CHAPTER 2

2.2-1. The rectangular rules for numerical integration are illustrated in Fig. P2.2-1. The left-side rule is depicted in Fig. P2.2-1(a), and the right-side rule is depicted in Fig. P2.2-1(b). The integral of $x(t)$ is approximated by the sum of the rectangular areas shown for each rule. Let $y(kT)$ be the numerical integral of $x(t)$, $0 \le t \le kT$.

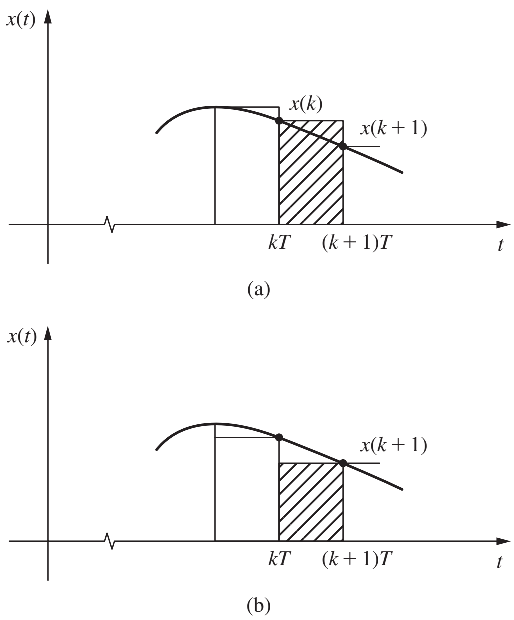


FIGURE P2.2-1 Rectangular rules for integration: (a) left side; (b) right side.

(a) Write the difference equation relating $y(k+1)$, $y(k)$, and $x(k)$ for the left-side rule.

(b) Find the transfer function $Y(z)/X(z)$ for part (a).

(c) Write the difference equation relating $y(k+1)$, $y(k)$, and $x(k+1)$ for the right-side rule.

(d) Find the transfer function $Y(z)/X(z)$ for part (c).

(e) Express $y(k)$ as a summation on $x(k)$ for the left-side rule.

(f) Express $y(k)$ as a summation on $x(k)$ for the right-side rule.

Solution:

(a) $y(k+1) = y(k) + Tx(k)$

(b) $zY(z) = Y(z) + TX(z) \Rightarrow \dfrac{Y(z)}{X(z)} = \dfrac{T}{z-1}$

(c) $y(k+1) = y(k) + Tx(k+1)$

(d) $zY(z) = Y(z) + TzX(z) \Rightarrow \dfrac{Y(z)}{X(z)} = \dfrac{Tz}{z-1}$

(e) $y(1) = y(0) + Tx(0)$

$$y(2) = y(1) + Tx(1) = y(0) + T\left(x(0) + x(1)\right)$$

$$y(3) = y(2) + Tx(2) = y(0) + T\left[x(0) + x(1) + x(2)\right]$$

$$\therefore\ y(k) = y(0) + T\sum_{n=0}^{k-1} x(n)$$

(f) $y(1) = y(0) + Tx(1)$

$$y(2) = y(1) + Tx(2) = y(0) + T\left[x(1) + x(2)\right]$$

$$\therefore\ y(k) = y(0) + T\sum_{n=1}^{k} x(n)$$

2.2-2. The trapezoidal rule (modified Euler method) for numerical integration approximates the integral of a function $x(t)$ by summing trapezoid areas as shown in Fig. P2.2-2. Let $y(t)$ be the integral of $x(t)$.

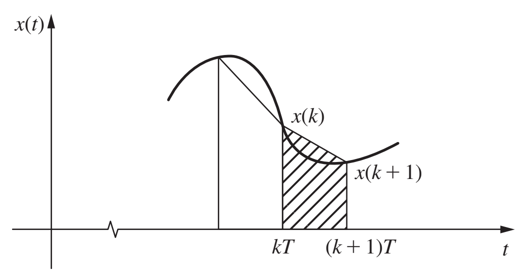


FIGURE P2.2-2 Trapezoidal rule for numerical integration.

(a) Write the difference equation relating $y[(k+1)T]$, $y(kT)$, $x[(k+1)T]$, and $x(kT)$ for this rule.

(b) Show that the transfer function for this integrator is given by

$$\frac{Y(z)}{X(z)} = \frac{(T/2)(z+1)}{z-1}$$

Solution:

(a) $y(k+1) = y(k) + T\dfrac{x(k)+x(k+1)}{2}$

(b) $zY(z) = Y(z) + \dfrac{T}{2}[X(z)+zX(z)] \Rightarrow Y(z) = \dfrac{T}{2}\dfrac{z+1}{z-1}X(z)$

2.2-3. (a) The transfer function for the right-side rectangular-rule integrator was found in Problem 2.2-1 to be $Y(z)/X(z) = Tz/(z-1)$. We would suspect that the reciprocal of this transfer function should yield an approximation to a differentiator. That is, if $w(kT)$ is a numerical derivative of $x(t)$ at $t = kT$,

$$\frac{W(z)}{X(z)} = \frac{z-1}{Tz}$$

Write the difference equation describing this differentiator.

(b) Draw a figure similar to those in Fig. P2.2-1 illustrating the approximate differentiation.

(c) Repeat part (a) for the left-side rule, where $W(z)/X(z) = T/(z-1)$.

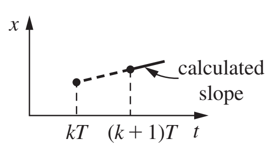
(d) Repeat part (b) for the differentiator of part (c).

Solution:

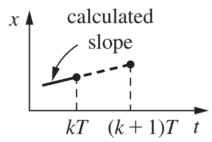
(a) $Tz\,W(z) = zX(z) - X(z)$

$$w(k+1) = \frac{1}{T}[x(k+1) - x(k)]$$

(b)



(c) $TW(z) = zX(z) - X(z)$



$$w(k) = \frac{1}{T}\left[x(k+1) - x(k)\right]$$

2.3-1. Find the z-transform of the number sequence generated by sampling the time function $e(t) = t$ every T seconds, beginning at $t = 0$. Can you express this transform in closed form?

Solution: $e(t) = t;\ E(z) = 0 + Tz^{-1} + 2Tz^{-2} + \cdots = \dfrac{Tz}{(z-1)^2}$

2.3-2. (a) Write, as a series, the z-transform of the number sequence generated by sampling the time function $e(t) = \varepsilon^{-t}$ every T seconds, beginning at $t = 0$. Can you express this transform in closed form?

(b) Evaluate the coefficients in the series of part (a) for the case that $T = 0.05\ s$.

(c) The exponential $e(t) = \varepsilon^{-bt}$ is sampled every $T = 0.2\ s$, yielding the z-transform

$$E(z) = 1 + \left(\frac{1}{2}\right)z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \left(\frac{1}{2}\right)^3 z^{-3} + \cdots$$

Evaluate b.

Solution:

(a) $E(z) = 1 + \varepsilon^{-T}z^{-1} + \varepsilon^{-2T}z^{-2} + \text{L}$

$$= 1 + (\varepsilon^{-T}z^{-1})^1 + (\varepsilon^{-T}z^{-1})^2 + \cdots = \frac{1}{1 - \varepsilon^{-T}z^{-1}} = \frac{z}{z - \varepsilon^{-T}}$$

(b) $E(z) = 1 + (0.9512z^{-1})^1 + (0.9512z^{-1})^2 + \cdots = \dfrac{z}{z - 0.9512}$

(c) $\varepsilon^{-bT}\Big|_{T=0.2} = \varepsilon^{-0.2b} = 0.5$

$$\therefore -0.2b = \ln(0.5) = -0.6931 \Rightarrow b = -3.466$$

2.3-3. Find the z-transforms of the number sequences generated by sampling the following time functions every T seconds, beginning at $t = 0$. Express these transforms in closed form.

(a) $e(t) = \varepsilon^{-at}$

(b) $e(t) = \varepsilon^{-(t-T)}u(t-T)$

(c) $e(t) = \varepsilon^{-(t-5T)}u(t-5T)$

Solution:

(a) $e(t) = \varepsilon^{-at} \Rightarrow E(z) = 1 + \varepsilon^{-aT}z^{-1} + \varepsilon^{-2aT}z^{-2} + \cdots = \dfrac{z}{z - \varepsilon^{-aT}}$ 2-3.

(b) $e(t) = \varepsilon^{-(t-T)}u(t-T)$

$$E(z) = z^{-1} + \varepsilon^{-T}z^{-2} + \varepsilon^{-2T}z^{-3} + \cdots = z^{-1}\left[\frac{z}{z-\varepsilon^{-T}}\right] = \frac{1}{z-\varepsilon^{-T}}$$

(c) $e(t) = \varepsilon^{-(t-5T)}u(t-5T)$

$$E(z) = z^{-5} + \varepsilon^{-T}z^{-6} + \varepsilon^{-2T}z^{-7} + \cdots = z^{-5}\left[\frac{z}{z-\varepsilon^{-T}}\right] = \frac{1}{z^4(z-\varepsilon^{-T})}$$

2.4-1. A function $e(t)$ is sampled, and the resultant sequence has the z-transform

$$E(z) = \frac{z^3 - 2z}{z^4 - 0.9z^2 + 0.8}$$

Solve this problem using $E(z)$ and the properties of the z-transform.

(a) Find the z-transform of $e(t-2T)u(t-2T)$.

(b) Find the z-transform of $e(t+2)u(t)$.

(c) Find the z-transform of $e(t-T)u(t-2T)$.

Solution:

(a) $\mathfrak{z}[e(t-2T)u(t-2T)] = \dfrac{(z^3-2z)z^{-2}}{z^4-0.9z^2+0.8}$

(b) $e(0)=0,\ e(1)=1$

$\therefore \mathfrak{z}[e(t+T)u(t)] = z[E(z)-e(0)-e(1)z^{-1}]$

$$= z\left[\frac{z^3-2z}{z^4-0.9z^2+0.8} - \frac{1}{z}\right] = \frac{-1.1z^2+0.8}{z^4-0.9z^2+0.8}$$

(c) $\mathfrak{z}[e(t-T)u(t-2T)] = e(T)z^{-2} + e(2T)z^{-3} + \cdots$

$= z^{-1}[E(z)-e(0)] = z^{-1}E(z)$, since $e(0)=0$

$$= \frac{z^2-z}{z^4-0.9z^2+0.8}$$

2.4-2. A function $e(t)$ is sampled, and the resultant sequence has the z-transform

$$E(z) = \frac{z-b}{z^2-cz^2+d}$$

Find the z-transform of $\varepsilon^{akT}e(kT)$. Solve this problem using $E(z)$ and the properties of the z-transform.

Solution:

By complex translation

$$\mathfrak{z}\left[\varepsilon^{akT}e(kT)\right] = E(z\varepsilon^{-aT}) = \frac{z\varepsilon^{-aT}-b}{z^2\varepsilon^{-2aT}-cz^2\varepsilon^{-2aT}+d}$$

2.5-1. From Table 2-3,

$$\mathfrak{z}\left[\cos akT\right] = \frac{z(z-\cos aT)}{z^2-2z\cos aT+1}$$

(a) Find the conditions on the parameter a such that $\mathfrak{z}[\cos akT]$ is first order (pole-zero cancellation occurs).

(b) Give the first-order transfer function in part (a).

(c) Find a such that $\mathfrak{z}[\cos akT] = \mathfrak{z}[u(kT)]$, where $u(kT)$ is the unit step function.

Solution:

(a) poles: $z = \dfrac{z\cos a \pm \sqrt{4\cos^2 a - 4}}{2} = \cos(a) \pm j\,\sin(a)$

$\therefore$ pole $= \cos a$, provided $\sin a = 0 \Rightarrow a = 0, \pm\pi, \pm 2\pi, \text{K}\ , \pm n\pi$

Then $\cos a = (-1)^n$ $\therefore$ poles = $\cos a$

(b) $E(z) = \dfrac{z(z-\cos a)}{(z-\cos a)(z-\cos a)} = \dfrac{z}{z-\cos a}, \quad a = \pm n\pi,\ n = 0, 1, \ldots$

(c) $E(z) = \dfrac{z}{z-\cos a} = \dfrac{z}{z-1}, \quad \therefore \cos a = 1,\ \ a = 0, \pm 2\pi, \pm 4\pi, \ldots$

2.5-2. Find the z-transform, in closed form, of the number sequence generated by sampling the time function $e(t)$ every T seconds beginning at $t = 0$. The function $e(t)$ is specified by its Laplace transform,

$$E(s) = \frac{2\left(1 - \varepsilon^{-5s}\right)}{s(s+2)}, \qquad T = 1s$$

Solution:

$$E_1(s) = \frac{2}{s(s+2)} = \frac{1}{s} + \frac{-1}{s+2}$$

$$\therefore e_1(t) = (1-\varepsilon^{-2t})u(t) \Rightarrow e_1(kT) = (1-\varepsilon^{-2kT})u(kT)$$

$$\therefore E_1(z) = (1 + z^{-1} + z^{-2} + \cdots) - (1 - \varepsilon^{-2T}z^{-1} + \varepsilon^{-4T}z^{-2} + \cdots)$$

$$= \frac{1}{1-z^{-1}} - \frac{1}{1-\varepsilon^{-2}z^{-1}} = \frac{z}{z-1} - \frac{z}{z-\varepsilon^{-2T}} = \frac{(1-\varepsilon^{-2})z}{(z-1)(z-\varepsilon^{-2})},\ T=1$$

$$E(z) = E_1(z) - z^{-5}E_1(z) = \frac{(1-\varepsilon^{-2})(z^5-1)}{z^4(z-1)(z-\varepsilon^{-2})} = \frac{0.8647(z^5-1)}{z^4(z-1)(z-0.1353)}$$

2.6-1. Solve the given difference equation for $x(k)$ using:

$$x(k) - 3x(k-1) + 2x(k-2) = e(k),\ \ e(k) = \begin{cases} 1,\ k = 0, 1 \\ 0,\ k \ge 2 \end{cases}$$

$$x(-2) = x(-1) = 0$$

(a) The sequential technique.

(b) The z-transform.

(c) Will the final-value theorem give the correct value of $x(k)$ as $k \to \infty$?

Solution:

(a) $x(0) = e(0) = 1$

$x(1) = e(1) + 3x(0) = 4$

$x(2) = e(2) + 3x(1) - 2x(0) = 10$

$x(3) = 0 + 3(10) - 2(4) = 22$

$x(4) = 0 + 3(22) - 2(10) = 46$

(b) $[1 - 3z^{-1} + 2z^{-2}]X(z) = E(z) = 1 + z^{-1} = \dfrac{z+1}{z}$

$$X(z) = \frac{z^2}{(z-1)(z-2)} \times \frac{z+1}{z} = \frac{z(z+1)}{(z-1)(z-2)} = z\left[\frac{-2}{z-1} + \frac{3}{z-2}\right]$$

$\therefore x(k) = -2 + 3(2)^k$

(c) *No*, since the final value does not exist.

2.6-2. Given the difference equation

$$y(k+2) - \frac{3}{4}y(k+1) + \frac{1}{8}y(k) = e(k)$$

where $y(0) = y(1) = 0,\ e(0) = 0$, and $e(k) = 1,\ k = 1,\ 2,\dots$.

(a) Solve for $y(k)$ as a function of k, and give the numerical values of $y(k),\ 0 \le k \le 4$.

(b) Solve the difference equation directly for $y(k),\ 0 \le k \le 4,$ to verify the results of part (a).

(c) Repeat parts (a) and (b) for $e(k) = 0$ for all k, and $y(0) = 1,\ y(1) = -2$.

Solution:

(a) $E(z) = \mathfrak{z}[u(k-1)] = z^{-1}\left[\frac{z}{z-1}\right] = \frac{1}{z-1}$

$$\left[z^2 - \frac{3}{4}z + \frac{1}{8}\right]Y(z) = E(z)$$

$$\frac{Y(z)}{z} = \frac{1}{z\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} \cdot \frac{1}{z-1} = \frac{-8}{z} + \frac{8/3}{z-1} + \frac{-16}{z-\frac{1}{2}} + \frac{64/3}{z-1/4}$$

$$\therefore y(k) = -8\delta(0) + \frac{8}{3} - 16\left(\frac{1}{2}\right)^k + \frac{64}{3}\left(\frac{1}{4}\right)^k$$

$$\therefore y(0) = 0;\ y(1) = 0;\ y(2) = 0;\ y(3) = 1;\ y(4) = \frac{7}{4}$$

(b) $y(k+2) = e(k) + \frac{3}{4}y(k+1) - \frac{1}{8}y(k)$

$$y(2) = 0 + \frac{3}{4}(0) - \frac{1}{8}(0) = 0$$

$$y(3) = 1 + \frac{3}{4}(0) - \frac{1}{8}(0) = 1$$

$$y(4) = 1 + \frac{3}{4}(1) - \frac{1}{8}(0) = 7/4$$

(c) (a) $y(k+2)-\frac{3}{4}y(k+1)+\frac{1}{8}y(k)=0$

$$\therefore z^2[Y(z)-y(0)-y(1)z^{-1}]-\frac{3}{4}z[Y(z)-y(0)]+\frac{1}{8}Y(z)=0$$

$$\therefore \left[z^2-\frac{3}{4}z+\frac{1}{8}\right]Y(z)=z^2-2z-\frac{3}{4}z$$

$$\therefore Y(z)=z\left[\frac{z-1/4}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right]=z\left[\frac{-9}{z-\frac{1}{2}}+\frac{10}{z-\frac{1}{4}}\right]\Rightarrow y(k)=-9\left(\frac{1}{2}\right)^k+10\left(\frac{1}{4}\right)^k$$

$y(0)=1,\ y(1)=-2,\ y(2)=-13/8,\ y(3)=-31/32,\ y(4)=-67/128$

(b) $y(k+2)=\frac{3}{4}y(k+1)-\frac{1}{8}y(k)$

$$y(2)=\frac{3}{4}(-2)-\frac{1}{8}(1)=-13/8$$

$$y(3)=\frac{3}{4}\left(-\frac{13}{8}\right)-\frac{1}{8}(-2)=-31/32$$

$$y(4)=\frac{3}{4}\left(-\frac{31}{32}\right)-\frac{1}{8}\left(-\frac{13}{8}\right)=-\frac{67}{128}$$

2.6-3. Given the difference equation

$$x(k)-x(k-1)+x(k-2)=e(k)$$

where $e(k)=1$ for $k\geq 0$.

(a) Solve for $x(k)$ as a function of k, using the z-transform. Give the values of $x(0), x(1)$, and $x(2)$.

(b) Verify the values $x(0)$, $x(1)$, and $x(2)$, using the power-series method.

(c) Verify the values $x(0)$, $x(1)$, and $x(2)$ by solving the difference equation directly.

(d) Will the final-value property give the correct value for $x(\infty)$?

Solution:

(a) $[1 - z^{-1} + z^{-2}]X(z) = E(z) = \dfrac{z}{z-1}$

$$X(z) = \frac{z^3}{(z-1)(z^2 - z + 1)}, \qquad \text{poles: } z = \frac{1}{2} \pm j\frac{\sqrt{3}}{2} = 1\angle \pm 60°$$

$$\frac{X(z)}{z} = \frac{1}{z-1} + \frac{k_1}{z - p_1} + \frac{k_1^*}{z - p_1^*} \quad \text{with } p = 1\angle 60°$$

$$k_1 = \left.\frac{z^2}{(z-1)(z - 1\angle -60°)}\right|_{z=1\angle 60°} = \frac{1\angle 120°}{(.5 + j.866 - 1)(.5 + j.866 - .5 + j.866)}$$

$$= \frac{1\angle 120°}{1\angle 120°\,[j2(0.866)]} = 0.5774\angle -90°$$

$$\therefore aT = \ln(|p_1|) = 0; \; bT = \arg p_1 = \frac{\pi}{3}$$

$$A = 2|k_1| = 1.155; \; \theta = \arg\, k_1 = -90°$$

$$\therefore x(k) = 1 + 1.155\cos\left(\frac{\pi}{3}k - 90°\right) = 1 + 1.155\sin\left(\frac{\pi}{3}k\right)$$

$$x(0) = 1, \; x(1) = 2, \; x(2) = 2$$

(b)
$$\begin{array}{r|l} & 1 + 2z^{-1} + 2z^{-2} + \cdots \\ \hline z^3 - 2z^2 + 2z - 1 & z^3 \\ & \underline{z^3 - 2z^2 + 2z - 1} \\ & \quad 2z^2 - 2z + 1 \\ & \quad \underline{2z^2 - 4z + 4 - 2z^{-1}} \\ & \qquad 2z + \cdots \end{array}$$

$\therefore x(0) = 1$

$x(1) = 2$

$x(2) = 2$

(c) $x(k) = 1 + x(k-1) - x(k-2)$

$x(0) = 1 + 0 - 0 = 1$

$x(1) = 1 + 1 - 0 = 2$

$x(2) = 1 + 2 - 1 = 2$

(d) *No*, 3 poles for $X(z)$ on the unit circle.

2.6-4. Given the difference equation

$$x(k+2)+3x(k+1)+2x(k)=e(k)$$

where

$$e(k)=\begin{cases}1, & k=0\\ 0, & \text{otherwise}\end{cases}$$

$$x(0)=1$$

$$x(1)=-1$$

(a) Solve for $x(k)$ as a function of k.

(b) Evaluate $x(0)$, $x(1)$, $x(2)$, and $x(3)$ in part (a).

(c) Verify the results in part (b) using the power-series method.

(d) Verify the results in part (b) by solving the difference equation directly.

Solution:

(a) $z^2[X(z)-x(0)-x(1)z^{-1}]+3z[X(z)-x(0)]+2X(z)=E(z)=1$

$$\therefore X(z)=\frac{1+z^2-z+3z}{z^2-3z+2}=\frac{z^2+2z+1}{z^2+3z+2}=\frac{z+1}{z+2}$$

$$\therefore X(z)=z\left[\frac{z+1}{z(z+2)}\right]=z\left[\frac{1/2}{z}+\frac{1/2}{z+2}\right]$$

$$\therefore x(k)=\frac{1}{2}\delta(k)+\frac{1}{2}(-2)k$$

(b) $x(0)=1,\ x(1)=-1,\ x(2)=2,\ x(3)=-4$

(c)

$$
\begin{array}{r l}
 & 1 - z^{-1} + 2z^{-2} - 4z^{-3} + \ldots \\
z+2 \,\big) & z+1 \\
 & \underline{z+2} \\
 & -1 \\
 & \underline{-1 - 2z^{-1}} \\
 & 2z^{-1} \\
 & \underline{2z^{-1} + 4z^{-2}} \\
 & -4z^{-2} \\
 & \ldots
\end{array}
$$

(d) $x(k+2) = e(k) - 3x(k+1) - 2x(k)$

$x(2) = 1 - 3(-1) - 2(1) = 2$

$x(3) = 0 - 3(2) - 2(-1) = -4$

2.6-5. Given the difference equation

$$x(k+3) - 2.2x(k+2) + 1.57x(k+1) - 0.36x(k) = e(k)$$

where $e(k) = 1$ for all $k \geq 0$, and $x(0) = x(1) = x(2) = 0$.

(a) Write a digital computer program that will calculate $x(k)$. Run this program solving for $x(3)$, $x(4)$, . . . , $x(25)$.

(b) Using the sequential technique, check the values of $x(k)$, $0 \leq k \leq 5$.

(c) Use the z-transform and the power-series method to verify the values $x(k)$, $0 \leq k \leq 5$.

Solution:

(a)
```
x0 = 0;

x1 = 0;

x2 = 0;

for k = 0:5;

    x3 = 2.2*x2 – 1.57*x1 + 0.36*x0 + 1

    x0 = x1;
```

```
        x1 = x2;
        x2 = x3;
    end
```

(b) $x(k+3) = e(k) + 2.2x(k+2) - 1.57x(k+1) + 0.36x(k)$

$x(3) = 1 + 0 - 0 + 0 = 1$

$x(4) = 1 + 2.2(1) - 0 + 0 = 3.2$

$x(5) = 1 + 2.2(3.2) - 1.57(1) = 6.47$

(c) $[z^3 - 2.2z^2 + 1.57z - 0.36]X(z) = E(z) = \dfrac{z}{z-1}$

$$X(z) = \frac{z}{(z-1)(z^3 - 2.2z^2 + 1.57z - 0.36)}$$

$$\begin{array}{r|l} & z^{-3} + 3.2z^{-4} + 6.47z^{-5} + \cdots \\ \hline z^4 - 3.2z^3 + 3.77z^2 - 1.93z + 0.36 & z \\ & \underline{z - 3.2 + 3.77z^{-1} - \cdots} \\ & 3.2 - 3.77z^{-1} \\ & \underline{3.2 - 10.24z^{-1} + \cdots} \\ & 6.47z^{-1} + \cdots \\ & \cdots \end{array}$$

$\therefore x(3) = 1$
$x(4) = 3.2$
$x(5) = 6.47$

2.7-1. (a) Find $e(0)$, $e(1)$, and $e(10)$ for

$$E(z) = \frac{0.1}{z(z-0.9)}$$

using the inversion formula.

(b) Check the value of $e(0)$ using the initial-value property.

(c) Check the values calculated in part (a) using partial fractions.

(d) Find $e(k)$ for $k = 0, 1, 2, 3,$ and 4 if $\mathfrak{z}\left[e(k)\right]$ is given by

$$E(z) = \frac{1.98z}{\left(z^2 - 0.9z + 0.9\right)\left(z - 0.8\right)\left(z^2 - 1.2z + 0.27\right)}$$

(e) Find a function $e(t)$ which, when sampled at a rate of 10 Hz $(T = 0.1s)$, results in the transform $E(z) = 2z/(z - 0.8)$.

(f) Repeat part (e) for $E(z) = 2z/(z + 0.8)$.

(g) From parts (e) and (f), what is the effect on the inverse z-transform of changing the sign on a real pole?

Solution:

(a) $e(k) = \sum_{\text{residues}} \dfrac{0.1z^{k-1}}{z(z-0.9)} = \sum_{\text{residues}} \dfrac{0.1z^{k-2}}{z-0.9}$

$k = 0:$ fcn $= \dfrac{0.1}{z^2(z-0.9)}$, $\therefore$ residue$\Big|_{z=0.9} = \dfrac{0.1}{(0.9)^2} = 0.1235$

residue$\Big|_{z=0} = \dfrac{d}{dz}\left[\dfrac{0.1}{z-0.9}\right]_{z=0} = \dfrac{-0.1(1)}{(z-0.9)^2}\Bigg|_{z=0} = \dfrac{-0.1}{(0.9)^2} = -0.1235$

$\therefore e(0) = 0$

$k = 1: e(1) = \dfrac{0.1}{z-0.9}\Bigg|_{z=0} + \dfrac{0.1}{z}\Bigg|_{z=0.9} = 0$

$k = 10:\ e(10) = 0.1(0.9)^8$

(b) $e(0) = \lim_{z\to\infty} E(z) = \lim_{z\to\infty} \dfrac{0.1}{z(z-0.9)} = 0$

(c) $\dfrac{E(z)}{z} = \dfrac{0.1}{z^2(z-0.9)} = \dfrac{k_1}{z^2} + \dfrac{k_2}{z} + \dfrac{k_3}{z-0.9}$

$k_1 = \dfrac{-0.1}{0.9} = -\dfrac{1}{9};\ k_3 \dfrac{0.1}{(0.9)^2} = \dfrac{1}{8.1}$

$k_2 = \dfrac{d}{dz}\left[\dfrac{0.1}{z-0.9}\right]_{z=0} = \dfrac{-1}{8.1}$, from (a)

$$\therefore e(k) = \frac{-1}{8.1}\delta(k) - \frac{1}{9}\delta(k-1) + \frac{1}{8.1}(0.9)^k$$

$$x(0) = -\frac{1}{8.1} + 0 + \frac{1}{8.1} = 0; \;\; x(1) = -0 - \frac{1}{9} + \frac{0.9}{8.1} = 0$$

$$x(10) = -0 - 0 + \frac{0.1}{(0.9)^2}(0.9)^{10} = 0.1(0.9)^8$$

(d) $E(z) = \dfrac{1.98z}{z^5 + \cdots} = 1.98z^{-4} + (\cdot)z^{-5} + (\cdot)z^{-6} + \cdots$

$$\therefore e(0) = e(1) = e(2) = e(3) = 0; \;\; e(4) = 1.98$$

(e) $E(z) = \dfrac{2z}{z - 0.8} = \dfrac{2z}{z - \varepsilon^{-aT}} \qquad \therefore \varepsilon^{-aT} = 0.8 \Rightarrow aT = 0.2231$

$$\therefore a = \frac{0.2231}{0.1} = 2.231, \quad \therefore e(t) = 2\varepsilon^{-2.231t}u(t)$$

(f) $E(z) = \dfrac{2z}{z - (-0.8)}; \;\; \therefore \varepsilon^{-aT}\varepsilon^{j\pi} = -0.8 \Rightarrow aT = 2.231$

$$\therefore e(t) = 2e^{-2.231t}\cos 10\pi t \;\text{ where }\; \frac{\omega_s}{2} = 10\pi$$

(g) (e) $e(k) = (0.8)^k$; (f) $e(k) = (-0.8)^k$

$\therefore$ sign alternates on $e(k)$.

2.7-2. For the number sequence $\{e(k)\}$,

$$E(z) = \frac{z}{(z+1)^2}$$

(a) Apply the final-value theorem to $E(z)$.

(b) Check your result in part (a) by finding the inverse z-transform of $E(z)$.

(c) Repeat parts (a) and (b) with $E(z) = z/(z-1)^2$.

(d) Repeat parts (a) and (b) with $E(z) = z/(z - 0.9)^2$.

(e) Repeat parts (a) and (b) with $E(z) = z/(z - 1.1)^2$.

Solution:

(a) $e(\infty) = \lim_{z \to 1} (z-1)E(z) = \left. \dfrac{z(z-1)}{(z+1)^2} \right|_{z=1} = 0$

(b) $e(k) = z^{-1}\left[\dfrac{z}{(z-1)^2}\right] = k(-1)^k, \;\; \therefore e(\infty)$ unbounded

(c) (a) $e(\infty) = \lim_{z \to 1} (z-1)\dfrac{z}{(z-1)^2}, \quad \therefore$ unbounded

(b) $e(k) = k, \quad \therefore$ unbounded

(d) (a) $e(\infty) = \lim_{z \to 1} (z-1)\dfrac{z}{(z-0.9)^2} = 0$

(b) $e(k) = k(0.9)^k; \;\; \therefore e(\infty) \to 0$

(e) (a) $e(\infty) = \lim_{z \to 1} (z-1)\dfrac{z}{(z-1.1)^2} = 0$

(b) $e(k) = k(1.1)^k; \;\; \therefore e(\infty)$ is unbounded.

2.7-3. Find the inverse z-transform of each $E(z)$ below by the four methods given in the text. Compare the values of $e(z)$, for $k = 0, 1, 2,$ and 3, obtained by the four methods.

(a) $E(z) = \dfrac{0.5z}{(z-1)(z-0.6)}$

(b) $E(z) = \dfrac{0.5}{(z-1)(z-0.6)}$

(c) $E(z) = \dfrac{0.5(z+1)}{(z-1)(z-0.6)}$

(d) $E(z) = \dfrac{z(z-0.7)}{(z-1)(z-0.6)}$

(e) Use MATLAB to verify the partial-fraction expansions.

Solution:

(a) (i)

$$\begin{array}{r|l} & 0.5z^{-1}+0.8z^{-2}+0.98z^{-3}+\cdots \\ z^2-1.6z+0.6 & 0.5z \\ & \underline{0.5z-0.8+0.3z^{-1}} \\ & 0.8-0.3z^{-1} \\ & \underline{0.8-1.28z^{-1}+\cdots} \\ & 0.98z^{-1}+\cdots \end{array}$$

(ii) $\dfrac{E(z)}{z}=\dfrac{0.5}{(z-1)(z-0.6)}=\dfrac{1.25}{z-1}+\dfrac{-1.25}{z-0.6};\quad \therefore E(z)=\dfrac{1.25z}{z-1}-\dfrac{1.25z}{z-0.6}$

$\therefore e(k)=1.25(1-0.6^k)u(k)$

(iii) $z^{k-1}E(z)=\dfrac{0.5z^k}{(z-1)(z-0.6)}$

$$e(k)=\frac{0.5(1)^k}{1-0.6}+\frac{0.5(0.6)^k}{0.6-1}=1.25(1-0.6^k)u(k)$$

(iv) $E_1(z)=\dfrac{0.5z}{z-0.6}\Rightarrow e_1(k)=0.5(0.6)^k$

$$E_2(z)=\frac{1}{z-1}\Rightarrow e_2(0)=0;\ \ e_2(k)=1,\ k\ge 1$$

$$e(0)=e_1(0)e_2(0)=(0.5)(0)=0$$

$$e(1)=e_1(0)e_2(1)+e_1(1)e_2(0)=(0.5)(1)+(0.3)(0)=0.5$$

$$e(2)=e_1(0)e_2(2)+e_1(1)e_2(1)+e_1(2)e_2(0)$$

$$=0.5\times 1+0.3\times 1+0.18\times 0=0.8$$

$$e(3)=0.5\times 1+0.3\times 1+0.18\times 1+0.108\times 0=0.98$$

(b) $e(0)=0$

$$e(k)=1.25-2.083(0.6)^k,\quad k\ge 1$$

$$E(z)=0.5z^{-2}+0.8z^{-3}+0.98z^{-4}+1.088z^{-5}+\cdots$$

(c) $e(0)=0;\ e(k)=2.5-3.33(0.6)^k,\ \ k\ge 1$

$$E(z)=0.5z^{-1}+1.30z^{-2}+1.78z^{-3}+2.068z^{-4}+2.2408z^{-5}+\cdots$$

(d) $e(k)=0.75+0.25(0.6)^k$

$$E(z) = 1 + 0.9z^{-1} + 0.84z^{-2} + 0.804z^{-3} + \cdots$$

(e) num=[0 0 0.5];

den=[1 –1.6 0.6];

[r, p, k] = residue (num, den)

2.8-1. Given in Fig. P2.8-1 are two digital-filter structures, or realizations, for second-order filters.

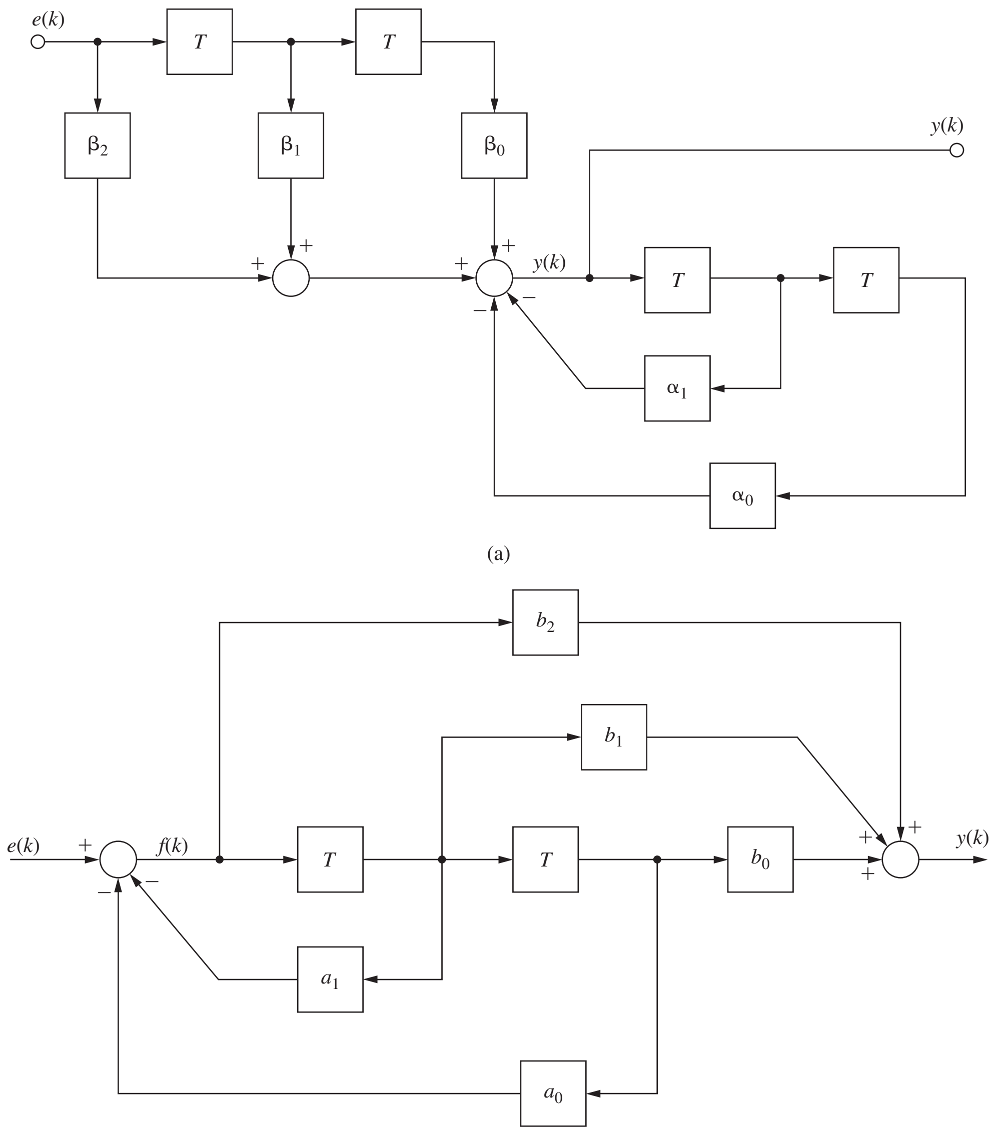


FIGURE P2.8-1 Digital-filter structures: (a) 3D; (b) 1D.

(a) Write the difference equation for the 3D structure of Fig. P2.8-1(a), expressing $y(k)$ as a function of $y(k-i)$ and $e(k-i)$.

(b) Derive the filter transfer function $Y(z)/E(z)$ for the 3D structure by taking the z-transform of the equation in part (a).

(c) Write the difference equation for the 1D structure of Fig. P2.8-1(b). Two equations are required, with one for $f(k)$ and one for $y(k)$.

(d) Derive the filter transfer function $Y(z)/E(z)$ for the 1D structure by taking the z-transform of the equations in part (c) and eliminating $F(z)$.

(e) From parts (b) and (d), relate the coefficients α_i, β_i to a_i, b_i such that the two filters realize the same transfer function.

(f) Write a computer-program segment that realizes the 3D structure. This program should be of the form used in Example 2.10.

(g) Write a MATLAB-program segment that realizes the 1D structure. This program should be of the form used in Example 2.10.

Solution:

(a) $y(k) = \beta_2 e(k) + \beta_1 e(k-1) + \beta_0 e(k-2) - \alpha_1 y(k-1) - \alpha_0 y(k-2)$

(b) $\left[1 + \alpha_1 z^{-1} + \alpha_0 z^{-2}\right] Y(z) = \left[\beta_2 + \beta_1 z^{-1} + \beta_0 z^{-2}\right] E(z)$

$$\frac{Y(z)}{E(z)} = \frac{\beta_2 z^2 + \beta_1 z + \beta_0}{z^2 + \alpha_1 z + \alpha_0}$$

(c) $f(k) = e(k) - a_1 f(k-1) - a_0 f(k-2)$

$y(k) = b_2 f(k) + b_1 f(k-1) + b_0 f(k-2)$

(d) $F(z) = E(z) - (a_1 z^{-1} + a_0 z^{-2}) F(z) \Rightarrow F(z) = \dfrac{E(z)}{1 + a_1 z^{-1} + a_0 z^{-2}}$

$$Y(z) = (b_2 + b_1 z^{-1} + b_0 z^{-2})\ F(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^2 + a_1 z + a_0} E(z)$$

(e) $\alpha_i = a_i$ and $\beta_i = b_i,\ \ i = 1, 2$

(f)

```
ykminus2 = 0;
ykminus1 = 0;
ekminus2 = 0;
ekminus1 = 0;
ek = 1;
for k = 0:5
    yk=b2*ek+b1*ekminus1+b0*ekminus2–a1*ykminus1-a0*ykminus2;
    [k, ek, yk]
    ekminus2 = ekminus1;
    ekminus1 = ek;
    ykminus2 = ykminus1;
    ykminus1 = yk;
end
```

(g)

```
fkminus2 = 0;
fkminus1 = 0;
ek = 1;
for k = 0:5
    fk=ek-a1*fkminus1–a0*fkminus2;
    yk = b2*fk+b1*fkminus1+b0*fkminus2;
    [k, ek, yk]
    fkminus2 = fkminus1;
    fkminus1 = fk;
end
```

2.8-2. Shown in Fig. P2.8-2 is the second-order digital-filter structure 1X.

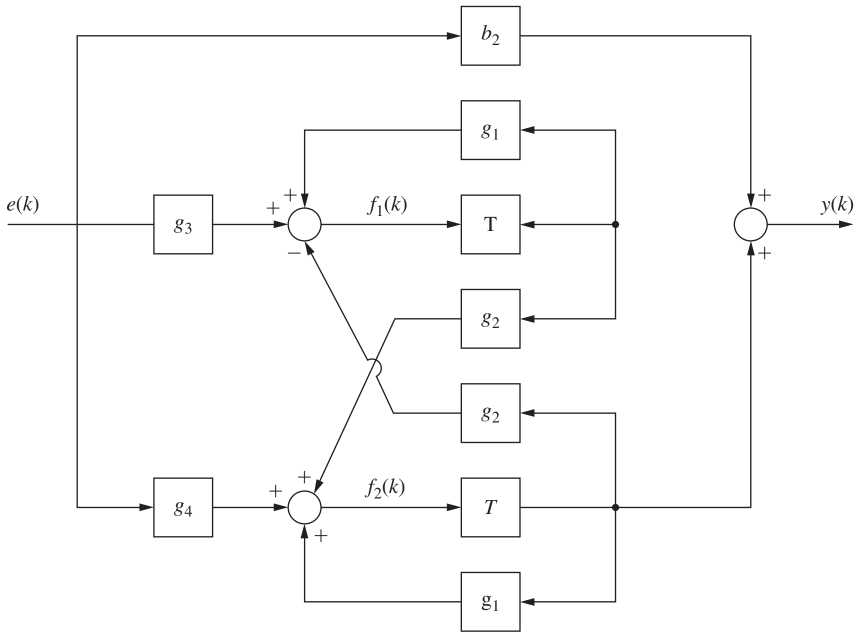


FIGURE P2.8-2 Digital-filter structure 1X.

This structure realizes the filter transfer function

$$D(z) = b_2 + \frac{A}{z-p} + \frac{A*}{z-p*}$$

where p and $p*$ (conjugate of p) are complex. The relationships between the filter coefficients and the coefficients in Fig. P2.8-2 are given by

$$g_1 = \text{Re}(p) \quad g_3 = -2\ \text{Im}(A)$$

$$g_2 = \text{Im}(p) \quad g_4 = 2\ \text{Re}(A)$$

(a) To realize this filter, difference equations are required for $f_1(k), f_2(k),$ and $y(k)$. Write these equations.

(b) Find the filter transfer function $Y(z)/E(z)$ by taking the z-transform of the equations of part (a) and eliminating $F_1(z)$ and $F_2(z)$.

(c) Verify the results in part (b) using Mason's gain formula.

(d) Write a MATLAB-program segment that realizes the 1X structure. This program should be of the form of that is used in Example 2.10.

Solution:

(a) $f_1(k) = g_1 f_1(k-1) - g_2 f_2(k-1) + g_3 e(k)$

$f_2(k) = g_1 f_2(k-1) + g_1 f_1(k-1) + g_4 e(k)$

$y(k) = b_2 e(k) + f_2(k-1)$

(b) (1) $F_1(z) = g_1 z^{-1} F_1(z) - g_2 z^{-1} F_2(z) + g_3 E(z)$

(2) $F_2(z) = g_1 z^{-1} F_2(z) + g_2 z^{-1} F_1(z) + g_4 E(z)$

(3) $Y(z) = b_0 E(z) + z^{-1} F_2(z)$

$\therefore$ (1) $(z - g_1) F_1(z) + g_2 F_2(z) = g_3 z E(z)$

(2) $-g_2 F_1(z) + (z - g_1) F_2(z) = g_4 z E(z)$

$$\therefore F_2(s) = \frac{\begin{vmatrix} z-g_1 & g_3 z E(z) \\ -g_2 & g_4 z E(z) \end{vmatrix}}{\begin{vmatrix} z-g_1 & g_2 \\ -g_2 & z-g_1 \end{vmatrix}} = \frac{(g_4 z^2 - g_1 g_4 z + g_2 g_3 z)}{(z-g_1)^2 + g_2^2} E(z)$$

$$\therefore \frac{Y(z)}{E(z)} = b_2 + \frac{g_4 z + g_2 g_3 - g_1 g_4}{(z-g_1)^2 + g_2^2}$$

$$also,\ D(z) = b_2 + \frac{\mathrm{Re}(A) + j\,\mathrm{Im}(A)}{z - \mathrm{Re}(p) - j\,\mathrm{Im}(p)} + \frac{\mathrm{Re}(A) - j\,\mathrm{Im}(A)}{z - \mathrm{Re}(p) + j\,\mathrm{Im}(p)}$$

$$= b_2 + \frac{\frac{1}{2}(g_4 - jg_3)}{z - g_1 - jg_2} + \frac{\frac{1}{2}(g_4 + jg_3)}{z - g_1 + jg_2}$$

$$= b_2 + \frac{g_4 z - g_1 g_4 + g_2 g_3}{(z-g_1)^2 + g_2^2}$$

(c) $$D(z) = b_0 + \frac{g_2 g_3 z^{-2} + g_4(1 - g_1 z^{-1})}{1 - g_1 z^{-1} - g_1 z^{-1} + g_1^2 z^{-2} + g_2^2 z^{-2}}$$

$$= b_0 + \frac{g_4 z + g_2 g_3 - g_1 g_4}{z^2 - 2 g_1 z + g_1^2 + g_2^2}$$

(d) f1kminus1 = 0;

f2kminus1 = 0;

```
ek = 1;
for k = 0:5
    yk = b0*ek+f2kminus1;
    [k, ek, yk]
    f1k = g1*f1kminus1 – g2*f2kminus1 + g3*ek;
    f2k = g1*f2kminus1 + g2*f1kminus1 + g3*ek;
    f1kminus1 = f1k;
    f2kminus1 = f2k;
end
```

2.8-3. Given the second-order digital-filter transfer function

$$D(z) = \frac{2z^2 - 2.4z + 0.72}{z^2 - 1.4z + 0.98}$$

(a) Find the coefficients of the 3D structure of Fig. P2.8-1 such that $D(z)$ is realized.

(b) Find the coefficients of the ID structure of Fig. P2.8-1 such that $D(z)$ is realized.

(c) Find the coefficients of the IX structure of Fig. P2.8-2 such that $D(z)$ is realized.

The coefficients are identified in Problem 2.8-2.

(d) Use MATLAB to verify the partial-fraction expansions in part (c).

(e) Verify the results in part (c) using Mason's gain formula.

Solution:

(a) $\beta_2 = 2,\ \beta_1 = -2.4,\ \beta_0 = 0.72,\ \alpha_1 = -1.4,\ \alpha_0 = 0.98$

(b) $b_2 = 2,\ b_1 = -2.4,\ b_0 = 0.72,\ a_1 = -1.4,\ a_0 = 0.98$

(c) poles: $z = \dfrac{1.4 \pm \left(1.4^2 - 4(0.98)\right)^{1/2}}{2} = 0.7 \pm j0.7 = 0.99\ \angle \pm 45°$

$$D(z) = 2 + \frac{A}{z - 0.7 - j0.7} + \frac{A^*}{z - 0.7 + j0.7}$$

$$\therefore A = \left.\frac{2z^2 - 2.4z + 0.72}{z - 0.7 + j0.7}\right|_{z=0.99\angle 45^\circ} = \frac{j1.96 - (1.68 + j1.68) + 0.72}{j1.4}$$

$$= 0.2 + j0.6857$$

$$\therefore g_1 = 0.7 \qquad g_3 = 1.371$$

$$g_2 = 0.7 \qquad g_4 = 0.4$$

(d)
```
num=[2  -2.4  .72];
den=[1  -1.4  0.98];
[r,p,k,]=residue(num, den)
```

(e) $\Delta = 1 - (0.7z^{-1} + 0.7z^{-1} + 0.4z^{-2}) + 0.49z^{-2}$

$$= 1 - 1.4z^{-1} + 0.98z^{-2}$$

$$D(z) = 2 + \frac{1}{\Delta}[1.371\ (0.7)z^{-2} + 0.4z^{-1}(1 + 0.7z^{-1})]$$

$$= 2 + \frac{0.4z - 1.24}{z^2 - 1.4z + 0.98} = \frac{2z^2 - 2.4z + 0.72}{z^2 - 1.4z + 0.98}$$

2.9-1. Find two different state-variable formulations that model the system whose difference equation is given by:

(a) $y(k+2) + 6y(k+1) + 5y(k) = 2e(k)$

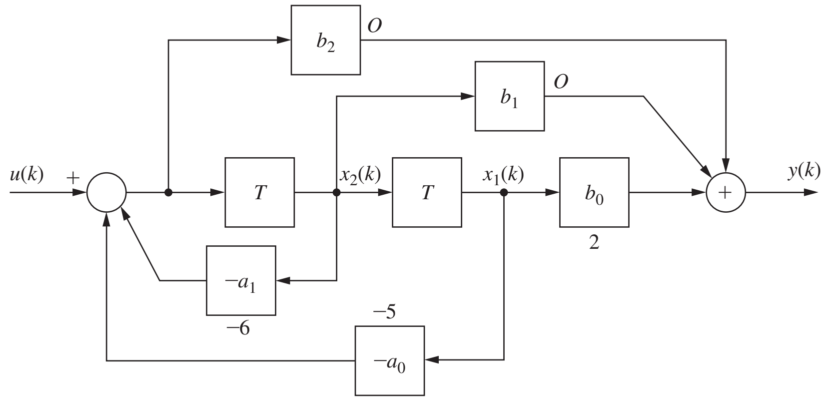
(b) $y(k+2) + 6y(k+1) + 5y(k) = e(k+1) + 2e(k)$

(c) $y(k+2) + 6y(k+1) + 5y(k) = 3e(k+2) + e(k+1) + 2e(k)$

Solution:

(a) $\dfrac{Y(z)}{U(z)} = \dfrac{2}{z^2 + 6z + 5}$

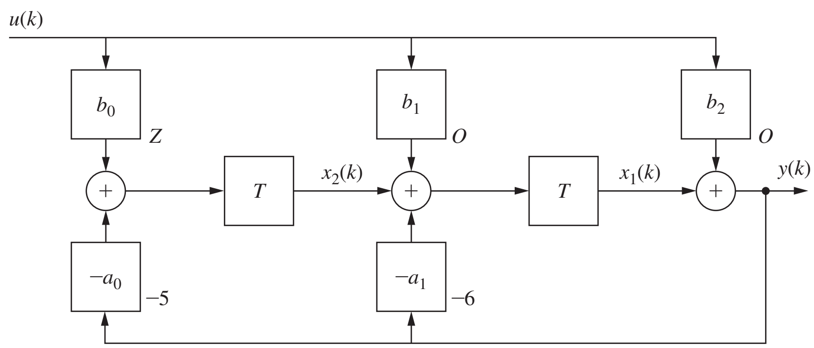
(1) control canonical:



$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [2 \quad 0]\mathbf{x}(k)$$

(2) observer canonical:



$$\mathbf{x}(k+1) = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0]\mathbf{x}(k)$$

(b) $\dfrac{Y(z)}{U(z)} = \dfrac{z+2}{z^2+6z+5}$ (1) control canonical: $\mathbf{x}(k+1) =$ same as (a)
$y(k) = [2 \quad 1]x(k)$

(2) observer canonical:

$$\mathbf{x}(k+1) = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0]\mathbf{x}(k)$$

(c) $\dfrac{Y(z)}{U(z)} = \dfrac{3z^2+z+2}{z^2+6z+5}$ (1) control canonical: $\mathbf{x}(k+1) =$ same as (a)
$y(k) = [-13 \quad -17]\mathbf{x}(k) + 3u(k)$

(2) observer canonical:

$$\mathbf{x}(k+1) = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(k) + 3u(k)$$

2.9-2. Write the state equations for the observer canonical form of a system, shown in Fig. 2-10, which has the transfer function given in (2-51) and (2-61)

$$G(z) = \frac{b_{n-1}z^{n-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}$$

Solution:

$$\mathbf{x}(k+1) = \begin{bmatrix} a_{n-1} & 1 & 0 & \cdots & 0 \\ a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}\mathbf{x}(k)$$

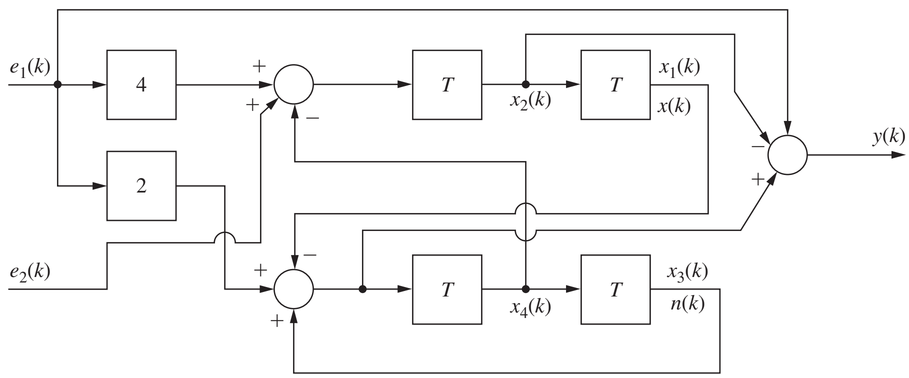
2.10-1. Find a state-variable formulation for the system described by the coupled second-order difference equations given. The system output is $y(k)$, and $e_1(k)$ and $e_2(k)$ are the system inputs. *Hint:* Draw a simulation diagram first.

$$x(k+2) + v(k+1) = 4e_1(k) + e_2(k)$$

$$v(k+2) - v(k) + x(k) = 2e_1(k)$$

$$y(k) = v(k+2) - x(k+1) + e_1(k)$$

Solution:



$$\mathbf{x}(k+1)=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 1\\ -1 & 0 & 1 & 0\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}0 & 0\\ 4 & 1\\ 0 & 0\\ 2 & 0\end{bmatrix}e(k)$$

$$y(k)=x_4(k+1)-x_2(k)+e_1(k)=-x_1(k)+x_3(k)-x_2(k)+e_1(k)$$

$$\therefore y(k)=[-1 \quad -1 \quad 1 \quad 0]\mathbf{x}(k)+[1 \quad 0]e(k)$$

2.10-2. Consider the system described by

$$\mathbf{x}(k+1)=\begin{bmatrix}0 & 1\\ 0 & 3\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}1\\ 1\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}-2 & 1\end{bmatrix}\mathbf{x}(k)$$

(a) Find the transfer function $Y(z)/U(z)$.

(b) Using any similarity transformation, find a different state model for this system.

(c) Find the transfer function of the system from the transformed state equations.

(d) Verify that $\mathbf{A}$ given and $\mathbf{A}_w$ derived in part (b) satisfy the first three properties of similarity transformations. The fourth property was verified in part (c).

Solution:

(a) $z\mathbf{I}-\mathbf{A}=\begin{bmatrix}z & -1\\ 0 & z-3\end{bmatrix}$; $\Delta=|z\mathbf{I}-\mathbf{A}|=z(z-3)=\Delta$

$$\frac{Y(z)}{U(z)} = \mathbf{C}[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B} = \frac{1}{\Delta}\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}z-3 & 1\\ 0 & z\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{\Delta}\begin{bmatrix}-2z+6 & z-2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{-z+4}{z(z-3)}$$

(b) $\mathbf{P} = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$; $\mathbf{P}^{-1} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$

$$\mathbf{A}_w = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 3\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 1\\ 3 & 3\end{bmatrix}$$

$$= \begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix}$$

$$\mathbf{B}_w = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\mathbf{C}_w = \mathbf{C}\mathbf{P} = \begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}-1 & 3\end{bmatrix}$$

$$\therefore \mathbf{w}(k+1) = \begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix}\mathbf{w}(k) + \begin{bmatrix}1\\0\end{bmatrix}\mathbf{u}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix}-1 & 3\end{bmatrix}\mathbf{w}(k)$$

(c) $z\mathbf{I}-\mathbf{A}_w = \begin{bmatrix}z-2 & -2\\ -1 & z-1\end{bmatrix}$; $\Delta = |z\mathbf{I}-\mathbf{A}_w| = z^2-3z+2-2 = z(z-3)$

$$\frac{Y(z)}{U(z)} = \mathbf{C}_w[z\mathbf{I}-\mathbf{A}_w]^{-1}\mathbf{B}_w = \frac{1}{\Delta}\begin{bmatrix}-1 & 3\end{bmatrix}\begin{bmatrix}z-1 & 2\\ 1 & z-2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \frac{1}{\Delta}\begin{bmatrix}-1 & 3\end{bmatrix}\begin{bmatrix}z-1\\ 1\end{bmatrix} = \frac{-z+4}{z(z-3)}$$

(d) $|z\mathbf{I}-\mathbf{A}| = \begin{vmatrix}z & 1\\ 0 & z-3\end{vmatrix} = z^2-3z$; $|z\mathbf{I}-\mathbf{A}_w| = \begin{vmatrix}z-2 & -2\\ -1 & z-1\end{vmatrix} = z(z-3)$

$$\therefore z_1 = 0,\ z_2 = 3$$

$$|\mathbf{A}| = \begin{vmatrix}0 & 1\\ 0 & 3\end{vmatrix} = 0 = z_1 z_2;\ |\mathbf{A}_w| = \begin{vmatrix}2 & 2\\ 1 & 1\end{vmatrix} = 0$$

$$\text{tr } \mathbf{A} = 3 = z_1 + z_2;\ \text{tr } \mathbf{A}_w = 3$$

2.10-3. Consider the system of Problem 2.10-2. A similarity transformation on these equations yields

$$\mathbf{w}(k+1) = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \mathbf{w}(k) + \mathbf{B}_w u(k)$$

$$y(k) = \mathbf{C}_w \mathbf{x}(k)$$

(a) Find d_1 and d_2.

(b) Find a similarity transformation that results in the $\mathbf{A}_w$ matrix given. Note that this matrix is diagonal.

(c) Find $\mathbf{B}_w$ and $\mathbf{C}_w$.

(d) Find the transfer functions of both sets of state equations to verify the results of this problem.

Solution:

(a) Let z_1, z_2 be the characteristic value of $\mathbf{A}$. $d_1 = z_1, \quad d_2 = z_2$

$$z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z & -1 \\ 0 & z-3 \end{bmatrix}, \quad \therefore\ |z\mathbf{I} - \mathbf{A}| = z(z-3); \ \therefore z_1 = 0,\ z_2 = 3$$

(b) $$(z_1\mathbf{I} - \mathbf{A})m_1 = \begin{bmatrix} 0 & -1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -m_{21} = 0 \\ -3m_{21} = 0 \end{matrix}$$

$$\therefore m_{21} = 0, \text{ let } m_{11} = 1, \ \therefore m_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(z_2\mathbf{I} - \mathbf{A})m_2 = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3m_{12} - m_{22} = 0$$

$$\therefore \text{ let } m_{12} = 1,\ m_{22} = 3, \ \therefore m_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\therefore \mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, |\mathbf{M}| = 3, \mathbf{M}^{-1} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix}$$

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

(c) $\mathbf{B}_w = \mathbf{M}^{-1}\mathbf{B} = \begin{bmatrix} 1 & -1/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$

$$\mathbf{C}_w = \mathbf{CM} = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

$$\therefore \mathbf{w}(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{w}(k) + \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} -2 & 1 \end{bmatrix} \mathbf{w}(k)$$

(d) See Problem 2.10-2(a) for the first transfer function.

$$z\mathbf{I} - \mathbf{A}_w = \begin{bmatrix} z & 0 \\ 0 & z-3 \end{bmatrix}; \left| z\mathbf{I} - \mathbf{A}_w \right| = z(z-3) = \Delta$$

$$\frac{Y(z)}{U(z)} = \mathbf{C}_w [z\mathbf{I} - \mathbf{A}_w]^{-1} \mathbf{B}_w = \frac{1}{\Delta} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} z-3 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} -2z+6 & z \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \frac{-\frac{4}{3}z + 4 + \frac{1}{3}z}{\Delta} = \frac{-z+4}{z(z-3)}$$

2.10-4. Repeat Problem 2.10-2 for the system described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{x}(k)$$

(a) Find the transfer function $Y(z)/U(z)$.

(b) Using any similarity transformation, find a different state model for this system.

(c) Find the transfer function of the system from the transformed state equations.

(d) Verify that $\mathbf{A}$ given and $\mathbf{A}_w$ derived in part (b) satisfy the first three properties of similarity transformations. The fourth property was verified in part (c).

Solution:

(a)

$$\frac{Y(z)}{U(z)} = \mathbf{C}[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B} = [1 \quad 2]\begin{bmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z-0.5} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= [1 \quad 2]\begin{bmatrix} \frac{2}{z-1} \\ \frac{1}{z-0.5} \end{bmatrix} = \frac{2}{z-1} + \frac{2}{z-0.5} = \frac{4z-3}{(z-1)(z-0.5)}$$

(b) $\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \ \mathbf{P}^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

$$\therefore \mathbf{A}_w = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ -1/2 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{bmatrix}$$

$$\mathbf{B}_w = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$\mathbf{C}_w = \mathbf{C}\mathbf{P} = [1 \quad 2]\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = [3 \quad 1]$$

$$\therefore \mathbf{w}(k+1) = \begin{bmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{bmatrix}\mathbf{w}(k) + \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}\mathbf{u}(k)$$

$$\mathbf{y}(k) = [3 \quad 1]\mathbf{x}(k)$$

(c) $z\mathbf{I}-\mathbf{A}_w = \begin{bmatrix} z-3/4 & 1/4 \\ 1/4 & z-3/4 \end{bmatrix}, \ |z\mathbf{I}-\mathbf{A}_w| = z^2 - 1.5z + \frac{9}{16} - \frac{1}{16} = z^2 - 1.5z + 0.5 = \Delta$

$$\frac{Y(z)}{U(z)} = \mathbf{C}_w[z\mathbf{I}-\mathbf{A}_w]^{-1}\mathbf{B}_w = [3 \quad 1]\frac{1}{\Delta}\begin{bmatrix} z-3/4 & -1/4 \\ -1/4 & z-3/4 \end{bmatrix}\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

$$= \frac{1}{\Delta}[3z \quad -2.5 \quad z-1.5]\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} = \frac{4z-3}{(z-1)(z-0.5)}$$

(d) $|z\mathbf{I}-\mathbf{A}| = \begin{vmatrix} z-1 & 0 \\ 0 & z-0.5 \end{vmatrix} = z^2 - 1.5z + 0.5; \ |z\mathbf{I}-\mathbf{A}_w| = z^2 - 1.5z + 0.5$

$$\therefore z_1 = 1, \ z_2 = 0.5$$

$$|\mathbf{A}| = \begin{vmatrix} 1 & 0 \\ 0 & 0.5 \end{vmatrix} = 0.5 = z_1 z_2; \ |\mathbf{A}_w| = \begin{vmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{vmatrix} = \frac{9}{16} - \frac{1}{16} = 0.5$$

$$\text{tr } \mathbf{A} = 1.5 = z_1 + z_2; \ \text{tr } \mathbf{A}_w = 1.5$$

2.11-1. Consider a system with the transfer function

$$G(z) = \frac{Y(z)}{U(z)} = \frac{2}{z(z-1)}$$

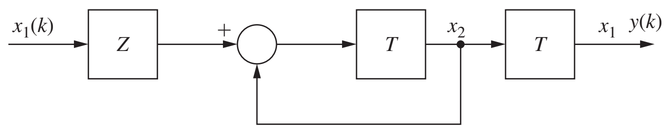
(a) Find three different state-variable models of this system.

(b) Verify the transfer function of each state model in part (a), using (2-84).

Solution:

(a) $G(z) = G_1(z)G_2(z) = \dfrac{2}{z^2 - z} = \dfrac{2z^{-2}}{1 - z^{-1}}$

(1)

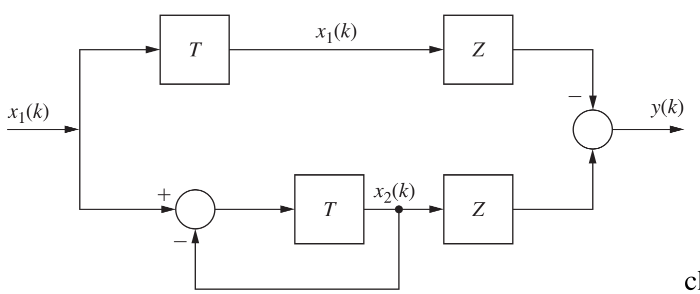


change Z to 2

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(k)$$

(2) $G(z) = \dfrac{2}{z(z-1)} = \dfrac{-2}{z} + \dfrac{2}{z-1} = G_1(z) + G_2(z)$

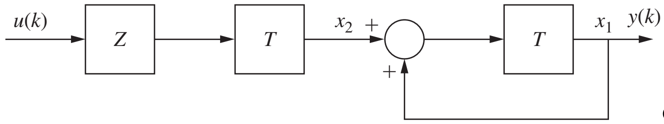


change Z to 2

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} -2 & 2 \end{bmatrix}\mathbf{x}(k)$$

(3)



change Z to 2

$$\mathbf{x}(k+1)=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}0\\ 2\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}(k)$$

(b) (1) $z\mathbf{I}-\mathbf{A}=\begin{bmatrix}z & -1\\ 0 & z-1\end{bmatrix}; \left|z\mathbf{I}-\mathbf{A}\right|=z^2-z=\Delta$

$$G(z)=\mathbf{C}[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}=\frac{1}{\Delta}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}z-1 & 1\\ 0 & z\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}z-1 & 1\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix}=\frac{2}{z(z-1)}$$

(2) $z\mathbf{I}-\mathbf{A}=\begin{bmatrix}z & 0\\ 0 & z-1\end{bmatrix}; \left|z\mathbf{I}-\mathbf{A}\right|=\Delta=z^2-z$

$$G(z)=\mathbf{C}[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}=\frac{1}{\Delta}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}z-1 & 0\\ 0 & z\end{bmatrix}\begin{bmatrix}2\\ 2\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}2z-2\\ 2z\end{bmatrix}=\frac{2}{z(z-1)}$$

(3) $z\mathbf{I}-\mathbf{A}=\begin{bmatrix}z-1 & -1\\ 0 & z\end{bmatrix}; \left|z\mathbf{I}-\mathbf{A}\right|=z^2-z=\Delta$

$$G(z)=\mathbf{C}[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}=\frac{1}{\Delta}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}z & 1\\ 0 & z-1\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}z & 1\end{bmatrix}\begin{bmatrix}0\\ 2\end{bmatrix}=\frac{2}{z(z-1)}$$

2.11-2. Consider a system described by the coupled difference equation

$$y(k+2)-v(k)=0$$

$$v(k+1)+y(k+1)=u(k)$$

where $u(k)$ is the system input.

(a) Find a state-variable formulation for this system. Consider the outputs to be $y(k+1)$ and $v(k)$.
Hint: Draw a simulation diagram first.

(b) Repeat part (a) with $y(k)$ and $v(k)$ as the outputs.

(c) Repeat part (a) with the single output $v(k)$.

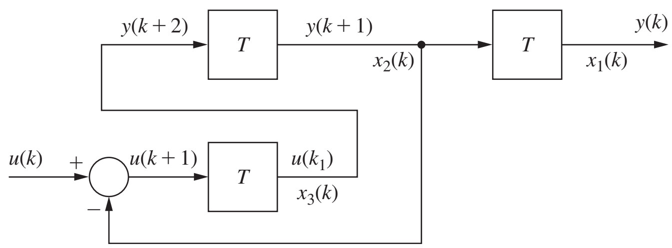
(d) Use (2-84) to calculate the system transfer function with $v(k)$ as the system output, as in part (c); that is, find $V(z)/U(z)$.

(e) Verify the transfer function $V(z)/U(z)$ in part (d) by taking the z-transform of the given system difference equations and eliminating $Y(z)$.

(f) Verify the transfer function $V(z)/U(z)$ in part (d) by using Mason's gain formula on the simulation diagram of part (a).

Solution:

(a)



$$\mathbf{x}(k+1)=\begin{bmatrix}0&1&0\\0&0&1\\0&-1&0\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}0\\0\\1\end{bmatrix}u(k)$$

$$y_0(k)=\begin{bmatrix}x_2(k)\\x_3(k)\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\mathbf{x}(k);\ \ y_0(k)=\text{output}$$

(b) $\mathbf{x}(k+1)=$ same as (a)

$$y_0(k)=\begin{bmatrix}x_1(k)\\x_3(k)\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}\mathbf{x}(k)$$

(c) $\mathbf{x}(k+1)=$ same as (a)

$$y_0(k)=x_3(k)=\begin{bmatrix}0&0&1\end{bmatrix}\mathbf{x}(k)$$

(d) $$z\mathbf{I}-\mathbf{A}=\begin{bmatrix}z&-1&0\\0&z&-1\\0&1&z\end{bmatrix};\ \ |z\mathbf{I}-\mathbf{A}|=z^3-(-z)=z^3+z=\Delta$$

$$\text{Cof}[z\mathbf{I}-\mathbf{A}] = \begin{bmatrix} z^2+1 & z^2 & 0 \\ z & z^2 & z \\ 1 & z & z^2 \end{bmatrix}; \; [z\mathbf{I}-\mathbf{A}]^{-1} = \frac{1}{\Delta}\begin{bmatrix} z^2+1 & z & 1 \\ z^2 & z^2 & z \\ 0 & z & z^2 \end{bmatrix}$$

$$\therefore \frac{Y_0(z)}{U(z)} = \mathbf{C}[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B} = \frac{1}{\Delta}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} z^2+1 & z & 1 \\ z^2 & z^2 & z \\ 0 & z & z^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

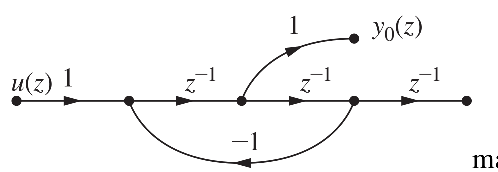
$$= \frac{1}{\Delta}\begin{bmatrix} 0 & z & z^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{z^2}{z^3-z} = \frac{z}{z^2+1}$$

(e) $z^2Y(z) - V(z) = 0 \Rightarrow Y(z) = \frac{1}{z^2}V(z)$

$$zV(z) + zY(z) = zV(z) + \frac{1}{z}V(z) = U(z)$$

$$\therefore \frac{V(z)}{U(z)} = \frac{Y_0(z)}{U(z)} = \frac{1}{z+\frac{1}{z}} = \frac{z}{z^2+1}$$

(f) From (a):



make *u* and *y* capital letters

$$\therefore \frac{Y_0(z)}{U(z)} = \frac{z^{-1}}{1+z^{-2}} = \frac{z}{z^2+1}$$

2.11-3. Given the system described by the state equations

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}(k)$$

(a) Calculate the transfer function $Y(z)/U(z)$, using (2-84).

(b) Draw a simulation diagram for this system, from the state equations given.

(c) Use Mason's gain formula and the simulation diagram to verify the transfer function found in part (a).

Solution:

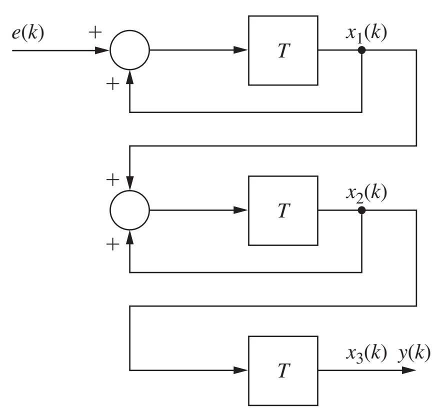
(a) $$z\mathbf{I}-\mathbf{A}=\begin{bmatrix} z-1 & 0 & 0 \\ -1 & z-1 & 0 \\ 0 & -1 & z \end{bmatrix};\ \Delta = z^3-2z^2+z = z(z-1)^2$$

$$\text{Cof }(z\mathbf{I}-\mathbf{A})=\begin{bmatrix} z(z-1) & z & 1 \\ 0 & z(z-1) & z-1 \\ 0 & 0 & (z-1)^2 \end{bmatrix},\ (z\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix} \dfrac{1}{z-1} & 0 & 0 \\ \dfrac{1}{(z-1)^2} & \dfrac{1}{z-1} & 0 \\ \dfrac{1}{z(z-1)^2} & \dfrac{1}{z(z-1)} & \dfrac{1}{z} \end{bmatrix}$$

$$G(z)=\mathbf{C}[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B}=\begin{bmatrix}0 & 0 & 1\end{bmatrix}[z\mathbf{I}-\mathbf{A}]^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{1}{z(z-1)^2} & \dfrac{1}{z(z-1)} & \dfrac{1}{z}\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\frac{1}{z(z-1)^2}=\frac{1}{z^3-2z^2+z}$$

(b)



(c) $\Delta = 1-z^{-1}-z^{-1}+z^{-2}=1-2z^{-1}+z^{-2}$

$$\therefore G(z) = \frac{z^{-3}}{\Delta} = \frac{1}{z^3 - 2z^2 + z}$$

2.11-4. Section 2.9 gives some standard forms for state equations (simulation diagrams for the control canonical and observer canonical forms). The MATLAB statement

$$[\mathrm{A,B,C,D}] = \mathrm{tf2ss}(\mathrm{num,den})$$

generates a standard set of state equations for the transfer function whose numerator coefficients are given in the vector *num* and denominator coefficients in the vector *den*.

(a) Use the MATLAB statement given to generate a set of state equations for the transfer function

$$G(z) = \frac{3z + 4}{z^2 + 5z + 6}$$

(b) Draw a simulation diagram for the state equations in part (a).

(c) Determine if the simulation diagram in part (b) is one of the standard forms in Section 2.9.

Solution:

(a) n = [0 3 4];

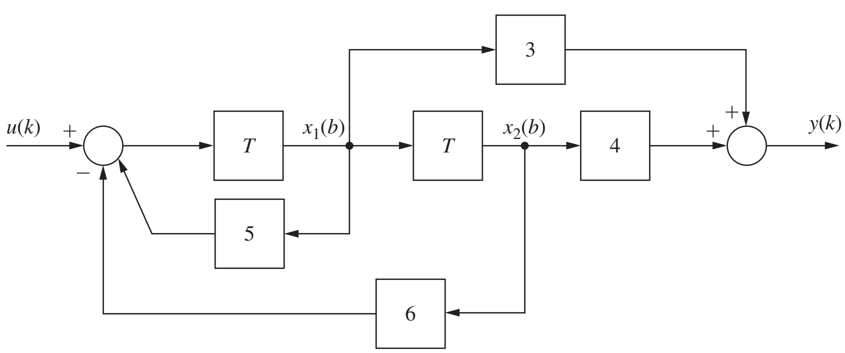
d = [1 5 6];

[A,B,C,D] = tf2ss(n, d)

$$\mathbf{x}(k+1) = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 3 & 4 \end{bmatrix}\mathbf{x}(k)$$

(b)



(c) Yes, it is the control canonical form with the states renumbered.

2.12-1. Consider the system described in Problem 2.10-2.

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -2 & 1 \end{bmatrix} \mathbf{x}(k)$$

(a) Find the transfer function of this system.

(b) Let $u(k) = 1,\ k \ge 0$ (a unit step function) and $\mathbf{x}(0) = 0$. Use the transfer function of part (a) to find the system response.

(c) Find the state transition matrix $\boldsymbol{\Phi}(k)$ for this system.

(d) Use (2-90) to verify the step response calculated in part (b). This calculation results in the response expressed as a summation. Then check the values $y(0)$, $y(1)$, and $y(2)$.

(e) Verify the results of part (d) by the iterative solution of the state equations.

Solution:

(a) $z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z & -1 \\ 0 & z-3 \end{bmatrix}$; $\Delta = |z\mathbf{I} - \mathbf{A}| = z(z-3) = \Delta$

$$\frac{Y(z)}{U(z)} = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} = \frac{1}{\Delta}[-2 \quad 1]\begin{bmatrix} z-3 & 1 \\ 0 & z \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\Delta}[-2z+6 \quad z-2]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{-z+4}{z(z-3)}$$

(b) $Y(z) = \dfrac{(-z+4)z}{z(z-3)(z-1)}$

$$\frac{Y(z)}{z} = \frac{-z+4}{z(z-1)(z-3)} = \frac{4/3}{z} + \frac{-3/2}{z-1} + \frac{1/6}{z-3}$$

$$\therefore y(k) = \begin{cases} 4/3 - 3/2 + 1/6 = 0, & k = 0 \\ -3/2 + 1/6(3)^k & k \geq 1 \end{cases}$$

$$\therefore y(0) = 0$$
$$y(1) = -\frac{3}{2} + \frac{1}{2} = -1$$
$$y(2) = -\frac{3}{2} + \frac{3}{2} = 0$$

(c)

$$\mathbf{\Phi}(z) = z(z\mathbf{I} - \mathbf{A})^{-1} = z\begin{bmatrix} \frac{z-3}{z(z-3)} & \frac{1}{z(z-3)} \\ 0 & \frac{z}{z(z-3)} \end{bmatrix} = z\begin{bmatrix} \frac{1}{z} & \frac{-1/3}{z} + \frac{1/3}{z-3} \\ 0 & \frac{1}{z-3} \end{bmatrix}$$

$$\therefore \mathbf{\Phi}(k) = \begin{bmatrix} \delta(k) & -\frac{1}{3}\delta(k) + \frac{1}{3}(3)^k \\ 0 & (3)^k \end{bmatrix}$$

(d) $$\mathbf{y}(k) = \sum_{j=0}^{k-1} \mathbf{C\Phi}(k-1-j)\mathbf{Bu}(j) = \sum_{j=0}^{k-1} \begin{bmatrix} -2 & 1 \end{bmatrix} \mathbf{\Phi}(k-1-j) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \sum_{j=0}^{k-1} \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3}\delta(k-1-j) + \frac{1}{3}(3)^{k-j-1} \\ (3)^{k-j-1} \end{bmatrix} = \sum_{j=0}^{k-1} \left[\frac{-4}{3}\delta(k-j-1) + \frac{1}{3}(3)^{k-j-1} \right]$$

$$= \sum_{j=0}^{k-1} \left[\frac{-4}{3}\delta(k-1-j) + \frac{1}{3}(3)^{k-1-j} \right]$$

$$y(0) = 0; \quad y(1) = -\frac{4}{3}\delta(0) + \frac{1}{3}(3)^0 = -\frac{4}{3} + \frac{1}{3} = -1$$

$$y(2) = -\frac{4}{3}\delta(1) + \frac{1}{3}(3)^1 - \frac{4}{3}\delta(0) + \frac{1}{3}(3)^0 = 1 - \frac{4}{3} + \frac{1}{3} = 0$$

(e) $$\mathbf{x}(1) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad y(1) = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1$$

$$\mathbf{x}(2) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}; \quad y(2) = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 0$$

2.12-2. The system described by the equations

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 2]\mathbf{x}(k)$$

is excited by the initial conditions $\mathbf{x}(0) = [\ -1 \quad 2\ ]^T$ with $u(k) = 0$ for all k.

(a) Use (2-89) to solve for $\mathbf{x}(k)$, $k \geq 0$.

(b) Find the output $y(z)$.

(c) Show that $\mathbf{\Phi}(k)$ in (a) satisfies the property $\mathbf{\Phi}(0) = \mathbf{I}$.

(d) Show that the solution in part (a) satisfies the given initial conditions.

(e) Use an iterative solution of the state equations to show that the values $y(k)$, for $k = 0, 1, 2,$ and 3, in part (b) are correct.

(f) Verify the results in part (e) using MATLAB.

Solution:

(a) $$z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z-1 & 0 \\ 0 & z-0.5 \end{bmatrix}; \ |z\mathbf{I} - \mathbf{A}| = \Delta = (z-1)(z-0.5)$$

$$(z\mathbf{I} - \mathbf{A}^{-1}) = \frac{1}{\Delta}\begin{bmatrix} z-0.5 & 0 \\ 0 & z-1 \end{bmatrix} = \begin{bmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z-0.5} \end{bmatrix}$$

$$\therefore \mathbf{\Phi}(k) = \mathcal{J}^{-1}\begin{bmatrix} \frac{z}{z-1} & 0 \\ 0 & \frac{z}{z-0.5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5^k \end{bmatrix}$$

$$\therefore \mathbf{x}(k) = \mathbf{\Phi}(k)\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5^k \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2(0.5)^k \end{bmatrix}$$

(b) $$y(k) = \mathbf{C}\mathbf{x}(k) = [1 \quad 2]\begin{bmatrix} 1 \\ 2(0.5)^k \end{bmatrix} = 1 + 4(0.5)^k$$

(c) $$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

(d) $x(k)\big|_{k=0} = \begin{bmatrix} 1 \\ 2(0.5)^k \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(e) From (b), $\begin{matrix} y(0)=5 & y(2)=2 \\ y(1)=3 & y(3)=1.5 \end{matrix}$

$$y(0) = \mathbf{C}\mathbf{x}(0) = [1 \quad 2]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5$$

$$x(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\; y(1) = [1 \quad 2]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3$$

$$x(2) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},\; y(2) = [1 \quad 2]\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = 2$$

$$x(3) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.25 \end{bmatrix},\; y(3) = [1 \quad 2]\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 1.5$$

(f)
```
A = [1 0;0 .5]; B = [2; 1]; C = [1 2];
x=[1; 2];
u = 0;
for k = 0:3
    x1 = A*x + B*u;
    y = C*x;
    [k,y]
    x = x1;
end
```

2.12-3. The system described by the equations

$$\mathbf{x}(k+1) = \begin{bmatrix} 1.1 & 1 \\ -0.3 & 0 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix}\mathbf{x}(k)$$

is excited by the initial conditions $\mathbf{x}(0) = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$ with $u(k) = 0$ for all k.

(a) Use (2-89) to solve for $\mathbf{x}(k)$, $k \geq 0$.

(b) Find the output $y(k)$.

(c) Show that $\mathbf{\Phi}(k)$ in part (a) satisfies the property $\mathbf{\Phi}(0) = \mathbf{I}$.

(d) Show that the solution in part (a) satisfies the given initial conditions.

(e) Use an iterative solution of the state equations to show that the values $y(k)$, for $k = 0, 1, 2,$ and 3, in part (b) are correct.

Solution:

(a) $$z\mathbf{I}-\mathbf{A} = \begin{bmatrix} z-1.1 & -1 \\ 0.3 & z \end{bmatrix}; \; |z\mathbf{I}-\mathbf{A}| = \Delta = z^2 - 1.1z + 0.3 = (z-0.5)(z-0.6)$$

$$(z\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{\Delta}\begin{bmatrix} z & 1 \\ -0.3 & z-1.1 \end{bmatrix}$$

$$\Phi(k) = \mathcal{Z}^{-1}[z(z\mathbf{I}-\mathbf{A})^{-1}] = \mathcal{Z}^{-1}\left(z\begin{bmatrix} \dfrac{z}{(z-0.5)(z-0.6)} & \dfrac{1}{(z-0.5)(z-0.6)} \\ \dfrac{-0.3}{(z-0.5)(z-0.6)} & \dfrac{z-1.1}{(z-0.5)(z-0.6)} \end{bmatrix}\right)$$

$$= \mathcal{Z}^{-1}\left(z\begin{bmatrix} \dfrac{-5}{z-.5}+\dfrac{6}{z-.6} & \dfrac{-10}{z-.5}+\dfrac{10}{z-.6} \\ \dfrac{3}{z-.5}+\dfrac{-3}{z-.6} & \dfrac{6}{z-.5}+\dfrac{-5}{z-.6} \end{bmatrix}\right)$$

$$= \begin{bmatrix} -5(0.5)^k + 6(0.6)^k & -10(0.5)^k + 10(0.6)^k \\ 3(0.5)^k - 3(0.6)^k & 6(0.5)^k - 5(0.6)^k \end{bmatrix}$$

$$\therefore \mathbf{x}(k) = \mathbf{\Phi}(k)\mathbf{x}(0) = \begin{bmatrix} -5(0.5)^k + 6(0.6)^k & -10(0.5)^k + 10(0.6)^k \\ 3(0.5)^k - 3(0.6)^k & 6(0.5)^k - 5(0.6)^k \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -15(0.5)^k + 14(0.6)^k \\ 9(0.5)^k - 7(0.6)^k \end{bmatrix}$$

(b) $$y(k) = \mathbf{C}\mathbf{x}(k) = [1 \quad -1]\begin{bmatrix} -15(0.5)^k + 14(0.6)^k \\ 9(0.5)^k - 7(0.6)^k \end{bmatrix} = -24(0.5)^k + 21(0.6)^k$$

(c) $$\mathbf{\Phi}(0) = \begin{bmatrix} -5+6 & -10+10 \\ 3-3 & 6-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

(d) $\mathbf{x}(k)\big|_{k=0} = \begin{bmatrix} -15+14 \\ 9-7 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(e) From (b), $\quad y(0) = -3 \quad y(2) = 1.56$
$\quad y(1) = 0.6 \quad y(3) = 1.536$

$$y(0) = \mathbf{C}\mathbf{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -3$$

$$\mathbf{x}(1) = \begin{bmatrix} 1.1 & 1 \\ -0.3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix}; \; y(1) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix} = 0.6$$

$$\mathbf{x}(2) = \begin{bmatrix} 1.1 & 1 \\ -0.3 & 0 \end{bmatrix} \begin{bmatrix} 0.9 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 1.29 \\ -0.27 \end{bmatrix}; \; y(2) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1.29 \\ -0.27 \end{bmatrix} = 1.56$$

$$\mathbf{x}(3) = \begin{bmatrix} 1.1 & 1 \\ -0.3 & 0 \end{bmatrix} \begin{bmatrix} 1.29 \\ -0.27 \end{bmatrix} = \begin{bmatrix} 1.149 \\ -0.387 \end{bmatrix}; \; y(3) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1.149 \\ -0.389 \end{bmatrix} = 1.536$$

MATLAB:

```
A = [1.1 1;–0.3 0]; B = [1; 1]; C = [1 –1];
x=[–1; 2];
u = 0;
for k = 0:3
    x1 = A*x + B*u;
    y = C*x;
    [k,y]
    x = x1;
end
```

2.12-4. Let $\mathbf{\Phi}(k)$ be the state transition matrix for the equations

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k)$$

Show that $\mathbf{\Phi}(k)$ satisfies the difference equation

$$\mathbf{\Phi}(k+1) = \mathbf{A}\mathbf{\Phi}(k)$$

Solution:

$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k)$; $\mathbf{x}(k) = \mathbf{\Phi}(k)\mathbf{x}(0)$

$\therefore \mathbf{\Phi}(k+1)\mathbf{x}(0) = \mathbf{A}\mathbf{\Phi}(k)\mathbf{x}(0)$

Since this is true for any $\mathbf{x}(0)$, $\therefore \mathbf{\Phi}(k+1) = \mathbf{A}\mathbf{\Phi}(k)$