Problem 1.15

[Difficulty: 5]

1.15 For Problem 1.14, the initial horizontal speed of the sky diver is 70 m/s. As she falls, the k value for the vertical drag remains as before, but the value for horizontal motion is $k = 0.05\ \mathrm{N \cdot s/m^2}$. Compute and plot the 2D trajectory of the sky diver.

Given: Data on sky diver: $M = 70 \cdot kg$ $\quad k_{vert} = 0.25 \cdot \frac{N \cdot s^2}{m^2}$ $\quad k_{horiz} = 0.05 \cdot \frac{N \cdot s^2}{m^2}$ $\quad U_0 = 70 \cdot \frac{m}{s}$

Find: Plot of trajectory.

Solution: Use given data; integrate equation of motion by separating variables.

Treat the sky diver as a system; apply Newton's 2nd law in horizontal and vertical directions:

Vertical: Newton's 2nd law for the sky diver (mass M) is (ignoring buoyancy effects): $M \cdot \frac{dV}{dt} = M \cdot g - k_{vert} \cdot V^2$ (1)

For $V(t)$ we need to integrate (1) with respect to t:

Separating variables and integrating: $$\int_0^V \frac{V}{\frac{M \cdot g}{k_{vert}} - V^2}\, dV = \int_0^t 1\, dt$$

so $$t = \frac{1}{2} \cdot \sqrt{\frac{M}{k_{vert} \cdot g}} \cdot \ln\left(\left|\frac{\sqrt{\frac{M \cdot g}{k_{vert}}} + V}{\sqrt{\frac{M \cdot g}{k_{vert}}} - V}\right|\right)$$

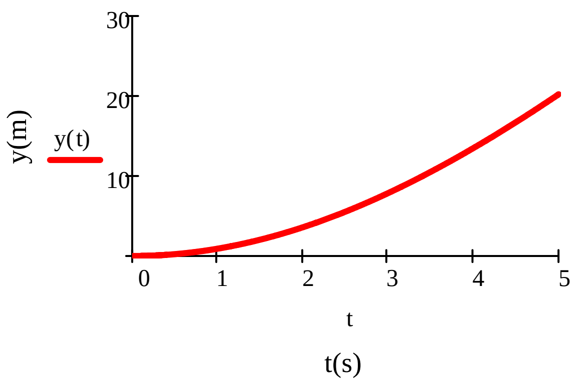
Rearranging $$V(t) = \sqrt{\frac{M \cdot g}{k_{vert}}} \cdot \frac{\left(e^{2 \cdot \sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t} - 1\right)}{\left(e^{2 \cdot \sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t} + 1\right)}$$ so $$V(t) = \sqrt{\frac{M \cdot g}{k_{vert}}} \cdot \tanh\left(\sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t\right)$$

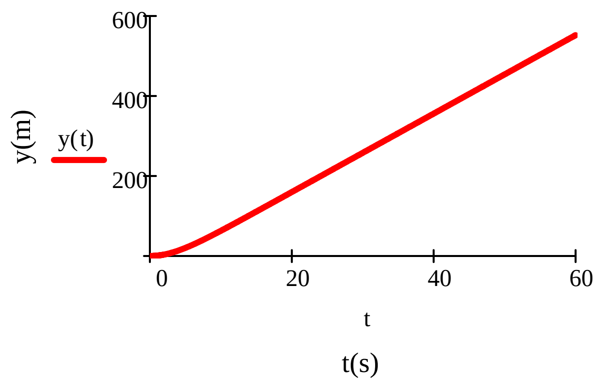
For $y(t)$ we need to integrate again: $\frac{dy}{dt} = V$ or $y = \int V\, dt$

$$y(t) = \int_0^t V(t)\, dt = \int_0^t \sqrt{\frac{M \cdot g}{k_{vert}}} \cdot \tanh\left(\sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t\right) dt = \sqrt{\frac{M \cdot g}{k_{vert}}} \cdot \ln\left(\cosh\left(\sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t\right)\right)$$

$$y(t) = \sqrt{\frac{M \cdot g}{k_{vert}}} \cdot \ln\left(\cosh\left(\sqrt{\frac{k_{vert} \cdot g}{M}} \cdot t\right)\right)$$

After the first few seconds we reach steady state:





Horizontal: Newton's 2nd law for the sky diver (mass M) is:

$$M \cdot \frac{dU}{dt} = -k_{horiz} \cdot U^2 \qquad (2)$$

For *U*(*t*) we need to integrate (2) with respect to *t*:

Separating variables and integrating:

$$\int_{U_0}^{U} \frac{1}{U^2}\,dU = \int_0^t -\frac{k_{horiz}}{M}\,dt \qquad \text{so} \qquad -\frac{k_{horiz}}{M} \cdot t = -\frac{1}{U} + \frac{1}{U_0}$$

Rearranging

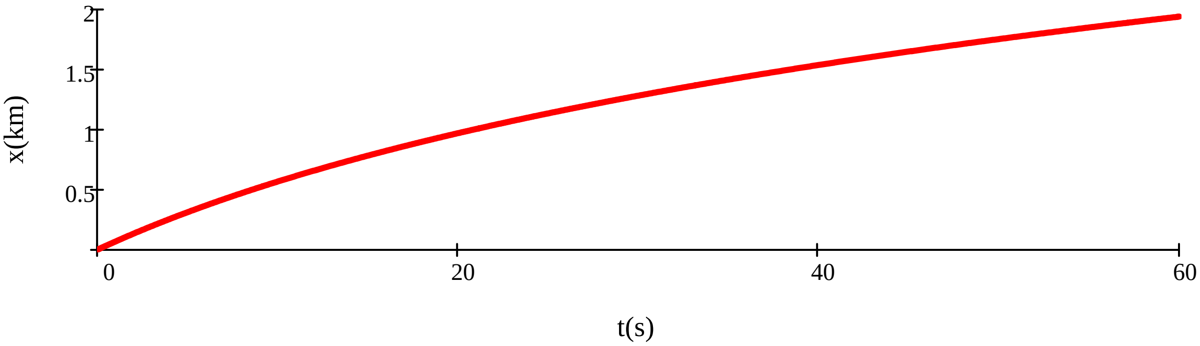
$$U(t) = \frac{U_0}{1 + \frac{k_{horiz} \cdot U_0}{M} \cdot t}$$

For *x*(*t*) we need to integrate again:

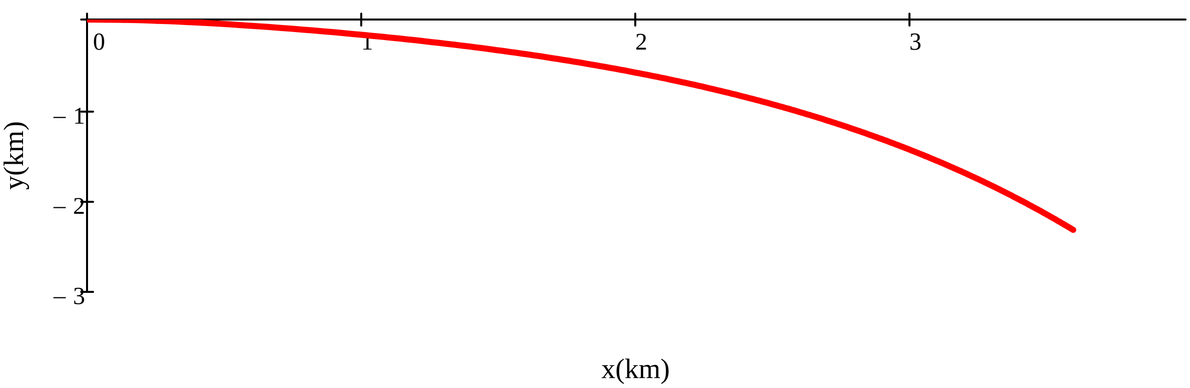
$$\frac{dx}{dt} = U \qquad \text{or} \qquad x = \int U\,dt$$

$$x(t) = \int_0^t U(t)\,dt = \int_0^t \frac{U_0}{1 + \frac{k_{horiz} \cdot U_0}{M} \cdot t}\,dt = \frac{M}{k_{horiz}} \cdot \ln\left(\frac{k_{horiz} \cdot U_0}{M} \cdot t + 1\right)$$

$$x(t) = \frac{M}{k_{horiz}} \cdot \ln\left(\frac{k_{horiz} \cdot U_0}{M} \cdot t + 1\right)$$



Plotting the trajectory:



These plots can also be done in Excel.

Problem 1.16

[Difficulty: 3]

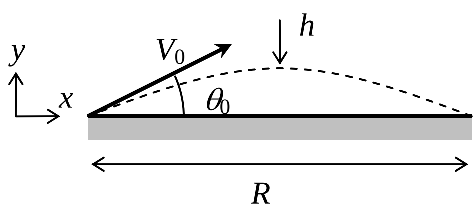
1.16 The English perfected the longbow as a weapon after the Medieval period. In the hands of a skilled archer, the longbow was reputed to be accurate at ranges to 100 m or more. If the maximum altitude of an arrow is less than $h = 10$ m while traveling to a target 100 m away from the archer, and neglecting air resistance, estimate the speed and angle at which the arrow must leave the bow. Plot the required release speed and angle as a function of height h.

Given: Long bow at range, R = 100 m. Maximum height of arrow is h = 10 m. Neglect air resistance.

Find: Estimate of (a) speed, and (b) angle, of arrow leaving the bow.

Plot: (a) release speed, and (b) angle, as a function of h

Solution: Let $\vec{V}_0 = u_0\hat{i} + v_0\hat{j} = V_0(\cos\theta_0\hat{i} + \sin\theta_0\hat{j})$



$\Sigma F_y = m\frac{dv}{dt} = -mg$, so $v = v_0 - gt$, and $t_f = 2t_{v=0} = 2v_0/g$

Also, $mv\dfrac{dv}{dy} = -mg$, $v\,dv = -g\,dy$, $0 - \dfrac{v_0^2}{2} = -gh$

Thus $h = v_0^2/2g$ (1)

$$\Sigma F_x = m\frac{du}{dt} = 0, \text{ so } u = u_0 = \text{const, and } R = u_0 t_f = \frac{2u_0 v_0}{g} \quad (2)$$

From Eq. 1: $v_0^2 = 2gh$ (3)

From Eq. 2: $u_0 = \dfrac{gR}{2v_0} = \dfrac{gR}{2\sqrt{2gh}} \quad \therefore\ u_0^2 = \dfrac{gR^2}{8h}$

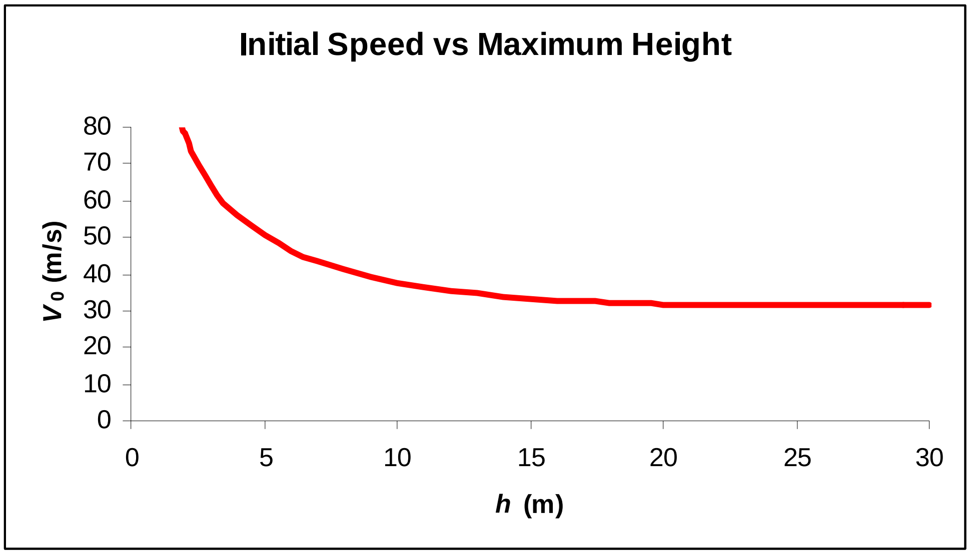
Then $V_0^2 = u_0^2 + v_0^2 = \dfrac{gR^2}{8h} + 2gh$ and $V_0 = \left(2gh + \dfrac{gR^2}{8h}\right)^{\frac{1}{2}}$ (4)

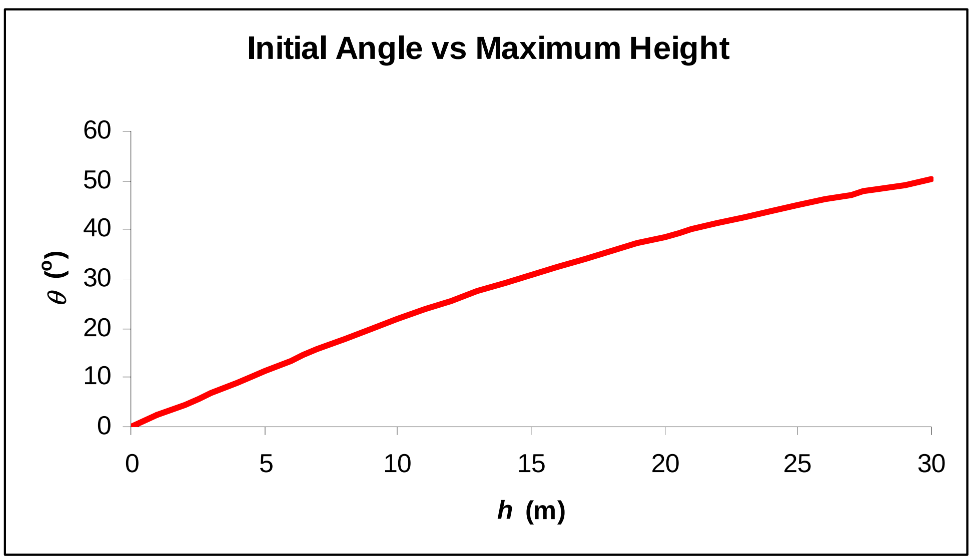
$$V_0 = \left(2 \times 9.81\frac{\text{m}}{\text{s}^2} \times 10\text{ m} + \frac{9.81}{8}\frac{\text{m}}{\text{s}^2} \times 100^2\text{ m}^2 \times \frac{1}{10\text{ m}}\right)^{\frac{1}{2}} = 37.7\frac{\text{m}}{\text{s}}$$

From Eq. 3: $v_0 = \sqrt{2gh} = V_0 \sin\theta$, $\theta = \sin^{-1}\dfrac{\sqrt{2gh}}{V_0}$ (5)

$$\theta = \sin^{-1}\left[\left(2\times 9.81\frac{\text{m}}{\text{s}^2}\times 10\ \text{m}\right)^{\frac{1}{2}}\times\frac{\text{s}}{37.7\ \text{m}}\right] = 21.8°$$

Plots of $V_0 = V_0(h)$ (Eq. 4) and $\theta_0 = \theta_0(h)$ (Eq. 5) are presented below:





Problem 1.17

[Difficulty: 2]

1.17 For each quantity listed, indicate dimensions using mass as a primary dimension, and give typical SI and English units:
(a) Power
(b) Pressure
(c) Modulus of elasticity
(d) Angular velocity
(e) Energy
(f) Moment of a force
(g) Momentum
(h) Shear stress
(i) Strain
(j) Angular momentum

Given: Basic dimensions M, L, t and T.

Find: Dimensional representation of quantities below, and typical units in SI and English systems.

Solution:

(a) Power

$$\text{Power} = \frac{\text{Energy}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \frac{F \cdot L}{t}$$

From Newton's 2nd law $\quad \text{Force} = \text{Mass} \times \text{Acceleration} \quad$ so $\quad F = \frac{M \cdot L}{t^2}$

Hence $\quad \text{Power} = \frac{F \cdot L}{t} = \frac{M \cdot L \cdot L}{t^2 \cdot t} = \frac{M \cdot L^2}{t^3} \qquad \frac{kg \cdot m^2}{s^3} \qquad \frac{slug \cdot ft^2}{s^3}$

(b) Pressure

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} = \frac{M \cdot L}{t^2 \cdot L^2} = \frac{M}{L \cdot t^2} \qquad \frac{kg}{m \cdot s^2} \qquad \frac{slug}{ft \cdot s^2}$$

(c) Modulus of elasticity

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} = \frac{M \cdot L}{t^2 \cdot L^2} = \frac{M}{L \cdot t^2} \qquad \frac{kg}{m \cdot s^2} \qquad \frac{slug}{ft \cdot s^2}$$

(d) Angular velocity

$$\text{AngularVelocity} = \frac{\text{Radians}}{\text{Time}} = \frac{1}{t} \qquad \frac{1}{s} \qquad \frac{1}{s}$$

(e) Energy

$$\text{Energy} = \text{Force} \times \text{Distance} = F \cdot L = \frac{M \cdot L \cdot L}{t^2} = \frac{M \cdot L^2}{t^2} \qquad \frac{kg \cdot m^2}{s^2} \qquad \frac{slug \cdot ft^2}{s^2}$$

(f) Moment of a force

$$\text{MomentOfForce} = \text{Force} \times \text{Length} = F \cdot L = \frac{M \cdot L \cdot L}{t^2} = \frac{M \cdot L^2}{t^2} \qquad \frac{kg \cdot m^2}{s^2} \qquad \frac{slug \cdot ft^2}{s^2}$$

(g) Momentum

$$\text{Momentum} = \text{Mass} \times \text{Velocity} = M \cdot \frac{L}{t} = \frac{M \cdot L}{t} \qquad \frac{kg \cdot m}{s} \qquad \frac{slug \cdot ft}{s}$$

(h) Shear stress

$$\text{ShearStress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} = \frac{M \cdot L}{t^2 \cdot L^2} = \frac{M}{L \cdot t^2} \qquad \frac{kg}{m \cdot s^2} \qquad \frac{slug}{ft \cdot s^2}$$

(i) Strain

$$\text{Strain} = \frac{\text{LengthChange}}{\text{Length}} = \frac{L}{L} \qquad \text{Dimensionless}$$

(j) Angular momentum

$$\text{AngularMomentum} = \text{Momentum} \times \text{Distance} = \frac{M \cdot L}{t} \cdot L = \frac{M \cdot L^2}{t} \qquad \frac{kg \cdot m^2}{s} \qquad \frac{slugs \cdot ft^2}{s}$$

Problem 1.18

[Difficulty: 2]

1.18 For each quantity listed, indicate dimensions using force as a primary dimension, and give typical SI and English units:

(a) Power
(b) Pressure
(c) Modulus of elasticity
(d) Angular velocity
(e) Energy
(f) Momentum
(g) Shear stress
(h) Specific heat
(i) Thermal expansion coefficient
(j) Angular momentum

Given: Basic dimensions F, L, t and T.

Find: Dimensional representation of quantities below, and typical units in SI and English systems.

Solution:

(a) Power

$$\text{Power} = \frac{\text{Energy}}{\text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Time}} = \frac{F \cdot L}{t} \qquad \frac{N \cdot m}{s} \qquad \frac{lbf \cdot ft}{s}$$

(b) Pressure

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} \qquad \frac{N}{m^2} \qquad \frac{lbf}{ft^2}$$

(c) Modulus of elasticity

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} \qquad \frac{N}{m^2} \qquad \frac{lbf}{ft^2}$$

(d) Angular velocity

$$\text{AngularVelocity} = \frac{\text{Radians}}{\text{Time}} = \frac{1}{t} \qquad \frac{1}{s} \qquad \frac{1}{s}$$

(e) Energy

$$\text{Energy} = \text{Force} \times \text{Distance} = F \cdot L \qquad N \cdot m \qquad lbf \cdot ft$$

(f) Momentum

$$\text{Momentum} = \text{Mass} \times \text{Velocity} = M \cdot \frac{L}{t}$$

From Newton's 2nd law $\quad \text{Force} = \text{Mass} \times \text{Acceleration} \quad$ so $\quad F = M \cdot \frac{L}{t^2} \quad$ or $\quad M = \frac{F \cdot t^2}{L}$

Hence $\quad \text{Momentum} = M \cdot \frac{L}{t} = \frac{F \cdot t^2 \cdot L}{L \cdot t} = F \cdot t \qquad N \cdot s \qquad lbf \cdot s$

(g) Shear stress

$$\text{ShearStress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{L^2} \qquad \frac{N}{m^2} \qquad \frac{lbf}{ft^2}$$

(h) Specific heat

$$\text{SpecificHeat} = \frac{\text{Energy}}{\text{Mass} \times \text{Temperature}} = \frac{F \cdot L}{M \cdot T} = \frac{F \cdot L}{\left(\frac{F \cdot t^2}{L}\right) \cdot T} = \frac{L^2}{t^2 \cdot T} \qquad \frac{m^2}{s^2 \cdot K} \qquad \frac{ft^2}{s^2 \cdot R}$$

(i) Thermal expansion coefficient

$$\text{ThermalExpansionCoefficient} = \frac{\frac{\text{LengthChange}}{\text{Length}}}{\text{Temperature}} = \frac{1}{T} \qquad \frac{1}{K} \qquad \frac{1}{R}$$

(j) Angular momentum

$$\text{AngularMomentum} = \text{Momentum} \times \text{Distance} = F \cdot t \cdot L \qquad N \cdot m \cdot s \qquad lbf \cdot ft \cdot s$$

Problem 1.19

[Difficulty: 1]

1.19 Derive the following conversion factors:
(a) Convert a viscosity of 1 m^2/s to ft^2/s.
(b) Convert a power of 100 W to horsepower.
(c) Convert a specific energy of 1 kJ/kg to Btu/lbm.

Given:
Viscosity, power, and specific energy data in certain units

Find:
Convert to different units

Solution:

Using data from tables (e.g. Table G.2)

(a) $$1 \cdot \frac{m^2}{s} = 1 \cdot \frac{m^2}{s} \times \left(\frac{\frac{1}{12} \cdot ft}{0.0254 \cdot m} \right)^2 = 10.76 \cdot \frac{ft^2}{s}$$

(b) $$100 \cdot W = 100 \cdot W \times \frac{1 \cdot hp}{746 \cdot W} = 0.134 \cdot hp$$

(c) $$1 \cdot \frac{kJ}{kg} = 1 \cdot \frac{kJ}{kg} \times \frac{1000 \cdot J}{1 \cdot kJ} \times \frac{1 \cdot Btu}{1055 \cdot J} \times \frac{0.454 \cdot kg}{1 \cdot lbm} = 0.43 \cdot \frac{Btu}{lbm}$$

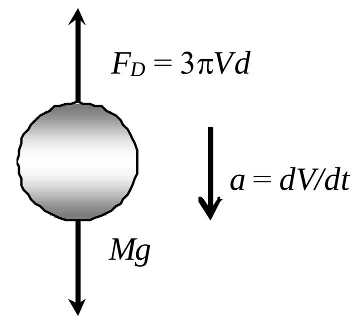
Problem 1.10

[Difficulty: 4]

1.10 In a combustion process, gasoline particles are to be dropped in air at 200°F. The particles must drop at least 10 in. in 1 s. Find the diameter d of droplets required for this. (The drag on these particles is given by $F_D = \pi\mu V d$, where V is the particle speed and μ is the air viscosity. To solve this problem, use *Excel's Goal Seek*.)

NOTE: Drag formula is in error: It should be:

$F_D = 3\cdot\pi\cdot V\cdot d$



Given: Data on sphere and formula for drag.

Find: Diameter of gasoline droplets that take 1 second to fall 10 in.

Solution: Use given data and data in Appendices; integrate equation of motion by separating variables.

The data provided, or available in the Appendices, are:

$$\mu = 4.48\times10^{-7}\cdot\frac{\mathrm{lbf\cdot s}}{\mathrm{ft}^2} \qquad \rho_w = 1.94\cdot\frac{\mathrm{slug}}{\mathrm{ft}^3} \qquad SG_{gas} = 0.72 \qquad \rho_{gas} = SG_{gas}\cdot\rho_w \qquad \rho_{gas} = 1.40\cdot\frac{\mathrm{slug}}{\mathrm{ft}^3}$$

Newton's 2nd law for the sphere (mass M) is (ignoring buoyancy effects) $\quad M\cdot\frac{dV}{dt} = M\cdot g - 3\cdot\pi\cdot\mu\cdot V\cdot d$

so

$$\frac{dV}{g - \frac{3\cdot\pi\cdot\mu\cdot d}{M}\cdot V} = dt$$

Integrating twice and using limits

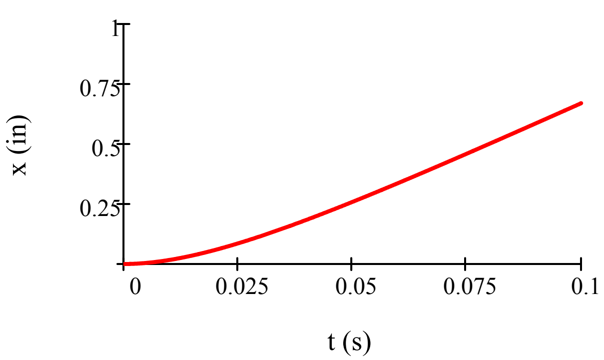
$$V(t) = \frac{M\cdot g}{3\cdot\pi\cdot\mu\cdot d}\cdot\left(1 - e^{\frac{-3\cdot\pi\cdot\mu\cdot d}{M}\cdot t}\right) \qquad x(t) = \frac{M\cdot g}{3\cdot\pi\cdot\mu\cdot d}\cdot\left[t + \frac{M}{3\cdot\pi\cdot\mu\cdot d}\cdot\left(e^{\frac{-3\cdot\pi\cdot\mu\cdot d}{M}\cdot t} - 1\right)\right]$$

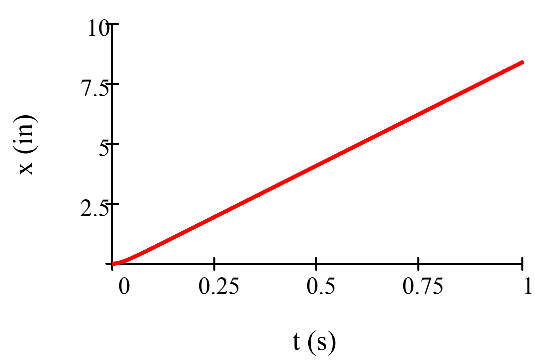
Replacing M with an expression involving diameter d

$$M = \rho_{gas}\cdot\frac{\pi\cdot d^3}{6} \qquad x(t) = \frac{\rho_{gas}\cdot d^2\cdot g}{18\cdot\mu}\cdot\left[t + \frac{\rho_{gas}\cdot d^2}{18\cdot\mu}\cdot\left(e^{\frac{-18\cdot\mu}{\rho_{gas}\cdot d^2}\cdot t} - 1\right)\right]$$

This equation must be solved for d so that $x(1\cdot s) = 10\cdot in$. The answer can be obtained from manual iteration, or by using *Excel's Goal Seek*.

$$d = 4.30\times10^{-3}\cdot in$$





Note That the particle quickly reaches terminal speed, so that a simpler approximate solution would be to solve $Mg = 3\pi\mu Vd$ for d, with $V = 0.25$ m/s (allowing for the fact that M is a function of d)!

Problem 1.11

[Difficulty: 3]

1.11 For a small particle of styrofoam (1 lbm/ft^3) (spherical, with diameter $d = 0.3$ mm) falling in standard air at speed V, the drag is given by $F_D = 3\pi\mu V d$, where μ is the air viscosity. Find the maximum speed starting from rest, and the time it takes to reach 95 percent of this speed. Plot the speed as a function of time.

Given: Data on sphere and formula for drag.

Find: Maximum speed, time to reach 95% of this speed, and plot speed as a function of time.

Solution: Use given data and data in Appendices, and integrate equation of motion by separating variables.

The data provided, or available in the Appendices, are:

$$\rho_{air} = 1.17\cdot\frac{kg}{m^3} \qquad \mu = 1.8\times 10^{-5}\cdot\frac{N\cdot s}{m^2} \qquad \rho_w = 999\cdot\frac{kg}{m^3} \qquad SG_{Sty} = 0.016 \qquad d = 0.3\cdot mm$$

Then the density of the sphere is $\rho_{Sty} = SG_{Sty}\cdot\rho_w$ $\qquad \rho_{Sty} = 16\,\frac{kg}{m^3}$

The sphere mass is

$$M = \rho_{Sty}\cdot\frac{\pi\cdot d^3}{6} = 16\cdot\frac{kg}{m^3}\times\pi\times\frac{(0.0003\cdot m)^3}{6} \qquad M = 2.26\times 10^{-10}\,kg$$

Newton's 2nd law for the steady state motion becomes (ignoring buoyancy effects) $\qquad M\cdot g = 3\cdot\pi\cdot V\cdot d$

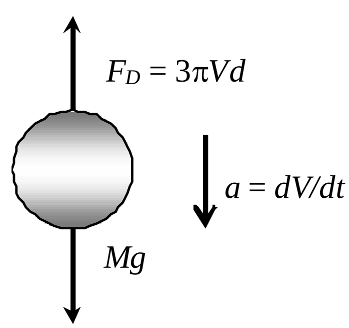
so

$$V_{max} = \frac{M\cdot g}{3\cdot\pi\cdot\mu\cdot d} = \frac{1}{3\cdot\pi}\times 2.26\times 10^{-10}\cdot kg\times 9.81\cdot\frac{m}{s^2}\times\frac{m^2}{1.8\times 10^{-5}\cdot N\cdot s}\times\frac{1}{0.0003\cdot m} \qquad V_{max} = 0.0435\,\frac{m}{s}$$

Newton's 2nd law for the general motion is (ignoring buoyancy effects) $\qquad M\cdot\frac{dV}{dt} = M\cdot g - 3\cdot\pi\cdot\mu\cdot V\cdot d$

so

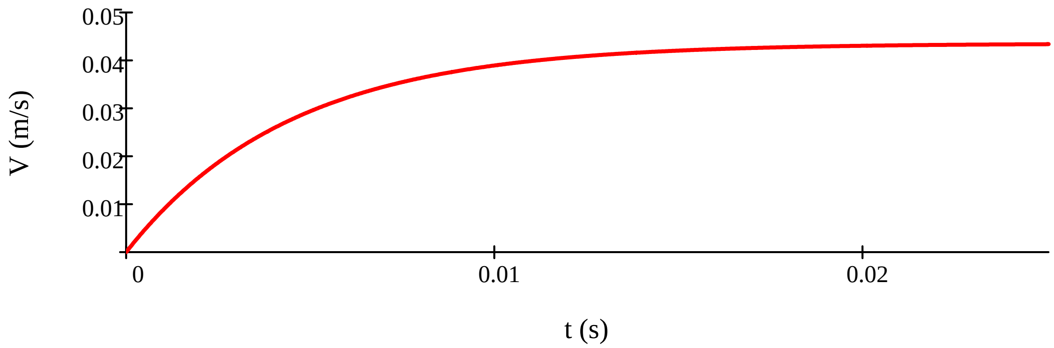
$$\frac{dV}{g - \frac{3\cdot\pi\cdot\mu\cdot d}{M}\cdot V} = dt$$



Integrating and using limits

$$V(t) = \frac{M\cdot g}{3\cdot\pi\cdot\mu\cdot d}\cdot\left(1 - e^{\frac{-3\cdot\pi\cdot\mu\cdot d}{M}\cdot t}\right)$$

Using the given data



The time to reach 95% of maximum speed is obtained from

$$\frac{M \cdot g}{3 \cdot \pi \cdot \mu \cdot d} \cdot \left(1 - e^{\frac{-3 \cdot \pi \cdot \mu \cdot d}{M} \cdot t} \right) = 0.95 \cdot V_{max}$$

so $$t = -\frac{M}{3 \cdot \pi \cdot \mu \cdot d} \cdot \ln\left(1 - \frac{0.95 \cdot V_{max} \cdot 3 \cdot \pi \cdot \mu \cdot d}{M \cdot g} \right)$$

Substituting values $t = 0.0133\,s$

The plot can also be done in *Excel*.

Problem 1.12

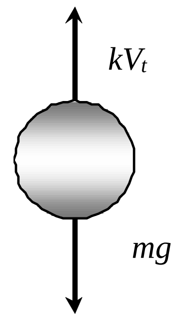
[Difficulty: 3]

1.12 In a pollution control experiment, minute solid particles (typical mass 1×10^{-13} slug) are dropped in air. The terminal speed of the particles is measured to be 0.2 ft/s. The drag of these particles is given by $F_D = kV$, where V is the instantaneous particle speed. Find the value of the constant k. Find the time required to reach 99 percent of terminal speed.

Given: Data on sphere and terminal speed.

Find: Drag constant *k*, and time to reach 99% of terminal speed.

Solution: Use given data; integrate equation of motion by separating variables.



The data provided are: $M = 1 \times 10^{-13} \cdot \text{slug}$ $\quad V_t = 0.2 \cdot \frac{\text{ft}}{\text{s}}$

Newton's 2nd law for the general motion is (ignoring buoyancy effects) $\quad M \cdot \frac{dV}{dt} = M \cdot g - k \cdot V \quad (1)$

Newton's 2nd law for the steady state motion becomes (ignoring buoyancy effects) $\quad M \cdot g = k \cdot V_t$ so $\quad k = \frac{M \cdot g}{V_t}$

$$k = 1 \times 10^{-13} \cdot \text{slug} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times \frac{\text{s}}{0.2 \cdot \text{ft}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \qquad k = 1.61 \times 10^{-11} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}}$$

To find the time to reach 99% of V_t, we need $V(t)$. From 1, separating variables $\quad \frac{dV}{g - \frac{k}{M} \cdot V} = dt$

Integrating and using limits $\quad t = -\frac{M}{k} \cdot \ln\left(1 - \frac{k}{M \cdot g} \cdot V\right)$

We must evaluate this when $\quad V = 0.99 \cdot V_t \qquad V = 0.198 \cdot \frac{\text{ft}}{\text{s}}$

$$t = -1 \times 10^{-13} \cdot \text{slug} \times \frac{\text{ft}}{1.61 \times 10^{-11} \cdot \text{lbf} \cdot \text{s}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \cdot \ln\left(1 - 1.61 \times 10^{-11} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}} \times \frac{1}{1 \times 10^{-13} \cdot \text{slug}} \times \frac{\text{s}^2}{32.2 \cdot \text{ft}} \times \frac{0.198 \cdot \text{ft}}{\text{s}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}\right)$$

$t = 0.0286\,\text{s}$

Problem 1.13

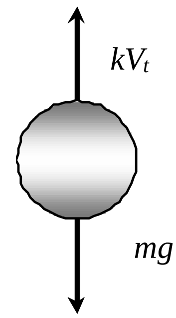
[Difficulty: 5]

1.13 For Problem 1.12, find the distance the particles travel before reaching 99 percent of terminal speed. Plot the distance traveled as a function of time.

Given: Data on sphere and terminal speed from Problem 1.12.

Find: Distance traveled to reach 99% of terminal speed; plot of distance versus time.

Solution: Use given data; integrate equation of motion by separating variables.



The data provided are: $M = 1 \times 10^{-13} \cdot \text{slug}$ $\quad V_t = 0.2 \cdot \frac{\text{ft}}{\text{s}}$

Newton's 2nd law for the general motion is (ignoring buoyancy effects) $\quad M \cdot \frac{dV}{dt} = M \cdot g - k \cdot V \quad (1)$

Newton's 2nd law for the steady state motion becomes (ignoring buoyancy effects) $\quad M \cdot g = k \cdot V_t \quad$ so $\quad k = \frac{M \cdot g}{V_t}$

$$k = 1 \times 10^{-13} \cdot \text{slug} \times 32.2 \cdot \frac{\text{ft}}{\text{s}^2} \times \frac{\text{s}}{0.2 \cdot \text{ft}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}} \qquad k = 1.61 \times 10^{-11} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}}$$

To find the distance to reach 99% of V_t, we need $V(y)$. From 1: $\quad M \cdot \frac{dV}{dt} = M \cdot \frac{dy}{dt} \cdot \frac{dV}{dy} = M \cdot V \cdot \frac{dV}{dy} = M \cdot g - k \cdot V$

Separating variables $\quad \frac{V \cdot dV}{g - \frac{k}{M} \cdot V} = dy$

Integrating and using limits $\quad y = -\frac{M^2 \cdot g}{k^2} \cdot \ln\left(1 - \frac{k}{M \cdot g} \cdot V\right) - \frac{M}{k} \cdot V$

We must evaluate this when $\quad V = 0.99 \cdot V_t \qquad V = 0.198 \cdot \frac{\text{ft}}{\text{s}}$

$$y = \left(1 \cdot 10^{-13} \cdot \text{slug}\right)^2 \cdot \frac{32.2 \cdot \text{ft}}{\text{s}^2} \cdot \left(\frac{\text{ft}}{1.61 \cdot 10^{-11} \cdot \text{lbf} \cdot \text{s}}\right)^2 \cdot \left(\frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}\right)^2 \cdot \ln\left(1 - 1.61 \cdot 10^{-11} \cdot \frac{\text{lbf} \cdot \text{s}}{\text{ft}} \cdot \frac{1}{1 \cdot 10^{-13} \cdot \text{slug}} \cdot \frac{\text{s}^2}{32.2 \cdot \text{ft}} \cdot \frac{0.198 \cdot \text{ft}}{\text{s}} \cdot \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{s}^2}\right) \ldots$$
$$+ 1 \cdot 10^{-13} \cdot \text{slug} \times \frac{\text{ft}}{1.61 \cdot 10^{-11} \cdot \text{lbf} \cdot \text{s}} \times \frac{0.198 \cdot \text{ft}}{\text{s}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}$$

$$y = 4.49 \times 10^{-3} \cdot \text{ft}$$

Alternatively we could use the approach of Problem 1.12 and first find the time to reach terminal speed, and use this time in y(t) to find the above value of y:

From 1, separating variables $\quad \frac{dV}{g - \frac{k}{M} \cdot V} = dt$

Integrating and using limits $\quad t = -\frac{M}{k} \cdot \ln\left(1 - \frac{k}{M \cdot g} \cdot V\right) \qquad (2)$

We must evaluate this when $V = 0.99 \cdot V_t$ $\quad V = 0.198 \cdot \frac{ft}{s}$

$$t = 1 \times 10^{-13} \cdot slug \times \frac{ft}{1.61 \times 10^{-11} \cdot lbf \cdot s} \cdot \frac{lbf \cdot s^2}{slug \cdot ft} \cdot \ln\left(1 - 1.61 \times 10^{-11} \cdot \frac{lbf \cdot s}{ft} \times \frac{1}{1 \times 10^{-13} \cdot slug} \times \frac{s^2}{32.2 \cdot ft} \times \frac{0.198 \cdot ft}{s} \times \frac{slug \cdot ft}{lbf \cdot s^2}\right)$$

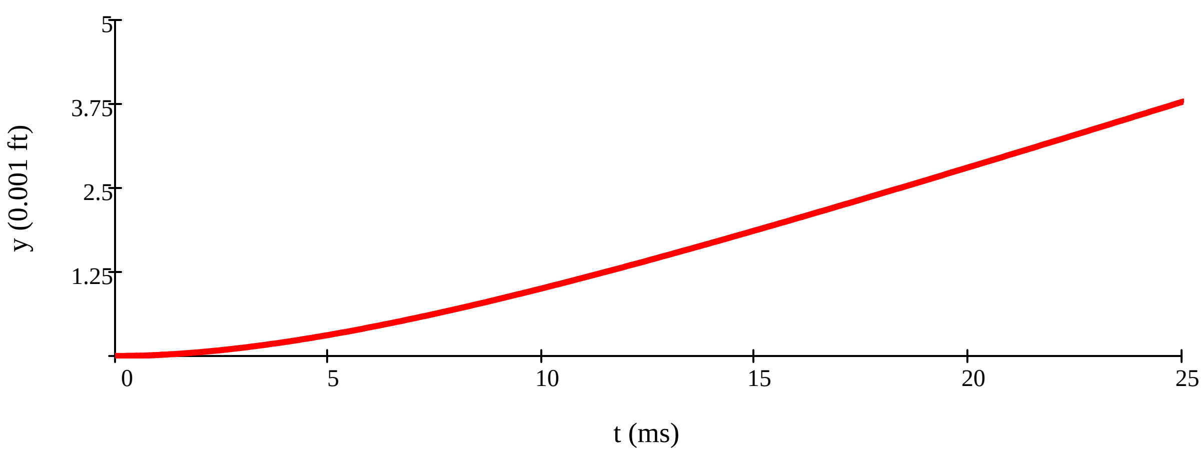
$t = 0.0286\,s$

From 2, after rearranging $\quad V = \frac{dy}{dt} = \frac{M \cdot g}{k} \cdot \left(1 - e^{-\frac{k}{M} \cdot t}\right)$

Integrating and using limits $\quad y = \frac{M \cdot g}{k} \cdot \left[t + \frac{M}{k} \cdot \left(e^{-\frac{k}{M} \cdot t} - 1\right)\right]$

$$y = 1 \times 10^{-13} \cdot slug \times \frac{32.2 \cdot ft}{s^2} \times \frac{ft}{1.61 \times 10^{-11} \cdot lbf \cdot s} \cdot \frac{lbf \cdot s^2}{slug \cdot ft} \cdot \left[0.0291 \cdot s \ldots + 10^{-13} \cdot slug \cdot \frac{ft}{1.61 \times 10^{-11} \cdot lbf \cdot s} \cdot \frac{lbf \cdot s^2}{slug \cdot ft} \cdot \left(e^{-\frac{1.61 \times 10^{-11}}{1 \times 10^{-13}} \cdot .0291} - 1\right)\right]$$

$y = 4.49 \times 10^{-3} \cdot ft$



This plot can also be presented in Excel.

Problem 1.14

[Difficulty: 4]

1.14 A sky diver with a mass of 70 kg jumps from an aircraft. The aerodynamic drag force acting on the sky diver is known to be $F_D = kV^2$, where $k = 0.25\ \mathrm{N \cdot s^2/m^2}$. Determine the maximum speed of free fall for the sky diver and the speed reached after 100 m of fall. Plot the speed of the sky diver as a function of time and as a function of distance fallen.

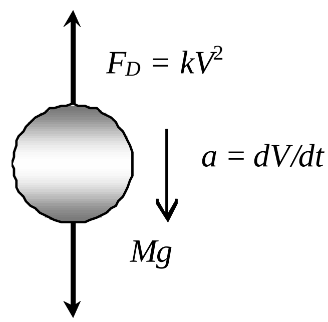
Given: Data on sky diver: $M = 70 \cdot kg$ $\quad k = 0.25 \cdot \dfrac{N \cdot s^2}{m^2}$

Find: Maximum speed; speed after 100 m; plot speed as function of time and distance.

Solution: Use given data; integrate equation of motion by separating variables.

Treat the sky diver as a system; apply Newton's 2nd law:

Newton's 2nd law for the sky diver (mass M) is (ignoring buoyancy effects): $M \cdot \dfrac{dV}{dt} = M \cdot g - k \cdot V^2 \quad (1)$



(a) For terminal speed V_t, acceleration is zero, so $M \cdot g - k \cdot V^2 = 0$ so $V_t = \sqrt{\dfrac{M \cdot g}{k}}$

$$V_t = \left(70 \cdot kg \times 9.81 \cdot \frac{m}{s^2} \times \frac{m^2}{0.25 \cdot N \cdot s^2} \cdot \frac{N \cdot s^2}{kg \times m}\right)^{\frac{1}{2}} \qquad V_t = 52.4 \frac{m}{s}$$

(b) For V at y = 100 m we need to find $V(y)$. From (1) $M \cdot \dfrac{dV}{dt} = M \cdot \dfrac{dV}{dy} \cdot \dfrac{dy}{dt} = M \cdot V \cdot \dfrac{dV}{dt} = M \cdot g - k \cdot V^2$

Separating variables and integrating:

$$\int_0^V \frac{V}{1 - \dfrac{k \cdot V^2}{M \cdot g}}\, dV = \int_0^y g\, dy$$

so

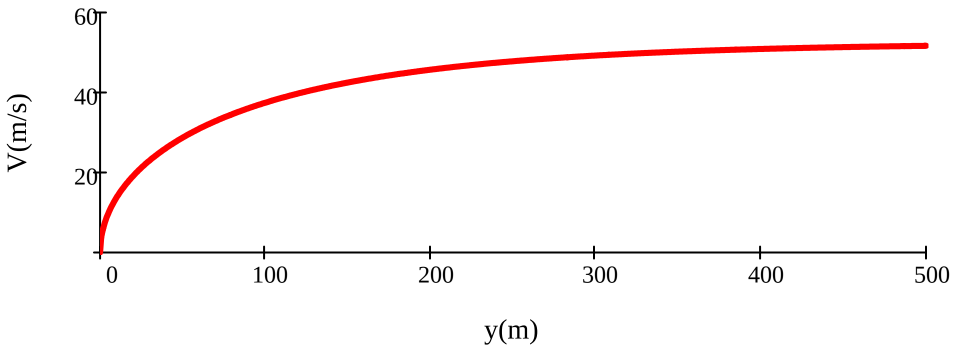
$$\ln\left(1 - \frac{k \cdot V^2}{M \cdot g}\right) = -\frac{2 \cdot k}{M} y \qquad \text{or} \qquad V^2 = \frac{M \cdot g}{k} \cdot \left(1 - e^{-\frac{2 \cdot k \cdot y}{M}}\right)$$

Hence

$$V(y) = V_t \cdot \left(1 - e^{-\frac{2 \cdot k \cdot y}{M}}\right)^{\frac{1}{2}}$$

For y = 100 m:

$$V(100 \cdot m) = 52.4 \cdot \frac{m}{s} \cdot \left(1 - e^{-2 \times 0.25 \cdot \frac{N \cdot s^2}{m^2} \times 100 \cdot m \times \frac{1}{70 \cdot kg} \times \frac{kg \cdot m}{s^2 \cdot N}}\right)^{\frac{1}{2}} \qquad V(100 \cdot m) = 37.4 \cdot \frac{m}{s}$$

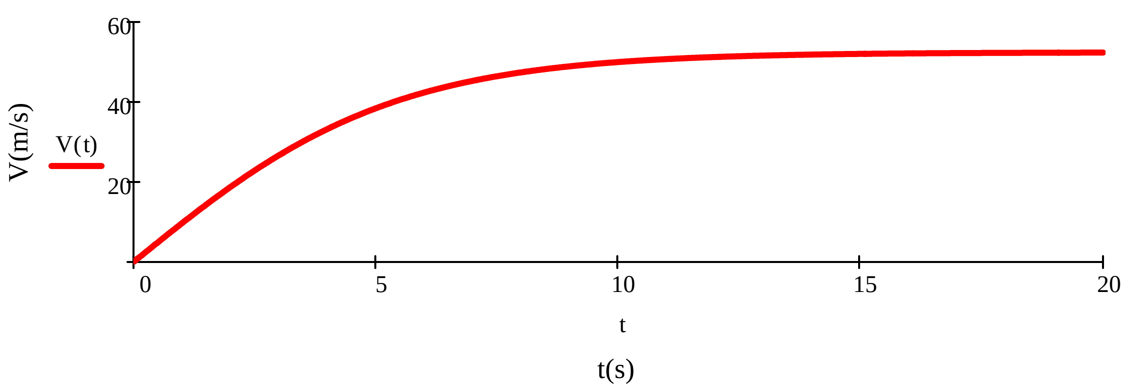


(c) For $V(t)$ we need to integrate (1) with respect to t: $M \cdot \frac{dV}{dt} = M \cdot g - k \cdot V^2$

Separating variables and integrating: $$\int_0^V \frac{V}{\frac{M \cdot g}{k} - V^2}\, dV = \int_0^t 1\, dt$$

so $$t = \frac{1}{2} \cdot \sqrt{\frac{M}{k \cdot g}} \cdot \ln\left(\left|\frac{\sqrt{\frac{M \cdot g}{k}} + V}{\sqrt{\frac{M \cdot g}{k}} - V}\right|\right) = \frac{1}{2} \cdot \sqrt{\frac{M}{k \cdot g}} \cdot \ln\left(\frac{\left|V_t + V\right|}{\left|V_t - V\right|}\right)$$

Rearranging $$V(t) = V_t \cdot \frac{\left(e^{2 \cdot \sqrt{\frac{k \cdot g}{M}} \cdot t} - 1\right)}{\left(e^{2 \cdot \sqrt{\frac{k \cdot g}{M}} \cdot t} + 1\right)}$$ or $$V(t) = V_t \cdot \tanh\left(V_t \cdot \frac{k}{M} \cdot t\right)$$



The two graphs can also be plotted in Excel.