2.1. Counting

Developing Ideas

1. **Muchos mangos.** There are 5 layers with approximately 18 mangos each for a total of 90.

2. **Packing balls.** In the bottom of the box, line up as many balls as will fit from one corner to another corner along one edge. Because the box is a perfect cube, this gives the number of layers of balls that will fit when stacked vertically. Multiply this number by the number of balls that fit in a single layer to get your estimate.

3. **Alternative rock.** With five CD's, each one can have its own shelf. With one more CD, some shelf must have two according to the Pigeonhole principle.

4. **The Byrds.** The answer to both questions is "no." If each shelf had 3 (or fewer) CD's, then the total number of CD's would be (at most) 15.

5. **For the birds.** The Pigeonhole principle states that if you have more pigeons than you have pigeonholes, and every pigeon must be in some pigeonhole, then there must be at least one pigeonhole with more than one pigeon.

Solidifying Ideas

6. **Treasure chest.** The weight of the bills alone is enough to consider rejecting the offer. Let's *underestimate* the weight in the following way. A typical ream of laser-printer paper (500 sheets) definitely weighs more than 5 lb. So a single sheet of paper weighs at least 5/500 = 1/100 pounds. Now we can almost fit 6 one dollar bills into the area contained by one 8.5"x11" piece of paper, so a single dollar bill weighs more than 1/600 pounds. Therefore, a million bills weighs 1000000/600 or roughly 1,666 pounds! Wait for a better offer.

7. **Order please.** States in the United States, honest congressmen (debatable), cars, telephones on the planet, people, grains of sand.

8. **Penny for your thoughts.** How many pieces would be placed on the last square of the checkerboard? The number of gold pieces doubles with each square and there are 64 total squares. The first square has $1 = 2^0$ pieces, the second square has $2 = 2^1$ pieces, the third square has $4 = 2^2$ pieces, and so on. So, the 64th square has 2^{63}, or more than 9×10^{18} gold pieces. Notice, too, that the number of pieces on each square is one more than the sum of all the gold pieces on the previous squares. So, in total, there are $2^{64} - 1$ pieces.

9. **Twenty-nine is fine.** Two possible candidates: First, 29 is prime. Secondly, 29 happens to be the sum of three consecutive squares, 29 = 4 + 9 + 16. Lest the number 27 feel left out, it should be noted that 27 = 3x3x3, a perfect cube. Is the set of prime numbers sparser than the set of perfect cubes? Does this make it less interesting in your eyes?

10. **Perfect numbers.** The next perfect number in line is 28 = 1 + 2 + 4 + 7 + 14.

11. **Many fold.** To get started let's estimate the thickness of an ordinary piece of paper by noting that packages of 200 sheets of paper are more than half an inch thick. So, a single piece of paper is at least 1/400 inches thick. Now, after one folding, the paper is twice the original thickness. After 2 foldings, the paper is $4 = 2^2$ times as thick. After 50 foldings, the paper will be 2^{50} times as thick. The resulting paper is then $2^{50}/400$ inches thick. (That's more than 2.8×10^{12} inches and more than 40 million miles!)

12. Only one cake. Because there are more people than possible birthdays, there must be at least two people that share the same birthday. To be more convincing, imagine that they all have different birthdays. Now select 366 people from the group. Because they all have different birthdays and because there are only 366 possible birthdays (including leap year), all the birthdays must be accounted for. The remaining four people must all share a birthday with someone else in the room.

13. For the birds. There must be some hole containing more than one pigeon. In the hairy-bodies question, the six billion people in the world play the role of the pigeons, and the 400 million hairs play the role of the holes. Just as there are at least two pigeons sleeping in the same hole, there are necessarily two people with the same total number of body hairs.

14. Sock hop. To guarantee one match, you need only pull out three socks. Either two will be black, or two will be blue. To get two matched pairs, you need at most seven socks. However, to guarantee a black pair, you need to pull out 12 socks, because you might be unlucky and pull out all the blue socks first!

15. The last one. 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. The sequences for 11 and 22 are within the sequence above.
30, 15, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1

Creating New Ideas

16. See the three. There are two ways to approach this question: (1) Count the numbers with 3's or (2) count the numbers without 3's. Method (1): There is only one number with three 3's in it, namely 333. How many numbers have exactly two 3's? There are 9 such numbers of the form 33*x*, 9 of the form 3*x*3, and 9 of the form *x*33, for a total of 27 "doubles". How many numbers have exactly one 3 in them? Let's overcount by saying that there are 100 numbers of the form *xx*3, *x*3*x*, and 3*xx*. Of the 300 numbers, we counted the 27 doubles twice, and 333 three times. So we have 300 – 27 – 2 = 271. The corresponding proportion is 0.271. Method (2): A number with no 3 could have any of 9 digits in each position, for a total of $9 \times 9 \times 9$, or 729 numbers. The remaining 271 numbers have a 3.

17. See the three II. There are two ways to approach this question: (1) Count the numbers with 3's or (2) Count the numbers without 3's. Method (1): There is a nonobvious way to keep track of all the overcounting. There are 1000 numbers of the forms *xxx*3, *xx*3*x*, *x*3*xx*, and 3*xxx*, for a total of 4000. There are 100 numbers of the form *xx*33, *x*3*x*3, 3*xx*3, *x*33*x*, 3*x*3*x*, and 33*xx*, for a total of 600. We have 10 each of the form *x*333, 3*x*33, 33*x*3, and 333*x* for a total of 40 such numbers. And lastly, there is only 1 number with four 3's. Here's the trick: 4000 – 600 + 40 – 1 = 3439 represents the number of numbers with 3's in them. The alternating signs account for all the overcounting! The corresponding proportion is 0.3439. Method (2): A number with no 3 could have any of 9 digits in each position, for a total of $9 \times 9 \times 9 \times 9$ or 6,561 numbers. The remaining 3,439 numbers have a 3.

18. See the three III. The proportion of million-digit numbers without a 3 is (9/10) raised to the millionth power.

19. Commuting. There are 100 people arriving at work between 8:00 and 8:30. Imagine slicing this time frame into 30 distinct intervals. Because we have more people than intervals, at least two people will arrive within the same interval. This also means that their arrival times differ by less than a minute.

20. RIP. Within the next 100 years, virtually all of the 6.2 billion people currently populating the earth will die. If fewer than 50 million people died each year, then at the end of 100 years, only 5 billion people would have died, which means that well over 100 billion people would live to at least 100. This contradiction shows that at some point more than 50 million people will die. Alternatively, you could say that the average number of people that will die each year is 6,200,000,000 / 100, or 62 million. And because this is the average, there must be at least some year in which at least 62 million people will die.

Further Challenges

21. Say the sequence. Reading the last number, "One 3, One 1, Two 2's, Two 1's" generates the next number in the sequence, namely, 13122221. At the beginning of the sequence, '1' is read, "One 1," which becomes "11." This in turn is read, "Two 1's" or "21," and so on.

22. Lemonade. You have two choices for the first option (yes or no), two for the second option, two for the third option, and four choices for the color. Therefore there are $2 \times 2 \times 2 \times 4 = 32$ different types of this particular model. Each of the 100,000 cars fits into one of these 32 categories. There is an average of 100,000/32 = 312.5 cars per category and so some category must have at least this many cars. You are guaranteed to find at least 312 identical cars.

For the Algebra Lover

24. Ramanujan noodles (H). We are given that $4^3 + x^3 = 2261$. Thus $64 + x^3 = 2261$, which yields $x^3 = 2197$. Using a calculator to take cube roots, we find $x = 13$.

25. Bird count. We are told that the total number of pigeons is $x^2 - 100$ which equals $x + x + 3x + 5x + x + 2x + 2x$. Simplifying, we obtain $x^2 - 100 = 15x$, which yields $x^2 - 15x - 100 = 0$. Factoring we obtain $(x - 20)(x + 5) = 0$. So $x = -5$ or $x = 20$. Because we cannot have a negative number of pigeons, we must have $x = 20$, which means we have $15x = 300$ pigeons.

26. Many pennies. After putting one penny on the first square, the number of pennies on each square is 3 times the number on the previous square. Thus the total number of pennies is $1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + 3^7 + 3^8$. This sum is 9841.

27. Park clean-up. Let x denote the number of volunteers needed for the largest park. Then the medium sized parks require $x/2$ volunteers each, the small parks require $x/4$ volunteers each (half the number needed for a medium-sized park), and the very small park requires $x/5$ volunteers. Thus the total number of volunteers required is $x + 2(x/2) + 2(x/4) + x/5$. This expression equals 108, so we have to solve the equation $x + x + x/2 + x/5 = 108$, or $2x + x/2 + x/5 = 108$. Multiply both sides of the equation by 10 to clear the denominators, obtaining $20x + 5x + 2x = 1080$. Thus $27x = 1080$, so $x = 40$. Therefore 40 volunteers go to the large park, 20 each to the medium parks, 10 each to the small parks, and 8 to the very small park.

28. Where's the birdie? At 3:00 pm the pigeon has been flying for $t = 3$ hours. Thus her distance from the coop is $D(3) = 5(3) - 3^2 = 6$ miles. At 5:00 she has been flying for $t = 5$ hours, so her distance from the coop is $D(5) = 5(5) - 5^2 = 0$ miles. Looks like she's back home, where the meaning of life awaited her.

2.2. Numerical Patterns in Nature

Developing Ideas

1. First Fibonaccis. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610.

2. Born φ. The symbol φ represents the infinitely long fraction expression $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}$.

It is also a solution to the equation $\varphi = 1 + \frac{1}{\varphi}$.

One sequence of numbers that approaches φ is the list of ratios of consecutive Fibonacci numbers.

3. Tons of ones. Simplifying, we see that $1+\cfrac{1}{1+\frac{1}{1}} = 1+\frac{1}{1+1} = \frac{3}{2}$.

4. Twos and threes. $2+\cfrac{2}{2+\frac{2}{2}} = 2+\frac{2}{2+1} = \frac{8}{3}$; $3+\cfrac{3}{3+\frac{3}{3}} = 3+\frac{3}{4} = \frac{15}{4}$.

5. The family of φ. If $x = 2 + \frac{1}{x}$, multiply through by x to get $x^2 = 2x + 1$. So $x^2 - 2x - 1 = 0$. This does not factor, so we use the quadratic formula to get $x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$. Similarly, multiply $x = 3 + \frac{1}{x}$ through by x to get $x^2 = 3x + 1$. So $x^2 - 3x - 1 = 0$. Again this does not factor, so we use the quadratic formula to get $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{13}}{2}$.

Solidifying Ideas

6. Baby bunnies.

Month		1	2	3	4	5	6	7	8
Pairs of adults	1	1	2	3	5	8	13	21	
Pairs of babies	0	1	1	2	3	5	8	13	
Total number of pairs	1	2	3	5	8	13	21	34	

After each month, the total number of pairs of bunnies becomes the number of mature pairs for the next month. Because all the mature pairs produce offspring, the number of mature pairs during one month becomes the number of new pairs of offspring in the next month. Each row contains the same sequence of numbers, but the sequences are offset from one another. Note the connection to Fibonacci.

7. Discovering Fibonacci relationships.

n		1	2	3	4	5	6	
$(F_n)^2$		1	1	4	9	25	64	…
$(F_{n+1})^2$	1	4	9	25	64	169	…	
Sum		2	5	13	34	89	233	
		F_3	F_5	F_7	F_9	F_{11}	F_{13}	

Note that we are getting all the odd Fibonacci numbers. This leads to the formula, $(F_n)^2 + (F_{n+1})^2 = F_{2n+1}$.

8. Discovering more Fibonacci relationships.

n		1	2	3	4	5	6
$(F_{n+1})^2$	1	4	9	25	64	169	…
$(F_{n-1})^2$	.	1	1	4	9	25	…
Difference		.	3	8	21	55	144
			F_4	F_6	F_8	F_{10}	F_{12}

Now we're getting all the even Fibonacci numbers (see Mindscape 7.). More compactly, $(F_{n+1})^2 - (F_{n-1})^2 = F_{2n}$

9. Late bloomers.

Month	2	3	4	5	6	7	8	9	10	11
Mature pairs	1	1	1	2	3	4	6	9	13	19
Pairs of new babies	0	1	1	1	2	3	4	6	9	13
Pairs of old babies	0	0	1	1	1	2	3	4	6	9
Total pairs	1	2	3	4	6	9	13	19	28	41

As in Mindscape 6, starting with Month 2, each row contains the same sequence of numbers (though shifted). If T_n represents the total number of pairs of bunnies at the end of the nth month, then $T_n = T_{n-1} + T_{n-3}$.

10. A new Start. 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843,… Because 843/521 = 1.61803…, it looks like the ratio of consecutive numbers still approaches the golden mean.

If we start with – 7 and 3, we get a sequence that includes negative numbers:
– 7, 3, – 4, – 1, – 5, – 6, – 11, – 17, – 28, – 45, – 73, – 118, – 191, 309, – 500,… Yet the ratios still converge to the Golden Mean, (– 500)/(– 309) = 1.61812… We can view the Golden Mean as defined by this infinite process independent of the starting numbers. In Chapter 6, we will see that images and pictures can be defined in a similar manner.

11. Discovering Lucas relationships.

n	1	2	3	4	5	6	7	8
$L(n-1)$.	2	1	3	4	7	11	18	
$L(n+1)$ 1	3	4	7	11	18	29	47	
Sum	.	5	5	10	15	25	40	65
		g_1	g_2	g_3	g_4	g_5	g_6	g_7

$L_{(n-1)} + L_{(n+1)} = g_{(n-1)}$, where g_n's are constructed as the Lucas numbers are, but with the first two numbers being 5 and 5 instead of 2 and 1. You can write the answer in terms of Fibonacci numbers by noting that $L_n = F_n + F_{(n-2)}$.

12. Still more Fibonacci relationships.
See the solution to Mindscape 11. By the same reasoning, we find that the sum is the Lucas sequence starting with 3 and 4.

$F_{(n-1)}$	.	1	1	2	3	5	8	13	21
$F_{(n+1)}$	1	2	3	5	8	13	21	34	55
Sum	.	3	4	7	11	18	29	47	76

13. Even more Fibonacci relationships.

$F_{(n+2)}$	2	3	5	8	13	21	34
$F_{(n-2)}$	.	.	1	1	2	3	5
Difference	.	.	4	7	11	18	29

Note that we get the same sequence as in Mindscape 12. This is because $F_{(n+2)} - F_{(n-2)} = F_{(n-1)} + F_{(n+1)}$, which is straightforward to prove using the definition of Fibonacci numbers.

14. Discovering Fibonacci and Lucas relationships.

N	1	2	3	4	5	6	7	8
F_n	1	1	2	3	5	8	13	21
L_n	2	1	3	4	7	11	18	…
Sum	3	2	5	7	12	19	31	…

See Mindscape 11 for more insights into these types of sequences.

15. The enlarging area paradox. If you look closely, you'll notice that the pieces don't line up exactly. Note that the little triangle with sides 3 and 8 appears to be similar to the big "triangle" with sides 5 and 13. If this were true, then the corresponding ratios would be equal. But 8/3 isn't 13/5. Because these are ratios of consecutive Fibonacci numbers, the ratios are close, which is why this is a convincing trick.

16. Sum of Fibonacci. Start with the largest Fibonacci number smaller than the given number and work your way backwards

52 = 34 + 13 + 5,
143 = 89 + 34 + 13 + 5 + 2
13 = 13,
88 = 55 + 21 + 8 + 3 + 1

17. Some more sums. 43 = 34 + 8 + 1; 90 = 89 + 1; 2000 = 1597 + 377 + 21 + 5; 609 = 377 + 144 + 55 + 21 + 8 + 3 + 1

18. Fibonacci nim: The first move. After mentally expressing 52 as a sum of non-consecutive Fibonacci numbers, (52 = 34 + 13 + 5), you remove five sticks from the pile.

19. Fibonacci nim: The first move II. Because 100 = 89 + 8 + 3, you need only remove three sticks.

20. Fibonacci nim: The first move III. Noting that 609 = 377 + 144 + 55 + 21 + 8 + 3 + 1, we remove only one stick.

21. Fibonacci nim: The next move. After the friend removes four, there are nine sticks left. Because 9 = 8 + 1, so we remove one stick to keep ourselves in a winning position.

22. Fibonacci nim: The next move II. A total of 26 sticks have been removed, leaving 24. Express 24 as a sum of non-consecutive Fibonacci numbers (24 = 21 + 3) and remove 3 sticks.

23. Fibonacci nim: The next move III. A total of 24 sticks have been removed leaving 66. Since 66 = 55 + 8 + 3, you can keep your winning position by removing only 3 sticks.

Creating New Ideas

26. Discovering still more Fibonacci relationships.

n	1	2	3	4	5	6	7
F_n	1	1	2	3	5	8	13
F_{n-1}	.	1	1	2	3	5	8
F_{n+1}	1	2	3	5	8	13	21
G_n	.	1	–1	1	–1	1	–1

Surprisingly, the expression is $(-1)^n$ (1 if n is even, and – 1 if n is odd!)

27. Finding Factors. Question 8 showed us that $F_{2n} = (F_{n+1})^2 - (F_{n-1})^2$, which can be factored as the product $(F_{n+1} - F_{n-1}) \times (F_{n+1} + F_{n+1})$. This means that except for 2, none of the even Fibonacci numbers are prime. Who would have guessed?

28. The rabbits rest.

Month	1	2	3	4	5	6	7	8	9
Pairs of kids	1	0	1	1	1	2	2	3	4
Pairs of parents 0	1	0	1	1	1	2	2	3	
Pairs of old parents	0	0	1	0	1	1	1	2	2
Total pairs	1	1	2	2	3	4	5	7	9

Thus, $T_n = T_{n-3} + T_{n-2}$, with $T_1 = T_2 = 1$, and $T_3 = 2$

29. Digging up Fibonacci roots. Once again, the limiting ratio is the Golden Mean. See the solution to Mindscape 40 for an argument as to why this is so.

30. Tribonacci. 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927; 504/274 = 1.8394…; 927/504 = 1.8392… In fact, the limiting ratio converges to a solution of the equation $x^4 - 2x^3 + 1 = 0$.

31. Fibonacci follies. There are nine sticks left and we would like to take (9 = 8 + 1) 1 stick to get us back in a winning situation. Because we are allowed to take one stick, we're in good shape. But a mistake was made during the second move. We should have taken 1 stick instead of two. Had our friend taken two instead of one, we would still be losing.

32. Fibonacci follies II. There are 18 sticks left (18 = 13 + 5). Because our friend took only two sticks, we can't take the five sticks we need to get back to a winning position. We take one stick and hope our friend makes a mistake. On the previous move, we should have taken two sticks instead of three.

33. Fibonacci follies III. There are 18 sticks left and we can remove at most 4 from the pile. 18 = 13 + 5, so we can't follow our winning strategy. The strategy doesn't apply here because we started with a number of sticks that equaled a Fibonacci number. The moral? When you're starting with a Fibonacci number, graciously let your friend play first.

34. A big fib. Let's suppose F is the kth Fibonacci number: $F = F_k$. Then N lies between F_k and F_{k+1}. Because $N < F_{k+1}$, then $N - F_k < F_{k+1} - F_k$; but this last difference is F_{k-1} because $F_{k-1} + F_k = F_{k+1}$. Putting this together, we have $N - F_k < F_{k-1}$.

35. Decomposing naturals. Let's prove this inductively. We know that we can trivially express 1 as a sum of distinct, non-consecutive Fibonacci numbers. We know that we can do this for the first few natural numbers, now let's assume that we can do this for all numbers less than some natural number k, and try to show that we can also write k as such a sum. If k is a Fibonacci number, then we're done; if it isn't then we will grab the largest Fibonacci number smaller than k, call it F, and use it in our sum. Because $(k - F)$ is less than k, we can invoke the induction hypothesis to express $(k - F)$ as a sum of distinct non-consecutive Fibonacci numbers. We add F to the list to complete the problem.

Mindscape 34 shows (k – F) is smaller than the next smaller Fibonacci number, so by adding F to the previous list, we still have a set of non-consecutive Fibonacci numbers.

Further Challenges

36. **How big is it?** In other words, can the ratio F_{k+1}/F_k equal 2? Rewrite the ratio as was done in the text, $(F_k + F_{k-1})/F_k = 1 + F_{k-1}/F_k$. Because the Fibonacci numbers keep getting bigger, the last fraction is always less than or equal to one. And this implies that the original expression is less than or equal to 2. It is only equal to 2 when $F_{k-1} = F_k$, i.e. when $F_{k+1} = 2$.

37. **Too small.** Let F_k and $F_{(k-1)}$ represent F and the Fibonacci number immediately preceding F. The largest Fibonacci numbers less than F are $F_{(k-2)}$ and $F_{(k-1)}$, and their sum is precisely F which is less than N. The sum of any other distinct Fibonacci numbers would be even smaller.

38. **Beyond Fibonacci.** 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782,… 80782/33461 = 2.414213562… Assume that consecutive ratios tend to some mystery number x. Use the recurrence relation to rewrite the quotient:

$$(G_{n+1}/G_n) = (2G_n + G_{n-1})/G_n = 2 + 1/(G_n/G_{n-1}).$$

The left side tends to x, while the right side tends to $2 + 1/x$. *So* x satisfies the equation $x = 2 + 1/x$. Equivalently, $x^2 = 2x + 1$, whose positive solution is $1 + \sqrt{2}$.

39. **Generalized sums.** No such luck. 3 = 1 + 2, which involves consecutive generalized Fibonacci numbers, and if we write 3 = 1 + 1 + 1, then we haven't used distinct generalized Fibonacci numbers. Even if we didn't mind using consecutive numbers, we would still run into problems. For example, 4 can be written as 1 + 1 + 1 + 1, 2 + 1 + 1, or 2 + 2, all of which include duplicate numbers.

40. **It's hip to be square.** Rewrite the fraction under the square root in the following way: $F_9/F_7 = F_9/F_8 \times F_8/F_7$. Note the last two fractions are ratios of consecutive Fibonacci numbers, and so should closely approximate φ, the Golden Mean. Therefore the original fraction tends to φ^2, and when we take the square root, we just get φ back.

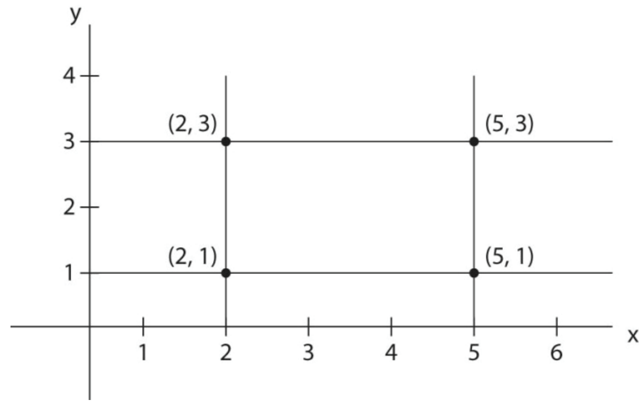
For the Algebra Lover

45. **Rabbit line.** The number of rabbits after t years is $2 + 5t$. If you have 102 rabbits, to find t you must solve $2 + 5t = 102$, that is, $5t = 100$. So $t = 20$ years have passed.

46. **Finding *x*.** Following the hint, we multiply both sides of $x = 1 + \frac{6}{x}$ by x to obtain $x^2 = x + 6$ which becomes $x^2 - x - 6 = 0$. Factoring we obtain $(x - 3)(x + 2) = 0$. So $x = 3$ or -2.

47. **Appropriate address.** If x is the first of the odd numbers, then the next one is $x + 2$, and the third one is $x + 4$. Thus their product, together with 107, is $107(x)(x + 2)(x + 4)$. Here are some possible house numbers for different values of x:
$x = 1$: house numbers is 107(1)(3)(5) = 1605;
$x = 3$: house numbers is 107(3)(5)(7) = 11235;
$x = 5$: house numbers is 107(5)(7)(9) = 33705.
The number 11235 looks like Fibonacci's address! So we must have $x = 3$.

48. Zen bunnies.



49. The power of gold. Letting $n = 1$, we find

$$y = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}-(1-\sqrt{5})}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

Letting $n = 2$, we find $y = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}}$ and we are faced with expanding quadratics in the numerator. Following the hint, before we FOIL, we factor out the1/2 to obtain

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{\left(1+\sqrt{5}\right)^2}{2^2} = \frac{1}{4}\left(1+\sqrt{5}\right)^2 = \frac{1}{4}\left(1+\sqrt{5}\right)\left(1+\sqrt{5}\right) = \frac{1}{4}\left(1+2\sqrt{5}+5\right) = \frac{1}{4}\left(5+2\sqrt{5}\right)$$

and

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{\left(1-\sqrt{5}\right)^2}{2^2} = \frac{1}{4}\left(1-\sqrt{5}\right)^2 = \frac{1}{4}\left(1-\sqrt{5}\right)\left(1-\sqrt{5}\right) = \frac{1}{4}\left(1-2\sqrt{5}+5\right) = \frac{1}{4}\left(5-2\sqrt{5}\right)$$

Now the numerator of our expression becomes

$$\frac{1}{4}\left(5+2\sqrt{5}\right) - \frac{1}{4}\left(5-2\sqrt{5}\right) = \frac{5}{4} + \frac{2\sqrt{5}}{4} - \frac{5}{4} + \frac{2\sqrt{5}}{4} = \frac{4\sqrt{5}}{4} = \sqrt{5}.$$

But the denominator in the expression for y is also $\sqrt{5}$. So the value of y simplifies to 1. Wow! For $n = 3$, we also follow the hint and use a calculator to discover that $y = 2$. So our first three values of y are 1, 1, and 2. Look familiar? When $n = 10$, y will indeed equal the 10th Fibonacci number, which is 55. Check this using a calculator.

2.3. Prime cuts of numbers

Developing Ideas

1. **Primal instincts**. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

2. **Fear factor.** $6 = 2 \cdot 3$, $24 = 2 \cdot 2 \cdot 2 \cdot 3$, $27 = 3 \cdot 3 \cdot 3$, $35 = 5 \cdot 7$, $120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$.

3. **Odd couple**. No, n+1 will be an even number greater than 2, and so will have 2 as a factor. If n = 1, then n+1 = 2 which is prime.

4. **Tower of power.** The first ten powers of 2 are: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024. The first five powers of 5 are: 5, 25, 625, 3125, 15,625.

5. **Compose a list.** The list of even numbers starting with 4 contains no primes. The list of powers of 5 starting with 25 contains no primes.

Solidifying Ideas

6. **A silly start.** It's a personal choice, but 51 has our vote. Nothing about the number screams that its factors are 3 and 17. Another favorite is 91 = 7 x 13.

7. **Waiting for a nonprime.** When n = 4, the resulting number is 25 which isn't prime. In fact, most of the time the constructed number won't be prime. The next prime number doesn't occur until n = 11.

8. **Always, sometimes, never.** By definition, a product of two numbers is not prime, so "Never" is the answer to both questions.

9. **The dividing line.** Sometimes. For example, 8/4 = 2 is prime, but 16/4 = 4 is not.

10. **Prime power.** No. Raising to a power stands for repeated multiplication, and so the resulting number would be represented as a product of numbers, a definite giveaway of its non-prime status.

11. **Nonprimes.** Besides 2, all the even numbers are non-primes. So there are infinitely many numbers that are not prime.

12. **Prime test.** No. The crux of the definition of *prime* is that no other numbers other than 1 and n divide into n. For example, 1 and 4 both divide into 4 evenly, but 4 is not prime. The numbers 1 and n will always divide into n evenly, for any number n.

13. **Twin primes.** (3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73), (101,103), (107,109), (137,139), (149,151), (179,181), (191,193), (197,199)

Do you think it becomes harder and harder to find twin primes as we look at larger and larger prime numbers? ... Or does their distribution appear random?

14. **Goldbach.** 4 = 2 + 2, 6 = 3 + 3, 8 = 3 + 5, 10 = 3 + 7, 12 = 5 + 7, 14 = 3 + 11, 16 = 3 + 13, 18 = 5 + 13, 20 = 3 + 17, 22 = 3 + 19, 24 = 5 + 19, 26 = 3 + 23, 28 = 5 + 23, 30 = 7 + 23

Note that as the numbers get larger, there are more ways to express the number as a sum of two primes. 32 = 3 + 29, 11 + 19, 13 + 17, etc..

15. **Odd Goldbach.** The smallest counter-example is 11. The sum of two odd primes is an even number, so if we are to represent 11 as such a sum, 2 will be one of the primes. The other number is then 11 – 2 = 9, but 9 isn't prime.

16. **Still the 1.** The harder question is, "Are any of these prime?" We can describe each element in the list by its number of digits. If it has an even number of digits, then the number is divisible by 11. If it has 3,6,9,... digits, then the number is at least divisible by 111. So the only candidates for prime numbers are those whose length is itself a prime! By computer search, the first three primes in the sequence have 19, 23, and 317 digits.

17. **Zeros and ones.** 1001 = 13 x 11 x 7 is the first non-prime in the sequence.

18. **Zeros, ones, and threes.** This sequence includes several primes, but they are still few and far between. First non-prime is 1003 = 17 x 59. The next three primes on the list are $10^5 + 3$, $10^6 + 3$, and $10^{11} + 3$.

19. **A rough count.** The prime number theorem states that the number of primes less than 10^{10} is about $10^{10} /\ln(10^{10})$ or just over 400 million.

20. **Generating primes.** The first non-prime given by the sequence $n^2 + n + 17$ occurs for $n = 16$. The resulting number is $289 = 17 \times 17$. The first non-prime for $n^2 - n + 41$ occurs for n = 41. What are the factors of the corresponding number?

21. **Generating primes II.** These are the Mersenne primes, and the first non-prime of this form is $2^{4} - 1 = 15$.

22. **Floating in factors.** The answer is the product of the three smallest prime numbers, $2 \times 3 \times 5 = 30$.

23. **Lucky 13 factor.** Call the mystery number X. The first statement allows us to express X as $13A + 7$ for some unknown A. The number less one, $X - 1 = 13A + 6$, is still not divisible by 13. If we subtract 7, then we get $X - 7 = 13A$, and this is divisible by 13. So the answer is 7.

24. **Remainder reminder.** As in Mindscape 23, we write the original number as $X = 13A + 7$. Adding 22 yields, $X + 22 = 13A + 7 + 22 = 13A + 29 = 13A + 13 \times 2 + 3 = 13(A + 2) + 3$. So 13 goes into our new number $(A + 2)$ times with a remainder of 3.

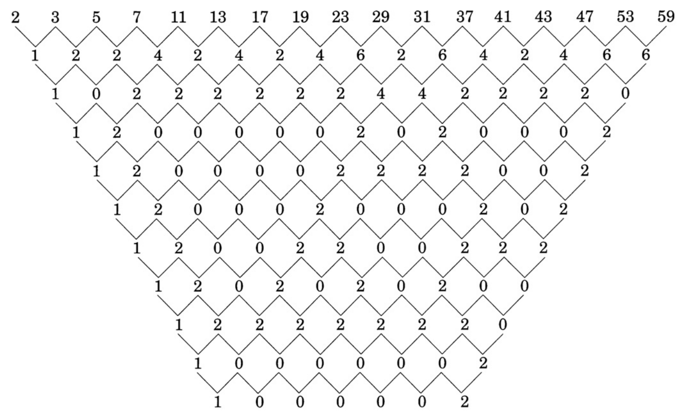
25. **Remainder roundup.** As in Mindscapes 23, 24, write $X = 91A + 52$. Then $X + 103 = 91A + 155$. Recognize that $91 = 7 \times 13$, and $155 = 22 \times 7 + 1$, so that we can write $X + 103 = 7(13A) + 7 \times 22 + 1 = 7(13A + 22) + 1$. Final answer is 1.

Creating New Ideas

26. **Related remainders.** The first line allows us to write our two numbers, X and Y, in the following way: $X = 57A + r$, and $Y = 57B + r$. So $(X - Y) = 57A - 57B = 57(A - B)$, and 57 definitely divides this number. Because $57 = 3 \times 19$, 3 and 19 will also divide $(X - Y)$.

    Suppose we divide two numbers by some integer m. The two numbers will have the same remainder upon division if and only if m is a factor of the difference.

27. Prime differences.



It appears that the first number of each row will be a one.

28. Minus two. If a prime number less two is also prime, then we call those numbers "twin primes." For example, 5 and 7 are twin primes, but 9 and 11 aren't. See Mindscape 8.

29. Prime neighbors. Because 2 is the only even prime, 2 and 3 are the only primes that differ by one.

30. Perfect squares. There are 6 perfect squares less than 36, 12 less than 144, and in general, there are n perfect squares less than n2. Turning this around, we have an estimate of √N perfect squares less than N.

31. Perfect squares versus primes. Using the results of Mindscape 30, there are roughly √1,000,000,000 = 31,622 perfect squares less than a billion. An estimate of the number of primes in this range is 1 billion/ ln(1 billion) = just over 48 million. Because 48 million / 32 thousand = 1500, there are roughly 1500 more prime numbers than perfect squares. So, yes, perfect squares are less common.

32. Prime pairs. This is the same question as Mindscape 29. If p is a prime greater than 2, then p is odd and p + 1 is even. Because 2 is the only even prime, p + 1 isn't prime.

33. Remainder addition. The two remainders are one and the same. We don't know how many times n goes into A, but let's call it something, say p, so that we can express A in equation form: A = np + a. Similarly, B = nq + b, where p and q are unknown. Finally, let's let (a + b) = ns + c, where both s and c are unknown. This means that the remainder after dividing (a + b) by n is c. Now, (A + B) = np + a + nq + b = n(p + q) + (a + b) = n(p + q) + (ns + c) = n(p + q + s) + c, which means that (A + B) will also have a remainder of c when divided by n.

34. Remainder multiplication. As we did in Mindscape 33, write A = np + a, B = nq + b and finally (ab) = ns + c. So AB = (np + a)(nq + b) = npnq + npb + naq + ab = nnpq + npb + naq + (ns + c) = n(npq + pb + aq + s) + c. This means dividing AB by n leaves a remainder of c, which is precisely the remainder when (ab) is divided by n.

35. A prime-free gap. By exhaustively looking at the difference between successive primes you will find that the first string of six non-primes appears between 89

Further Challenges

36. Prime-free gaps. Mindscape 35 shows that you can find six composite numbers in a row after the number (7! + 1). Similarly, you can find n composite numbers in a row after $(n + 1)! + 1$.

37. Three primes. Other than 2 and 3, you cannot even have two consecutive integers both of which are prime, because one of them would be even and the only even prime is 2. So to have three in a row, 2 must be included. This leaves us with {1,2,3} or {2,3,4}, neither of which contains only prime numbers.

38. Prime plus three. In this question, 11 is included as a distraction. Except for 2, all primes are odd, and if you add 3 to an odd prime, you get an even number larger than 2, so the sum won't be prime. If the 11 were replaced with a 2, the answer wouldn't change.

39. A small factor. Take 60 for example: $\sqrt{60} = 7.745...$. Let's suppose that 60 has just two prime factors, p and q, so that $60 = pq$. If both p and q were greater than 7.745, then pq would be greater than $7.745 \times 7.745 = 60$. This is impossible because pq is equal to 60, not greater and not less. So, one of the factors has to be less than or equal to 7.745... By the same reasoning, one of the factors has to be greater than or equal to 7.745... Similarly, if a number N has three factors, then its smallest prime factor is less than the cube root of N.

40. Prime products. If $N = 2 \times 5 \times 17 + 1$, then 2 doesn't divide N because 2 goes into N (5×17) times with a remainder of 1. Similarly, 5 and 17 don't evenly divide into N. All the prime factors of N are necessarily different from 2, 5, and 17. (N doesn't need to be prime though). Suppose there were only finitely many primes, $p_1, p_2, p_3, ..., p_L$. We can arrive at a contradiction to this statement by forming the product $N = p_1p_2p_3... p_L + 1$. The number N is larger than any of the primes $p_1, p_2, p_3, ..., p_L$, so it can't be prime. Therefore, N can be expressed as a product of primes. By the argument above, all the factors of N must be different from $p_1, p_2, p_3, ..., p_L$. This contradicts the idea that the primes $p_1, p_2, p_3, ..., p_L$ were a complete list.

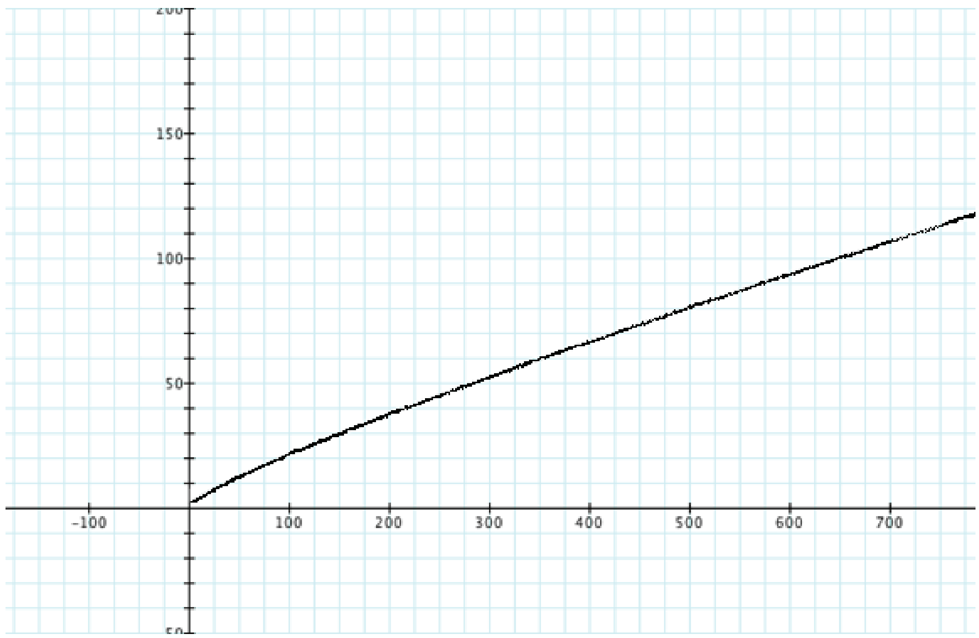
For the Algebra Lover

45. Seldom prime. When $x = 2$, $y = 3$, which is prime. Now suppose x is a natural number other than 2. Observe that $y = x^2 - 1 = (x - 1)(x + 1)$. If $x = 1$, then $y = 0$, which is not prime. If x is at least 3, then $x - 1$ is at least 2 and $x + 1$ is at least 4, so y has divisors other than itself and 1. Thus, again, y is not prime.

46. A special pair of twins. We are told that p and q are twin primes with $p < q$. So we must have $q = p + 2$. Substituting for q into the "cool" equation, $3q - 2p = 17$, yields $3(p + 2) - 2p = 17$. Thus $3p + 6 - 2p = 17$, and we obtain $p = 11$. Therefore $q = 13$ and, finally, $x = 11x13 = 143$.

47. Special ~~K~~ *p*. Factoring $p^3 - 4p$ we obtain $p(p^2 - 4) = p(p - 2)(p + 2)$. Thus 105 is the product of three primes, $p - 2, p, p + 2$. Using guess and check, we find the three primes are 3, 5, and 7, and thus, $p = 5$. There are no other trios of three such primes. (It can be shown that when given three numbers, $x - 2, x, x + 2$, one of the three must be divisible by 3. In order for all three numbers to be prime, one of them has to actually equal 3, as in the case above.)

48. **Prime estimate.** The graph of *y* = *n*/*Ln(n)* below shows that the number of primes less than or equal to 500 is approximately 80.



49. **One real root.** We know in advance that *x* cannot be a natural number because by Fermat's Last Theorem, there are no whole number solutions to an equation of the form $x^3 + y^3 = z^3$. Given that $x^3 + 1^3 = 2^3$, we know that $x^3 = 7$. Thus the exact answer is $x = \sqrt[3]{7}$.

2.4. Crazy Clocks and Checking Out Bars

Developing Ideas

1. A flashy timepiece. Twelve hours after 3:00, your watch will again show 3:00. Because $14 = 12 + 2$, in 14 hours your watch will show 5:00, 2 hours after 3:00. Because $25 = 2 \times 12 + 1$, in 25 hours your watch will show 4:00, just 1 hour after 3:00. Because $240 = 20 \times 12$, in 240 hours your watch will show 3:00 again.

2. Living in the past. Twenty-four hours before 8:00 your watch showed 8:00. Because $10 + 2 = 12$, 10 hours earlier it showed 10:00; 25 hours earlier, it showed 7:00; 2400 hours earlier, it showed 8:00.

3. Mod prods. $16 \equiv 2 \pmod 7$; $24 \equiv 3 \pmod 7$; $16 \times 24 = 384 \equiv 6 \pmod 7$; $[16 \pmod 7 \times 24 \pmod 7] = 2 \times 3 = 6$. The last two quantities are equal.

4. Mod power. $7 \equiv 1 \pmod 3$; $7^2 \equiv 1 \pmod 3$; $[7 \pmod 3]^2 = (1)^2$, which equals $7^2 \pmod 3$. $7^{1000} \pmod 3 \equiv [7 \pmod 3]^{1000} \equiv 1^{1000} \equiv 1 \pmod 3$.

5. A tower of mod power. $13 \equiv 2 \pmod{11}$; $13^2 \pmod{11} \equiv 169 \pmod{11} \equiv 4 \pmod{11}$. Note that $[13 \pmod{11}]^2 = 13^2 \pmod{11}$. Also, $13^3 \pmod{11} \equiv [13 \pmod{11}]^3 \equiv 2^3 \pmod{11} \equiv 8 \pmod{11}$. Finally, $13^4 \pmod{11} \equiv [13 \pmod{11}]^4 \equiv 2^4 \pmod{11} \equiv 16 \pmod{11} \equiv 5 \pmod{11}$.

Solidifying Ideas

6. Hours and hours. Because $96 = 8 \times 12$, the clock will complete 8 full revolutions after 96 hours leaving the hand positions unchanged. Because $1063 = 88 \times 12 + 7$, after 1063 hours the clock will spin completely around 12 times, and then spin 7 more hours' worth, leaving the hands at 5:45. Because $-23 = -2 \times 12 + 1$, 23 hours before 7:10 the clock read 8:10. Similarly, 108 hours earlier, the clock read 7:10 because $-108 = -9 \times 12$.

7. Days and days. $3724 = 532 \times 7$ and $365 = 52 \times 7 + 1$. So in 3724 days it will still be Saturday, while the 365th day from now will fall on a Sunday.

8. Months and months. Express each number as a simpler number mod 12. ($219 = 18 \times 12 + 3$; $120{,}963 = 10080 \times 12 + 3$; $-89 = -7 \times 12 - 5$; or $-8 \times 12 + 7$...) 219 months from now will be October (July + 3), and so will 120,963 months from now. Because -89 divided by 12 has a remainder of -5, we need only go back 5 months to February.

9. Celestial seasonings. Compute $3 \times 0 + 1 \times 7 + 3 \times 1 + 1 \times 7 + 3 \times 3 + 1 \times 4 + 3 \times 0 + 1 \times 0 + 3 \times 0 + 1 \times 2 + 3 \times 1 + 1 \times 8 = 43$. Because the sum is not evenly divisible by 10, it is not a correct UPC. The corresponding sum for the next two codes is 40 and 42 respectively. So the second code is the correct one.

10. SpaghettiOs. (See Mindscape 9.) Because the sums are 41, 49, and 50 respectively, the third number is correct.

11. Progresso. (See Mindscape 9.) Because the sums are 50, 24, and 68 respectively, the first number is correct.

12. Tonic water. (See Mindscape 9.) Because the sums are 50, 51, and 19 respectively, the first number is correct.

13. Real mayo. If the covered digit were D, then the sum would be $55 + 3 \times D$. The goal is to find a value for D that will make the sum divisible by 10, which is easily found by trial and error. $D = 5$ is the only digit between 0 and 9 that works. (Note that we are solving the equation $55 + 3D \equiv 0 \pmod{10}$, or equivalently, $3D \equiv 5 \pmod{10}$.)

14. **Applesauce.** (See Mindscapes 13–18.) If D represents the missing digit, then the computed sum is $54 + D$. The only way to make the sum a multiple of 10 is to choose $D = 6$.

15. Grand Cru. (See Mindscapes 13–18.) If the missing digit were 0, the sum would be 89. If the missing digit were D, the sum would be $89 + 3D$. We want to make this sum evenly divisible by 10. Because $89 \equiv 9 \pmod{10}$, we need $3D \equiv 1 \pmod{10}$, and the only solution is $D = 7$.

16. Mixed nuts. (See Mindscapes 13–18.) Suppose that the missing digit were K, then the sum would be $80 + K$. To make this sum a multiple of 10, we only need to choose $K = 0$, and indeed, this is the missing digit.

17. Blue chips. (See Mindscapes 13–18.) Just as in Mindscape 16, the sum is $50 + 3M$, where M is the covered digit. The only way to make this sum divisible by 10 is to choose $M = 0$.

18. Lemon. (See Mindscapes 13–18.) If M is the missing digit, then the sum is $49 + M$, so $M = 1$.

19. Decoding. There are three unknown digits, the 9, the 1, and the 7. Because each digit could be one of two different numbers, we have 8 possible combinations in all to try. 903068823517, 903068823511, 903068823577, 903068823571, 403068823517, 403068823511, 403068823577, 403068823571. And of all these numbers, only 903068823577 is a valid code. This is your best guess.

20. Check your check. Look up your bank code on your check. Use the technique on text page 104 to verify that it is a valid bank code.

21. Bank checks. As with Mindscapes 13 – 18, let the missing digit be D, and compute the sum. The resulting sum for the first bank code is $170 + 9D$, so D must be 0 to keep the sum divisible by 10. The second sum is $136 + 9 \times D$. We need $9D \equiv 4 \pmod{10}$, so $D = 6$.

22. More bank checks. (See Mindscape 21.) With the missing digit represented by D, the sums are $171 + 9D$ and $84 + 9D$ respectively. So the correct codes are 6 2 9 1 0 0 2 7 1 and 5 5 0 3 1 0 1 1 4. In the second example, $84 \equiv 4 \pmod{10}$ so we need $9D \equiv 6 \pmod{10}$ and the only value of D that satisfies this equation is $D = 4$.

23. UPC your friends. Answers will vary.

24. Whoops. In each example, two changes were made, and they canceled each other out. In the first code, the 9th and 11th digits were switched. Because the sum is computed by multiplying the 9th digit by 3 and the 11th digit by 3, the sum doesn't change. Similarly, for the second example, the 3rd and 8th digits are changed. Instead of the sum equaling .... + 3 × 1 + ... + 1 × 2 + ... , we have ... + 3 × 0 + ... + 1 × 5 + ..., where the ... represents parts of the sum that are unchanged. Because $3 \times 1 + 1 \times 2 = 3 \times 0 + 1 \times 5$, the sum remains unchanged.

25. Whoops again. (See Mindscape 24.) In the first code, the 1st and 4th digits are changed, so instead of 7 × 0 + ... + 7 × 7 + ... we have 7 × 7 + ... + 7 × 0 + ..., and so the sum remains unchanged. The same type of mistake occurs in the second example where the 6th and 9th terms are interchanged. Because the 6th and 9th terms are both multiplied by the same weight, 9, the total sum will remain unchanged.

Creating New Ideas

26. Mod remainders. $129 = 9 \times 13 + 12$, so 12 is the remainder when 129 is divided by 13. We can also say $129 \equiv 12 \pmod{13}$. A quick way to see this is $129 = 130 - 1 = 10 \times 13 - 1 = 9 \times 13 + 13 - 1 = 9 \times 13 + 12$. You would spin around 13 times and then move the clock ahead 12 hours more.

27. More mod remainders. $2015 = 287 \times 7 + 6$. So $2015 \equiv 6 \pmod{7}$. If m divided by n gives a remainder r, then we can say $m \equiv r \pmod{n}$. If we had a clock with n hour positions (0 through $n - 1$), then after moving the hour hand of the clock m places, the hand will be sitting in the r^{th} positions.

28. Money orders. Because 6830910275 is divisible by 7, the check digit is 0.

29. Airline tickets. We have $10061559129884 = 1437365589983 \times 7 + 3$, so the check digit is 3.

30. UPS. (See Mindscapes 28–29.) $84200912 = 12028701 \times 7 + 5$, so the check digit is 5.

31. Check a code. Check the identification number on your example using the technique in Mindscape 28 or 29.

32. ISBN. Verify this check method for the ISBN of this book.

33. ISBN check. The first code has a sum of 152 + D where D is the check digit. Because 152 = 8 (mod 11), we want 8 + D = 0 (mod 11), that is, D = 3. Similarly the second code has sum 107 + D. To solve the equation 107 + $D \equiv$ 0(mod 11), note that 107 = 9 × 11 + 8, simplifying our equation to 8 + $D \equiv$ 0 (mod 11). Once again the check digit is 3.

34. ISBN error. The current number corresponds to a sum of 192 ≡ 5(mod 11). If the 1st and 2nd digits were transposed, what would the new sum be? The old sum can be expressed as 10 × 3 + 9 × 5 + ..., and the new sum would be 10 × 5 + 9 × 3 + ... The difference between the two sums is 10 × (5 – 3) + 9 × (3 – 5) or (5 – 3) × (10 – 9) = (5 – 3) = 2. Without recomputing the sum, we can deduce that the new sum would total 194. Notice the pattern: if we interchange the ith and (i + 1)th digits, the difference will be ($d_{i+1} - d_i$). We want to find a difference that is equal to 6 so that the new sum will be 192 + 6 ≡ 198 (mod 11). The only pair of adjacent digits that differs by 6 is the 5th and 6th digits. The correct number is 3-540-60395-6.

35. Brush up your Shakespeare. Find a book with a play by Shakespeare and check its ISBN number using the technique described in Mindscape 32.

Further Challenges

36. Mods and remainders. Example: 23 divided by 7 is 3 with a remainder of 2. When performing long division, 7 is outside, 23 is inside the division sign, 3 is on top, and 2 is at the very bottom, the last number computed. This means that 23 = 3 × 7 + 2, or equivalently 23 ≡ 2 mod 7. In terms of clocks, if we had a mod 7 clock with hand initially at 0, moving 23 units is equivalent to spinning around completely 3 times, and then moving 2 more units. This puts the clock hand in position 2. Generalize this example, where $n = qm + r$, where q is the quotient and r is the remainder.

37. Catching errors. Extreme examples are usually simpler to understand. Because the weights of any adjacent pair of digits are either 3 or 1, let's focus on transposing the first two digits. 1600000001 and 6100000001 are both valid. Similarly, transposing the first digits of the following will still result in a valid number: 2700000007, 3800000003, 4900000009, 5000000005, etc.. In summary, if the difference between adjacent digits is divisible by 5, transposing the digits still represents a valid code.

38. Why three? The key insight is that you get 10 different remainders (mod 10) when multiplying by 3 and only 5 when multiplying by 6. Turns out that any number relatively prime to 10, such as, 1,3,7,9,11,13…, will provide 10 distinct remainders. With 10 distinct remainders, each digit contributes a different amount to the total sum. For example, when multiplying by 6, both 4 and 9 contribute the same because 4 × 6 = 24 ≡ 4 (mod 10) and 9 × 6 = 54 ≡ 4(mod 10). If the 4 were scratched out, you could only tell that the number was either a 4 or a 9. With ten different remainders, you can always recover the covered digit.

39. A mod surprise. It's surprising that n^4 (mod 5) equals 1 for every number. Section 2.5 discusses this in detail.

40. A prime magic trick. The mystery number that you write down will always be 1, so don't play the trick too many times on the same person!

For the Algebra Lover

45. **One congruence, two solutions.** Use guess-and-check to find $x = 4$ is a solution. Then add 7 to get $x = 11$ as another solution.

46. **Chinese remainder.** The first congruence implies that x be even; the second implies that x be 1 greater than a multiple of 3. So $x = 4$ will work.

47. **More remainders.** The congruences imply that $z - 1$ is congruent to 0 mod 2, mod 3, and mod 5. Therefore $z - 1$ is divisible by 2, 3, and 5. So we can let $z - 1 = 2\times3\times5 = 30$. Thus $z = 31$ is a solution.

48. **Quotient coincidence.** We are given that $x = 7q + 6$ and $x = 11q + 2$. Therefore $11q + 2 = 7q + 6$, and we have $4q = 4$. Therefore $q = 1$ and $x = 13$.

49. **Mod function.** The values $y = 10$ and $y = 17$ both satisfy the congruence $y \equiv 3 \text{ mod } 7$. Notice that 10 is 7 more than 3. If we add 14 to 3, we get 17, which also satisfies $y \equiv 3 \text{ mod } 7$. So we conjecture that adding any multiple of 7 to 3 will yield a workable value of y. Thus, y = $7x + 3$ should work for all natural numbers x. We check by dividing $7x + 3$ by 7. The result is x1 remainder 3. Thus $7x + 3 \equiv 3 \text{ mod } 7$.

2.5. Secret Codes and How to Become a Spy

Developing Ideas

1. **What did you say?** THIS IS THE CORRECT MESSAGE.

2. Secret admirer. The message encodes to: B WXAU GXL.

3. **Setting up secrets.** The numbers $p = 7$ and $q = 17$ are both prime because each has no factor other than itself and 1. The number $m = (p\text{-}1)(q\text{-}1) = 6 \times 16 = 96$. The number e = 5 has no factors in common with m = 96. Finally, $5 \times 77 - 96 \times 4 = 385 - 384 = 1$

4. Second secret setup. The numbers $p = 5$ and $q = 19$ are both prime because each has no factor other than itself and 1. The number $m = (p\text{-}1)(q\text{-}1) = 4 \times 18 = 72$. The number $e = 11$ has no factors in common with $m = 72$. Finally, $11 \times 59 - 72 \times 9 = 649 - 648 = 1$.

5. **Secret squares.** We find $2^2 = 4 \equiv 1 (\text{mod } 3)$; $3^2 = 9 \equiv 0$ (mod 3); $4^2 = 8 \equiv 2$ (mod 3); $5^2 = 25 \equiv 1$ (mod 3). As you successive integers, the result (mod 3) cycles through the pattern 1, 0, 2, 1, 0, 2, ….

Solidifying Ideas

6. Petit Fermat 5. The expressions are all of the form $n^{(p-1)}(\text{mod } p)$, and so by Fermat's Little Theorem, they are equal to 1 (mod p). e.g. $4^4 = (4^2)^2 = (16)^2 = 1^2 = 1$ (mod 5), where the second to last equality results because $16 \equiv 1 (\text{mod } 5)$.

7. Petit Fermat 7. As in question 1, the numbers are all of the form $k^{(p-1)}(\text{mod } p)$, and so are equal to 1 (mod 7).

8. Top secret. The encoded word is 4^7 (mod 143) $\equiv 82$. To decode the number, raise the encrypted information to the 103rd power and compute the remainder (mod 143).

9. Middle secret. You don't need to compute 3^7 explicitly. $3^5 \equiv 243 \equiv 100$ (mod 143), so $3^6 \equiv 3 \times 100 \equiv 14$ (mod 143) and finally $3^7 \equiv 3 \times 14 \equiv 42$ (mod 143). As in Mindscape 8, the information is decoded by the computation $42^{103} \equiv 3$ (mod 143).

10. Bottom secret. We need to compute 11^7 (mod 143). (11^7 = 19,487,171). Because 19,487,171 ÷ 143 = 136,273 with a remainder of 132, we have 11^7 (mod 143) $\equiv 132$. The original '101 can be recovered by computing $132^{103} \equiv 11$ (mod 143). Even though the encoded number is identical to the original number, it's still a secret because you are the only person who knows that they are one and the same.

11. Creating your code. Note first $m = (3 - 1) \times (5 - 1) = 8$. Because e must be relatively prime to m, we need only consider the values $e = 1, 3, 5$, and 7. For each possible value of e, find d and y that satisfy $de - 8y = 1$. For example, for $e = 1$, fill in the following blanks: __ x 1 – 8 x __ = 1. Because $1 \times 1 - 8 \times 0 = 1$, ($e = 1, d = 1$) is a pair. Similarly, because $3 \times 3 - 8 \times 1 = 1$, $5 \times 5 - 8 \times 2 = 1$, and $7 \times 7 - 8 \times 6 = 1$, ($e = 3, d = 3$), ($e = 5, d = 5$), and ($e = 7, d = 7$) are all pairs.

12. Using your code. "HI" becomes (08) (09). To use the coding scheme ($p = 3$, $q = 5$, e = 3, $d = 3$), we need to compute $8^3 \equiv 2$ (mod 15) and $9^3 \equiv 9$ (mod 15). So the code is (02)(09) or "BI". Because $2^3 \equiv 8$ (mod 15) and $9^3 \equiv 9$ (mod 15) we get the original message back upon decoding. Note that you can only use the first 14 letters of the alphabet!

13. Public secrecy. Using 83^7 □ 8 mod 143, the encoded version is '8'. One deciphers this message with the formula 8^{103} □ 83 mod 143.

14. Going public. You encode '61' by computing 61^7 □ 74 mod 143, and you decode '74' by computing 74^{103} □ 61 mod 143.

15. Secret says. Use 38^{103} □ 103 mod 143 to obtain the original message, '103'.

Creating New Ideas

16. Big Fermat. The hint asks you to recall that $5^6 \equiv 1$ (mod 7). This means that $(5^6)^k \equiv 1^k \equiv 1$ (mod 7) for any integer k. In particular, because $600 = 6 \times 100$, it is convenient to choose $k = 100$, giving us $5^{600} \equiv (5^{6 \times 100}) \equiv (5^6)^{100} \equiv 1^{100} \equiv 1$ (mod 7). Similarly, because $1000000 = 10 \times 100000$, $8^{1000000} \equiv 1$ (mod 11).

17. Big and powerful Fermat. (See also solution to Mindscape 16.) Our building block is the formula $5^6 \equiv 1$ (mod 7). Now after dividing 668 by 6 we can represent $668 = 6 \times 111 + 2$. Therefore,

$$5^{668} \equiv 5^{6 \times 111 + 2} \equiv 5^{6 \times 111} \times 5^2 \equiv (5^6)^{111} \times (25) \equiv 1^{111} \times 4 \equiv 4 \pmod{7}.$$

18. The value of information. You would have to answer the following questions: Who am I keeping this from? How much time would they be willing to spend trying to break the code? With their resources, what size numbers can they factor in that time? As a reference point, you might note that Maple, a standard mathematical computer package, can factor the product of two 29 digit primes in 30 seconds on a Linux workstation! For every 3 digits you tack on, Maple takes *twice* as long to complete the factorization. With two 32-digit primes, it takes 1 minute; 35-digit primes, 2 minutes; 50-digit primes, 1 hour! (How large would the primes need to be in order for Maple to require 100 years' worth of computation time?)

19. Something in common. Because p divides n, p will also divide $n^{(p-1)}$, so that when we divide $n^{(p-1)}$ by p, the remainder will be zero. This means exactly, $n^{(p-1)} \equiv 0 \bmod p$.

20. Faux pas Fermat. Fermat's theorem doesn't hold here because 6 isn't prime. We get $1^5 \equiv 1$, $2^5 \equiv 2$, $3^5 \equiv 3$, $4^5 \equiv 4$, $5^5 \equiv 5 \pmod 6$. However, $1^2 \equiv 1$, $2^2 \equiv 4$, $3^5 \equiv 3$, $4^5 \equiv 4$, $5^5 \equiv 1$, (mod 6). We also have $1^6 \equiv 1$, $2^6 \equiv 1$, $3^6 \equiv 0$, $4^6 \equiv 1$, $5^6 \equiv 1$, $6^6 \equiv 0$, $7^6 \equiv 1$, $8^6 \equiv 1 \pmod 9$. Note that the pattern of ones is broken only when the base shared a factor with 9. Note also that there are 6 numbers sharing no factors with 9 and the exponent is also 6. Similarly, 2 numbers relatively prime to 6 (1 and 5), and when these are raised to the 2nd power, they equal 1 (mod 6). Yes, if k and m are relatively prime, and n is the number of relatively prime numbers less than m, then $k^n \equiv 1 \pmod m$. For prime numbers p, there are $(p-1)$ numbers less than p that are also relatively prime to p (why?). Thus is a generalization of Fermat's Little Theorem.

Further Challenges

21. Breaking the code. We know only the public numbers e and the product pq, but we want to find d, the decoding number. After getting p and q, construct $m = (p-1)(q-1)$. The numbers e, and m satisfy the following equation __e – __m = 1, where the blanks are integers, and the first blank represents the decoding number d. Here is an outline of the systematic process that is the Euclidean Algorithm. Divide e into m, getting remainder r_1. Divide r_1 into e, giving r_2, etc.. until you get down to a remainder of $r_n = 1$. Now go backwards, and express 1 as a linear combination of r_{n-1} and r_n. Then because r_n can be expressed as a linear combination of r_{n-1} and r_{n-2}, we can write 1 as a linear combination of r_{n-1} and r_{n-2}. Repeat this process until you have written 1 as a linear combination of e and m. At this point you will have filled in the blanks, and found d.

22. Signing your name. Joseph writes a separate message, 'Really, this is me Joseph. Pork kidneys are the wave of the future.' He then scrambles it by decoding it as he would for any incoming encoded messages and inserts the new text into his letter. You decode this secret message by encoding it with his public keys as if you were going to send him a secret note. When you raise the text to the e power, his original message will appear. Irving Satan would have to know Joseph's private key in order to forge this extra personal message.

For the Algebra Lover

27. **Powers of 2.** We find $2^2 = 4 \equiv 1 \bmod 3$, $2^4 = 16 \equiv 1 \bmod 5$, and $2^6 = 128 \equiv 1 \bmod 7$. Each power of 2 is equivalent to 1 as predicted by Fermat's Little Theorem.

28. **FOILed!** We FOIL $(a-1)(b-1)$ to obtain $ab - a - b + 1$. Thus we have $323 - a - b + 1 = 288$ which simplifies to $324 - 288 - a - b = 0$, or $36 = a + b$. Knowing that both a and b are prime, we can guess and check to find $a = 19$ and $b = 17$. (Or the other way around – it doesn't matter. These are the only two primes that have sum 36 and product 323)

29. **FOILed again!** We FOIL $(x - 1)(y - 1)$ to obtain $xy - x - y + 1$. Thus we have $91 - x - y + 1 = 72$ which simplifies to $92 - 72 - x - y = 0$, or $20 = x + y$. Knowing that both x and y are prime, we guess and check to find $x = 7$ and $y = 13$. (Or the other way around – it doesn't matter. These are the only two primes that have sum 20 and product 91.)

30. **Secret primes.** From $q - p = 2$, we get $q = p + 2$. Substituting into $(p - 1)(q - 1) = 24$, we obtain $(p - 1)(p + 2 - 1) = 24$, or $(p - 1)(p + 1) = 24$. FOILing the left side and simplifying we get $p^2 - 1 = 24$. Thus $p^2 = 25$, so the prime p is 5 and the prime q is 7.

31. **More secrets.** From $q - p = 4$, we get $q = p + 4$. Substituting into $(p - 1)(q - 1) = 60$, we obtain $(p - 1)(p + 4 - 1) = 60$, or $(p - 1)(p + 3) = 60$. FOILing the left side and simplifying we get $p^2 + 2p - 3 = 60$, or $p^2 + 2p - 63 = 0$. Factoring we obtain $(p - 7)(p + 9) = 0$. Thus the prime p is 7 and the prime q is 11.

2.6 The Irrational Side of Numbers

Developing Ideas

1. **A rational being.** A rational number is a number that can be expressed as a fraction - the ratio (or quotient) of two whole numbers.

2. **Fattened fractions.** 6/24 = 1/4, 15/9 = 5/3, -14/42 = -1/3, 125/10 = 25/2, -121/11 = -11.

3. **Rational arithmetic**. $\frac{1}{2}+\frac{5}{2}=\frac{6}{2}=3$; $\frac{1}{2}-\frac{2}{3}=\frac{3}{6}-\frac{4}{6}=-\frac{1}{6}$; $\frac{1}{2}\times\frac{6}{5}=\frac{6}{10}=\frac{3}{5}$;

$\frac{1/2}{2/3}=\frac{1}{2}\times\frac{3}{2}=\frac{3}{4}$; $\frac{5/2\times 6/5}{2/3}=\frac{30/10}{2/3}=\frac{3}{1}\times\frac{3}{2}=\frac{9}{2}$

4. **Decoding decimals.** 0.02 = 2/100, 6.23 = 623/100, 2.71828 = 271,828/100,000, -168.5 = -1685/10, -0.00005 = -5/10,0000.

5. **Odds and ends**. The squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. The even numbers have even squares and the odd numbers have odd squares.

Solidifying Ideas

6. Irrational rationalization. No, 3√2/5√2 = 3/5, which is rational. The product or quotient of an irrational and a rational is always irrational, so both 3√2 and 5√2 are irrational. But the quotient (or product) of two irrationals is not always irrational.

7. Rational rationalization. Yes, the quotient of two rationals is rational. If a,b,c, and d are integers, then (a/b) / (c/d) = (ad)/(bc) which is rational by definition; it is the quotient of two integers.

8. Rational or not. √2/14 is the only irrational number in the list. As the ratio of two integers, 4/9 is rational by definition. 1.75 = 1 + 3/4 = 7/4, √20/(3√5) = (√20/√5)/3 = √4/3 = 2/3, 3.14159 = 314159/100000. We could reason that 3.14159 and 1.75 are rational because they each have a repeating decimal expansion (1.7500000..., 3.13159000...)

9. Irrational or not. All but √3/3 are rational. √16/20 = 4/5, 12/7.5 = 120/75, – 147 = – 147/1, 0 = 0/1. √3/3 is the quotient of an irrational and a rational number and therefore irrational.

10. √5. The proof is identical to the proof of the irrationality of √2, except that the notions *even* and *odd* are replaced with *divisible by 5* and *not divisible by 5*, respectively. Assume √5 = b/c with b and c having no common factors. We have $b^2 = 5c^2$, implying b is divisible by 5 (because 5 is prime and must therefore appear as one of the prime factors of b; we're using the uniqueness of the prime factorization here). Expressing b = 5d gives, $25d^2 = 5c^2$ or $5d^2 = c^2$ implying c is *also* divisible by 5, a contradiction. This same idea can be applied to the square root of any prime number.

11. 2√3. Though you can mimic the proof that √2 is irrational, it is simpler to note that because 3 is prime, if follow that √3 is irrational (see Mindscape 10), and then argue that a rational multiplied by an irrational is irrational.

12. √7. Identical in spirit to Mindscape 10.

13. √3 + √5. An alternate style of proof: $(\sqrt{3} + \sqrt{5})^2 = (\sqrt{3} + \sqrt{5})(\sqrt{3} + \sqrt{5}) = 3 + 2\sqrt{3}\sqrt{5} + 5 = 8 + 2\sqrt{15}$. First argue that √15 is irrational (see Mindscape 15). Use the fact that a rational times an irrational is irrational to show that 2 √15 is irrational. Similarly, the sum of a rational and an irrational is also irrational, which implies that $8 + 2\sqrt{15} = (\sqrt{3} + \sqrt{5})^2$ is irrational. If (√3 + √5) were rational, then $(\sqrt{3} + \sqrt{5})^2$ would be rational too. Because $(\sqrt{3} + \sqrt{5})^2$ is *not* rational, (√3 + √5) is not rational either, completing the proof.

14. √2 + √7. Model the text's proof that √2 + √3 is irrational. Assume √2 + √7 = a/b (in lowest terms). $(\sqrt{2} + \sqrt{7})^2 = 9 + \sqrt{14} = a^2/b^2$, so that $\sqrt{14} = a^2/b^2 - 9 = (a^2 - 9b^2)/b^2$ contradicting the fact that √14 is irrational (see Mindscape 15).

15. √10. We need a slight modification of the proof in Mindscape 10. Begin the same way: Assume √10 = *c*/*d* with *c* and *d* having no common factors. Squaring gives $c^2 = 10d^2$. Because the right hand side is divisible by 5, the left hand side is divisible by 5 as well. This means that *c* is divisible by 5. (If *c* weren't divisible by 5, then c^2 wouldn't be divisible by 5 either.) Write *c* = 5*n*, substituting gives $25n^2 = 10d^2$, or $5n^2 = 2d^2$. Now we must work harder to show that 5 divides *d*. Imagine writing out the prime factorization for the left and right sides of the equation. On the left we have all the prime factors of *n* (listed twice) and 5. On the right we have 2 and all the prime factors of *d* (listed twice). Because both sides represent the same number, we call upon the uniqueness of prime factorizations to argue that the list of primes on both sides are the same. Because the prime 5 appears on the left side, it must also appear on the right side. And because it can only come from the prime factorization of *d*, we must have that 5 is a prime factor of *d*. So d is divisible by *5* and we have our contradiction.

16. 1 + √10. If 1 + √10 = *a/b*, then √10 = *a*/*b* – 1 = (*a* – *b*)/*b* is rational. Mindscape 15 shows that this is not the case. This contradiction shows that our assumption was wrong, proving that 1 + √10 is irrational.

17. An irrational exponent. Assume that E is rational, that is, $E = n/m$. Substituting gives, $12^{(n/m)} = 7$, or $12^n = 7^m$. Look at the prime factorization of both sides of this last equation. Because both sides represent the same number, the prime factorizations must agree (Uniqueness of prime factorization). On the left, we have n 3's and twice as many 2's. On the right we have m 7's. These two lists can't be the same, regardless of the values of n and m. This represents a contradiction, and because our only assumption was the rationality of E, we conclude E is irrational.

18. Another irrational exponent. If $E = n/m$, then $13^n = 8^m$. The prime factorization of the left hand side is just a bunch of 13's, while the prime factorization of the right hand side is composed solely of 2's. Because this represents two different prime factorizations of the same number, we have derived a contradiction to the uniqueness of prime factorizations. Therefore, our assumption was wrong, E is not rational.

19. Still another exponent. Identical in spirit to Mindscapes 17 and 18.

20. Another rational exponent. $E = 2/3$. Raising 8 to the 2/3 power is equivalent to taking the cube root of 8 and then squaring the result. Applying the reasoning behind solutions to Mindscapes 17 and 18. Assume $E = n/m$, giving $8^n = 4^m$, so that the prime factorization of the left side has $3n$ 2's while the right hand side has $2m$ 2's. But these two factorizations could be exactly the same if the number of twos on both sides were the same. We need only $3n = 2m$, equivalently, $n/m = 2/3$.

21. Rational exponent. Because the notion of a prime factorization is relevant only to integers, let's rewrite the equation as $2^{E/2} = 2^{3/2}$. From this equation, it is immediately apparent that $E = 3$ works. So E is rational after all.

22. Rational sums. The two rationals are a/b and c/d, where a,b,c and d are all integers. $a/b + c/d = (ad + bc)/(bd)$. Because the product and sum of two integers is just another integer, we have expressed the sum as a ratio of two integers. So the sum is rational.

23. Rational Products. Let a/b and c/d represent the two rational numbers, where a,b,c and d are all integers. Because the product $(a/b)(c/d)$ can be expressed as the quotient of two integers, $(ac)/(bd)$, the product is rational.

24. Root of a rational. Rewrite $\sqrt{(1/2)}$ as $1/\sqrt{2}$ and use the fact that the quotient of a rational and an irrational is irrational. See Mindscape 25 for an alternate approach to this problem.

25. Root of a rational. Adapt the "$\sqrt{2}$ is irrational proof." Assume $\sqrt{(2/3)} = a/b$ (with no factors in common). Squaring gives $2b^2 = 3a^2$. At this point, it doesn't matter whether you choose 2 or 3, but you must stick with whatever you choose! (I'll choose 3.) The right side is divisible by 3, and so $2b^2$ is divisible by 3. Because 3 is prime and because 2 isn't divisible by 3, b^2 must be divisible by 3. Again, because 3 is prime, we conclude that b is divisible by 3. Writing b = $3n$ and substituting it into the equation gives $18n^2 = 3a^2$, or $6n^2 = a^2$. Using the same reasoning, we conclude that 3 divides a. So 3 divides both a and b, contradicting the fact that a and b had no common factors. We conclude that our original assumption was wrong and therefore $\sqrt{(2/3)}$ is irrational.

Creating New Ideas

26. ***π*****.** Use an indirect proof. Assume that the sum is rational – that is, suppose that $\pi + 3 = a/b$, with a and b representing integers. Rewriting yields $\pi = a/b - 3 = (a - 3b)/b$, which contradicts the fact that π cannot be written as the quotient of two integers. Because our only assumption was that the sum was rational, the contradiction allows us to conclude that the assumption was wrong. The only alternative is that the sum is irrational.

27. 2*π*. As in Mindscape 26, we prove this indirectly. If the product were rational, then we could write $2\pi = m/n$, where m and n were integers. Solving for π then yields $\pi = m/(2n)$, which is the form of a *rational* number. Thus we have shown that if 2π is rational, then π is rational. Because we know that π is *not* rational, we conclude 2 π is also *not* rational.

28. $\boldsymbol{\pi^2}$**.** This is identical in spirit to Mindscapes 26 and 27. If we assume that π is rational, we can write $\pi = n/m$ (where n and m are integers). So $\pi^2 = n^2/m^2$ which means that we have expressed π^2 as a rational number, a contradiction. We must conclude that our original assumption is wrong. The only alternative is that π is *not* rational!

29. A rational in disguise. By one useful property of exponents, we have $(x^a)^b = x^{ab}$. So $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\sqrt{2})} = \sqrt{2}^2 = \sqrt{2}\,\sqrt{2} = 2$. After this simplification, we can easily classify it as a rational number.

30. Cube roots. Use the "√2 is irrational proof" as a template. Assume $\sqrt[3]{2}$ is rational, that is, $\sqrt[3]{2} = a/b$ where a and b have no common factors. *Cube* both sides and multiply through by b^3 to get $2b^3 = a^3$. This implies that a^3 is even, which in turn implies that a is even allowing us to write $a = 2n$. Substituting gives, $2b^3 = 8n^3$, or $b^3 = 4n^3$. By the same reasoning we can argue that b is even contradicting the fact that a and b share no common factors. Thus our original assumption is wrong, which implies $\sqrt[3]{2}$ is irrational.

31. More cube roots. See Mindscapes 11 and 30. Assume $\sqrt[3]{3} = a/b$ where a and b are reduced to lowest terms. Cubing and rearranging gives $3b^3 = a^3$. This means that 3 divides a^3. Because 3 is prime 3 must divide a as well. (At the core of this reasoning is the Prime Factorization Theorem.) Write $a = 3n$, and substitute this into the last equation to get $3b^3 = 27n^3$, or $b^3 = 9n^3$. By the same reasoning, we can assert that because b^3 is divisible by 3, b is also divisible by 3. But now both a and b share the common factor 3, which is a contradiction. Thus our original assumption was wrong. Therefore, $\sqrt[3]{3}$ is irrational.

32. One-fourth root. This is identical in nature to Mindscape 31.

33. Irrational sums. Not always. The not-so-satisfying counterexample is that π and (– π) are both irrationals, yet their sum is zero, which is rational. The numbers 1.01001000100001… and 0.10110111011110… are both irrational because the tail end of their decimal expansion can't be represented as a repeating segment. Their sum is 1.11111111111111… = 10/9, a rational number. Keep in mind that sometimes the sum is irrational, e.g. $\pi + \pi = 2\pi$

34. Irrational products. Sometimes the product is irrational, as in $\sqrt{2}\,\sqrt{5} = \sqrt{10}$: but sometimes the product is rational, as in $\sqrt{2}\,\sqrt{2} = 2$. Another simple example: both π and 1/ π are irrational, but their product is 1, which is rational.

35. Irrational plus rational. This is a generalization of Mindscape 26. An indirect approach is best. Assume that the sum of an irrational and a rational is *rational* and try to derive a contradiction. Let q represent the irrational number, let a/b represent the rational, and because we are assuming that the sum is rational, let c/d represent the sum. We then have $q + a/b = c/d$. Solving for q gives $q = c/d - a/b = (cb - ad)/(db)$. We've just expressed q as the quotient of two integers, which contradicts the fact that q represented an irrational number. This contradiction proves our assumption wrong. The only alternative is that the original sum is irrational.

Further Challenges

36. $\sqrt{p}$. (This generalizes Mindscape 10.) Assume $\sqrt{p} = a/b$ where a and b share no common factors. Squaring and rearranging gives $pb^2 = a^2$ implying that a^2 is divisible by p. Because p is prime, it appears in the prime factorization of a^2. Because the prime factorization of a^2 contains two copies of all the primes appearing in the prime factorization of a (uniqueness of prime factorization), p must also be in the prime factorization of a, and so a is also divisible by p. So we can write $a = pn$ and put this back into the equation giving $pb^2 = p^2n^2$ or $b^2 = pn^2$. The exact same reasoning shows that b is divisible by p contradicting the fact that a and b share no factors. This contradiction implies that our assumption was wrong, which in turn means that $\sqrt{p}$ is irrational.

37. $\sqrt{(pq)}$. (This generalizes Mindscapes 15 and 36.) Assume $\sqrt{(pq)} = a/b$ where a and b share no common factors. Squaring and rearranging gives $pqb^2 = a^2$, implying that p divides a^2. Because p is prime, we can use the reasoning given in Mindscape 36 to show that p also divides a. Now replace a with np in the equation above. $pqb^2 = p^2a^2$ and so $qb^2 = pa^2$. Because p divides the right side of the equation, it must also divide the left side of the equation. Equivalently, p must appear as one of the primes in the prime factorization of qb^2. The prime factors of qb^2 are just q and all the prime factors of b listed twice. Because p is in the collection of primes, and because p doesn't equal q, p must be in the prime factorization of b. Thus p divides b, which is contrary to our assumption that a and b have no common factors. This contradiction proves that $\sqrt{(pq)}$ is irrational.

38. $\sqrt{p} + \sqrt{q}$. Break this problem into two cases: Case I: $p = q$. Case II: $p \neq q$. (Always do the easy one first!) If p = q, then we need only show that $2\sqrt{p}$ is irrational. By Mindscape 36, we know $\sqrt{p}$ is irrational, and because the product of a rational and an irrational is always irrational, we have that $2\sqrt{p}$ is also irrational (done with Case I). Now demand that p and q are different. Let's assume $\sqrt{p} + \sqrt{q}$ is rational and try to derive a contradiction, that is, assume $\sqrt{p} + \sqrt{q} = a/b$. Squaring both sides yields $p + 2\sqrt{pq} + q = a^2/b^2$. Solving for $\sqrt{pq}$ gives, $\sqrt{pq} = ((a^2/b^2) - p - q)/2$. The right side represents a rational number, but in Mindscape 37, we proved that the left side was irrational. This contradiction shows that our assumption about the rationality of $\sqrt{p} + \sqrt{q}$ was wrong. Therefore, $\sqrt{p} + \sqrt{q}$ is irrational.

39. **√4**. Assume $\sqrt{4} = a/b$, square both sides giving $4b^2 = a^2$. At this point we typically say, "4 divides a^2, so 4 must then also divide a." That's the mistake! For example, 4 divides 6^2, but 4 does not divide 6. All we can say is that because 2 divides a^2, then 2 also divides a. We can say this because 2 is prime. Because 2 appears in the prime factorization of a^2, it appear in the prime factorization of a. This isn't enough to derive a contradiction. Write a = $2n$, substituting gives, $4b^2 = 4n^2$, or $b^2 = n^2$. We can't conclude anything about the factors of b, so the argument breaks down.

40. **Sum or difference.** We want to show that either $a + b$ or $a – b$ is irrational. What's the alternative? What if this were *not* true? The only way the conclusion could be false is if $a + b$ and $a – b$ were *both* rational. In other words, the world is divided into two situations. Situation I: at least one of $a + b$ or $a – b$ is irrational. Situation II: Both $a + b$ and $a – b$ are rational. Let's explore the consequences of the second situation. If $a + b = m/n$ and $a – b = r/s$, then (solving for a) $a = (m/n + r/s)/2$. This contradicts the fact that a is irrational. So Situation II does not happen, and we are left with the fact that either the sum or the difference is irrational.

For the Algebra Lover

45. **Rational x.**

$$x = \frac{\frac{3}{5}+\frac{3}{5}}{\frac{17}{5}} = \frac{\frac{6}{5}}{\frac{17}{5}} = \frac{6}{5}\times\frac{5}{17} = \frac{6}{17}. \qquad x = \frac{\frac{5}{3}}{1+\frac{11}{4}} = \frac{\frac{5}{3}}{\frac{4}{4}+\frac{11}{4}} = \frac{\frac{5}{3}}{\frac{15}{4}} = \frac{5}{3}\times\frac{4}{15} = \frac{20}{45} = \frac{4}{9}$$

$$x = \frac{4x^2-100}{(3x+15)(x-5)} = \frac{4(x^2-25)}{3(x+5)(x-5)} = \frac{4(x+5)(x-5)}{3(x+5)(x-5)} = \frac{4}{3}$$

46. **High 5.** The positive number solution to $x^2 = 5$ is $x = \sqrt{5}$, which is an irrational number. If $\sqrt{5}$ were rational, then we could write $\sqrt{5} = a/b$, where a and b are positive whole numbers with no common factors. Thus $5 = a^2/b^2$, so $5b^2 = a^2$. Following the style of the argument in the text, we find that both a and b must be divisible by 5, which contradicts our assumption about a and b. Thus $\sqrt{5}$ is irrational.

47. **Don't be scared.** We can rewrite the equation as $7x^3 - 19x^2 + 10x - 5 = \sqrt{2}$. If x were a rational number, then the left side of the equation could be simplified into a rational number, giving us $\sqrt{2}$ as a rational number. This cannot happen, so we must have x irrational.

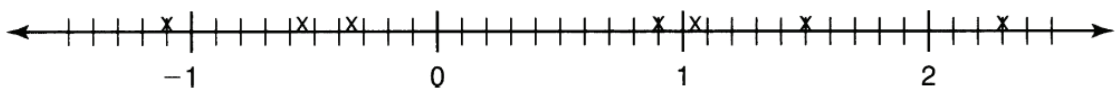
48. **A hunt for irrationals.** Factoring the left side we obtain $x^3 - 3x = x(x^2 - 3) = x(x - \sqrt{3})(x + \sqrt{3}) = 0$. Thus the solutions are $x = 0$, $x = \sqrt{3}$ and $x = -\sqrt{3}$. The first solution is rational, the remaining two are irrational.

49. **A hunt for rationals.** The given identity shows us how to factor the left side of the equation $2x^2 - x - 3 = 0$. We obtain $(2x - 3)(x + 1) = 0$. Thus $2x - 3 = 0$ or $x + 1 = 0$, so the solutions are $x = 3/2$ and $x = -1$. Both solutions are rational. In general, numbers r and s will always be solutions to the equation $(x - r)(x - s) = 0$. Expanding the left side, we get that r and s are solutions to the equation $x^2 - (r + s)x + rs = 0$. If r and s are rational, then so are rs and $r + s$. Thus, the coefficients of the quadratic equation are rational, as requested.

2.7. Get Real

Developing Ideas

1. X marks the "X-act" spot. The X's on the number line below mark the approximate locations, from left to right, of the numbers –1.1, -0.55, -1/3, 0.9, 1.05, 3/2, and 2.3,



2. **Moving the point**. Simplifying we get 10 × (3.14)=31.4, 1000x(0.123123…)=123.123123…, 10 × (0.4999…)=4.999…, $\frac{98.6}{100} = 0.986$, $\frac{0.333...}{10} = 0.0333...$

3. **Watch out for ones!** Using long division we find 1/9 = 0.111….

4. **Real redundancy.** If M = 0.4999…, then 10M = 4.999…. We find 10M – M = 9M and it also equals 4.999… - 0.4999… = 4.5. Then 9M = 4.5, so M = 4.5/9 = 0.5. Thus 0.4999… = 0.5.

5. Being irrational. A number is irrational if it is not rational, i.e., it *cannot* be written as a ratio of two integers.

Solidifying Ideas

6. Always, sometimes, never. Sometimes. By 'an unending decimal expansion' we mean a number whose decimal doesn't end in a trail of zeros All numbers ending in a trail of zeros are rational, but the converse is not true. for example, 9/7 = 1.28571428571428571428571428571... is rational, but 1.0100100010000100000100000001... is irrational.

7. Square root of 5. False: if the decimal expansion for √5 eventually repeated, we could use the ideas in the text to express √5 as the ratio of two integers and so prove that √5 is rational. Because we proved √5 irrational in the Section 2.6, this can't happen; so the only alternative is that the decimal expansion for √5 does not repeat.

8. A rational search. The first nine digits of our mystery number are predetermined, 12.0345691…. As long as the remaining decimals are not all zeros, the mystery number will sit between the given numbers. To keep the number rational, it's necessary to put a sequence of decimals that eventually repeats. 12.034569110000000… and 12.034569133333333…. both work. We can also take the number halfway between the two, 12.034569150000…

9. Another rational search. Let's create, digit by digit, a number X that sits in between the two numbers. The first six digits are 3.14159. Any smaller six digit number would be less than 3.14159 and any larger six digit number would be greater than 3.14159001, regardless of the remaining digits of X. The next three digits must be zero to keep the number less than 3.14159001. After that we are free to choose any repeating sequence or ending sequence, such as, $X = 3.141590005$ works.

10. An irrational search. Take an irrational number that you know and stick the non-repeating decimals after 5.70. A simple and useful irrational number is 0.0100100010000100001…; the irrationality follows because there is no fixed string of numbers that repeats forever. (There is an obvious pattern that repeats, but no fixed string repeats! See Mindscape 25.) By the same reasoning 5.701010010001000010000001… is irrational, and it lies between the two numbers above.

11. Another irrational search. Let's use an irrational number that we know and stick it on the tail of the smaller number. We proved the irrationality of $\sqrt{2} = 1.4142135623730950488016887242 1$… in Section 2.6. 0.00010000141421356237309504880168872421… is therefore an irrational number sitting between the two given numbers.

12. Your neighborhood. The smallest number comes by inserting all zeros, 10.039800000, and the largest number is formed by inserting all nines, 10.039899999.

13. Another neighborhood. We don't just have five X's to replace, we have infinitely many X's to replace. Regardless, the smallest number is formed by replacing all the X's with zeros, 5.550100000… = 5.5501 and the largest number is formed by replacing all the X's with nines, 5.550199999… = 5.5502

14. 6/7 . Use long division (or a calculator). 7 goes into 60 eight times with a remainder of 4; bring down the zero. Then 7 goes into 40 five times, with a remainder of 5; and so on. 6/7 = .857142857142857142857142857143...

15. 17/20 . Use long division as in Mindscape 14, or rewrite the fraction in a simpler form by multiplying numerator and denominator by 5. 17/20 = 85/100 = 0.85

16. 21.5/15 . Either perform long division directly, 15 divided into 21.5, or first multiply numerator and denominator by 10 to get rid of the pesky decimal point in the numerator and reduce to lowest terms. 21.5/15 = 215/150 = 43/30 now divide 30 into 43. 21.5/15 = 1.433333…

17. 1.28901. First write this as 1.28901/1.00000, and multiply both top and bottom by 100000 in order to get rid of the decimal. So 1.28901 = 128901/100000. This method works for any decimal that terminates.

18. 20.4545 . Note that this decimal stops or ends in a trail of zeros (20.454500000…). Thus the method of Mindscape 17 will work here, too. 20.4545 = 204545/10000. It isn't necessary to reduce the fraction to lowest terms, but if you were curious, $X = 40909/2000$.

19. 12.999 . Because the decimal ends, we can eliminate the decimal by multiplying it by 1000. So write 12.999 = 12.999/1 = (12.999/1) × (1000/1000) = 12999/1000.

20. 2.22... . X = 2.22222…. Because the number has a segment of length one that repeats, multiply the number by 10 to shift the decimal point by exactly one digit. $10X = 22.22222\ldots$ Now subtract, $10X - X = 22.22222\ldots - 2.22222\ldots = 20$ (Note that all digits to the right cancel exactly.) So $9X = 20$, and $X = 20/9$.

21. 43.12... . Call our elusive number X, so that $X = 43.121212\ldots$ Because there are two digits in our repeating segment, multiply X by 100 to shift the decimal points by two digits. $100X = 4312.121212\ldots$ Subtracting gives $100X - X = 4312.121212\ldots - 43.121212 = 4269$. Together we get 99X = 4269 or $X = 4269/99 = 1423/33$. (Again simplifying fractions is not necessary!)

22. 5.6312... . Follow the reasoning in Mindscape 21. $X = 5.63121212\ldots$, $100X = 563.12121212\ldots$ $100X - X = 563.121212\ldots - 5.6312121212\ldots = 557.49$ We still have a decimal number, but at least it stops! Solving for X in $99X = 557.49$ gives $X = 557.49/99$. Now multiply both numerator and denominator of the fraction by 100 to eliminate the decimal points. $X = 55749/9900 = 18583/3300$.

23. 0.01... . $X = 0.010101\ldots$ Because the repeating segment has length 2, multiply by 100 to shift the decimal point two digits to the left. $100X = 1.010101\ldots$ Subtracting gives $100X - X = 1.010101 - 0.010101 = 1$ so that $99X = 1$, or $X = 1/99$.

24. 71.2399... . Note that this number can also be represented as 71.24 which is equal to 7124/1000. However, we could still use the ideas from Mindscape 22 to get this fraction. $X = 71.239999\ldots$ $10X = 712.39999\ldots$, $10X - X = 712.39999\ldots - 71.239999\ldots = 641.16$ so that $9X = 641.16$, or $X = 641.16/9 = 64116/900 = 7124/1000 = 1781/25$.

25. Just not rational. This number has a pattern (one 0, one 1, two 0s, one 1, three 0s, one 1, etc.), but that does not mean that it's rational. A decimal is rational if and only if there exists a fixed string of digits that repeats forever. This number has no repeating sequence. Suppose there was a repeating sequence of length N. If the repeating sequence were all zeros, then we'd end up with a rational number. If the repeating sequence were not all zeros, then eventually we would see a non-zero digit after every N digits. But this isn't the case; we see arbitrarily large sequences of zeros. This implies that there is no repeating sequence and so the number is irrational.

Creating New Ideas

26. Farey fractions. Let F_n be the collection of all rational numbers between 0 and 1 (we write 0 as 0/1 and 1 as 1/1) whose numerators and denominators do not exceed n. So, for example,

$F_1 = \{0/1, 1/1\}$, $F_2 = \{0/1, 1/2, 1/1\}$, $F_3 = \{0/1, 1/3, 1/2, 2/3, 1/1\}$.

F_n is called the nth *Farey fractions*. List F_4, F_5, F_6, F_7, and F_8. Make a large number line segment between 0 and 1 and write in the Farey fractions. How can you generate F_8 using F_7? Generalize your observations and describe how to generate F_n. (HINT: Try adding fractions a wrong way)

F_4 = {0/1,**1/4**,1/3,1/2,2/3,**3/4**,1/1}
F_5 = {0/1,**1/5**,1/4,1/3,**2/5**,1/2,**3/5**,2/3,3/4,**4/5**,1/1}
F_6 = {0/1,**1/6**,1/5,1/4,1/3,2/5,1/2,3/5,2/3,3/4,4/5,**5/6**,1/1}
F_7 = {0/1,**1/7**,1/6,1/5,1/4,**2/7**,1/3,2/5,**3/7**,1/2,**4/7**,3/5,2/3,**5/7**,3/4,4/5,5/6,**6/7**,1/1}
F_8 = {0/1,**1/8**,1/7,1/6,1/5,1/4,2/7,1/3,**3/8**,2/5,3/7,1/2,
4/7,3/5,**5/8**,2/3,5/7,3/4,4/5,5/6,6/7,**7/8**,1/1}

Create F_8 from F_7 by adding in fractions that (a) have an 8 in the denominator and (b) are already in lowest terms. We don't add '6/8' because '3/4' is already in the list.

27. Even irrational. Stringing together all the even positive integers creates this number. Its irrationality follows from the same logic used to solve Mindscape 25; that is, there exist arbitrarily large sequences of zeros. If the number were rational, then there would be a sequence of length *N* that repeated (and this repeating sequence is obviously not all zeros). So after the decimal starts repeating, every *N* digits will contain a non-zero digit. Regardless where we look in the decimal expansion, there will always be arbitrarily large sequences of zeros to the right. (10^{10} has 10 zeros, 10^{100} has 100 zeros, etc.) This contradiction proves that our number is irrational.

28. Odd irrational. This problem is essentially identical to Mindscape 27. Note that the number (10^{10} + 1) is an odd number with 9 zeros. (10^{100} + 1) has 99 adjacent zeros. Because we have arbitrarily large sequences of zeros to the right (interspersed with nonzero digits), there can be no sequence of digits that repeats forever.

29. A proof for π. It may be that decimal expansion of a number repeats after the trillionth place. To prove the rationality or irrationality of π, we need to show that it repeats forever after some point or it *never* repeats, neither of which can be done by looking at a finite number of digits. For example, $1/(10^{10^{12}} - 1)$ is a number that repeats every trillion digits. The first 999,999,999,999 digits are zeros but the trillionth digit is a one, and then it repeats!

30. Irrationals and zero. Build irrational numbers from something you *know* is irrational, like √2. We showed that dividing irrational numbers by rational numbers leaves an irrational number. Therefore, √2/2, √2/3, √2/4, √2/5, … are all irrational numbers that get closer and closer to zero. So no, there is no smallest irrational number, just as there is no smallest rational number. Alternatively, because √2 = 1.1414213… is irrational, the decimal expansion never repeats. Therefore 0.01414213…, 0.001414213…, 0.0001414213…, etc.. are irrational numbers that get closer and closer to zero.

31. Irrational with 1's and 2's. In Mindscape 25, we showed that $x = 0.01001000100001\ldots$ was irrational. By the same reasoning, $y = 0.21221222122221\ldots$ is also irrational. You could also argue that because $y = 2/9 - x$, the irrationality of x implies the irrationality of y because the sum of an irrational and a rational is always irrational. And finally, for fun, a more interesting, more random looking irrational number with only 1's and 2's: List all the rational numbers, and apply Cantor's diagonalization argument with a rule like, "If the nth digit is a 1, put a 2, otherwise put a 1."

32. Irrational with 1's and Some 2's. No; if only a finite number of 2's appeared in the decimal expansion, then after the last 2, the decimal tail would be all 1's and therefore repeating. So the number would be rational.

33. Half steps. This is Zeno's paradox. You will land on the numbers 1/2, 1/4, 1/8, $1/2^4$, $1/2^5$, $1/2^6$, … , $1/2^n$, … The nth step takes you to $1/2^n$, so you will never get to zero in a finite amount of time. You can get arbitrarily close, but you will never actually get there because $1/2^n$ doesn't equal zero for any finite number n. The limit of this sequence of numbers is zero, but none of the numbers themselves are zero.

34. Half steps again. Suppose the left half of your segment has length L. L may be small, but it is a positive number, and because the sequence $1/2^n$ tends to zero as n grows without bound, there exists some N such that $1/2^N < L$. This means that after N steps, your segment will contain the origin. Note that your center will never hit the origin, but at least some part of you will get to where you want to go.

35. Cutting π. This is an alternate way of asking whether π might be a rational number. Suppose we divided the interval into N pieces. The endpoints land on $3 + 1/N$, $3 + 2/N$, $3 + 3/N$, etc... all of which are rational numbers. Because π is irrational, there is no way that we can represent π as $3 + I/N$ for any integers I and N.

Further Challenges

36. From infinite to finite. How about our favorite irrational number $\sqrt{2}$? Because we proved it irrational, we know that the decimal is unending and non-repeating. By definition its square is 2, which has a terminating decimal representation.

37. Rationals. Assume x and y are two different positive numbers and that y is bigger than x. The sequence 1/2, 1/3, 1/4, 1/5, … gets arbitrarily small; thus, for some number N, the value of $1/N$ is smaller than the difference $y - x$. Now imagine cutting up the real number line with hash marks every $1/N$ units apart. So you mark 0, $1/N$, $2/N$, $3/N$, … and so on. All these hash marks are on rational numbers, but at least one of the hash marks lies between the numbers x and y because $y - x$ is greater than $1/N$.

38. Irrationals. The argument used in Mindscape 37 could be used here as well. Instead of using hash marks at $1/N$, $2/N$, $3/N$, … use hash marks at $1/N - \sqrt{2}$, $2/N - \sqrt{2}$, $3/N - \sqrt{2}$, etc. But there is a simpler argument: Assume that x and y are positive real numbers with x smaller than y, and let N be such that $1/N$ is smaller than $y - x$. If x is irrational, then $x + 1/N$ is also irrational and lies between x and y; done. If x is rational, then $x + (1/N)/\sqrt{2}$ is an irrational number lying between x and y. (Because $\sqrt{2}$ is bigger than 1, $(1/N)/\sqrt{2}$ is less than $(1/N)$ and, thus, less than $y - x$.)

39. Terminator. Mindscapes 17 and 18 show how to express terminating decimals as fractions, namely, remove the decimal point, and divide by a power of 10 equal to the number of non-zero decimal digits. For example, $12.3456 = 123456/10^4 = 123456/10000$. Because the only factors of 10 are 2 and 5, the only factors of the denominator 10^n are 2's and 5's, which completes the argument.

40. Terminator II. Let's turn the solution to Mindscape 39 around. We have a number of the form $x/(2^n5^m)$ (such as 101/400, where $400 = 2^45^2$.) If the denominator is a power of 10, then we are done, because we can immediately write down the terminating decimal. If not, we try to make the denominator a power of 10 by multiplying both the numerator and denominator by the appropriate number of 2's or 5's. If $m > n$, that is, if there are more 5's than 2's, multiply by $2^{(m-n)}$, otherwise multiply by $5^{(n-m)}$. For example, $101/400 = 101/2^45^2$ (need two extra 5's) = (25/25)(101/400) = 2525/10000 = 0.2525.

For the Algebra Lover

45. An unknown digit. Rewrite the equation to obtain $10 = x + 0.xxxxx\ldots$, which becomes $10 = x.xxxxx\ldots$. Therefore $1 = 0.xxxxx\ldots$, and following the example done in the text, we find $x = 9$.

46. Is *x* rational? Suppose x is rational, say, $x = a/b$ for whole numbers a and b. Then we have $4(a/b) - I = 2/3$. Solving for I we obtain $I = 2/3 + 4a/b = (2b + 12a)/3b$, which implies that I is rational. This is not the case, so x must be irrational.

47. Is *y* irrational? According to the table of values below, $y = 0.471013161922\ldots$ Thus the tenth digit of y is 9. It appears that y will be irrational. The formula $3n + 1$ will not generate repeating values as n increases.

n	1	2	3	4	5	6	7	. . .
$f(n) = 3n + 1$	4	7	10	13	16	19	22	. . .

48. Is *z* irrational? According to the table of values below, $z = 0.371321314357\ldots$ Thus the tenth digit of z is 3. It appears that z will be irrational. The formula $n^2 + n + 1$ will not generate repeating values as n increases.

n	1	2	3	4	5	6	7	. . .
$g(n) = n^2 + n + 1$	3	7	13	21	31	43	57	. . .

49. **Triple digits.** Following the hint, we add the third equation to the first and second equations to obtain $5a + 2c = 29$ and $3a + 4c = 37$. Subtract the second new equation from twice the first new equation to obtain $7a = 21$. So $a = 3$. Reverse substituting yields $c = 7$ and $b = 2$.