)\}\,du = [b\eta/(b+\gamma)]\exp(-\gamma\tau).$$

For $\tau<0$:

$$R_{X,Y}(\tau) = \int_0^{-\tau} b\exp(-bu)\,\eta\exp\{+\gamma(\tau+u)\}\,du + \int_{-\tau}^{\infty} b\exp(-bu)\,\eta\exp\{-\gamma(\tau+u)\}\,du$$

$$= [b\eta/(b-\gamma)][\exp(\gamma\tau)-\exp(b\tau)\} + [b\eta/(b+\gamma)]\exp(b\tau)$$

$$= [b\eta/(b-\gamma)]\exp(\gamma\tau) - [2\gamma/(b^2-\gamma^2)]b\eta\exp(b\tau).$$

3.7 In Exercise 3-7 (set the parameters β and α of this exercise equal to $b = 1/RC$) we found $f = h * \tilde{h}$ is given by $f(t) = (b/2)\exp\{-b|t|\}$. Because the input autocorrelation function is also a two-sided exponential—namely, $R_X(t) = \eta\exp\{-\gamma|\tau|\}$—we convolve two two-sided exponentials to find the output autocorrelation function (assume $\tau \geq 0$ and the limits are $-\infty$ to ∞ unless otherwise noted):

$$R_Y(\tau) = \int f(\tau-u)R_X(u)\,du = (\eta b/2)\int \exp\{-b|\tau-u|\}\exp\{-\alpha|u|\}\,du$$

$$= (\eta b/2)\left\{ \int_{-\infty}^{0} \exp\{-b(\tau-u)\}\ \exp\{\gamma u\}\,du + \int_{0}^{\tau}\exp\{-b(\tau-u)\}\exp\{-\gamma u\}\,du + \int_{\tau}^{\infty} \exp\{b(\tau-u)\}\ \exp\{-\gamma u\}\,du \right\}$$

Evaluation of the integrals and use of the fact that the autocorrelation is symmetrical gives $R_Y(\tau) = \eta b[b\exp(-\gamma|\tau|) - \gamma\exp(-b|\tau|)]/(b^2-\gamma^2)$.

3.8 The input autocorrelation function depends only on time differences and the linear filter is time invariant, so the output autocorrelation function depends only on time differences. Since h is rectangular, the function $f = h * \tilde{h}$ is triangular (Exercise 3-5). To determine $R_Y(\tau)$ for all τ, we would have to convolve two triangles. However, the problem asks for $R_Y(t_0,t_0)$ only (which is just $R_Y(0)$), so the solution requires evaluation of the following integral:

$$R_Y(0) = \int_{-\infty}^{\infty} f(-u)R_X(u)\,du = 2\int_{0}^{\infty} f(u)R_X(u)\,du = (2\beta/T)\int_{0}^{\xi}(\lambda-u)(T-u)\,du, \text{ where } \xi = \min\{\lambda,T\}.$$

Evaluation of the integrals gives

$$R_Y(0) = \begin{cases} \beta\lambda^2[T-(\lambda/3)]/T, & \lambda < T \\ 2\beta\lambda^2/3, & \lambda = T \\ \beta T[\lambda-(T/3)], & \lambda > T. \end{cases}$$

3.9 (a) $E\{X_i\} = 2-2p-q$ for each i, so $E\{Z_k\} = 3[2-2p-q]$ for each k.

First note that $E\{Z_k{}^2\}$ does not depend on k, so

$$E\{Z_k{}^2\} = E\{Z_1{}^2\} = E\{[X_1+X_2+X_3][X_1+X_2+X_3]\}$$

which, when the product is multiplied out and the expectation taken term by term, consists of three terms of the form $E\{X_i{}^2\}$ and six terms of the form $E\{X_iX_j\}$ with $i \neq j$. Let

$$m = E\{X_i{}^2\} = q + 4(1-p-q) = 4-4p-3q$$

and notice that if $i \neq j$,

$$E\{X_iX_j\} = \mu_X{}^2 = (2-2p-q)^2,$$

so we conclude that

$$E\{Z_k{}^2\} = 3m + 6\mu_X{}^2 = 3(4-4p-3q) + 6(2-2p-q)^2.$$