# INSTRUCTOR'S SOLUTIONS MANUAL

ROGER LIPSETT

TO ACCOMPANY

# CALCULUS FOR BIOLOGY AND MEDICINE

FOURTH EDITION

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# Contents

# Chapter 1

# Preview and Review

## 1.1 Precalculus Skills Diagnostic Test

**1.** **(a)** $y = x + 5$

**(b)** $y = 2x - 1$

**(c)** $y = 3x - 5$

(Review 1.2.2)

**2.** **(a)** $\frac{5}{9}(80 - 32) = \frac{240}{9} \approx 26.7\,°\mathrm{C}$

**(b)** Inverting the given formula gives $x = \frac{9}{5}y + 32$, so that

$$\frac{9}{5}(-10) + 32 = -18 + 32 = 14\,°\,\mathrm{F}$$

**(c)** Set $y = x$ in the conversion formula and solve for $x$:

$$x = \frac{5}{9}(x - 32) = \frac{5}{9}x - \frac{160}{9} \quad \Rightarrow \quad \frac{4}{9}x = -\frac{160}{9} \quad \Rightarrow \quad 4x = -160 \quad \Rightarrow \quad x = -40.$$

(Review 1.2.2)

**3.** A circle with center $(x, y) = (-1, 5)$ and with radius 3. (Review 1.2.3)

**4.** **(a)** $\frac{180}{\pi} \cdot \frac{\pi}{7} = \frac{180}{7} \approx 25.7°$.

**(b)** $x = -\frac{\pi}{3} + 2n\pi$ and $x = -\frac{2\pi}{3} + 2n\pi$ for any $n \in \mathbb{Z}$.

**(c)** Divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$:

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \text{ so } \tan^2\theta + 1 = \sec^2\theta.$$

**(d)** $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$

(Review 1.2.4)

**5.** **(a)** **(i)** $2^{7/3}$

**(ii)** $2^{8/3}$

**(b)** Take base 2 logarithms of both sides: $x \log_2 4 = \log_2 \frac{1}{2}$, so $2x = -1$ and therefore $x = -\frac{1}{2}$.

**(c)** Since $10\,000 = 10^4$, we have $\log_{10} 10\,000 = 4$.

**(d)** $\log_{10} 3x + \log_{10} 5x = \log_{10}(3x \cdot 5x) = \log_{10}(15x^2)$.

**(e)** $\ln\left(x^2\right) + \ln x = 2\ln x + \ln x = 3\ln x$. Solving $3\ln x = 2$ gives $\ln x = \frac{2}{3}$, so that $x = e^{2/3}$.

**(f)** If $\ln x = 3$, then $x = e^3$, so that $\log_{10} x = \log_{10}\left(e^3\right) = 3\log_{10} e$.

(Review 1.2.5)

**6. (a)** Using the quadratic formula, the roots are

$$\frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}.$$

**(b)** $(1+i)(2-i) = 1 \cdot 2 + 1 \cdot (-i) + i \cdot 2 - i^2 = 3 + i$.

**(c)** $z + \overline{z} = (a+bi) + (a-bi) = 2a \in \mathbb{R}$

(Review 1.2.6)

**7. (a) (i)** $f(x) \in [0,1]$

**(ii)** $f(x) \in [0,\infty)$

**(b) (i)** $f(3) = \sqrt{3}$

**(ii)** $(f \circ g)(x) = f\left((x+1)^2\right) = \sqrt{(x+1)^2} = |x+1|$.

**(iii)** $(g \circ f)(4) = g(\sqrt{4}) = g(2) = (2+1)^2 = 9$.

**(c)** Since $f(x) = |x| \geq 0$, then $(f \circ f)(x) = |f(x)| = f(x)$

(Review 1.3.1)

**8. (a)** For large values of $x$, (iii) grows much faster due to the presence of the $x^3$ term. For example, when $x = 6$, $p_1(x) = 6$, $p_2(x) = 18$, and $p_3(x) = 72$.

**(b) (i)** The highest power of $r$ that appears is 2, so this is a degree 2 polynomial in $r$.

**(ii)** Since the velocity should not be negative we must have $1 - \frac{r^2}{a^2} \geq 0$, so that $r \leq a$. Since $r$ is a distance, it must be nonnegative. Putting these together gives $0 \leq r \leq a$ for the domain.

**(iii)** Since $1 - \frac{r^2}{a^2} \leq 1$ for the allowable values of $r$, the range is $0 \leq u(r) \leq u_0$.

**(iv)** The maximum value occurs when $r = 0$; that value is $u_0$. $u(r) = \frac{1}{2}u_0$ when $1 - \frac{r^2}{a^2} = \frac{1}{2}$, which happens when $r = \frac{a}{\sqrt{2}}$.

(Review 1.3.2)

**9. (a)** For the three values of $c$, we have

$$r(1) = \frac{1}{1+10} \approx 0.091, \quad r(2) = \frac{2}{2+10} \approx 0.167, \quad r(3) = \frac{3}{3+10} \approx 0.231.$$

The rate of metabolization is largest when $c = 3$.

**(b)** Since

$$r(5) = \frac{5}{5+10} = \frac{1}{3}, \quad r(10) = \frac{10}{10+10} = \frac{1}{2},$$

the rate of metabolization increases by less than double.

**(c)** From part (b), $r(5) = \frac{1}{3}$; to get $r(c) = \frac{2}{3}$ we need

$$r(c) = \frac{c}{c+10} = \frac{2}{3},$$

so that $3c = 2(c+10)$. Solving gives $c = 20$.

**(d)** No. $r(10) = \frac{1}{2}$. Doubling the rate of metabolization would increase it to 1. But since the denominator of $r(c)$ is always strictly greater than the numerator, its value can never be 1.

(Review 1.3.3)

**10.** The relationship is $N = kA^{1/5}$.

**(a)** We want to find $A$ so that $kA^{1/5} = \frac{1}{2}kA_0^{1/5}$. Cancelling the $k$'s gives $A^{1/5} = \frac{1}{2}A_0^{1/5}$. Now raise both sides to the fifth power, giving $A = \frac{A_0}{32}$. The new habitat area is $\frac{A_0}{32}$.

**(b)** We want to find $A$ so that $kA^{1/5} = \frac{1}{3}kA_0^{1/5}$. Cancelling the $k$'s gives $A^{1/5} = \frac{1}{3}A_0^{1/5}$. Now raise both sides to the fifth power, giving $A = \frac{A_0}{243}$. The new habitat area is $\frac{A_0}{243}$.

**(c)** We want to find $A$ so that $kA^{1/5} = 2kA_0^{1/5}$. Cancelling the $k$'s gives $A^{1/5} = 2A_0^{1/5}$. Now raise both sides to the fifth power, giving $A = 32A_0$. The new habitat area is $32A_0$.

(Review 1.3.4)

**11.** **(a)** Since $1000 = N(0) = N_0e^{r\cdot 0} = N_0$ we get $N_0 = 1000$. Since $2000 = N(2) = 1000e^{r\cdot 2}$, we have $2 = e^{2r}$. Take natural logarithms, giving $2r = \ln 2$ so that $r = \frac{\ln 2}{2}$. The formula is therefore $N(t) = 1000e^{t\ln 2/2}$.

**(b)** Solving $N(t) = 3000$ gives $e^{t\ln 2/2} = 3$, so that $t\frac{\ln 2}{2} = \ln 3$, and then $t = \frac{2\ln 3}{\ln 2} \approx 3.17$.

**(c)** Solving $N(t) = 4000$ gives $e^{t\ln 2/2} = 4$, so that $t\frac{\ln 2}{2} = \ln 4$, and then $t = \frac{2\ln 4}{\ln 2} = 4$.

(Review 1.3.5)

**12.** **(a)** $f^{-1}(x) = \sqrt{x-1}$

**(b)** $f^{-1}(x) = -1 + e^{x/2}$

**(c)** $f^{-1}(x) = x^{1/5}$

(Review 1.3.6)

**13.** **(a)** **(i)** $\ln x + \ln(x^2+1) = \ln(x\cdot(x^2+1))$.

**(ii)** $\log\left(x^{1/3}\right) - \log\left((x+1)^{1/3}\right) = \frac{1}{3}\log x - \frac{1}{3}\log(x+1) = \frac{1}{3}(\log x - \log(x+1)) = \frac{1}{3}\log\frac{x}{x+1}$.

**(iii)** $2 + \log_2 x = \log_2 4 + \log_2 x = \log_2(4x)$.

**(b)** **(i)** $\frac{\ln 7}{\ln 2}$

**(ii)** $\frac{\ln 6}{\ln 10}$

**(iii)** $\frac{\ln 2}{\ln x}$

(Review 1.3.7)

**14.** **(a)** **(i)** The amplitude is 2, and the period is $2\pi$.

**(ii)** The amplitude is 2, and the period is $\frac{2\pi}{3}$.

**(iii)** The amplitude is 3, and the period is $\frac{2\pi}{\pi/2} = 4$.

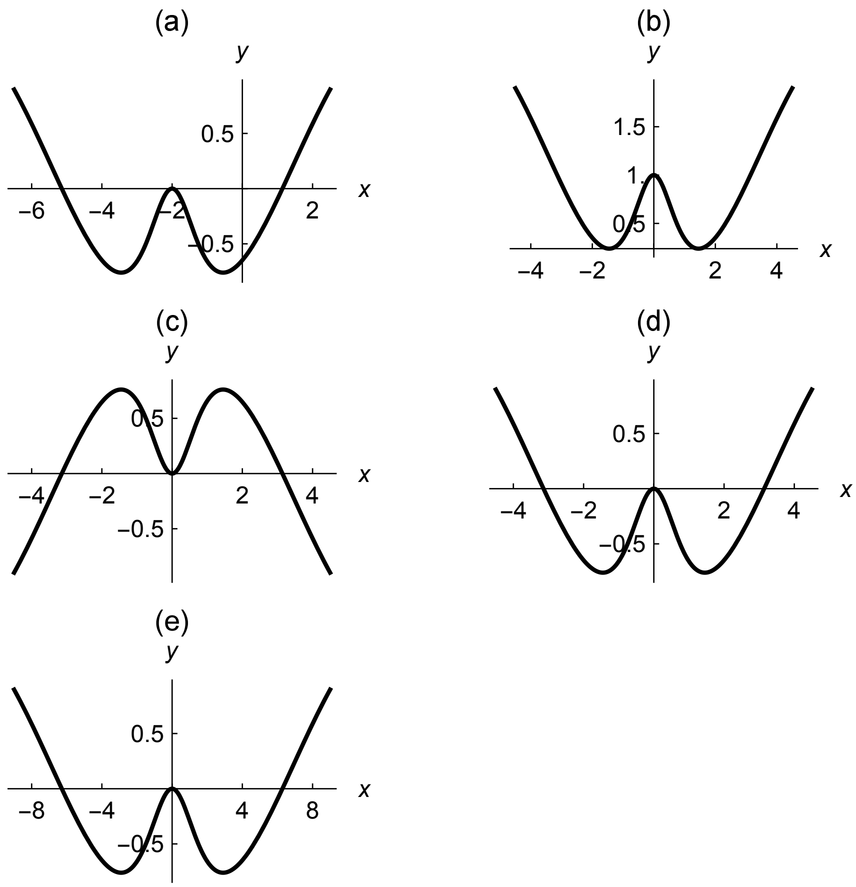
**(b)** **(i)** The range is all of $\mathbb{R}$. The maximum domain is all of $\mathbb{R}$ except for the points $\frac{(2n+1)}{2}\pi$ where $n$ is an integer (since tangent is undefined at odd multiples of $\pi/2$).

**(ii)** The range is $[-1, 1]$, since $-1 \le \cos x \le 1$ for all $x$. The domain is $\mathbb{R}$.

**(c)** Compress $\cos x$ by a factor of 2 in the $x$-direction and stretch it by a factor of 3 in the $y$-direction to get $3\cos 2x$.

(Review 1.3.8)

**15.**




(Review 1.4.1)

**16.** **(a)** **(i)** 2 kg

**(ii)** 30 kg

**(iii)** 100 kg

**(b)** $\frac{\text{Adult weight}}{\text{Puppy weight}} = \frac{20\,\text{kg}}{0.4\,\text{kg}} = 50$

**(c)** The point representing the cat will lie one tick below the Jack Russell Terrier.

(Review 1.4.2)

**17.** **(a)** Since both axes are log scales, this is a power relationship; since the slope of the line is 0.41, the relationship is $D = k\,A^{0.41}$.

**(b)** Since only the vertical axis is a log scale, this is an exponential relationship; since the slope of the line is $-1$, the relationship is $N = k\,10^{-m}$.

**(c)** Since only the vertical axis is a log scale, this is an exponential relationship; since the slope of the line is 0.86, the relationship is $N = k\,10^{0.86t} \approx k\,7.24^t$.

**(d)** Since only the vertical axis is a log scale, this is an exponential relationship; since the slope of the line is $-0.09$, the relationship is $N = k\,10^{-0.09t} \approx k\,0.813^t$.

(Review 1.4.3)

**18.** **(a)** This is the graph of patient 1 shifted two hours to the right, so it is graph 4.

**(b)** This graph should start from zero at 8AM just as for patient 1, but it should have a longer tail since it takes longer to enter the bloodstream. This is graph 3.

**(c)** This graph should have the same shape as the graph of patient A, which is graph 4, but is half as high. This is graph 2.

(Review 1.4.4)

## 1.2 Preliminaries

### ■ 1.2.1

**1.** **(a)** Walking 4 units to the right and to the left from $-1$ we get the numbers 3 and $-5$, respectively.

**(b)** $|x-(-1)| = 4$, so $x+1 = \pm 4$, yielding $x = 3$ or $x = -5$.

**2.** With three numbers there will be three pairwise distances: $|-5-2| = |-7| = 7, |2-7| = |-5| = 5$ and $|-5-7| = |-12| = 12$.

**3.** **(a)** $2x+4 = \pm 6$, so either $2x = 2$ giving the solution $x = 1$ or $2x = -10$ and the other solution is $x = -5$.

**(b)** $x-3 = \pm 2$, from which we see that $x = 5$ or $x = 1$.

**(c)** $2x-3 = \pm 5$, so $2x = 8$ giving $x = 4$ or $2x = -2$ and $x = -1$.

**(d)** $1-5x = \pm 6$, so that $5x = 1 \pm 6$. Then $5x = -5$, so $x = -1$, or $5x = 7$, so $x = \frac{7}{5}$.

**4.** **(a)** The equation implies $\pm(2x+4) = \pm(5x-2)$, which reduces to two equations $2x+4 = 5x-2$ and $2x+4 = -(5x-2)$, so the solutions are: $x = 2$ and $x = -\frac{2}{7}$.

**(b)** As in part (a) we get $1+2u = \pm(5-u)$, and solving both gives $1+2u = 5-u$ with solution $u = \frac{4}{3}$, and $1+2u = -(5-u) = u-5$ with solution $u = -6$.

**(c)** As in part (a) we get the equations $4+\frac{t}{2} = \pm(\frac{3}{2}t-2)$ that solves to $t = 6$ and $t = -1$.

**(d)** As in part (a) $2s-6 = \pm(3-s)$; solving both equations gives $s = 3$ and $s = 3$. So $s = 3$ is the only solution.

**5.** **(a)** We can rewrite the absolute value as two inequalities

$$-4 \le 5x-2 \le 4 \quad\Rightarrow\quad -4+2 \le 5x \le 4+2 \quad\Rightarrow\quad -2 \le 5x \le 6 \quad\Rightarrow\quad -\frac{2}{5} \le x \le \frac{6}{5}.$$

**(b)** Rewriting as a pair of inequalities gives $3-4x < -8$ or $3-4x > 8$. Solving the first we get: $4x > 11$, or $x > \frac{11}{4}$, and solving the second we get $4x < -5$, or $x < -\frac{5}{4}$. So $x < -\frac{5}{4}$ or $x > \frac{11}{4}$.

**(c)** Rewriting as a pair of inequalities gives $7x+4 \ge 3$, which solves as $x \ge -\frac{1}{7}$, and $7x+4 \le -3$ which gives $7x \le -7$ or $x \le -1$. So $x \le -1$ or $x \ge -\frac{1}{7}$.

**(d)** Rewriting gives

$$-7 < 3 + 2x < 7 \quad \Rightarrow \quad -7 - 3 < 2x < 7 - 3 \quad \Rightarrow \quad -10 < 2x < 4 \quad \Rightarrow \quad -5 < x < 2.$$

**6.** **(a)** $-6 < 2x + 3 < 6$; subtracting 3 gives $-9 < 2x < 3$, so that $-\frac{9}{2} < x < \frac{3}{2}$.

**(b)** Rewriting as a pair of inequalities gives $3 - 4x \geq 2$ or $3 - 4x \leq -2$, so that $x \leq \frac{1}{4}$ or $x \geq \frac{5}{4}$.

**(c)** $-1 \leq x + 5 \leq 1$, which is the same as $-6 \leq x \leq -4$.

**(d)** Since no absolute value can be negative, there are no values of $x$ that will satisfy the equation.

## ■ 1.2.2

**7.** Use the point slope formula $y - y_0 = m(x - x_0)$. In this case we get

$$y - 2 = -2(x - 3) \quad \Rightarrow \quad y - 2 = -2x + 6 \quad \Rightarrow \quad 2x + y - 8 = 0.$$

**8.** Use the point slope formula $y - y_0 = m(x - x_0)$. In this case we get

$$y - (-1) = \frac{1}{4}(x - 2) \quad \Rightarrow \quad y + 1 = \frac{1}{4}x - \frac{1}{2} \quad \Rightarrow \quad -x + 4y + 6 = 0.$$

**9.** Use the point slope formula $y - y_0 = m(x - x_0)$. In this case we get

$$y - (-2) = -3(x - 0) \quad \Rightarrow \quad y + 2 = -3x \quad \Rightarrow \quad 3x + y + 2 = 0.$$

**10.** Use the point slope formula $y - y_0 = m(x - x_0)$. In this case we get

$$y - 5 = \frac{1}{2}(x - (-3)) \quad \Rightarrow \quad 2y - 10 = x + 3 \quad \Rightarrow \quad -x + 2y - 13 = 0.$$

**11.** First compute the slope using the two given points:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-3)}{1 - (-2)} = \frac{7}{3}.$$

Now use the point slope formula $y - y_0 = m(x - x_0)$ with $(x_0, y_0) = (1, 4)$ (we could equally well use the other point, $(-2, -3)$):

$$y - 4 = \frac{7}{3}(x - 1) \quad \Rightarrow \quad 3(y - 4) = 7(x - 1) \quad \Rightarrow \quad 3y - 12 = 7x - 7 \quad \Rightarrow \quad -7x + 3y - 5 = 0.$$

**12.** First compute the slope using the two given points:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 4}{1 - (-1)} = -4.$$

Now use the point slope formula $y - y_0 = m(x - x_0)$ with $(x_0, y_0) = (-1, 4)$ (we could equally well use the other point, $(1, -4)$):

$$y - 4 = -4(x - (-1)) \quad \Rightarrow \quad y - 4 = -4x - 4 \quad \Rightarrow \quad 4x + y = 0.$$

**13.** First compute the slope using the two given points:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-3}{2-0} = -1.$$

Now use the point slope formula $y - y_0 = m(x - x_0)$ with $(x_0, y_0) = (0, 3)$ (we could equally well use the other point, $(2, 1)$):

$$y - 3 = -1(x - 0) \quad \Rightarrow \quad y - 3 = -x \quad \Rightarrow \quad x + y - 3 = 0.$$

**14.** First compute the slope using the two given points:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-1)}{4-1} = \frac{6}{3} = 2.$$

Now use the point slope formula $y - y_0 = m(x - x_0)$ with $(x_0, y_0) = (1, -1)$ (we could equally well use the other point, $(4, 5)$):

$$y - (-1) = 2(x - 1) \quad \Rightarrow \quad y + 1 = 2x - 2 \quad \Rightarrow \quad 2x - y - 3 = 0.$$

**15.** Horizontal lines are always of the form $y = k$; since this line goes through $\left(4, \frac{1}{4}\right)$, its equation is $y = \frac{1}{4}$. The standard form is $4y - 1 = 0$.

**16.** Horizontal lines are always of the form $y = k$; since this line goes through $(0, -1)$, its equation is $y = -1$. The standard form is $y + 1 = 0$.

**17.** Vertical lines are always of the form $x = k$; since this line goes through $(-2, 0)$, its equation is $x = -2$. The standard form is $x + 2 = 0$.

**18.** Vertical lines are always of the form $x = k$; since this line goes through $(2, -3)$, its equation is $x = 2$. The standard form is $x - 2 = 0$.

**19.** Use the slope-intercept form $y = mx + b$. We are given $m = 3$ and $b = 2$, so the equation is $y = 3x + 2$, which in standard form is $3x - y + 2 = 0$.

**20.** Use the slope-intercept form $y = mx + b$. We are given $m = -1$ and $b = 5$, so the equation is $y = -x + 5$, which in standard form is $x + y - 5 = 0$.

**21.** Use the slope-intercept form $y = mx + b$. We are given $m = \frac{1}{2}$ and $b = 2$, so the equation is $y = \frac{1}{2}x + 2$, or $2y = x + 4$, which in standard form is $x - 2y + 4 = 0$.

**22.** Use the slope-intercept form $y = mx + b$. We are given $m = -\frac{1}{3}$ and $b = \frac{1}{3}$, so the equation is $y = -\frac{1}{3}x + \frac{1}{3}$, which in standard form is $x + 3y - 1 = 0$.

**23.** Use the point slope formula $y - y_0 = m(x - x_0)$ where $(x_0, y_0) = (1, 0)$ is the given point. In this case we get

$$y - 0 = -2(x - 1) \quad \Rightarrow \quad y = -2x + 2 \quad \Rightarrow \quad 2x + y - 2 = 0.$$

**24.** Use the point slope formula $y - y_0 = m(x - x_0)$ where $(x_0, y_0) = (-2, 0)$ is the given point. In this case we get

$$y - 0 = 1(x - (-2)) \quad \Rightarrow \quad y = x + 2 \quad \Rightarrow \quad x - y + 2 = 0.$$

**25.** Use the point slope formula $y - y_0 = m(x - x_0)$ where $(x_0, y_0) = \left(-\frac{1}{2}, 0\right)$ is the given point. In this case we get

$$y - 0 = -\frac{1}{2}\left(x - \left(-\frac{1}{2}\right)\right) \quad \Rightarrow \quad y = -\frac{1}{2}x - \frac{1}{4} \quad \Rightarrow \quad 2x + 4y + 1 = 0.$$

**26.** Use the point slope formula $y - y_0 = m(x - x_0)$ where $(x_0, y_0) = \left(-\frac{1}{2}, 0\right)$ is the given point. In this case we get

$$y - 0 = \frac{1}{5}\left(x - \left(-\frac{1}{2}\right)\right) \quad \Rightarrow \quad y = \frac{1}{5}x + \frac{1}{10} \quad \Rightarrow \quad 2x - 10y + 1 = 0.$$

**27.** Since the line we want is parallel to $x+2y-4=0$, it must have the same slope. Now, $x+2y-4=0$ implies that $2y = -x + 4$, or $y = -\frac{1}{2}x + 2$, so the line has slope $m = -\frac{1}{2}$. Now use the point slope formula $y - y_0 = m(x - x_0)$ where $(x_0, y_0) = (2, -3)$ is the given point. In this case we get

$$y - (-3) = -\frac{1}{2}(x - 2) \quad \Rightarrow \quad y + 3 = -\frac{1}{2}x + 1 \quad \Rightarrow \quad 2y + 6 = -x + 2 \quad \Rightarrow \quad x + 2y + 4 = 0.$$

**28.** Since the line we want is parallel to $x-2y+4=0$, it must have the same slope. Now, $x-2y+4=0$ implies that $2y = x + 4$, or $y = \frac{1}{2}x + 2$, so the line has slope $m = \frac{1}{2}$. Now use the point slope formula $y - y_0 = m(x - x_0)$ where $(x_0, y_0) = (1, 2)$ is the given point. In this case we get

$$y - 2 = \frac{1}{2}(x - 1) \quad \Rightarrow \quad y - 2 = \frac{1}{2}x - \frac{1}{2} \quad \Rightarrow \quad x - 2y + 3 = 0.$$

**29.** The line passing through $(0, 2)$ and $(3, 0)$ has slope

$$m = \frac{0 - 2}{3 - 0} = -\frac{2}{3},$$

so the line we want has the same slope. Now use the point slope formula $y - y_0 = m(x - x_0)$ where $(x_0, y_0) = (-1, -1)$ is the given point. In this case we get

$$y - (-1) = -\frac{2}{3}(x - (-1)) \quad \Rightarrow \quad y + 1 = -\frac{2}{3}x - \frac{2}{3} \quad \Rightarrow \quad 2x + 3y + 5 = 0.$$

**30.** The line passing through $(0, -4)$ and $(2, 1)$ has slope

$$m = \frac{1 - (-4)}{2 - 0} = \frac{5}{2},$$

so the line we want has the same slope. Now use the point slope formula $y - y_0 = m(x - x_0)$ where $(x_0, y_0) = (2, -1)$ is the given point. In this case we get

$$y - (-1) = \frac{5}{2}(x - 2) \quad \Rightarrow \quad y + 1 = \frac{5}{2}x - 5 \quad \Rightarrow \quad 5x - 2y - 12 = 0.$$

**31.** The line $2y - 5x + 7 = 0$ can be written as $2y = 5x - 7$, or $y = \frac{5}{2}x - \frac{7}{2}$, so it has slope $\frac{5}{2}$. Since the line we want is perpendicular to it, it must have slope $m_\perp = -\frac{1}{m} = -\frac{2}{5}$. It passes through $(1, 4)$, so using the point slope formula $y - y_0 = m(x - x_0)$ we get

$$y - 4 = -\frac{2}{5}(x - 1) \quad \Rightarrow \quad 5y - 20 = -2x + 2 \quad \Rightarrow \quad 2x + 5y - 22 = 0.$$

**32.** The line $x - 2y + 3 = 0$ can be written as $2y = x + 3$, or $y = \frac{1}{2}x + \frac{3}{2}$, so it has slope $m_0 = \frac{1}{2}$. Since the line we want is perpendicular to it, it must have slope $m = -2$ (since for perpendicular lines we must have $m \cdot m_0 = -1$). It passes through $(1, -1)$, so using the point slope formula $y - y_0 = m(x - x_0)$ we get

$$y - (-1) = -2(x - 1) \quad \Rightarrow \quad y + 1 = -2x + 2 \quad \Rightarrow \quad 2x + y - 1 = 0.$$

**33.** The given line, passing through $(-2,1)$ and $(1,-2)$, has slope

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2) - 1}{1 - (-2)} = -\frac{3}{3} = -1.$$

Therefore the slope of the perpendicular line is $m_\perp = -\frac{1}{m} = 1$. It passes through $(5,-1)$, so using the point slope formula $y - y_0 = m(x - x_0)$ we get

$$y - (-1) = 1(x - 5) \quad \Rightarrow \quad y + 1 = x - 5 \quad \Rightarrow \quad x - y - 6 = 0.$$

**34.** The given line, passing through $(-2,0)$ and $(1,1)$, has slope

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{1 - (-2)} = \frac{1}{3}.$$

Therefore the slope of the perpendicular line is $m_\perp = -\frac{1}{m} = -3$. It passes through $(4,-1)$, so using the point slope formula $y - y_0 = m(x - x_0)$ we get

$$y - (-1) = -3(x - 4) \quad \Rightarrow \quad y + 1 = -3x + 12 \quad \Rightarrow \quad 3x + y - 11 = 0.$$

**35.** The line we want is horizontal and passes through $(1,3)$. Since all horizontal lines are of the form $y = k$, here we must have $y = 3$, or $y - 3 = 0$.

**36.** The line we want is horizontal and passes through $(1,5)$. Since all horizontal lines are of the form $y = k$, here we must have $y = 5$, or $y - 5 = 0$.

**37.** The line we want is vertical and passes through $(-2,3)$. Since all vertical lines are of the form $x = k$, here we must have $x = -2$, or $x + 2 = 0$.

**38.** The line we want is vertical and passes through $(3,1)$. Since all vertical lines are of the form $x = k$, here we must have $x = 3$, or $x - 3 = 0$.

**39.** A line perpendicular to a horizontal line is a vertical line. Since all vertical lines are of the form $x = k$, and the line passes through $(1,-3)$, we must have $x = 1$, or $x - 1 = 0$.

**40.** A line perpendicular to a horizontal line is a vertical line. Since all vertical lines are of the form $x = k$, and the line passes through $(1,3)$, we must have $x = 1$, or $x - 1 = 0$.

**41.** A line perpendicular to a vertical line is a horizontal line. Since all horizontal lines are of the form $y = k$, and the line passes through $(7,3)$, we must have $y = 3$, or $y - 3 = 0$.

**42.** A line perpendicular to a vertical line is a horizontal line. Since all horizontal lines are of the form $y = k$, and the line passes through $(-2,5)$, we must have $y = 5$, or $y - 5 = 0$.

**43.** **(a)** We convert each of these into centimeters, remembering that the formula $y = 30.5x$ requires that $x$ be measured in feet:

- **(i)** $y = 30.5 \cdot 6\,\text{ft} = 183\,\text{cm}$
- **(ii)** $y = 30.5 \cdot \frac{4}{12}\,\text{ft} \approx 30.5 \cdot 0.1667\,\text{ft} \approx 10.2\,\text{cm}$
- **(iii)** $1\,\text{ft}, 7\,\text{in} = 1 + \frac{7}{12}\,\text{ft} \approx 1.583\,\text{ft}$, so $y \approx 30.5 \cdot 1.583\,\text{ft} \approx 48.3\,\text{cm}$.
- **(iv)** $y = 30.5 \cdot \frac{20.5}{12}\,\text{ft} \approx 30.5 \cdot 1.708\,\text{ft} \approx 52.1\,\text{cm}$

**(b)** Divide both sides of $y = 30.5x$ by 30.5, giving $\frac{y}{30.5} = x$, or $x = \frac{y}{30.5}$. Since $x$ is measured in feet and $y$ in centimeters, this gives a way to convert from centimeters to feet.

**(c)** **(i)** $x = \frac{195}{30.5} \approx 6.39\,\text{ft}$
**(ii)** $x = \frac{12}{30.5} \approx 0.39\,\text{ft}$
**(iii)** $x = \frac{48}{30.5} \approx 1.57\,\text{ft}$

**44.** **(a)** Since 1 pound equals 2.20 kg, let $y = 2.20x$, where $x$ is kilograms and $y$ is pounds.
**(b)** **(i)** $y = 2.20x$, so $63 = 2.20x$, or $x \approx 28.6\,\text{kg}$.
**(ii)** $y = 2.20x$, so $5 = 2.20x$, or $x \approx 2.27\,\text{kg}$.
**(iii)** $y = 2.20 \cdot 2.5 = 5.5$ pounds.
**(iv)** $y = 2.20 \cdot 76 \approx 167$ pounds.

**45.** Distance = rate · time; here time $= 15\,\text{min} = \frac{1}{4}\,\text{h} = 0.25\,\text{h}$, and the distance is 10 mi. The constant of proportionality is miles per hour or "mph". Using the given values gives

$$10\,\text{mi} = \text{speed} \cdot 0.25\,\text{h} \quad \Rightarrow \quad \text{speed} = \frac{10}{0.25} = 40\,\text{mph}.$$

The speed (constant of proportionality) is 40 mph.

**46.** Let $y$ be the number of seeds produced, and $x$ be the biomass. Since $y$ is proportional to $x$, we have $y = mx$ where $m$ is the constant of proportionality. Then $13 = m \cdot 213$, so that $m = \frac{13}{213} \approx 0.061$.

**47.** $1\,\text{ft} = 0.305\,\text{m}$, so $3.279\,\text{ft} = 1\,\text{m}$. Then

$$1\,\text{m}^2 = (1\,\text{m})(1\,\text{m}) = (3.279\,\text{ft})(3.279\,\text{ft}) \approx 10.75\,\text{ft}^2.$$

**48.** $1\,\text{ha} = 10\,000\,\text{m}^2$, and $1\,\text{acre} = 4046.86\,\text{m}^2$. Therefore the area ratio of a hectare to an acre (that is, the number of acres in a hectare) is

$$\frac{1\,\text{ha}}{1\,\text{acre}} = \frac{10\,000\,\text{m}^2}{4046.86\,\text{m}^2} \approx 2.471\,\text{acre}.$$

**49.** **(a)** Since the relationship is linear, if we let $y$ be the number of liters and $x$ the number of ounces, we have $x = my$ where $m = 33.81$. So the relationship is $x = 33.81y$, or $y \approx 0.0296x$.
**(b)** With $x = 12$ ounces, we get $y = 0.0296 \cdot 12 \approx 0.355\,\text{l}$.

**50.** **(a)** Since the relationship is linear, if we let $y$ be the number of kilometers and $x$ the number of miles, we have $y\,\text{km} = mx\,\text{mi}$ where $m = 1.609$. So the relationship is $y = 1.609x$.
**(b)** We have $434 = 1.609x$, so that $x = \frac{434}{1.609} \approx 270\,\text{mi}$.

**51.** **(a)** Since one cup of flour weighs 120 g, it follows that 2.5 cups of flour weighs $2.5 \cdot 120 = 300\,\text{g}$.
**(b)** Since one cup of flour weighs 120 g, one gram of flour is $\frac{1}{120}$ cups. Thus 225 g of flour is $225 \cdot \frac{1}{120} = 1.875$ cups.
**(c)** Let $y$ be the number of cups and $x$ the weight in grams. Then from part (b), $y = \frac{1}{120}x$.

**52.** **(a)** Let $x$ be the temperature in °C and let $y$ be the temperature in °F. Then we have the points $(0, 32)$ and $(100, 212)$. First find the slope between these two pts:

$$m = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{18}{10} = \frac{9}{5}$$

The $y$-intercept is 32. A linear equation that relates Celsius and Fahrenheit then is $y = \frac{9}{5}x + 32$, or $F = \frac{9}{5}C + 32$

**(b)** Solving the equation from part (a) for $C$ gives

$$C = \frac{5}{9}(F - 32).$$

Then the lower and upper normal body temperatures in humans are

$$97.6\,^\circ\text{F corresponds to } \frac{5}{9}(97.6 - 32) = \frac{5}{9} \cdot 65.6 \approx 36.44\,^\circ\text{C}$$
$$99.6\,^\circ\text{F corresponds to } \frac{5}{9}(99.6 - 32) = \frac{5}{9} \cdot 67.6 \approx 37.56\,^\circ\text{C}$$

**(c)** Setting $C = F$ in the equation from part (a) gives

$$F = \frac{9}{5}F + 32 \quad \Rightarrow \quad -\frac{4}{5}F = 32 \quad \Rightarrow \quad F = -40.$$

The two scales read the same at a temperature of $-40°$.

**53. (a)** Since the relationship is linear, these two points, $(0.16, 0.52)$ and $(1.0, 1.0)$, must lie on a line. The slope of that line must be

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1.0 - 0.52}{1.0 - 0.16} = \frac{0.48}{0.84} \approx 0.5714.$$

The equation of the line is then (using the point slope form with the point $(1.0, 1.0)$)

$$y - 1.0 = 0.5714(x - 1.0) \quad \Rightarrow \quad y = 0.5714x + 0.4286.$$

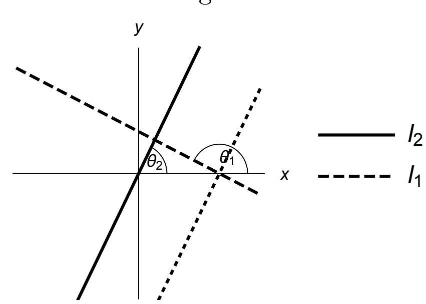
**(b)** Substitute each value of $x$ into the equation from part (a):

**(i)** $y = 0.5714 \cdot 0.5 + 0.4286 = 0.7143$
**(ii)** $y = 0.5714 \cdot 0.9 + 0.4286 = 0.9429$
**(iii)** $y = 0.5714 \cdot 0 + 0.4286 = 0.4286$

**(c)** Solving the equation in part (a) for $x$, we get

$$y = 0.5714x + 0.4286 \quad \Rightarrow \quad y - 0.4286 = 0.5714x \quad \Rightarrow$$
$$x = \frac{1}{0.5714}y - \frac{0.4286}{0.5714} \quad \Rightarrow \quad x = 1.75y - 0.75.$$

**54. (a)** A plot of two perpendicular lines, with their inclinations labeled, is below. $\ell_2$ has also been translated so we can see what the relative angles of inclination are:




From the graph, comparing $\ell_1$ with the dashed line, we see that $\theta_1 = \theta_2 + \frac{\pi}{2}$.

**(b)** We see that the slopes of the two lines are

$$m_2 = \frac{\triangle y}{\triangle x} = \tan\theta_2$$
$$m_1 = \frac{\triangle y}{\triangle x} = -\tan(\pi - \theta_1) = \tan\theta_1$$

**(c)** Now using the fact from (a) that $-\theta_2 = \frac{\pi}{2} - \theta_1$, we have

$$m_1 = \tan\theta_1 = \frac{1}{\cot\theta_1} = \frac{1}{\tan(\frac{\pi}{2} - \theta_1)} = \frac{1}{\tan(-\theta_2)} = \cot(-\theta_2) = -\cot\theta_2.$$

**(d)** $m_1 m_2 = -\cot\theta_2 \cdot \tan\theta_2 = -1.$

## ■ 1.2.3

**55.** Using the general equation $r^2 = (x - x_0)^2 + (y - y_0)^2$ with $r = 2$ and $(x_0, y_0) = (1, -2)$ gives

$$4 = (x-1)^2 + (y - (-2))^2 \quad \Rightarrow \quad 4 = (x-1)^2 + (y+2)^2.$$

**56.** Using the general equation $r^2 = (x - x_0)^2 + (y - y_0)^2$ with $r = 4$ and $(x_0, y_0) = (2, 3)$ gives

$$4^2 = (x-2)^2 + (y-3)^2 \quad \Rightarrow \quad 16 = (x-2)^2 + (y-3)^2.$$

**57.** **(a)** Using the general equation $r^2 = (x - x_0)^2 + (y - y_0)^2$ with $r = 4$ and $(x_0, y_0) = (2, 5)$ gives

$$4^2 = (x-2)^2 + (y-5)^2 \quad \Rightarrow \quad 16 = (x-2)^2 + (y-5)^2.$$

**(b)** When $x = 0$ the circle is on the $y$-axis. Setting $x = 0$ gives

$$16 = (0-2)^2 + (y-5)^2 \;\Rightarrow\; 16 = 4 + (y-5)^2 \;\Rightarrow\; 12 = (y-5)^2 \;\Rightarrow\; \pm 2\sqrt{3} = y - 5 \;\Rightarrow\; y = 5 \pm 2\sqrt{3}.$$

The points of intersection are $(0, 5 \pm 2\sqrt{3})$.

**(c)** When $y = 0$ the circle is on the $x$-axis. Setting $y = 0$ gives

$$16 = (x-2)^2 + (0-5)^2 \quad \Rightarrow \quad 16 = (x-2)^2 + 25 \quad \Rightarrow \quad -9 = (x-2)^2.$$

But the right hand side is the square of a real number, which cannot be negative, so this is impossible and the circle does not intersect the $x$-axis.

**58.** **(a)** If we start with a point at $(2, -5)$, and think of increasing the radius of a circle centered at that point, it will touch one axis when the radius becomes 2, and will touch the second axis when the radius becomes 5, so $2 \le r < 5$.

**(b)** As in part (a), it will intersect both axes when $r \ge 5$.

**59.** $(x+2)^2 + y^2 = 25$, so $(x - (-2))^2 + (y - 0)^2 = 5^2$. This is a circle with center $(-2, 0)$ and radius 5.

**60.** $(x+1)^2 + (y-3)^2 = 9$, so $(x - (-1))^2 + (y-3)^2 = 3^2$. This is a circle with center $(-1, 3)$ and radius 3.

**61.** Completing the square, we get

$$\begin{aligned} 0 &= x^2 + y^2 + 6x + 2y - 12 \\ 0 &= (x^2 + 6x + 9) + (y^2 + 2y + 1) - 12 - 9 - 1 \\ 22 &= (x+3)^2 + (y+1)^2 = (x-(-3))^2 + (y-(-1))^2, \end{aligned}$$

which is a circle with center $(-3, -1)$ and radius $\sqrt{22}$.

**62.** Completing the square, we get

$$\begin{aligned} 0 &= x^2 + y^2 + 2x - 4y + 1 \\ 0 &= (x^2 + 2x + 1) + (y^2 - 4y + 4) + 1 - 1 - 4 \\ 4 &= (x+1)^2 + (y-2)^2, \end{aligned}$$

which is a circle with center $(-1, 2)$ and radius 2.

## ■ 1.2.4

**63.** **(a)** Convert to radian measure: $65^\circ \cdot \frac{\pi}{180^\circ} = \frac{13}{36}\pi\,\text{rad}$.

**(b)** Convert to degree measure: $\frac{11}{12}\pi \cdot \frac{180^\circ}{\pi} = \frac{11 \cdot 180^\circ}{12} = 165^\circ$.

**64.** **(a)** $-15^\circ \cdot \frac{\pi}{180^\circ} = -\frac{\pi}{12}$ radians or $\frac{23}{12}\pi$ radians.

**(b)** $\frac{7\pi}{4} \cdot \frac{180^\circ}{\pi} = \frac{1260^\circ}{4} = 315^\circ$.

**65.** **(a)** $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

**(b)** $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

**(c)** $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$.

**66.** **(a)** $\sin\left(\frac{5}{4}\pi\right) = \frac{\sqrt{2}}{2}$.

**(b)** $\cos\left(-\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

**(c)** $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.

**67.** **(a)** From the definition of cosine as adjacent divided by hypotenuse, it is clear that reflecting $\alpha$ across the $y$-axis negates the cosine; that is, $\cos(\pi - \alpha) = -\cos\alpha$. Since $\cos\frac{\pi}{3} = \frac{1}{2}\sqrt{3}$, we get

$$\cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}\sqrt{3}.$$

Then, since $\cos(-\alpha) = \cos\alpha$, we also get

$$\cos\frac{4\pi}{3} = \cos\left(2\pi - \frac{2\pi}{3}\right) = \cos\left(-\frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}\sqrt{3}.$$

So the two solutions are $\alpha = \frac{2\pi}{3}, \frac{4\pi}{3}$.

**(b)** From the table in the text, we see that $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ is equal to $\frac{1}{\sqrt{3}}$ when $\alpha = \frac{\pi}{6}$. The other solution is $\alpha = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$, since for that angle both sine and cosine have the same numerical values as for $\frac{\pi}{6}$, but they are both negative.

**68.** **(a)** From the definition of cosine as adjacent divided by hypotenuse, it is clear that reflecting $\alpha$ across the $y$-axis negates the cosine; that is, $\cos(\pi - \alpha) = -\cos\alpha$. Since $\cos\frac{\pi}{4} = \frac{1}{2}\sqrt{2}$, we get

$$\cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{2}\sqrt{2}.$$

Then, since $\cos(-\alpha) = \cos\alpha$, we also get

$$\cos\frac{5\pi}{4} = \cos\left(2\pi - \frac{3\pi}{4}\right) = \cos\left(-\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} = -\frac{1}{2}\sqrt{2}.$$

So the two solutions are $\alpha = \frac{3\pi}{4}, \frac{5\pi}{4}$.

**(b)** Since $\sec\alpha = \frac{1}{\cos\alpha}$, we are looking for solutions to $\cos\alpha = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$. From the table in the text, one solution is $\alpha = \frac{\pi}{4}$; since $\cos(-\alpha) = \cos\alpha$, $\alpha = -\frac{\pi}{4}$ is another solution. Adding $2\pi$ to get it into the required range gives $\alpha = \frac{7\pi}{4}$. The two solutions are $\alpha = \frac{\pi}{4}, \frac{7\pi}{4}$.

**69.** Using the facts that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and that $\sec\theta = \frac{1}{\cos\theta}$, we get

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \text{(Divide both sides by} \cos^2\theta)$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \text{(Distribute)}$$

$$\tan^2\theta + 1 = \sec^2\theta \quad \text{(Make substitution)}$$

**70.** Using the facts that $\cot\theta = \frac{\cos\theta}{\sin\theta}$ and that $\csc\theta = \frac{1}{\sin\theta}$, we get

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \quad \text{(Divide both sides by} \sin^2\theta)$$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \quad \text{(Distribute)}$$

$$1 + \cot^2\theta = \csc^2\theta \quad \text{(Make substitution)}$$

**71.** Using the identity $\sin^2\theta + \cos^2\theta = 1$, the equation $\cos^2\theta - 2 = 2\sin\theta$ is the same as $(1 - \sin^2\theta) - 2 = 2\sin\theta$, or $\sin^2\theta + 2\sin\theta + 1 = (\sin\theta + 1)^2 = 0$. Thus $\sin\theta = -1$, so that $\theta = \frac{3\pi}{2}$.

**72.** Use the identity $1 + \tan^2 x = \sec^2 x$ to rewrite the equation:

$$\sec^2 x = \sqrt{3}\tan x + 1 \quad \Rightarrow \quad 1 + \tan^2 x = \sqrt{3}\tan x + 1 \quad \Rightarrow \quad \tan x(\tan x - \sqrt{3}) = 0.$$

The solutions to this equation are the solutions to $\tan x = 0$ or $\tan x = \sqrt{3}$ for $0 \le x < \pi$, which are

$$x = 0\ (\tan x = 0), \quad x = \frac{\pi}{3}\ (\tan x = \sqrt{3}).$$

## ■ 1.2.5

**73.** **(a)** $2^4 8^{-2/3} = 2^4 \left(2^3\right)^{-2/3} = 2^4 2^{3(-2/3)} = 2^4 2^{-2} = 2^2 = 4.$

**(b)** $\dfrac{3^3 3^{-1/2}}{3^{1/2}} = 3^3 3^{-1/2} 3^{-1/2} = 3^{3-1/2-1/2} = 3^2 = 9.$

**(c)** $\dfrac{5^k (25)^{k-1}}{5^{2-k}} = 5^k \left(5^2\right)^{k-1} 5^{-(2-k)} = 5^k 5^{2(k-1)} 5^{k-2} = 5^{k+(2k-2)+(k-2)} = 5^{4k-4}.$

**74.** **(a)** $(2^4 \cdot 2^{-3/2})^2 = (2^{8/2} \cdot 2^{-3/2})^2 = (2^{8/2-3/2})^2 = (2^{5/2})^2 = 2^5 = 32.$

**(b)**

$$\left(\frac{6^{5/2} \cdot 6^{2/3}}{6^{1/3}}\right)^3 = \left(\frac{6^{5/2+2/3}}{6^{1/3}}\right)^3 = \left(\frac{6^{15/6+4/6}}{6^{2/6}}\right)^3 = \left(\frac{6^{19/6}}{6^{2/6}}\right)^3$$
$$= \left(6^{(19/6)-(2/6)}\right)^3 = \left(6^{17/6}\right)^3 = 6^{(17/6)\cdot 3} = 6^{17/2}.$$

**(c)** $\left(\dfrac{3^{-2k+3}}{3^{4+k}}\right)^3 = \left(3^{-2k+3-4-k}\right)^3 = \left(3^{-3k-1}\right)^3 = 3^{(-3k-1)(3)} = 3^{-9k-3}.$

**75.** **(a)** $\log_4 x = -2$, so that $x = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$.

**(b)** $\log_{1/3} x = -3$, so that $x = \left(\frac{1}{3}\right)^{-3} = \frac{1}{\left(\frac{1}{3}\right)^3} = \frac{1}{\frac{1}{27}} = 27.$

**(c)** $\log_{10} x = -2$, so that $x = 10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

**76.** **(a)** $\log_2 x = -3$, so that $x = 2^{-3} = \frac{1}{8}$.

**(b)** $\log_{1/4} x = -\frac{1}{2}$, so that $x = \left(\frac{1}{4}\right)^{-1/2} = \frac{1}{(1/4)^{1/2}} = 4^{1/2} = 2.$

**(c)** $\log_3 x = 0$, so that $x = 3^0 = 1$.

**77.** **(a)** $\log_{1/2} 32 = x$, so that $32 = \left(\dfrac{1}{2}\right)^x = \dfrac{1^x}{2^x} = 2^{-x}$. Therefore $2^{-x} = 32 = 2^5$, so that $x = -5$.

**(b)** $\log_{1/3} 81 = x$, so that $81 = \left(\dfrac{1}{3}\right)^x = \dfrac{1^x}{3^x} = 3^{-x}$. Therefore $3^{-x} = 81 = 3^4$, so that $x = -4$.

**(c)** $\log_{10} 0.001 = x$, so that $10^x = 0.001 = 10^{-3}$. It follows that $x = -3$.

**78.** **(a)** $\log_3 81 = x$, so that $81 = 3^4 = 3^x$. It follows that $x = 4$.

**(b)** $\log_5 \frac{1}{25} = x$, so that $\frac{1}{25} = 5^{-2} = 5^x$. It follows that $x = -2$.

**(c)** $\log_{10} 1000 = x$, so that $1000 = 10^3 = 10^x$. It follows that $x = 3$.

**79.** **(a)** $-\ln\left(\frac{1}{3}\right) = \ln\left[\left(\frac{1}{3}\right)^{-1}\right] = \ln 3.$

**(b)** $\log_4(x^2 - 4) = \log_4[(x-2)(x+2)] = \log_4(x-2) + \log_4(x+2).$

**(c)** $\log_2 4^{3x-1} = (3x-1)\log_2(2^2) = (3x-1)(2 \cdot \log_2 2) = (3x-1)(2 \cdot 1) = 6x - 2.$

**80.** **(a)** $-\log_3 \frac{1}{4} = \log_3 \frac{1}{1/4} = \log_3 4.$

**(b)** $\log\left(\frac{x^3-x}{x-1}\right) = \log\left(\frac{x(x^2-1)}{x-1}\right) = \log\left(\frac{x(x-1)(x+1)}{x-1}\right) = \log(x(x+1)) = \log(x^2+x).$

**(c)** $\ln\left(e^{x-2}\right) = (x-2)\ln e = x - 2.$

**81.** **(a)** Since $e^{3x-1} = 2$, we have $(3x - 1)\ln e = \ln 2$, so that $3x - 1 = \ln 2$ and therefore $x = \frac{1+\ln 2}{3}$.

**(b)** Since $e^{-2x} = 10$, we have $-2x \ln e = \ln 10$, so that $-2x = \ln 10$ and therefore $x = -\frac{\ln 10}{2}$.

**(c)** Since $e^{x^2-1} = 10$, we have $(x^2 - 1)\ln e = \ln 10$, so that $x^2 - 1 = \ln 10$ and therefore $x = \pm\sqrt{1 + \ln 10}$.

**82.** **(a)** Since $625 = 5^4$, we have $5^x = 5^4$ so that $x = 4$.

**(b)** Since $256 = 4^4$, we have $4^{4x} = 4^4$, so that $4x = 4$ and thus $x = 1$.

**(c)** Since $0.0001 = 10^{-4}$, we have $10^{2x} = 10^{-4}$, so that $2x = -4$ and thus $x = -2$.

**83.** **(a)** Since $\ln(x - 3) = 5$, we have $e^{\ln(x-3)} = e^5$, so that $x - 3 = e^5$. Therefore $x = e^5 + 3$.

**(b)** Since $\ln(x + 2) + \ln(x - 2) = \ln((x + 2)(x - 2))\ln(x^2 - 4)$, we have $\ln(x^2 - 4) = 1$, so that $e^{\ln(x^2-4)} = e^1 = e$. Therefore $x^2 - 4 = e$ so that $x = \pm\sqrt{e + 4}$. However, $x = -\sqrt{e + 4}$ is invalid, since then $x - 2 < 0$ so that $\ln(x - 2)$ is undefined. The only solution is $x = \sqrt{e + 4}$.

**(c)** We have $\log_3 x^2 - \log_3 2x = \log_3 \frac{x^2}{2x} = \log_3 \frac{x}{2}$, so that $\log_3 \frac{x}{2} = 2$. Therefore $3^{\log_3(x/2)} = 3^2 = 9$, so that $\frac{x}{2} = 9$ and then $x = 18$.

**84.** **(a)** $\log_3(2x - 1) = 2 \quad\Rightarrow\quad 2x - 1 = 3^2 = 9$, so that $2x = 10$ and then $x = 5$.

**(b)** $\ln(2 - 3x) = 0 \quad\Rightarrow\quad 2 - 3x = e^0 = 1$, so that $3x = 1$ and then $x = \frac{1}{3}$.

**(c)** $\log(x) - \log(x + 1) = \log \frac{x}{x+1}$. Therefore $\log \frac{x}{x+1} = \log \frac{2}{3}$, so that $\frac{x}{x+1} = \frac{2}{3}$. It follows that $x = 2$.

## ■ 1.2.6

**85.** $(3 - 2i) - (-5 + 2i) = 3 - 2i + 5 - 2i = 8 - 4i$

**86.** $(6 + i) - 4i = 6 + i - 4i = 6 - 3i$

**87.** $(4 - 2i) + (9 + 4i) = 4 + 9 - 2i + 4i = 13 + 2i$

**88.** $(6 - 4i) + (2 + 5i) = 6 + 2 - 4i + 5i = 8 + i$

**89.** $4(5 + 3i) = 20 + 12i$

**90.** $(2 - 3i)(3 + 2i) = 6 + 4i - 9i - 6i^2 = 6 - 5i + 6 = 12 - 5i$ since $i^2 = -1$.

**91.** $(6 - i)(6 + i) = 36 - 6i + 6i - i^2 = 36 - (-1) = 37$.

**92.** $(-4-3i)(4+3i) = -4\cdot 4 - 4\cdot 3i - 3i\cdot 4 - 9i^2 = -16 - 12i - 12i - 9(-1) = -16 - 24i + 9 = -7 - 24i$.

**93.** Since conjugation replaces $i$ by $-i$, we get $\overline{z} = \overline{1 + 2i} = 1 - 2i$.

**94.** $z + u = (1 + 2i) + (2 - 3i) = 1 + 2i + 2 - 3i = 3 - i$.

**95.** $\overline{z + v} = \overline{(1 + 2i) + (1 - 5i)} = \overline{1 + 2i + 1 - 5i} = \overline{2 - 3i} = 2 + 3i$.

**96.** $\overline{v - w} = \overline{(1 - 5i) - (1 + i)} = \overline{1 - 5i - 1 - i} = \overline{-6i} = 6i$.

**97.** $\overline{vw} = \overline{(1 - 5i)(1 + i)} = \overline{6 - 4i} = 6 + 4i$.

**98.** $\overline{uz} = \overline{(2 - 3i)(1 + 2i)} = \overline{8 + i} = 8 - i$.

**99.** Using the quadratic formula with $a = 2$, $b = -3$, $c = 2$, we get

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = \frac{3 \pm \sqrt{-7}}{4} = \frac{3 \pm \sqrt{7}\, i}{4}.$$

**100.** Using the quadratic formula with $a = 1$, $b = 1$, $c = 1$, we get

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}.$$

**101.** Using the quadratic formula with $a = -1$, $b = 1$, $c = 2$, we get

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot (-1) \cdot 2}}{2 \cdot (-1)} = \frac{-1 \pm \sqrt{9}}{-2} = \frac{-1 \pm 3}{-2},$$

so that the two roots are $x_1 = \frac{-1+3}{-2} = -1$ and $x_2 = \frac{-1-3}{-2} = 2$.

**102.** Using the quadratic formula with $a = 1$, $b = 2$, $c = 3$, we get

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm 2\sqrt{-2}}{2} = -1 \pm \sqrt{2}\, i.$$

**103.** Using the quadratic formula with $a = 1$, $b = 1$, $c = 6$, we get

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 6}}{2 \cdot 1} = \frac{-1 \pm \sqrt{-23}}{2} = \frac{-1 \pm \sqrt{23}\, i}{2}.$$

**104.** Using the quadratic formula with $a = -2$, $b = 4$, $c = -3$, we get

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot (-2) \cdot (-3)}}{2 \cdot (-2)} = \frac{-4 \pm \sqrt{-8}}{4} = \frac{-4 \pm 2i\sqrt{2}}{4} = -1 \pm \frac{\sqrt{2}}{2} i.$$

**105.** The discriminant is

$$b^2 - 4ac = (-4)^2 - 4 \cdot 3 \cdot (-7) = 16 + 84 = 100,$$

so there are two real roots, which are

$$x_{1,2} = \frac{4 \pm \sqrt{100}}{6} = \frac{4 \pm 10}{6} = -1,\ \frac{7}{3}.$$

**106.** The discriminant is

$$b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot (-1) = 1 + 4 = 5,$$

so there are two real roots, which are

$$x_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

**107.** The discriminant is

$$b^2 - 4ac = (-1)^2 - 4 \cdot 3 \cdot (-4) = 49,$$

so there are two real roots, which are

$$x_{1,2} = \frac{1 \pm \sqrt{49}}{6} = \frac{1 \pm 7}{6} = -1,\ \frac{4}{3}.$$

**108.** The discriminant is

$$b^2 - 4ac = (-1)^2 - 4 \cdot 4 \cdot 1 = -15,$$

so there are two complex roots, which are complex conjugates. They are

$$x_{1,2} = \frac{1 \pm \sqrt{-15}}{8} = \frac{1 \pm \sqrt{15}\, i}{8}.$$

**109.** The discriminant is

$$b^2 - 4ac = (-5)^2 - 4 \cdot 3 \cdot 6 = -47,$$

so there are two complex roots, which are complex conjugates. They are

$$x_{1,2} = \frac{5 \pm \sqrt{-47}}{6} = \frac{5 \pm \sqrt{47}\, i}{6}.$$

**110.** The discriminant is

$$b^2 - 4ac = (-1)^2 - 4 \cdot (-3) \cdot (-4) = -47,$$

so there are two complex roots, which are complex conjugates. They are

$$x_{1,2} = \frac{1 \pm \sqrt{-47}}{-6} = \frac{-1 \pm \sqrt{47}\, i}{6}.$$

**111.** If $z = a + bi$, then $\overline{z} = a - bi$, so that

$$\begin{aligned} z + \overline{z} &= (a + bi) + (a - bi) = a + bi + a - bi = 2a \\ z - \overline{z} &= (a + bi) - (a - bi) = a + bi - a + bi = 2bi. \end{aligned}$$

**112.** If $z = a + bi$, then $\overline{z} = a - bi$. Since taking complex conjugates negates the imaginary portion of the number we also have

$$\overline{(\overline{z})} = \overline{a - bi} = a + bi = z.$$

**113.** If $z = a + bi$, then

$$\overline{(\overline{z})} = \overline{a - bi} = a + bi = z.$$

**114.** Let $z = a + bi$ and $w = c + di$. Then

$$\overline{z + w} = \overline{(a + bi) + (c + di)} = \overline{(a + c) + (b + d)i} = (a + c) - (b + d)i = (a - bi) + (c - di) = \overline{z} + \overline{w}.$$

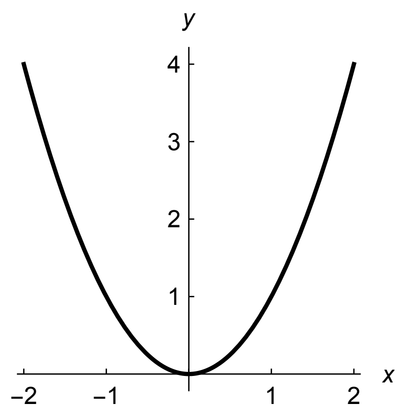
**115.** Let $z = a + bi$ and $w = c + di$. Then

$$\overline{zw} = \overline{(a + bi)(c + di)} = \overline{(ac - bd) + (ad + bc)i} = (ac - bd) - (ad + bc)i = (a - bi)(c - di) = \overline{z} \cdot \overline{w}.$$

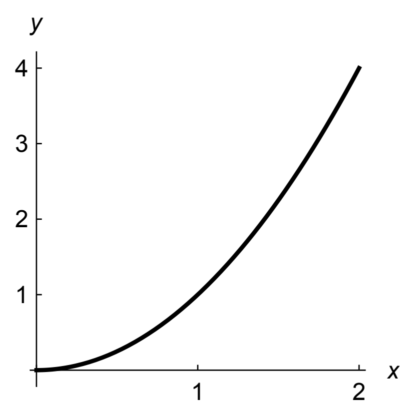
# 1.3 Elementary Functions

## ■ 1.3.1

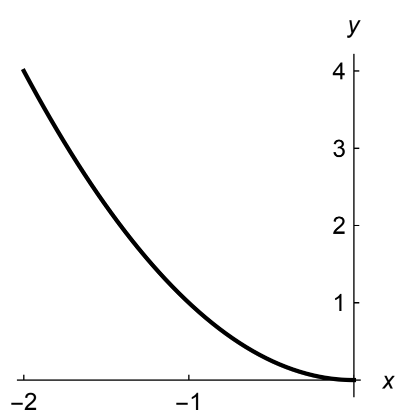
**1.** The range is $y \geq 0$.



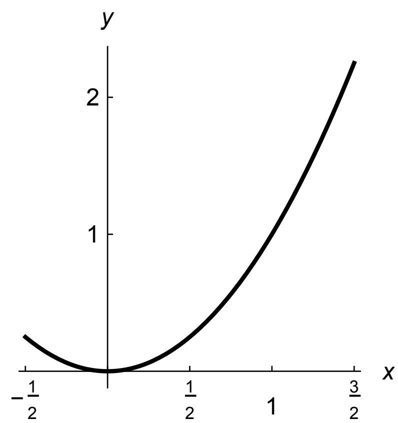

**2.** The range is $0 \leq y \leq 4$.




**3.** The range is $0 \leq y < 4$ (note that $y = 4$ is excluded since the domain does not include $x = -2$).



**4.** The range is $0 \le x < \left(\frac{3}{2}\right)^2$, or $0 \le x < \frac{9}{4}$.




**5.** **(a)** Factoring the numerator, we get

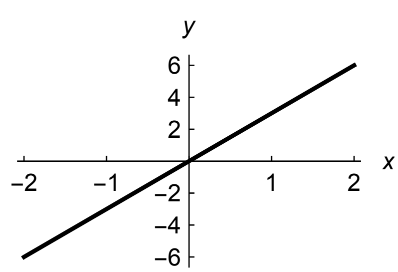
$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1 \text{ if } x \neq 1 \text{ (so that the function is defined).}$$

**(b)** No the functions are not equal; they have different domains. The function $f$ is not defined at the point $x = 1$ while $g$ is defined for all of $\mathbb{R}$.

**6.** **(a)** For $x \ge 2$ the quantity inside the absolute value is non-negative, so it is equal to $2(x-2)$ . For $x < 2$, the quantity inside the absolute value is negative, and so the absolute value will be equal to $-2(x-2) = 2(2-x)$. This shows the equality. (Note that the case $x = 2$ is given in both branches on the right. However, the value is in any case 0, so the two definitions agree.)

**(b)** Indeed $f$ and $g$ are the same function because they are defined on the same domain and $f(x) = g(x)$ for all $x$ in the domain. This can be seen as follows: for $0 \le x \le 2$, $g(x) = 2\,|x-2| = -2(x-2) = 4-2x$, and for $2 \le x \le 3$, $g(x) = 2\,|x-2| = 2(x-2) = 2x-4$.

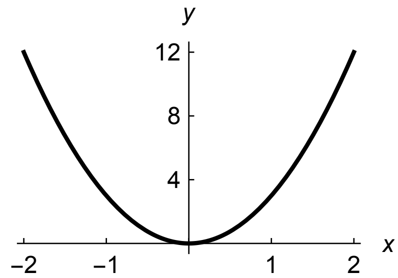
**7.** The function is odd, as the graph below indicates:




To see this algebraically, notice that

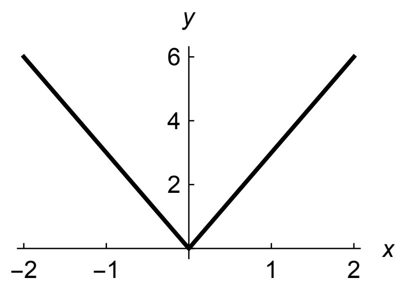
$$f(-x) = 3(-x) = -3x = -f(x).$$

**8.** The function is even, as the graph below indicates:



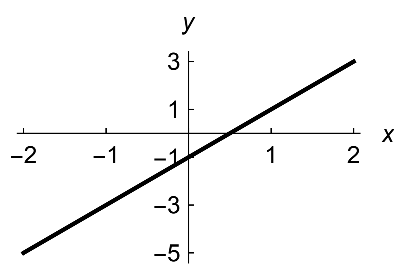
To see this algebraically, notice that $f(-x) = 3(-x)^2 = 3x^2 = f(x)$.

**9.** The function is even, as the graph below indicates:



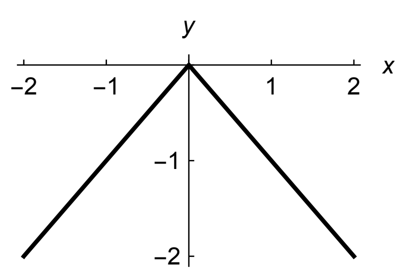
To see this algebraically, notice that $f(-x) = |3(-x)| = |-3x| = |3x| = f(x)$.

**10.** The function is neither even nor odd, as the graph below implies:

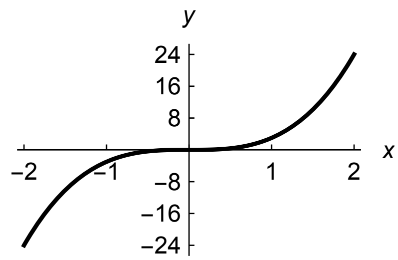


To see this algebraically, notice that for example $f(-1) = 2(-1) - 1 = -3$, while $f(1) = 2 \cdot 1 - 1 = 1$, so that $f(-1) \neq f(1)$ and $f(-1) \neq -f(1)$.

**11.** The function is even, as the graph below implies. To see this algebraically, notice that $f(-x) = -|-x| = -|x| = f(x)$.



**12.** The function is odd, as the graph below implies:




To see this algebraically, notice that

$$f(-x) = 3(-x)^3 = -3x^3 = -f(x).$$

**13.** **(a)** $(f \circ g)(x) = f(g(x)) = f(3+x) = (3+x)^2$.

**(b)** $(g \circ f)(x) = g(f(x)) = g(x^2) = 3 + x^2$.

**14.** **(a)** $(f \circ g)(x) = f(g(x)) = f(1-2x) = \sqrt{1-2x}$.

**(b)** $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = 1 - 2\sqrt{x}$.

**15.** **(a)** $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 1 - \sqrt{x}$. The domain of this function is $\{x \in \mathbb{R} : x \geq 0\}$, since otherwise $\sqrt{x}$ is undefined.

**(b)** $(g \circ f)(x) = g(f(x)) = g(1-x) = \sqrt{1-x}$. The domain of this function is $\{x \in \mathbb{R} : x \leq 1\}$, since otherwise $\sqrt{1-x}$ is undefined.

**16.** **(a)** $(f \circ g)(x) = f(g(x)) = f(2x^2) = \frac{1}{2x^2+1}$. The domain of this function is $\mathbb{R}$, since the denominator is never zero and is therefore defined everywhere, and since $g(x)$ is defined for all $x$.

**(b)** $(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x+1}\right) = 2 \cdot \left(\frac{1}{x+1}\right)^2 = \frac{2}{(x+1)^2}$. The domain of this function is $\{x \in \mathbb{R} : x \neq -1\}$, since the denominator vanishes for $x = -1$.

**17.** **(a)** $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \frac{1}{\sqrt{x}}$. The domain of this function is $\{x \in \mathbb{R} : x > 0\}$, since $\sqrt{x}$ is undefined for $x < 0$, and the fraction is undefined for $x = 0$.

**(b)** $(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \sqrt{\frac{1}{x}} = \frac{1}{\sqrt{x}}$. This is the same function as in part (a), so its domain is also $\{x \in \mathbb{R} : x > 0\}$.

**18.** $(f \circ g)(x) = f(g(x)) = f(\sqrt{x+1}) = (\sqrt{x+1})^4 = (x+1)^2$. In order for the composition to be defined, we must have $x \geq 3$ (since this is the domain of $g$), and also $\sqrt{x+1} \geq 3$ (since this is the domain of $f$). These two taken together give $x \geq 8$. Since $(x+1)^2$ is defined for $x \geq 8$, the domain is in fact $x \geq 8$.

**19.**

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$
$$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|\,.$$

In addition, both compositions have the same domain, namely $\{x \in \mathbb{R} : x \geq 0\}$. As a result, $|x| = x$ on that domain, so that $f \circ g = g \circ f$.

**20.** **(a)**

$$(f \circ g)(x) = f(g(x)) = f(x+1) = (x+1)^4$$
$$(g \circ f)(x) = g(f(x)) = g(x^4) = x^4 + 1.$$

Since these expressions are not equal, $(f \circ g)(x) \neq (g \circ f)(x)$.

**(b)**

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^4 = x^2$$
$$(g \circ f)(x) = g(f(x)) = g(x^4) = \sqrt{x^4} = x^2.$$

While these expressions are equal, note that $(f \circ g)(x)$ is defined only for $x \geq 0$, while $(g \circ f)(x)$ is defined for any $x \in \mathbb{R}$. Since the domains of $f \circ g$ and $g \circ f$ are unequal, the functions are unequal.

**(c)**

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^4 = \frac{1}{x^4}$$
$$(g \circ f)(x) = g(f(x)) = g(x^4) = \frac{1}{x^4}.$$

Since these expressions are equal, $(f \circ g)(x) = (g \circ f)(x)$.

**(d)**

$$(f \circ g)(x) = f(g(x)) = f(-x) = (-x)^4$$
$$(g \circ f)(x) = g(f(x)) = g(x^4) = -x^4.$$

Since $(-x)^4 = x^4 \neq -x^4$, these expressions are not equal, so $(f \circ g)(x) \neq (g \circ f)(x)$.

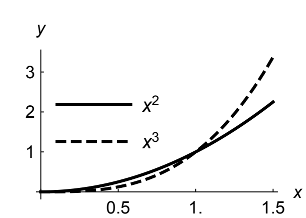
**(e)**

$$(f \circ g)(x) = f(g(x)) = f(|x|) = |x|^4$$
$$(g \circ f)(x) = g(f(x)) = g(x^4) = \left|x^4\right|.$$

Since $|x|^4 = x^4 = \left|x^4\right|$, these expressions are equal, so $(f \circ g)(x) = (g \circ f)(x)$.

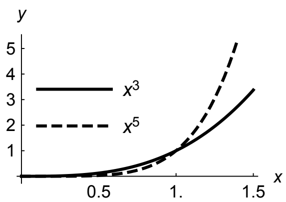
## ■ 1.3.2

**21.**



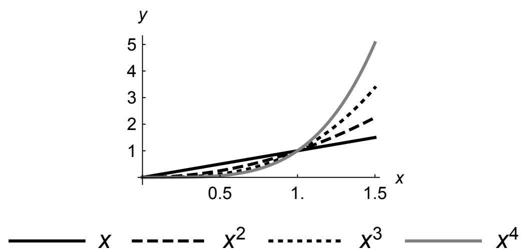

$f(x) > g(x)$ for $0 < x < 1$, and $f(x) < g(x)$ for $x > 1$.

**22.**



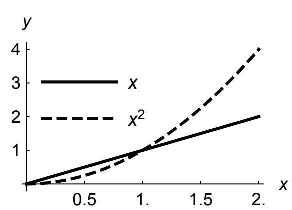

$f(x) > g(x)$ for $0 < x < 1$, and $f(x) < g(x)$ for $x > 1$.

**23.**



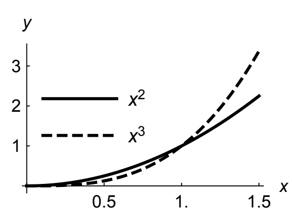

The curves intersect at $x = 0$ and again at $x = 1$.

**24.** **(a)**




**(b)** $f(x) \geq g(x)$ for $0 \leq x \leq 1$, and $f(x) \leq g(x)$ for $x = 0$ and $x \geq 1$.

**25.** **(a)**




**(b)** Multiply both sides of $x \leq 1$ by $x$ to get $x^2 \leq x$. Note that we do not have to worry about reversing the sign of the inequality since we also know $x \geq 0$.

**(c)** Multiply both sides of $x \geq 1$ by $x$ to get $x^2 \geq x$. Again note that we do not have to worry about reversing the sign of the inequality since we must have $x > 0$.

**26.** First suppose $0 \leq x \leq 1$. Multiplying by $x$, we have $0 \leq x^2 \leq x \leq 1$. Multiplying by $x$ again, we have $0 \leq x^3 \leq x^2 \leq x \leq 1$. Repeating this process $n - 1$ times, we get the chain of inequalities

$$0 \leq x^n \leq x^{n-1} \cdots \leq x^m \leq x^{m-1} \cdots \leq x \leq 1$$

Thus $x^n \leq x^m$ if $n \geq m$.

Next suppose $x \geq 1$. Multiplying by $x$, we have $x^2 \geq x \geq 1$. Multiplying by $x$ again, we have $x^3 \geq x^2 \geq x \geq 1$. Repeating this process $n-1$ times, we get the chain of inequalities

$$x^n \geq x^{n-1} \cdots \geq x^m \geq x^{m-1} \cdots \geq x \geq 1$$

Thus $x^n \geq x^m$ if $n \geq m$.

**27.** **(a)** Let $f(x) = y = x^4$; then $f(-x) = (-x)^4 = x^4 = f(x)$, so that $f$ is even.

**(b)** Let $f(x) = y = x^3$; then $f(-x) = (-x)^3 = -x^3 = -f(x)$, so that $f$ is odd.

**28.** **(a)** If $n$ is even, write $n = 2k$ where $k$ is an integer. Then letting $f(x) = y = x^n = x^{2k}$ we get

$$f(-x) = (-x)^n = (-x)^{2k} = \left((-x)^2\right)^k = \left(x^2\right)^k = x^{2k} = f(x),$$

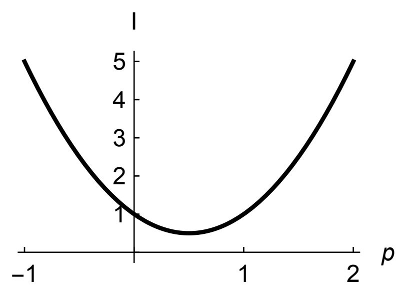
so that $f$ is even.

**(b)** If $n$ is odd, write $n = 2k+1$ where $k$ is an integer. Then letting $f(x) = y = x^n = x^{2k+1}$ we get

$$\begin{aligned} f(-x) = (-x)^n = (-x)^{2k+1} = (-x)^{2k} \cdot (-x) &= -x\left((-x)^2\right)^k \\ &= -x\left(x^2\right)^k = -x \cdot x^{2k} = -x^{2k+1} = -f(x), \end{aligned}$$

so that $f$ is odd.

**29.** **(a)** $I(p)$ is a probability, and all probabilities must be between 0 and 1. A graph of $I$ is below:




From the graph we see that $I(p) > 0$ everywhere. To see when $I(p) = 1$, solve $I(p) = 2p^2 - 2p + 1 = 1$, giving $2p(p-1) = 0$, so that $p = 0$ or $p = 1$. Thus for $0 \leq p \leq 1$, the value of $I(p) \in [0, 1]$; it follows that the domain of $I$ is $p \in [0, 1]$.

**(b)** See part (a).

**(c)** The minimum value of $I(p)$ occurs for $p = \frac{1}{2}$, where it is $2 \cdot \left(\frac{1}{2}\right)^2 - 2 \cdot \frac{1}{2} + 1 = \frac{1}{2}$, so the range $I([0, 1]) = \left[\frac{1}{2}, 1\right]$.

**30.** **(a)** The reaction rate $R$ is proportional to the amount of the two reactants that are left; that is, $R \propto AB$. Denote by $k$ the constant of proportionality. If $x$ is the concentration of $AB$, then we have used up $x$ units of both $A$ and $B$, so that

$$R(x) = k([A] - x)([B] - x) = k(3 - x)(1 - x)$$

since there were three units of $A$ and one unit of $B$ to start with.

**(b)** We cannot use up more of either component than there was to start with; since there is only one unit of $B$, and we use a nonnegative amount of $B$, the domain is $x \in [0, 1]$. Note that for $x \leq 1$ that $R(x) \leq 3k$, and $R(x) \geq 0$ everywhere, so that the range of $R(x)$ is $[0, 3k]$.

**(c)** Substituting $[A] = 6$ and $[B] = 3$ in the equation in part (a) gives $R(x) = k(6 - x)(3 - x)$. Here, by the same argument as in part (b), the domain of $R(x)$ is $x \in [0, 3]$, and its range is $[0, 18k]$, since the maximum reaction rate occurs when $x = 0$, at which time it is $18k$.

**31.** The beetle has a constant speed of 1 m/h. From

$$\text{rate} \cdot \text{time} = \text{distance},$$

we see that in one hour the beetle goes

$$(1\,\text{m/h}) \cdot (1\,\text{h}) = 1\,\text{m}.$$

Similarly,

$$\text{In two hours the beetle goes } (1\,\text{m/h}) \cdot (2\,\text{h}) = 2\,\text{m}$$
$$\text{In three hours the beetle goes } (1\,\text{m/h}) \cdot (3\,\text{h}) = 3\,\text{m}$$

Let the distance in meters be $d$ and the time in hours be $t$; then from the distance-rate-time equation we get a polynomial equation of degree 1 in $t$:

$$d = \text{rate} \cdot t \quad \Rightarrow \quad d = 1 \cdot t \quad \Rightarrow \quad d = t.$$

**32.** The fungal disease spreads radially at the speed of 3 ft/day. To calculate the area of the affected part of the orchard, we need to know its radius. Using the fact that speed · time = distance, we get for the radius

$$(3\,\text{ft/day})(2\,\text{day}) = 6\,\text{ft on day two.}$$

Thus, the area of infection on the $2^{\text{nd}}$ day is

$$A = \pi \cdot 6^2 \approx 113.1\,\text{ft}^2.$$

To find the radius on the fourth and eighth days, use the same idea:

$$(3\,\text{ft/day})(4\,\text{day}) = 12\,\text{ft on day four}$$
$$(3\,\text{ft/day})(8\,\text{day}) = 24\,\text{ft on day eight}$$

Therefore the area on these days is

$$A = \pi \cdot 12^2 = 144\pi \approx 452.4\,\text{ft}^2 \text{ on day four}$$
$$A = \pi \cdot 24^2 = 576\pi \approx 1809.6\,\text{ft}^2 \text{ on day eight}$$

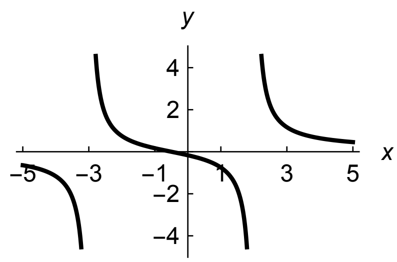
An equation that expresses the area as a function of time: we know that the circular area is $A = \pi r^2$ where $r = 3\,t$; substituting for $r$ gives a polynomial equation of degree 2 in $t$:

$$A = \pi r^2 = \pi(3t)^2 = 9\pi t^2,$$

## ■ 1.3.3

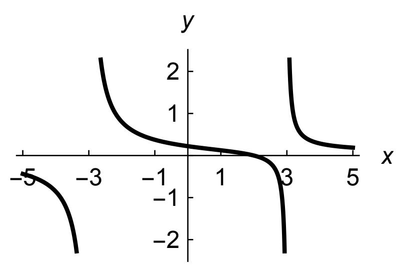
**33.** The domain is $(-\infty,\ 1) \cup (1,\ \infty)$, and the range is $(-\infty,\ 0) \cup (0,\ \infty)$

**34.** The domain consists of all real numbers such that the denominator does not vanish, so the domain is $\{x \in \mathbb{R} : x \neq 2, x \neq -3\}$. A plot of the function is below:

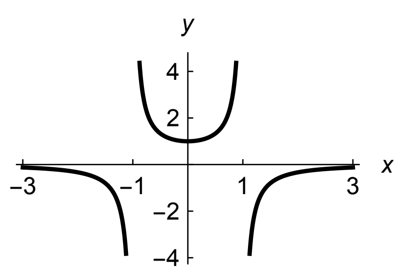



From this plot, it is clear that the range is $x \in \mathbb{R}$, since the function gets arbitrarily large positively as $x$ approaches $-3$ from the right, and arbitrarily large negatively as $x$ approaches 2 from the left.

**35.** The domain is $\{x : x \neq -3, 3\}$, or $(-\infty, -3) \cup (-3, +3) \cup (3, \infty)$, since the denominator vanishes at $x = \pm 3$. The range is $(-\infty, \infty)$, as seen from the plot below:

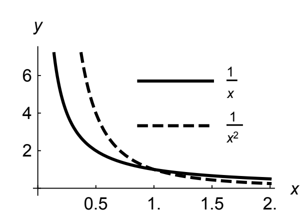



**36.** The domain consists of all real numbers such that the denominator does not vanish, so the domain is $\{x \in \mathbb{R} : x \neq \pm 1\}$. A plot of the function is below:




From this, we see that as $x$ gets arbitrarily large, $1-x^2$ is negative but also gets very large negatively, so that $\frac{1}{1-x^2}$ gets arbitrarily close to zero. As $x$ approaches 1 from the right, $1 - x^2$ is negative but gets very close to zero, so that $\frac{1}{1-x^2}$ gets arbitrarily large negatively. Between $-1$ and 1, the denominator is always positive, and it is always at most 1, so that $\frac{1}{1-x^2}$ is always at least 1. Thus the range is $x \in (-\infty, 0) \cup [1, \infty)$.

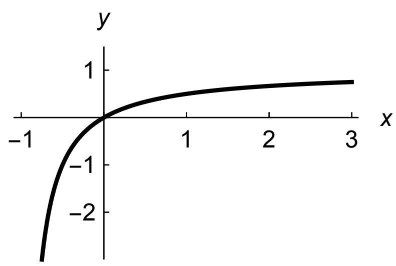
**37.** A plot of the function is below:



From the graphs, we see that the curves intersect at $x = 1$. For small values of $x$ $(x < 1)$, $\frac{1}{x^2} > \frac{1}{x}$, while for $x > 1$ the reverse holds: $\frac{1}{x} > \frac{1}{x^2}$.

**38.** If $n = m$, then the two curves are the same. So we assume that $m < n$, and we set $x^{-n} = x^{-m}$ to find their intersection. Simplifying, we have $x^{-n}x^n = x^{-m}x^n \Rightarrow 1 = x^{n-m} \Rightarrow 1 = x$. Thus the curves intersect at the point (1, 1). For values of $x < 1$, the function $x^{-n}$ is greater because $x^n < x^m$. When $x > 1$, the function $x^{-m}$ is greater because now $x^m < x^n$.

**39.** **(a)**

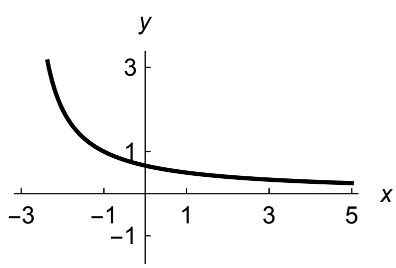



**(b)** From the graph, the range of $f(x)$ is $(-\infty, 1)$.

**(c)** Solving $f(x) = \frac{3}{4}$, we get $\frac{x}{x+1} = \frac{3}{4}$, so that $4x = 3x + 3$ and therefore $x = 3$.

**(d)** Since any horizontal line $y = a$ for $a \in (-\infty, 1)$ intersects the graph exactly once, we see that $f(x) = a$ has exactly one solution if $a$ is in the range of $f$.

**40.** **(a)**



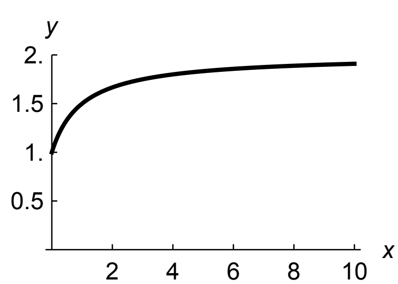

**(b)** From the graph, the range of $f(x)$ is $(0, \infty)$.

**(c)** Solving $f(x) = 1$, we get $\frac{2}{3+x} = 1$, so that $2 = 3 + x$ and therefore $x = -1$.

**(d)** Since any horizontal line $y = a$ for $a \in (0, \infty)$ intersects the graph exactly once, we see that $f(x) = a$ has exactly one solution if $a$ is in the range of $f$. Solving $f(x) = a$, we get

$$\frac{2}{3+x} = a \quad \Rightarrow \quad 2 = 3a + ax \quad \Rightarrow \quad ax = 2 - 3a \quad \Rightarrow \quad x = \frac{2-3a}{a}.$$

**41.** **(a)**



**(b)** From the graph, the range of $f(x)$ is $[1, 2)$ for $x \geq 0$.

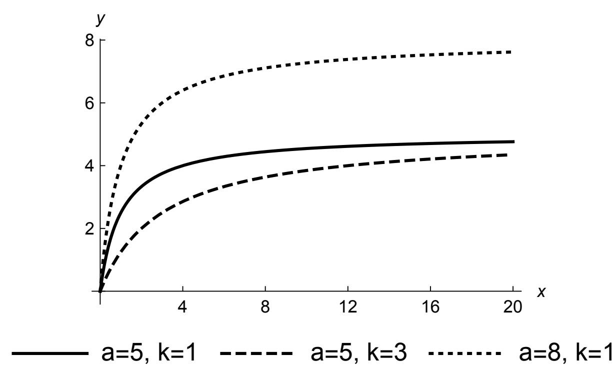
**(c)** Solving $f(x) = \frac{5}{4}$, we get

$$\frac{2x+1}{1+x} = \frac{5}{4} \quad \Rightarrow \quad 4(2x+1) = 5(1+x) \quad \Rightarrow \quad 8x+4 = 5x+5 \quad \Rightarrow \quad x = \frac{1}{3}.$$

**(d)** Since any horizontal line $y = a$ for $a \in [1, 2)$ intersects the graph exactly once, we see that $f(x) = a$ has exactly one solution if $a$ is in the range of $f$. Solving $f(x) = a$, we get

$$\frac{2x+1}{1+x} = a \quad \Rightarrow \quad 2x+1 = a + ax \quad \Rightarrow \quad (a-2)x = 1-a \quad \Rightarrow \quad x = \frac{1-a}{a-2}.$$

**42.** **(a)**




**(b)** Changing $a$ changes the asymptotic value of the function.

**(c)** Increasing $k$ will decrease the value of the function, although the asymptotic value stays the same.

**43.** The percentage increases are

$$\frac{r(0.2) - r(0.1)}{r(0.1)} = \frac{5/6 - 5/11}{5/11} = \frac{5}{6} \approx 0.8333 = 83.33\%$$
$$\frac{r(40) - r(20)}{r(20)} = \frac{200/41 - 100/21}{100/21} = \frac{1}{41} \approx 0.0244 = 2.44\%.$$

**44.** **(a)** **(i)** To solve for $k$, cross-multiply the given equation and simplify:

$$\frac{2(k+2)}{k+4} = \frac{4}{3} \quad \Rightarrow \quad 6k+12 = 4k+16 \quad \Rightarrow \quad 2k = 4 \quad \Rightarrow \quad k = 2.$$

**(ii)** Substituting $k = 2$ back into the equation for $r(2)$ gives

$$\frac{2a}{2+2} = 1.5 \quad \Rightarrow \quad 2a = 4 \cdot 1.5 = 6 \quad \Rightarrow \quad a = 3.$$

**(iii)** If we substitute into the equation for $r(4)$, we get

$$\frac{4a}{2+4} = 2 \quad \Rightarrow \quad 4a = 6 \cdot 2 = 12 \quad \Rightarrow \quad a = 3.$$

The two values for $a$ are the same.

**(b)** The relevant equations are

$$\begin{aligned} r(0):&\quad 0 = 0 \\ r(1):&\quad \frac{a}{k+1} = 1 \\ r(3):&\quad \frac{3a}{k+3} = 2.25 \end{aligned}$$

Dividing $r(3)$ by $r(1)$ gives

$$\frac{3a/(k+3)}{a/(k+1)} = \frac{2.25}{1} \quad \Rightarrow \quad \frac{3(k+1)}{k+3} = 2.25 \quad \Rightarrow \quad 3k+3 = 2.25k+6.75 \quad \Rightarrow \quad k = 5.$$

Substituting that value into the equation for $r(1)$ gives

$$\frac{a}{5+1} = 1 \quad \Rightarrow \quad a = 6.$$

The Monod growth function is

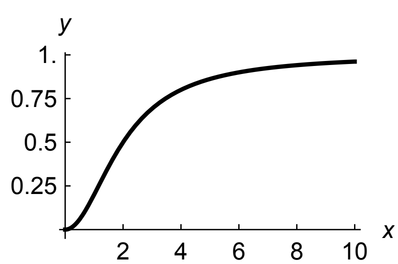
$$r(N) = \frac{6N}{5+N}.$$

**(c)** The relevant equations are

$$\begin{aligned} r(0):&\quad 0 = 0.5 \\ r(1):&\quad \frac{a}{k+1} = 1 \\ r(3):&\quad \frac{3a}{k+3} = 1.5 \end{aligned}$$

Since the first equation is $0 = 0.5$, which is false for any values of $a$ and $k$, there are no values that will fit the Monod growth function to this data.

**45. (a)**




**(b)** The range of $f(x)$ is $[0, 1)$

**(c)** As $x \to \infty$, $f(x)$ approaches 1 from below.

**46. (a) (i)** We get

$$\begin{aligned} r(0.1) &= \frac{0.1}{2+0.1} = \frac{0.1}{2.1} \approx 0.0476 \\ r(0.2) &= \frac{0.2}{2+0.2} = \frac{0.2}{2.2} \approx 0.0909. \end{aligned}$$

The increase from $N = 0.1$ to $N = 0.2$ is therefore $0.0909 - 0.0476 = 0.0433$.

**(ii)** We get

$$r(2) = \frac{2}{2+2} = \frac{2}{4} = 0.5$$
$$r(2.1) = \frac{2.1}{2+2.1} = \frac{2.1}{4.1} \approx 0.5122.$$

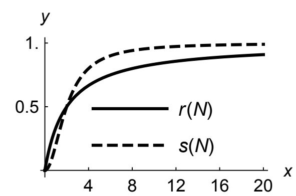
The increase from $N = 2$ to $N = 2.1$ is $0.5122 - 0.5 = 0.0122$, which is less than the increase from $N = 0.1$ to $N = 0.2$.

**(iii)** We get

$$r(4) = \frac{4}{2+4} = \frac{4}{6} \approx 0.6667$$
$$r(4.1) = \frac{4.1}{2+4.1} = \frac{4.1}{6.1} \approx 0.6721.$$

The increase from $N = 4$ to $N = 4.1$ is $0.6721 - 0.6667 \approx 0.0054$, which is less than the increase from $N = 2$ to $N = 2.1$.

**(b)** **(i)** The plots of $s(N)$ and $r(N)$ are below:




The low end of the range for both $r(N)$ and $s(N)$ is zero, and it appears that both graphs approach (but do not reach) the line $y = 1$, so that the range for both functions is $[0, 1)$.

**(ii)** Computing $s(N)$ gives

$$s(0) = 0$$
$$s(0.1) = \frac{0.1^2}{4+0.1^2} = \frac{0.001}{4.001} \approx 0.002494$$
$$s(2) = \frac{2^2}{4+2^2} = \frac{4}{8} = 0.5$$
$$s(2.1) = \frac{2.1^2}{4+2.1^2} = \frac{4.41}{8.41} \approx 0.52438$$
$$s(4) = \frac{4^2}{4+4^2} = \frac{16}{20} = 0.8$$
$$s(4.1) = \frac{4.1^2}{4+4.1^2} = \frac{16.81}{20.81} \approx 0.80778.$$

The increases for the three different increments of 0.1 are

$$s(0.1) - s(0) \approx 0.002494$$
$$s(2.1) - s(2) \approx 0.52438 - 0.5 = 0.02438$$
$$s(4.1) - s(4) \approx 0.80778 - 0.8 = 0.00778.$$

Thus $s$ increases more rapidly at $N = 2$ than it does at $N = 0$. The rate of increase at $N = 4$ is slower than that at $N = 2$ but still faster than that at $N = 0$.

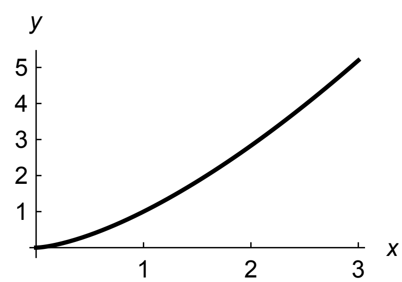
**(iii)** Note that

$$s(N+0.1)-s(N)=\frac{(N+0.1)^2}{4+(N+0.1)^2}-\frac{N^2}{4+N^2}=\frac{0.04+0.8N}{(4+(N+0.1)^2)(4+N^2)}.$$

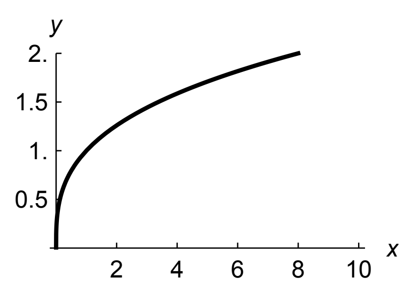
Now, if $N$ is large, increasing $N$ by one increases the numerator by only 0.8, but the denominator increases by quite a bit due to the factor of $N^4$. So the rate of increase for $N+1$ is less than the rate of increase for $N$, and thus the increase of $s(N)$ is decelerating for large $N$.

## ■ 1.3.4

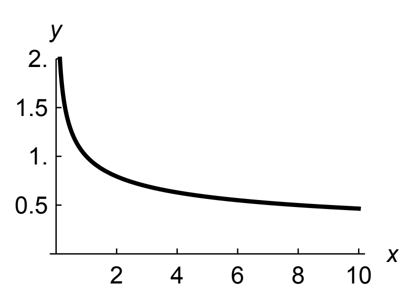
**47.**



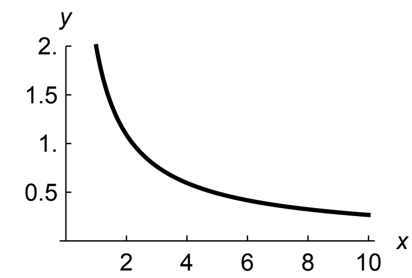

**48.**



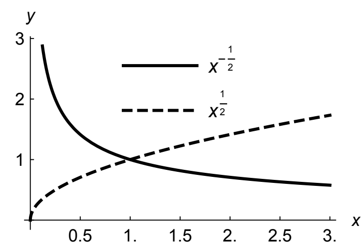

**49.**



**50.**



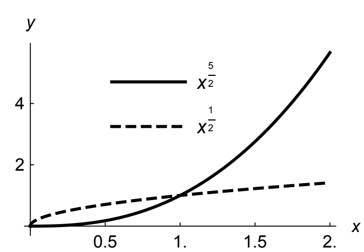

**51.** **(a)**




**(b)** Note that $\dfrac{x^{1/2}}{x^{-1/2}} = x^{1/2} \cdot x^{1/2} = x$, so if $x \leq 1$ we have $\dfrac{x^{1/2}}{x^{-1/2}} \leq 1$; multiplying both sides by the positive number $x^{-1/2}$ gives $x^{1/2} \leq x^{-1/2}$.

**(c)** As before, note that $\dfrac{x^{1/2}}{x^{-1/2}} = x^{1/2} \cdot x^{1/2} = x$, so if $x \geq 1$ we have $\dfrac{x^{1/2}}{x^{-1/2}} \geq 1$; multiplying both sides by the positive number $x^{-1/2}$ gives $x^{1/2} \geq x^{-1/2}$.

**52.** **(a)**




**(b)** If $0 \leq x \leq 1$, then multiplying both sides of the inequality $x \leq 1$ by $x$ gives $x^2 \leq x$; since $x \leq 1$ we have $x^2 \leq 1$. Multiply both sides of this last inequality by $x^{1/2}$, giving $x^{5/2} \leq x^{1/2}$.

**(c)** If $x \geq 1$, then multiplying both sides of the inequality $x \geq 1$ by $x$ gives $x^2 \geq x$; since $x \geq 1$ we have $x^2 \geq 1$. Multiply both sides of this last inequality by $x^{1/2}$, giving $x^{5/2} \geq x^{1/2}$.

**53.** **(a)** Converting 25.4 kg to 25 400 g and applying the formula gives

$$R(25\,400) = 1180 \cdot 25\,400^{-1/4} \approx 93.5 \text{ beats/min.}$$

**(b)** Converting 78 kg to 78 000 g and applying the formula gives

$$R(56\,000) = 1180 \cdot 78\,000^{-1/4} \approx 70.6 \text{ beats/min.}$$

**(c)** The relative errors are

$$\text{Molly:} \quad \left|\frac{40-93.8}{93.8}\right| = \frac{53.8}{93.8} \approx 0.573$$
$$\text{Professor R:} \quad \left|\frac{60-70.6}{70.6}\right| = \frac{10.6}{70.6} \approx 0.150.$$

**54.** **(a)** Applying the formula, we get $45 = k \cdot 275^{3.13} \approx 4.316 \times 10^7 \cdot k$, so that $k \approx \frac{45}{4.316\times 10^7} \approx 1.04 \times 10^{-6}$.

**(b)** Using the value of $k$ found in part (a), we get for the mass of the sabretooth tiger

$$1.04 \times 10^{-6} \cdot 350^{3.13} \approx 95.5\,\text{kg}.$$

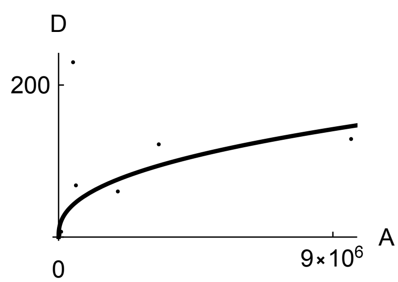
**(c)** **(i)** We have $7980 = c \cdot 200^{0.96} \approx 161.8c$, so that $c \approx \frac{7980}{161.8} \approx 49.3$.

**(ii)** Using the value of $c$ found in part (b), we get for the bite force of the sabretooth tiger

$$49.3 \cdot 95.7^{0.96} \approx 3931\,\text{N}.$$

**55.** **(a)** We get $D = 0.2 \cdot (9.857 \times 10^6)^{0.41} \approx 147$.

**(b)** Plots of the equation together with the points given are below:




**(i)** All of the countries but one lie reasonably close to the line.

**(ii)** The outlier is the country with over 200 languages, which from the table is Cameroon.

**(iii)** Applying the formula gives $D = 0.2 \cdot 1.7^{0.41} \approx 0.25$. This clearly is not close to the correct answer.

**56.** **(a)** Use the two data points to estimate $c$:

$$3.55 = c \cdot 4.8^{-1/6} \approx 0.77c \quad \Rightarrow \quad c \approx \frac{3.55}{0.77} \approx 4.61$$
$$1.70 = c \cdot 175.0^{-1/6} \approx 0.423c \quad \Rightarrow \quad c \approx \frac{1.70}{0.423} \approx 4.02.$$

We choose a value for $c$ midway between the two estimates, namely $c = 4.3$.

**(b)** We get $f = 4.3 \cdot 6.4^{-1/6} \approx 3.16/\text{s}$.

**(c)** We get $f = 4.3 \cdot 40^{-1/6} \approx 2.33/\text{s}$.

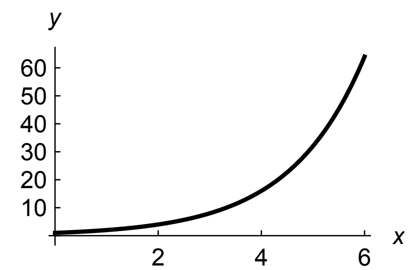
**(d)** If the lapping frequency is $1/\text{s}$, then we have $1 = 4.3 \cdot m^{-1/6}$; solving for $m$ gives $m = 4.3^6 \approx 6300\,\text{kg}$!

## 1.3.5

**57.** **(a)** $N(t) = 2^t$, so we get

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $N(t)$ | 1 | 2 | 4 | 8 | 16 |

**(b)**




**58.** **(a)** $N(0) = 20 \cdot 2^0 = 20$

**(b)** We have $e^{t \ln 2} = \left(e^{\ln 2}\right)^t = 2^t$; substituting in the formula for $N(t)$ gives $N(t) = 20e^{t \ln 2}$.

**(c)** We must solve the equation $1000 = 20 \cdot e^{t \ln 2}$ for $t$.

$$1000 = 20 \cdot e^{t \ln 2} \quad \Rightarrow \quad 50 = e^{t \ln 2} \quad \Rightarrow \quad \ln 50 = t \ln 2 \quad \Rightarrow \quad t = \frac{\ln 50}{\ln 2} \approx 5.64.$$

**59.** Since $\lambda = \frac{\ln 2}{T_h} = \frac{\ln 2}{5730}$, the amount left after 2000 years is

$$20 \exp\left[-\frac{\ln 2}{5730} 2000\right] = 20 \cdot 2^{-200/573} \approx 15.70\,\mu\text{g}.$$

**60.** **(a)** It will take one half-life to get to $20\,\mu$g, and another to get to $10\,\mu$g, so $5730 \cdot 2 = 11460$ years.

**(b)** It will take another half-life to go from $10\,\mu$g to $5\,\mu$g, so a total of $11460 + 5730 = 17190$ years.

**61.** The half-life is 7 days, so $\lambda = \frac{\ln 2}{7\,\text{days}} \approx 0.099\,/\text{day}$

**62.** If $w_0$ is the initial amount, then

$$\frac{30}{100} w_0 = w_0 e^{-4\lambda},$$

so that $-4\lambda = \ln \frac{3}{10}$ and $\lambda = \frac{1}{4} \ln \frac{10}{3}$. Thus the half-life is

$$T_h = \frac{\ln 2}{\lambda} = \frac{4 \ln 2}{\ln \frac{10}{3}} \approx 2.302 \text{ days}.$$

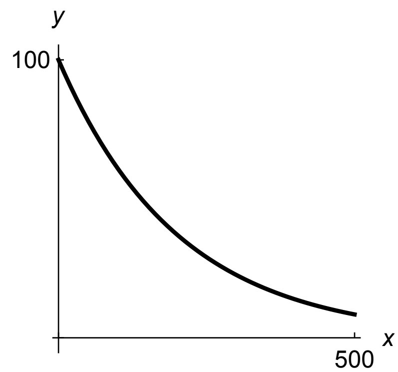
**63.** **(a)** Since the half-life is 140 days, the formula for the amount remaining after $t$ days from an initial amount of 100 micrograms is

$$W(t) = 100e^{-t \ln 2/140}.$$

**(b)** We want $W(t) = 10$; solving $10 = 100e^{-t \ln 2/140}$ gives $-t\frac{\ln 2}{140} = \ln \frac{1}{10}$, so that

$$t = \frac{140 \ln 10}{\ln 2} \approx 465 \text{ days}.$$

**(c)**




**64.** Since 35% of $W_0$ equals $W_0e^{-\lambda t}$, we get

$$0.35W_0 = W_0e^{-\lambda t} \quad \Rightarrow \quad 0.35 = e^{-\lambda t} \quad \Rightarrow \quad -\lambda t = \ln 0.35,$$

where $\lambda = \frac{\ln 2}{5730}$. This gives

$$t = -\frac{\ln 0.35}{\lambda} = -\frac{5730 \ln 0.35}{\ln 2} \approx 8678\,\text{years.}$$

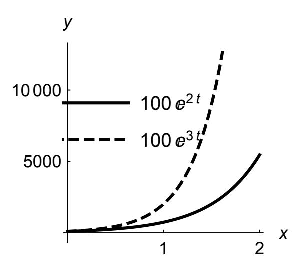
The wood was cut about 8678 years ago.

**65.** $\frac{W(t)}{W(0)} = \exp[-\frac{\ln 2}{5730} \cdot 10\,000] \approx 29.8\%$

**66.** The total percentage of $Ar^{40}$ (based on their $K^{40}$ content) is $\frac{79}{470}\%$, so computing the age of the rock we have

$$\frac{79}{470} = e^{-\lambda t} \quad \Rightarrow \quad \lambda t = \ln\frac{470}{79} \quad \Rightarrow \quad t = \frac{1}{5.335 \times 10^{-10}} \ln\frac{470}{79} \approx 3\,\text{billion years}$$

**67. (a)**




The graph for $r = 3$ grows faster.

**(b) (i)** Using the value for $t = 0$ gives $100 = N_0e^{0\cdot t} = N_0$, so that $N_0 = 100$. Using the value for $t = 2$, we now get

$$300 = 100e^{2r},$$

so that $2r = \ln 3$ and $r = \frac{\ln 3}{2}$. The equation is $N(t) = 100e^{t \ln 3/2}$.

**(ii)** $N(t) = 1000$ when $1000 = 100e^{t \ln 3/2}$, or $t\frac{\ln 3}{2} = \ln 10$. This occurs when $t = \frac{2 \ln 10}{\ln 3} \approx 4.19$.

**(iii)** $N(t) = 10\,000$ when $10\,000 = 100e^{t \ln 3/2}$, or $t\frac{\ln 3}{2} = \ln 100$. This occurs when $t = \frac{2 \ln 100}{\ln 3} \approx 8.38$.

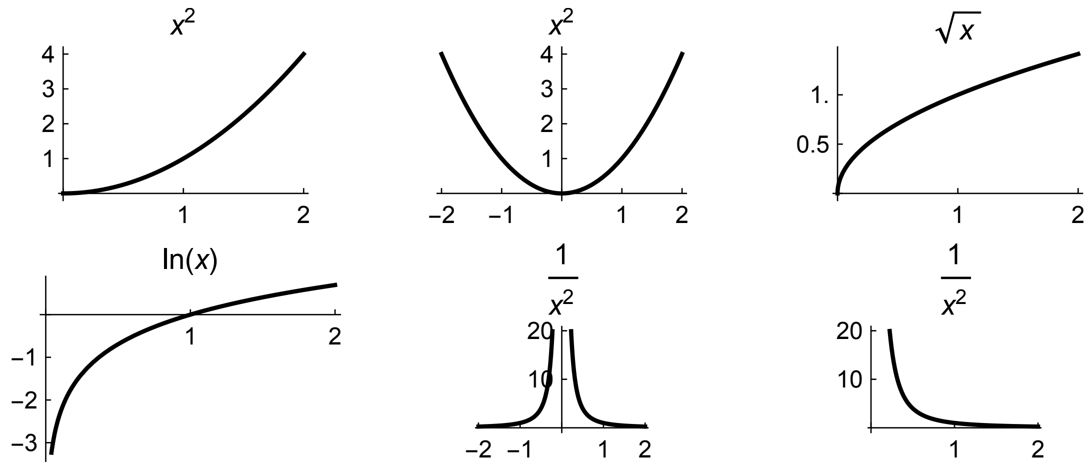
**68. (a) (i)** Since $N(5) = 10$, we get $10 = c \cdot 10^{-5}$, so that $c = 10 \cdot 10^5 = 10^6$.

**(ii)** $N(2) = 10^6 \cdot 10^{-2} = 10^4$, so there will be about 10 000 such earthquakes.

**(iii)** $N(6) = 10^6 \cdot 10^{-6} = 1$, so we expect about one such earthquake.

## 1.3.6

**69.** Apply the horizontal line test to the graphs below:



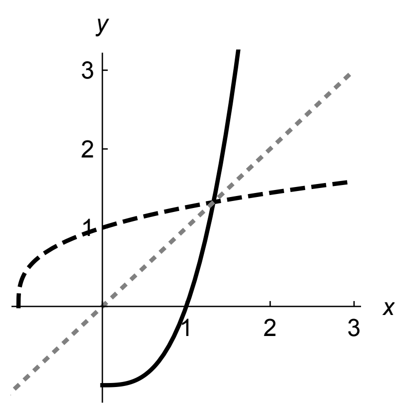

The answers are: (a) yes (b) no (c) yes (d) yes (e) no (f) yes

**70.** **(a)** $f$ is strictly increasing, so it is one to one. To find its inverse, we start with $y = x^3 - 1$ and solve for $x$:

$$y = x^3 - 1 \quad \Rightarrow \quad x^3 = y + 1 \quad \Rightarrow \quad x = \sqrt[3]{y+1}.$$

The inverse function is $f^{-1}(x) = \sqrt[3]{x+1}$, with domain $\mathbb{R}$.

**(b)** In the graph below, $f(x)$ is the solid line and $f^{-1}(x)$ is the dashed line. The dotted line, $y = x$, is shown to make it clear that $f^{-1}(x)$ is the reflection of $f(x)$ across this line.

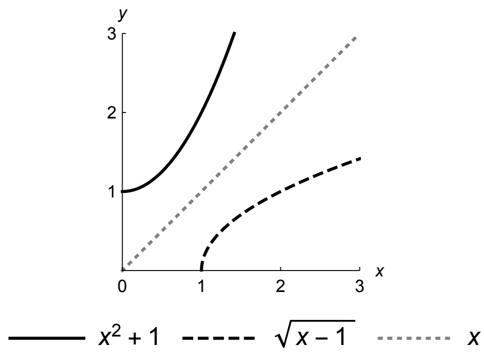



**71.** **(a)** Suppose that $f(a) = f(b)$ for $a, b \geq 0$; then $a^2 + 1 = b^2 + 1$, so that $a^2 = b^2$. It follows that $a = \pm b$; since $a, b \geq 0$ we have $a = b$ so that $f$ is one to one. To find the inverse, solve $y = x^2 + 1$ for $x$, giving $x = \pm\sqrt{y-1}$. Since $x$ must be nonnegative, we choose the positive square root, so the inverse is

$$f^{-1}(x) = \sqrt{x-1}.$$

The domain of this function is $x \geq 1$, since otherwise $\sqrt{x-1}$ is not defined.

**(b)** In the graph below, $f(x)$ is the solid line and $f^{-1}(x)$ is the dashed line. The dotted line, $y = x$, is shown to make it clear that $f^{-1}(x)$ is the reflection of $f(x)$ across this line.




**72.** **(a)** Suppose that $f(a) = f(b)$ with $a \neq b$; then $a^2 - a = b^2 - b$, so that $a^2 - b^2 = a - b$. Therefore $(a - b)(a + b) = a - b$. Since $a \neq b$ we can divide through by $a - b$, giving $a + b = 1$. But since both $a$ and $b$ are at least $\frac{1}{2}$ and are unequal, this is impossible. Thus if $f(a) = f(b)$, then $a = b$, so that $f$ is one to one for $x \geq \frac{1}{2}$. To compute the inverse, we solve $y = x^2 - x$ for $x$, giving

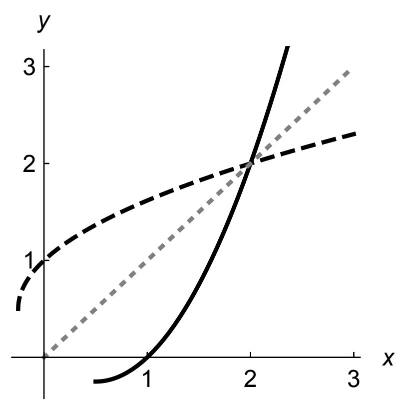
$$x = \frac{1}{2}(1 \pm \sqrt{1 + 4y}).$$

Since $f(2) = 2$, it follows that we must get equality when $x = y = 2$, so the plus sign is the proper choice. Therefore

$$f^{-1}(x) = \frac{1}{2}(1 + \sqrt{1 + 4x}).$$

The domain of this function is $x \geq -\frac{1}{4}$, since otherwise $\sqrt{1 + 4x}$ is not defined.

**(b)** In the graph below, $f(x)$ is the solid line and $f^{-1}(x)$ is the dashed line. The dotted line, $y = x$, is shown to make it clear that $f^{-1}(x)$ is the reflection of $f(x)$ across this line.

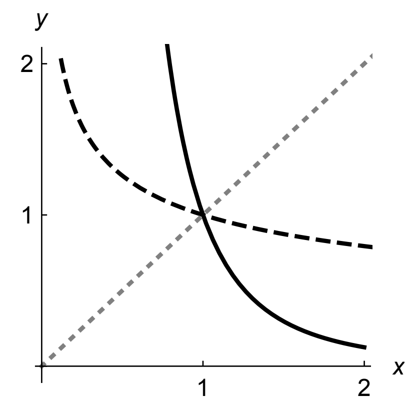



**73.** **(a)** Suppose that $f(a) = f(b)$. Then $a^{-3} = b^{-3}$; taking the cube root of both sides gives $a^{-1} = b^{-1}$, so that $a = b$. Therefore $f$ is one to one on $x > 0$. To find its inverse, we solve $y = x^{-3}$ for $x$: take the cube root of both sides to get $y^{1/3} = x^{-1}$, so that $x = y^{-1/3}$. The inverse function is

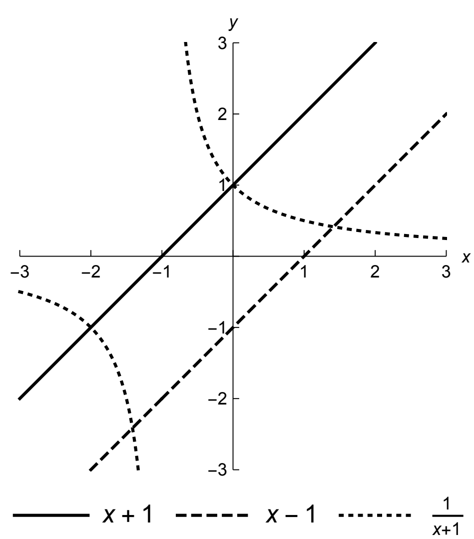
$$f^{-1}(x) = x^{-1/3}.$$

The domain of the function $f^{-1}(x)$ is $\{x \in \mathbb{R} \mid x > 0\}$.

**(b)** In the graph below, $f(x)$ is the solid line and $f^{-1}(x)$ is the dashed line. The dotted line, $y = x$, is shown to make it clear that $f^{-1}(x)$ is the reflection of $f(x)$ across this line.

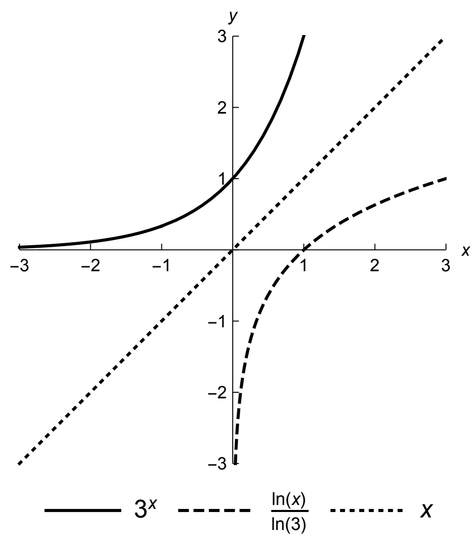



**74.** If $f(x) = y = x + 1$, then $x = y - 1$, so that $f^{-1}(x) = x - 1$. $[f(x)]^{-1} = \frac{1}{x+1}$. Graphing all three functions together gives

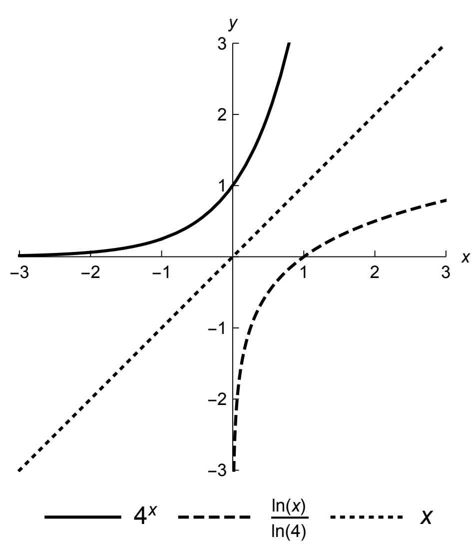


## ■ 1.3.7

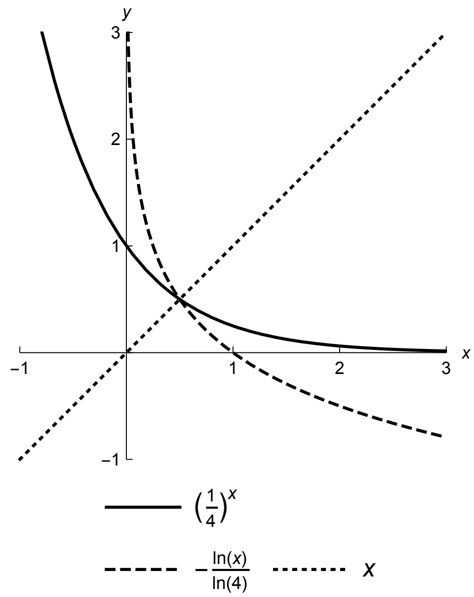
**75.** Solving $y = 3^x$ for $x$, we get $\ln y = x \ln 3$, so that $x = \frac{\ln y}{\ln 3}$. Therefore $f^{-1}(x) = \frac{\ln x}{\ln 3} = \log_3 x$. The domain of this function is $x > 0$, since otherwise $\ln x$ is undefined. A plot of $f$ and $f^{-1}$ together is below:



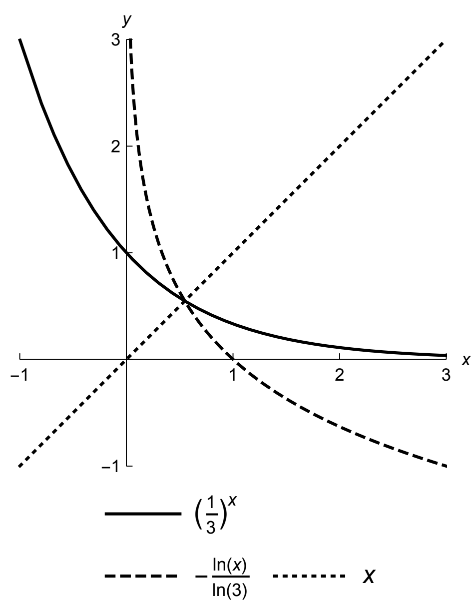

**76.** Solving $y = 4^x$ for $x$, we get $\ln y = x \ln 4$, so that $x = \frac{\ln y}{\ln 4}$. Therefore $f^{-1}(x) = \frac{\ln x}{\ln 4} = \log_4 x$. The domain of this function is $x > 0$, since otherwise $\ln x$ is undefined. A plot of $f$ and $f^{-1}$ together is below:



**77.** Solving $y = \left(\frac{1}{4}\right)^x$ for $x$, we get $\ln y = x \ln \frac{1}{4} = -x \ln 4$, so that $x = -\frac{\ln y}{\ln 4}$. Therefore $f^{-1}(x) = -\frac{\ln x}{\ln 4} = -\log_4 x$. The domain of this function is $x > 0$, since otherwise $\ln x$ is undefined. A plot of $f$ and $f^{-1}$ together is below:




**78.** Solving $y = \left(\frac{1}{3}\right)^x$ for $x$, we get $\ln y = x \ln \frac{1}{3} = -x \ln 3$, so that $x = -\frac{\ln y}{\ln 3}$. Therefore $f^{-1}(x) = -\frac{\ln x}{\ln 3} = -\log_3 x$. The domain of this function is $x > 0$, since otherwise $\ln x$ is undefined. A plot of $f$ and $f^{-1}$ together is below:




**79.** **(a)** $2^{5 \log_2 x} = 2^{\log_2 x^5} = x^5$

**(b)** $4^{3 \log_4 x} = 4^{\log_4 x^3} = x^3$

**(c)** $5^{5\log_{1/5}x} = 5^{-\log_{1/5}x^{-5}} = (\frac{1}{5})^{\log_{1/5}x^{-5}} = x^{-5}$

**(d)** $4^{3\log_2 x} = 2^{6\log_2 x} = 2^{\log_2 x^6} = x^6$

**(e)** $2^{3\log_{1/2}x} = 2^{-\log_{1/2}x^{-3}} = (\frac{1}{2})^{\log_{1/2}x^{-3}} = x^{-3}$

**(f)** $8^{\log_{1/2}x} = 2^{3\log_{1/2}x} = 2^{-\log_{1/2}x^{-3}} = (\frac{1}{2})^{\log_{1/2}x^{-3}} = x^{-3}$

**80.** **(a)** $\log_4 16^x = x\log_4 16 = x\log_4 4^2 = 2x\log_4 4 = 2x$

**(b)** $\log_2 16^x = \log_2(2^4)^x = \log_2 2^{(4x)} = 4x\log_2 2 = 4x$

**(c)** $\log_3 27^x = \log_3 3^{3x} = 3x\log_3 3 = 3x$

**(d)** $\log_{1/2} 4^x = \log_{1/2}(1/2)^{-2x} = -2x\log_{1/2} 1/2 = -2x$

**(e)** $\log_{1/2} 8^{-x} = \log_{1/2}(1/2)^{(-3)(-x)} = 3x\log_{1/2} 1/2 = 3x$

**(f)** $\log_3 9^{-x} = \log_3 3^{-2x} = -2x\log_3 3 = -2x$

**81.** **(a)** $\ln x^2 + \ln x^{-1} = \ln\left(x^2 \cdot x^{-1}\right) = \ln x$

**(b)** $\ln x^4 - \frac{1}{3}\ln x^{-2} = 4\ln x + \frac{2}{3}\ln x = \frac{14}{3}\ln x$

**(c)** $\ln(x^2-1) - \ln(x+1) = \ln\frac{x^2-1}{x+1} = \ln\frac{(x+1)(x-1)}{x+1} = \ln(x-1)$

**(d)** $\frac{1}{2}\ln x^{-1} + \ln x^{-3} = -\frac{1}{2}\ln x - 3\ln x = -\frac{7}{2}\ln x$

**82.** **(a)** $e^{3\ln x} = (e^{\ln x})^3 = x^3$

**(b)** $e^{-\ln(x^2+1)} = \frac{1}{e^{\ln(x^2+1)}} = \frac{1}{x^2+1}$

**(c)** $e^{-2\ln(1/x)} = (e^{\ln(1/x)})^{-2} = (\frac{1}{x})^{-2} = x^2$

**(d)** $e^{-2\ln x} = (e^{\ln x})^{-2} = x^{-2} = \frac{1}{x^2}$

**83.** **(a)** $e^{x\ln 3}$

**(b)** $e^{(x^2-1)\ln 4}$

**(c)** $e^{-(x+1)\ln 2}$

**(d)** $e^{(-4x+1)\ln 3}$

**84.** **(a)** $\log_2(x^2-1) = \frac{\ln(x^2-1)}{\ln 2}$

**(b)** $\log_3(5x+1) = \frac{\ln(5x+1)}{\ln 3}$

**(c)** $\log(x+2) = \frac{\ln(x+2)}{\ln 10}$

**(d)** $\log_2(2x^2-1) = \frac{\ln(2x^2-1)}{\ln 2}$

**85.** We have $y = (1/2)^x = e^{x\ln(1/2)} = e^{-x\ln 2}$, and so $\mu = \ln 2$.

**86.** We have $a^x = e^{\ln a^x} = e^{x\ln a}$. Now if $a > 1$ then $\ln a$ is positive, so we can take $\mu = \ln a > 0$.

**87.** **(a)** We get $m = \log 100 - 2.48 + 2.76\log 100 = 2 - 2.48 + 5.52 = 5.04$.

**(b)** Since $1\,\text{mm} = 10^3\,\mu\text{m}$, we get $m = \log 10^3 - 2.48 + 2.76\log 10 = 3 - 2.48 + 2.76 = 3.28$.

**(c)** **(i)** Plugging in the values gives

$$7.2 = \log A - 2.48 + 2.76\log 10 = \log A + 0.28 \quad\Rightarrow\quad \log A = 6.92 \quad\Rightarrow\quad A \approx 8.318 \times 10^6\,\mu\text{m}.$$

**(ii)** Plugging in the values gives

$$7.2 = \log A - 2.48 + 2.76 \log 100 = \log A + 3.04 \quad \Rightarrow \quad \log A = 4.16 \quad \Rightarrow \quad A \approx 14\,454\,\mu\text{m}.$$

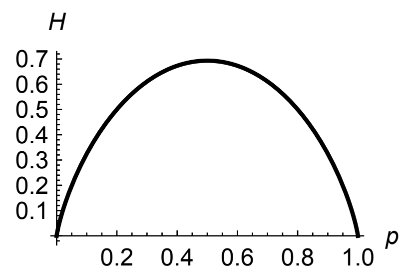
**(d)** Let $A_0$ be the amplitude of shaking at a magnitude of $m$, and $A_1$ the amplitude at a magnitude of $m+1$. Subtracting the equations for $m$ and $m+1$ gives

$$(m+1) - m = \log A_1 - 2.48 + 2.76 \log D - (\log A_0 - 2.48 + 2.76 \log D) = \log A_1 - \log A_0.$$

Therefore $\log A_1 - \log A_0 = \log \frac{A_1}{A_0} = 1$, so the amplitude of shaking is ten times larger when the magnitude increases by one.

**88. (a)** In order for $H(p)$ to be defined, both $\ln p$ and $\ln(1-p)$ must be defined, so we must have $p > 0$ and $1 - p > 0$. Therefore the domain is $0 < p < 1$.

**(b)** A plot of $H(p)$ is below:




It appears visually that $H(p)$ is at a maximum when $p = \frac{1}{2}$.

**(c)** Computing $H(1-p)$ gives

$$\begin{aligned} H(1-p) &= -(1-p)\ln(1-p) - (1-(1-p))\ln(1-(1-p)) \\ &= -(1-p)\ln(1-p) - p\ln p \\ &= -p\ln p - (1-p)\ln(1-p) = H(p). \end{aligned}$$

If we swap the labels on the two types of organisms, then a proportion $1-p$ are of type 1 and a proportion p are of type 2, so the diversity index is $H(1-p) = H(p)$. Thus the labeling does not affect the index.

**(d)** **(i)** $H(0)$ does not make sense because 0 is not in the domain of $H$ ($\ln 0$ is undefined).

**(ii)** Computing gives $H(0.001) \approx 0.0079$.

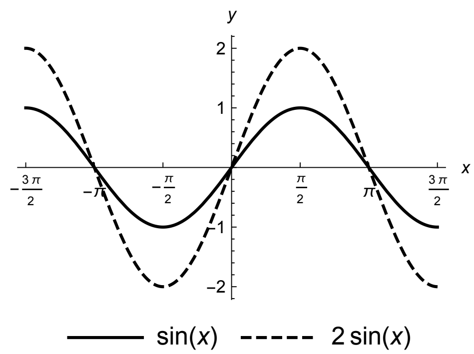
**(iii)** The table of values looks like this:

| $p$ | 0.001 | $10^{-4}$ | $10^{-5}$ | $10^{-6}$ |
|---|---|---|---|---|
| $H(p)$ | 0.0079 | 0.0010 | 0.00013 | 0.000015 |

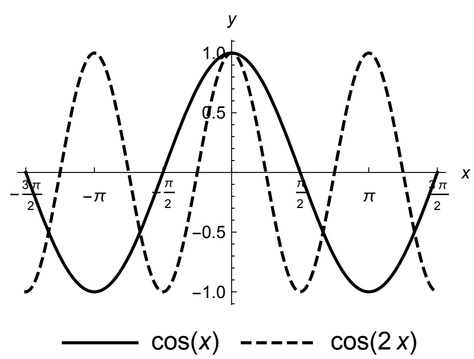
**(e)** Since the range of $H$ in the given equation is $0 < p < 1$, if we also define $H(0) = H(1) = 0$, the range of the new function is $0 \leq p \leq 1$.

## 1.3.8

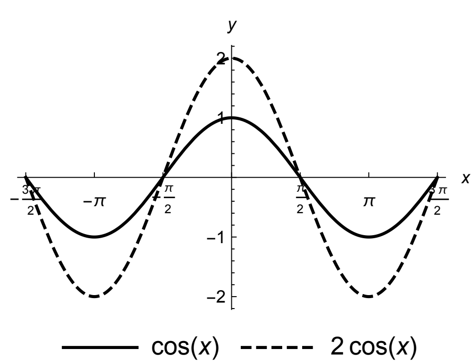
**89.** Same period, but $2\sin x$ has twice the amplitude of $\sin x$:



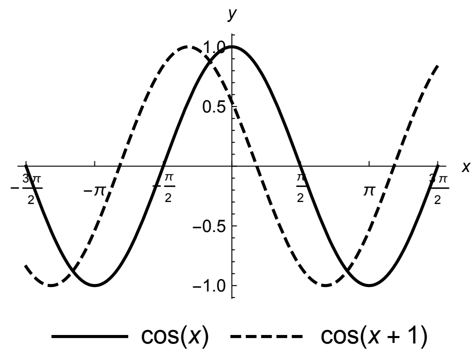

**90.** Same amplitude, but $\cos 2x$ has half the period of $\cos x$.



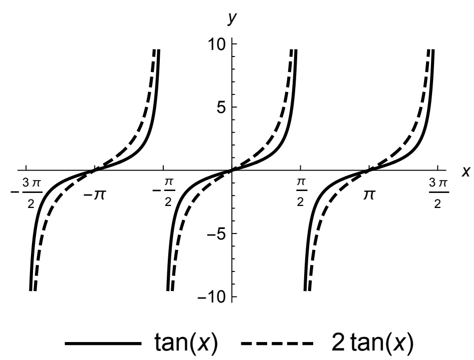

**91.** Same period, but $2\cos x$ has twice the amplitude of $\cos x$:



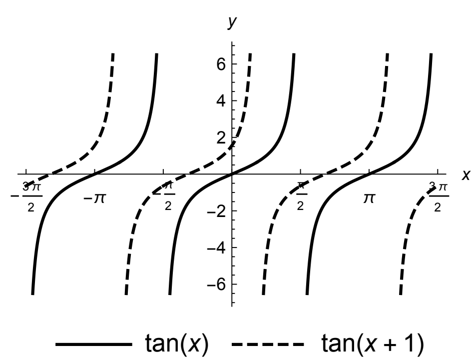
**92.** $\cos(x+1)$ is $\cos x$ shifted left by 1:




**93.** Same period, but $2\tan x$ has twice the amplitude of $\tan x$:




**94.** $\tan(x+1)$ is $\tan x$ shifted left by 1:




**95.** The amplitude is 2, and the period is twice the period of $\sin x$, so it is $4\pi$.

**96.** The amplitude is 3, and the period is $\frac{1}{4}$ times the period of $\cos x$, so it is $\frac{\pi}{2}$.

**97.** The amplitude is 4 (the minus sign does not affect the amplitude; it simply denotes a reflection of the graph through the $x$-axis). The period is $\frac{1}{\pi}$ times the period of $\sin x$, so the period is 2.

**98.** Amplitude is $\frac{3}{2}$ (the minus sign does not affect the amplitude; it simply denotes a reflection of the graph through the $x$-axis). The period is $\frac{3}{\pi}$ times the period of $\sin x$, so the period is 6.

**99.** This looks like a cosine curve whose amplitude and period have been modified, and that has been shifted up. The maximum value of the curve is about 4.8, and the minimum is about 1.6, so the amplitude is $\frac{4.8-1.6}{2} = 1.6$. The period is 12 (months). Therefore this is the graph of $1.6\cos\left(\frac{2\pi}{12}t\right)$ shifted up. The upwards shift is $4.8 - 1.6 = 3.2$. Thus the equation is

$$p(t) = 3.2 + 1.6\cos\left(\frac{2\pi}{12}t\right).$$

**100.** **(a)** Since the growth rate varies over the course of a day, the period is 24 (hours).

**(b)** This looks like a cosine curve whose amplitude and period have been modified, that has been shifted upwards by 3 units, and that has been reflected around the line $y = 3$. The maximum value of the curve is about 3.6, and the minimum is about 2.4, so the amplitude is $\frac{3.6-2.4}{2} = 0.6$. The period is 24 (hours), so without the upward shift we get $-0.6\cos\left(\frac{2\pi}{24}t\right)$ shifted up. The upwards shift is $3.6 - 0.6 = 3$. Thus the equation is

$$G(t) = 3 - 0.6\cos\left(\frac{2\pi}{24}t\right) = 3 - 0.6\cos\left(\frac{\pi}{12}t\right).$$

**101.** Since $\sec x = \dfrac{1}{\cos x}$, it is undefined for $\cos x = 0$, and $\cos x = 0$ for

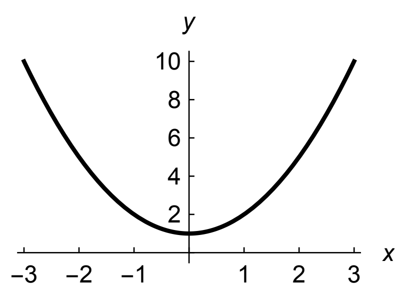
$$x = \frac{\pi}{2} + \pi n = \frac{\pi}{2}(1 + 2n), n \in \mathbb{Z}.$$

**102.** Since $\csc x = \dfrac{1}{\sin x}$, it is undefined for $\sin x = 0$, and $\sin x = 0$ when $x = n\pi$, $n \in \mathbb{Z}$.

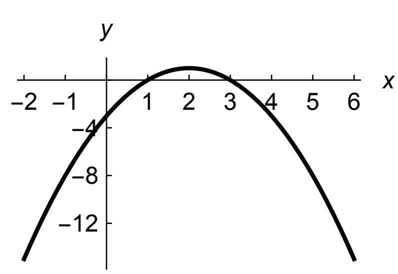
## 1.4 Graphing

### ■ 1.4.1

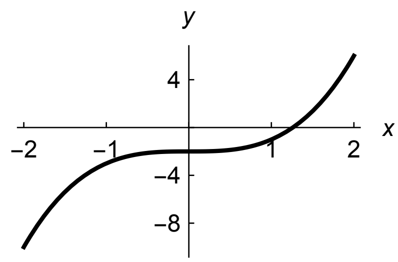
**1.** This is the graph of $x^2$ shifted up by one unit.




**2.** This is the graph of $x^2$, shifted right by two units, reflected about the $x$-axis, and then shifted up by one unit.

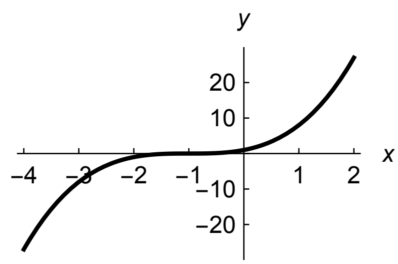


**3.**



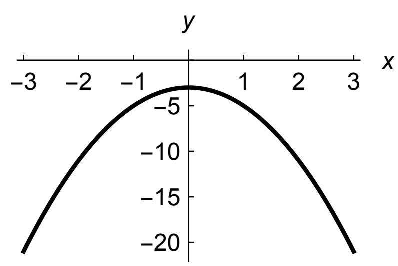

This is the graph of $x^3$ shifted down by two units.

**4.**



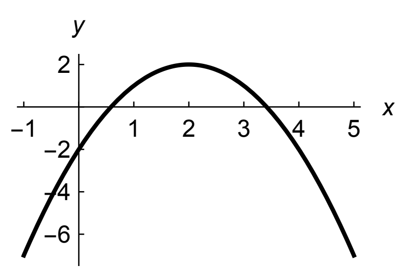

This is the graph of $x^3$ shifted left by one unit.

**5.**



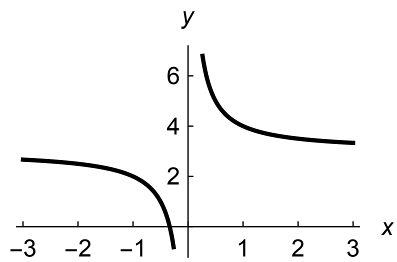

This is the graph of $x^2$, stretched vertically by a factor of two, reflected through the $x$-axis, and then shifted down by three units.

**6.**



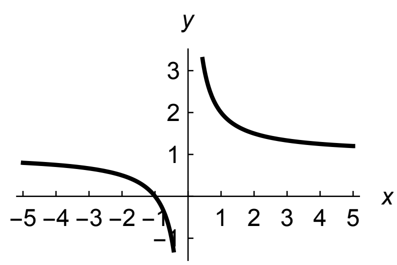

This is the graph of $x^2$ shifted right by two units, reflected in the $x$-axis, and then shifted up by two units (note that $(2-x)^2 = (x-2)^2$).

**7.**



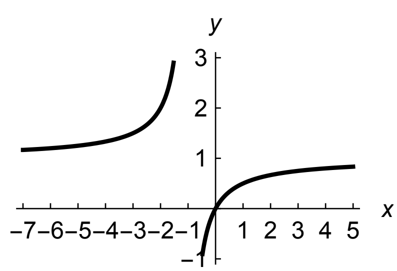

This is the graph of $\frac{1}{x}$ shifted up by three units.

**8.**



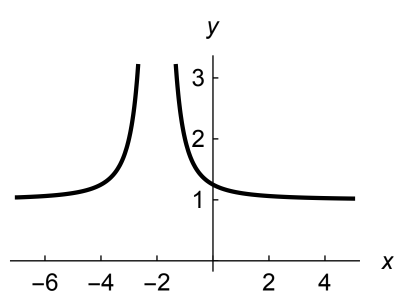

Since $\frac{x+1}{x} = 1 + \frac{1}{x}$, this is the graph of $\frac{1}{x}$ shifted up by one unit.

**9.**



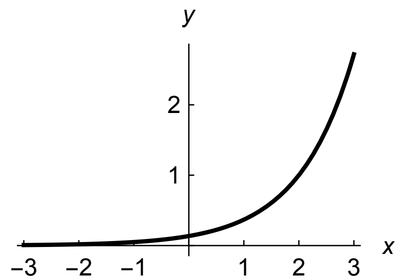

Since $\frac{x}{x+1} = 1 - \frac{1}{x+1}$, this is the graph of $\frac{1}{x}$ shifted left by one unit, reflected through the $x$-axis, and then shifted up by one unit.

**10.**



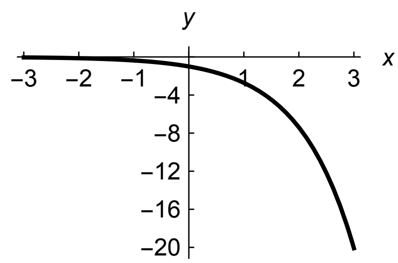

This is the graph of $\frac{1}{x^2}$ shifted left by two units and up by one unit.

**11.**



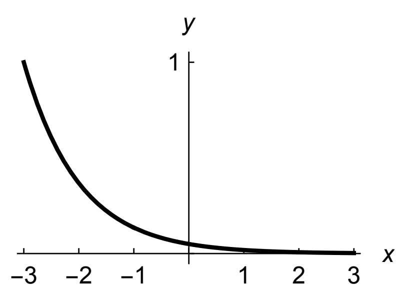

This is the graph of $e^x$ shifted right by two units.

**12.**



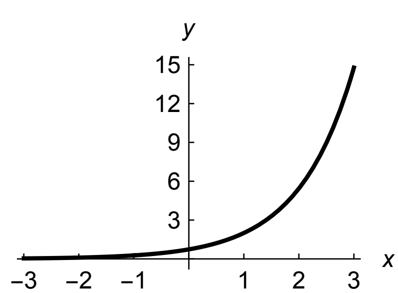

This is the graph of $e^x$ reflected through the $x$-axis.

**13.**



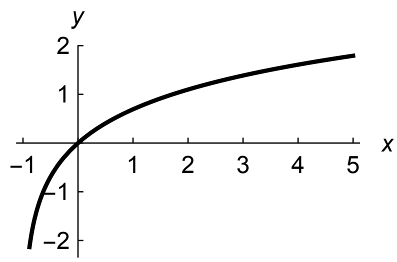

This is the graph of $e^{-x}$ shifted left three units.

**14.**



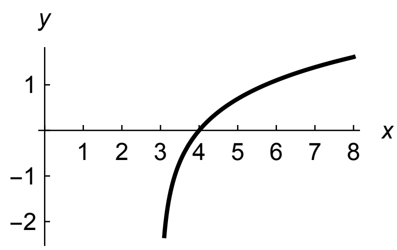

This is the graph of $e^x$ shifted right one unit and stretched vertically by a factor of 2.

**15.**



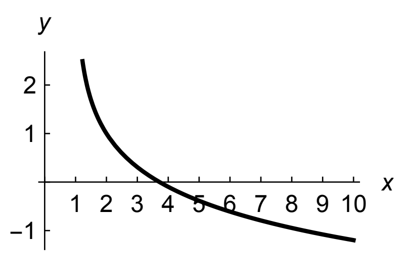

This is the graph of $\ln x$ shifted left by one unit.

**16.**



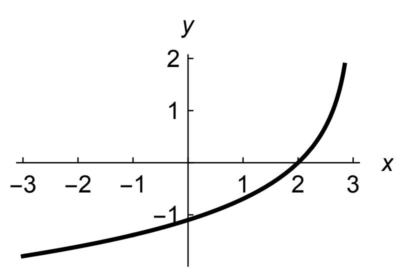

This is the graph of $\ln x$ shifted right by three units.

**17.**



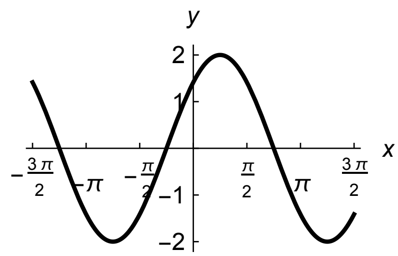

This is the graph of $\ln x$ shifted right by one unit, reflected through the $x$-axis, and then shifted up by one unit.

**18.**

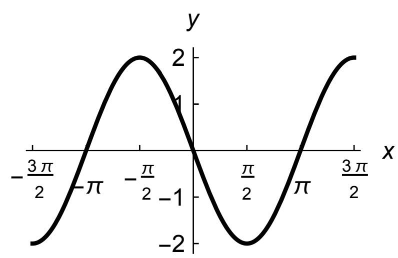



This is the graph of $\ln x$ reflected through the $y$-axis, then shifted right by 3 units, and finally reflected through the $x$-axis.

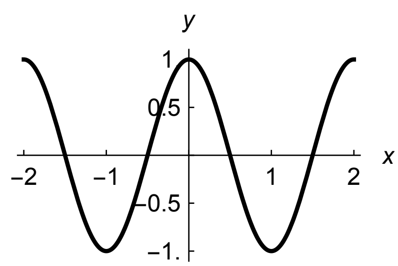
**19.** This is the graph of $\sin x$ shifted left by $\frac{\pi}{4}$ and expanded vertically by a factor of two.



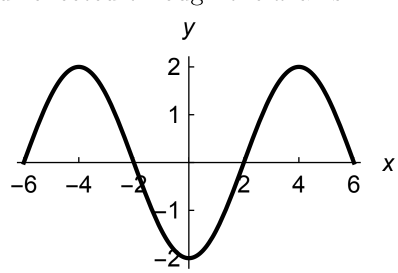

**20.** This is the graph of $\sin x$ reflected through the $y$-axis and expanded vertically by a factor of two.




**21.** This is the graph of $\cos x$ compressed horizontally by a factor of $\pi$.




**22.** This is the graph of $\cos x$ stretched horizontally by a factor of $\frac{\pi}{4}$ (so that its period is 8), expanded vertically by a factor of two, and reflected through the $x$-axis.




**23.** **(a)** Shift $y = x^2$ two units down.

**(b)** Shift $y = x^2$ one unit to the right and then one unit up.

**(c)** Shift $y = x^2$ two units to the left, stretch by a factor of 2, and reflect about the $x$-axis.

**24.** **(a)** Shift $y = x^3$ down by one unit.

**(b)** Shift $y = x^3$ one unit up and then reflect about the $x$-axis.

**(c)** Shift $y = x^3$ one unit up, stretch vertically by a factor of 2, and then reflect about the $x$-axis.

**25.** **(a)** First reflect $\frac{1}{x}$ about the $x$-axis, and then shift up one unit.

**(b)** First shift $\frac{1}{x}$ one unit to the right, and then reflect about the $x$-axis.

**(c)** We re-write the function as follows: $y = \frac{x}{x+1} = 1 - \frac{1}{x+1}$. First shift $y = \frac{1}{x}$ one unit to the left, then reflect about $x$-axis and finally shift up one unit.

**26.** **(a)** Shift $\frac{1}{x^2}$ up by 1.

**(b)** Shift $\frac{1}{x^2}$to the left by 1 then reflect about the $x$-axis.

**(c)** Reflect $\frac{1}{x^2}$ about the $x$-axis then shift down by 2.

**27.** **(a)** Shift $y = e^x$ up by three units.

**(b)** Reflect $y = e^x$ about the $y$-axis.

**(c)** Shift $y = e^x$ right by two units, stretch vertically by a factor of 2, and then shift up three units.

**28.** **(a)** Reflect about the $y$-axis then shift down by 1.

**(b)** Reflect about the $x$-axis then shift up by 1.

**(c)** First shift horizontally to the right by 3, then reflect about the $x$-axis and finally shift down by 2.

**29.** **(a)** Shift $y = \ln x$ one unit to the right.

**(b)** Reflect $y = \ln x$ about the $x$-axis, then shift up one unit.

**(c)** Shift $y = \ln x$ three units to the left, then down one unit.

**30.** **(a)** Reflection about the $y$-axis then horizontal translation to the right by 1.

**(b)** Horizontal translation to the left by 2 and vertical translation down by 1.

**(c)** Reflection about the $y$-axis, then horizontal translation to the right by 2, then reflection about the $x$-axis, and finally vertical translation up by 1.

**31.** **(a)** Compress $y = \sin x$ by a factor of $\pi$ in the horizontal direction.

**(b)** Shift $y = \sin x$ left by $\frac{\pi}{4}$ units.

**(c)** Since $-2\sin(\pi x + 1) = -2\sin\left(\pi\left(x + \frac{1}{\pi}\right)\right)$, first shift $y = \sin x$ left by $\frac{1}{\pi}$ units, then compress horizontally by a factor of $\pi$. Then stretch vertically by a factor of 2 and finally reflect about the $x$-axis.

**32.** **(a)** Stretch vertically by a factor of 2 and shift up by 1.

**(b)** Horizontal translation to the left by $\frac{\pi}{4}$ and reflection about the $x$-axis.

**(c)** Reflection about the $y$-axis, followed by horizontal translation to left by $\frac{\pi}{2}$ and finally reflection around the $x$-axis. (Alternatively, shift right by $\frac{\pi}{2}$ and reflect about the $y$ and $x$ axes.)

## ■ 1.4.2

**33.** To locate the points, we have to compute the log of each of the numbers, which are (approximately):

| $x$ | 0.003 | 0.03 | 3 | 5 | 30 | 50 | 1000 | 3000 | 30 000 |
|---|---|---|---|---|---|---|---|---|---|
| $\log x$ | −2.52 | −1.52 | 0.48 | 0.70 | 1.48 | 1.70 | 3 | 3.48 | 4.48 |

**34.** To locate the points, we have to compute the log of each of the numbers, which are (approximately):

| $x$ | 0.03 | 0.7 | 1 | 2 | 5 | 10 | 17 | 100 | 150 | 2000 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\log x$ | −1.52 | −0.15 | 0 | 0.30 | 0.70 | 1 | 1.23 | 2 | 2.18 | 3.30 |

**35.** **(a)** Since these are powers of ten, their logarithms are just the exponents. So the positions on a number line are 2, −3, −4, −7, and −10 respectively.

**(b)** No, since $\log 0$ is undefined.

**(c)** No, since logarithms of negative numbers are undefined.

**36.** **(a)** **(i)** $\log 10^{-3} = -3$, $\log(2 \times 10^{-3}) = \log 2 - 3 \approx -2.7$, $\log(3 \times 10^{-3}) = \log 3 - 3 \approx -2.52$.

**(ii)** $\log 10^{-1} = -1$, $\log(2 \times 10^{-1}) = \log 2 - 1 \approx -0.7$, $\log(3 \times 10^{-1}) = \log 3 - 1 \approx -0.52$.

**(iii)** $\log 10^{2} = 2$, $\log(2 \times 10^{2}) = \log 2 + 2 \approx 2.3$, $\log(3 \times 10^{2}) = \log 3 + 2 \approx 2.48$.

**(b)** Each of these is 0.3 units away from the other.

**(c)** Each of these is 0.48 units away from the other.

**37.** $\frac{6.7\,\text{m}}{0.51\,\text{mm}} = \frac{6.7 \times 10^3\,\text{mm}}{0.51\,\text{mm}} = \frac{6.7 \times 10^3}{0.51} \approx 13137$, so this is about four orders of magnitude. Alternatively, note that 6.7 m is four orders of magnitude larger than 0.67 mm and that the latter is close to the length of the shortest worm.

**38.** A sperm whale's weight has about three more zeros on the end, so it is about three orders of magnitude heavier.

**39.** The ratio of the weights is

$$\frac{190 \times 10^3\,\text{kg}}{1.8\,\text{g}} = \frac{190 \times 10^6\,\text{g}}{1.8\,\text{g}} \approx 10^8.$$

The blue whale is about eight orders of magnitude heavier.

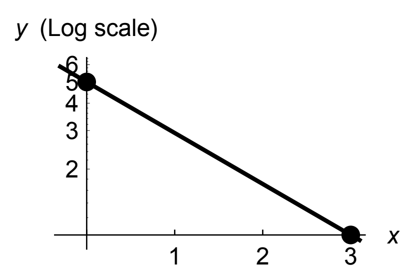
**40.** $A_1 = 1^2\,\text{m}^2 = 1\,\text{m}^2$ and $A_2 = (100\,\text{m})^2 = 100^2\,\text{m}^2 = 10^4\,\text{m}^2$, so 4 orders of magnitude.

**41.** If the shape of a typical bacterium is spherical with a radius of 0.5 to 1 $\mu$m, then the volume of a typical bacterium is between $\frac{4}{3}\pi \cdot 0.5^3 \approx 0.52\,\mu\text{m}^3$ and $\frac{4}{3}\pi \cdot 1^3 \approx 4.19\,\mu\text{m}^3$. Approximating the shape of *E. fishelsoni* as a cylinder, its volume is $\pi \cdot 40^2 \cdot 600 \approx 3.02 \times 10^6\,\mu\text{m}^3$. The ratio of these numbers is six to seven orders of magnitude.

**42.** Since volume varies as the cube of the radius, and there is one order of magnitude difference in the radius, there are three orders of magnitude difference in the volume.

## ■ 1.4.3

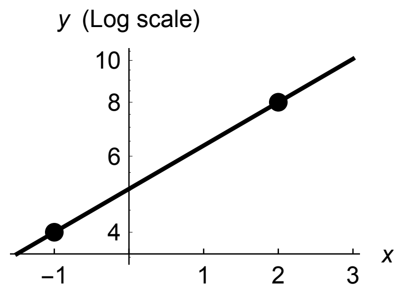
**43.** A graph of $\log y$ as a function of $x$ is below:



The slope of the line on the log-linear plot is $\frac{\log 5 - \log 1}{0-3} = -\frac{1}{3}\log 5$, so using the point-slope form at the point $(3, 1)$, we get for the equation of the line

$$\log y - \log 1 = -\frac{1}{3}\log 5 \cdot (x-3) \quad \Rightarrow \quad y = 10^{-\frac{1}{3}(x-3)\log 5} = 5^{-\frac{1}{3}(x-3)} = 5 \cdot 5^{-1/3x} \approx 5 \cdot 0.58^x.$$

**44.** A graph of $\log y$ as a function of $x$ is below:




The slope of the line on the log-linear plot is

$$\frac{\log 8 - \log 4}{2-(-1)} = \frac{1}{3}\log 2,$$

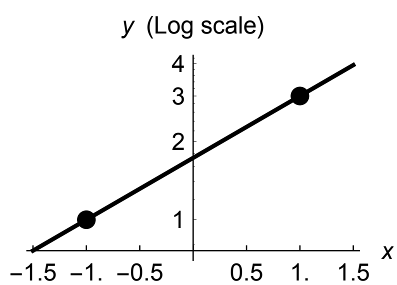
so using the point-slope form at the point $(-1, 4)$, we get for the equation of the line

$$\log y = \frac{1}{3}\log 2 \cdot (x+1) + \log 4 = \left(2 + \frac{1}{3}(x+1)\right)\log 2.$$

Therefore the original functional relation is, exponentiating both sides,

$$y = 10^{\left(2+\frac{1}{3}(x+1)\right)\log 2} = 2^{2+\frac{1}{3}(x+1)}.$$

**45.** A graph of $\log y$ as a function of $x$ is below:



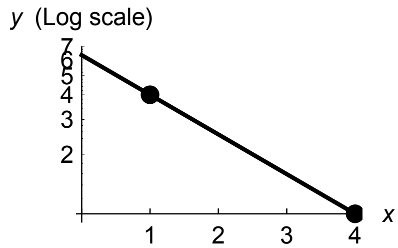

The slope of the line on the log-linear plot is

$$\frac{\log 3 - \log 1}{1-(-1)} = \frac{1}{2}\log 3,$$

so using the point-slope form at the point $(-1, 1)$, we get for the equation of the line

$$\log y - \log 1 = \frac{1}{2}\log 3 \cdot (x+1) \quad \Rightarrow \quad y = 10^{\frac{1}{2}(x+1)\log 3} = 3^{\frac{1}{2}(x+1)}.$$

**46.** A graph of $\log y$ as a function of $x$ is below:



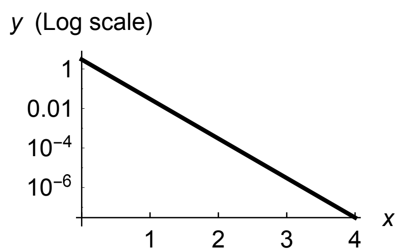

The slope of the line on the log-linear plot is

$$\frac{\log 4 - \log 1}{1-4} = -\frac{1}{3}\log 4,$$

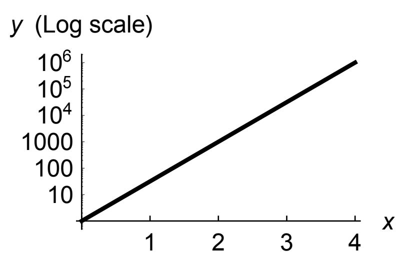
so using the point-slope form at the point $(4, 1)$, we get for the equation of the line

$$\log y - \log 1 = -\frac{1}{3}\log 4 \cdot (x-4) \quad \Rightarrow \quad y = 10^{-\frac{1}{3}(x-4)\log 4} = 4^{-\frac{1}{3}(x-4)}.$$

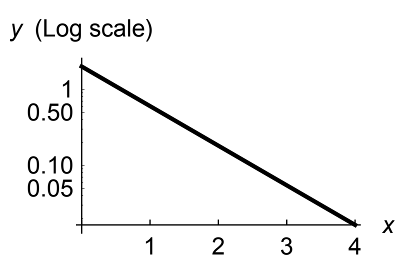
**47.** $\log y = \log\left(3 \times 10^{-2x}\right) = \log 3 + \log\left(10^{-2x}\right) = \log 3 - 2x$. A plot is below:



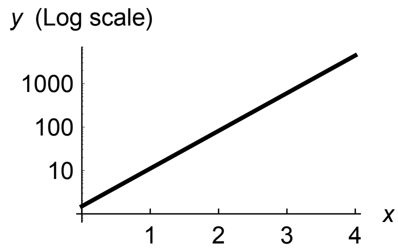

**48.** $\log y = \log\left(10^{1.5x}\right) = 1.5x$. A plot is below:



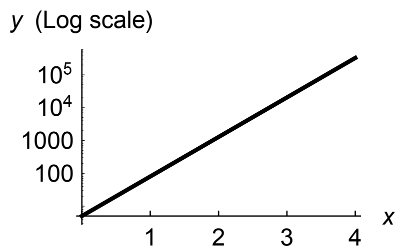

**49.** $\ln y = \ln\left(2e^{-1.2x}\right) = \ln 2 + \ln\left(e^{-1.2x}\right) = \ln 2 - 1.2x$. A plot is below:



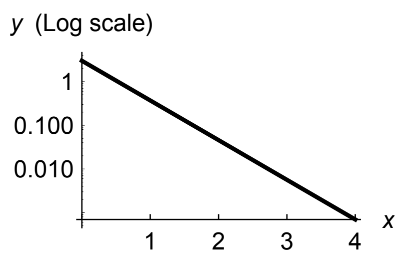
**50.** $\ln y = \ln\left(1.5e^{2x}\right) = \ln 1.5 + \ln e^{2x} = 2x + \ln 1.5$. A plot is below:



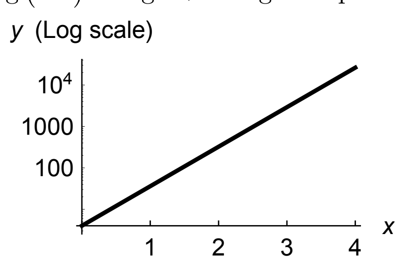

**51.** $\log y = \log\left(5 \times 2^{4x}\right) = \log 5 + \log\left(2^{4x}\right) = \log 5 + 4x \log 2$. A plot is below:



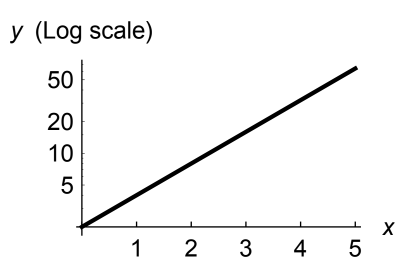

**52.** $\log y = \log\left(3 \times 5^{-1.3x}\right) = \log 3 + \log 5^{-1.3x} = \log 3 - 1.3x \log 5$. A plot is below:



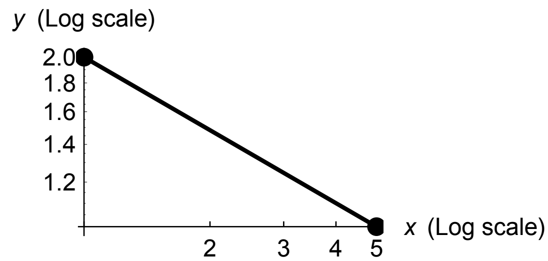

**53.** $\log y = \log\left(4 \times 3^{2x}\right) = \log 4 + \log\left(3^{2x}\right) = \log 4 + 2x \log 3$. A plot is below:




**54.** $\log y = \log\left(2^{x+1}\right) = (x+1) \log 2$. A plot is below:



**55.** A graph of $\log y$ as a function of $\log x$ is below:




The slope of the line on the log-log plot is

$$\frac{\log 2 - \log 1}{\log 1 - \log 5} = -\frac{\log 2}{\log 5},$$

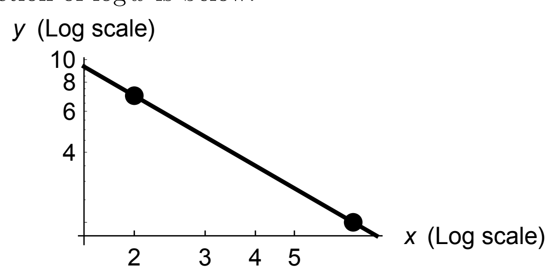
so using the point-slope form at the point $(1, 2)$, we get for the equation of the line

$$\log y = -\frac{\log 2}{\log 5} \cdot (\log x - \log 1) + \log 2 = -\frac{\log 2}{\log 5} \log x + \log 2.$$

Exponentiating both sides gives

$$y = 10^{-\frac{\log 2}{\log 5} \log x + \log 2} = 2 \times 10^{-\frac{\log 2}{\log 5} \log x} = 2 \times x^{-\log 2/\log 5}.$$

**56.** A graph of $\log y$ as a function of $\log x$ is below:




The slope of the line on the log-log plot is

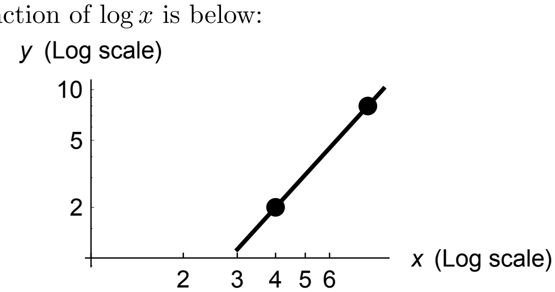
$$\frac{\log 2 - \log 7}{\log 7 - \log 2} = -1,$$

so using the point-slope form at the point $(7, 2)$, we get for the equation of the line

$$\log y - \log 2 = -(\log x - \log 7) = -\log \frac{x}{7} = \log \frac{7}{x}.$$

Note that $\log y - \log 2 = \log \frac{y}{2}$, and then exponentiate both sides, giving $\frac{y}{2} = \frac{7}{x}$, or $y = \frac{14}{x}$.

**57.** A graph of $\log y$ as a function of $\log x$ is below:



The slope of the line on the log-log plot is

$$\frac{\log 8 - \log 2}{\log 8 - \log 4} = \frac{3\log 2 - \log 2}{2\log 2 - \log 2} = 2,$$

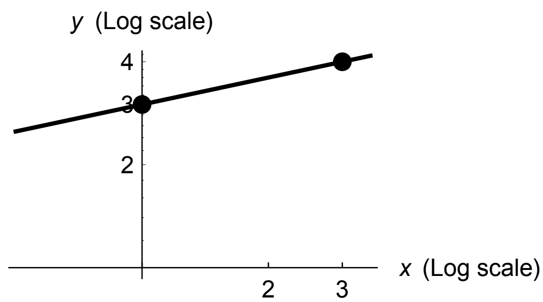
so using the point-slope form at the point $(4, 2)$, we get for the equation of the line

$$\log y - \log 2 = 2(\log x - \log 4) \quad \Rightarrow \quad \log\frac{y}{2} = 2\log\frac{x}{4}.$$

Exponentiating both sides gives

$$\frac{y}{2} = 10^{2\log(x/4)} = \left(\frac{x}{4}\right)^2 = \frac{x^2}{16} \quad \Rightarrow \quad y = \frac{x^2}{8}.$$

**58.** A graph of $\log y$ as a function of $\log x$ is below:




The slope of the line on the log-log plot is

$$\frac{\log 4 - \log 3}{\log 3 - \log 1} = \frac{\log 4 - \log 3}{\log 3},$$

so using the point-slope form at the point $(1, 3)$, we get for the equation of the line

$$\log y - \log 3 = \frac{\log 4 - \log 3}{\log 3}\log x \quad \Rightarrow \quad \log\frac{y}{3} = \frac{\log 4 - \log 3}{\log 3}\log x.$$

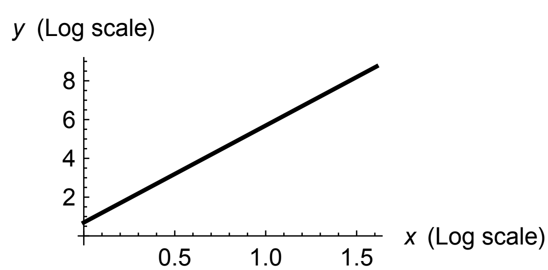
Exponentiating both sides gives

$$\frac{y}{3} = 10^{\frac{\log 4 - \log 3}{\log 3}\log x} = x^{\frac{\log 4 - \log 3}{\log 3}},$$

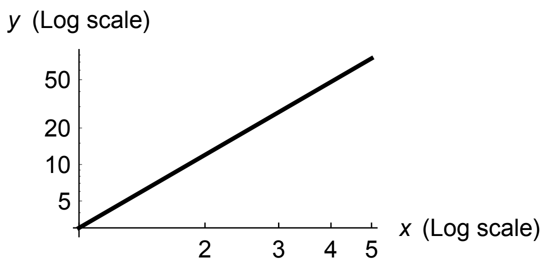
so that

$$y = 3x^{(\log 4 - \log 3)/\log 3}.$$

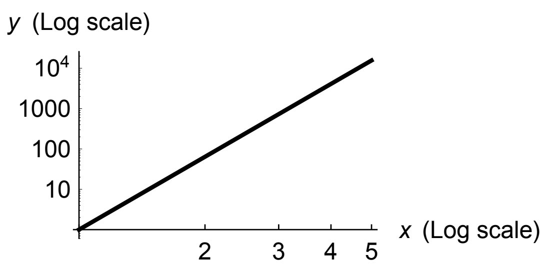
**59.** Taking logarithms on both sides gives $\log y = \log\left(2x^5\right) = \log 2 + \log x^5 = \log 2 + 5\log x$. Thus the linear relationship is $Y = 5X + \log 2$; a graph is below:



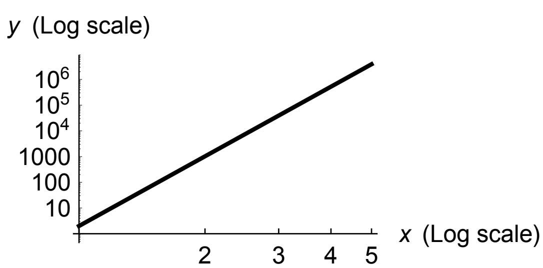
**60.** Taking logarithms on both sides gives $\log y = \log\left(3x^2\right) = \log 3 + \log x^2 = \log 3 + 2\log x$. Thus the linear relationship is $Y = 2X + \log 3$; a graph is below:



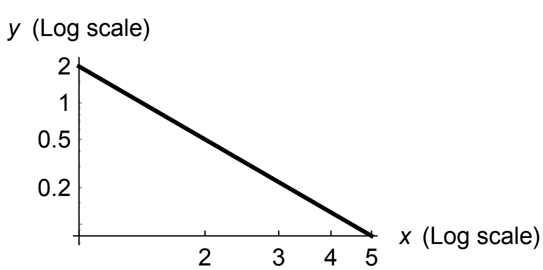

**61.** Taking logarithms on both sides gives $\log y = \log\left(x^6\right) = 6\log x$. Thus the linear relationship is $Y = 6X$; a graph is below:



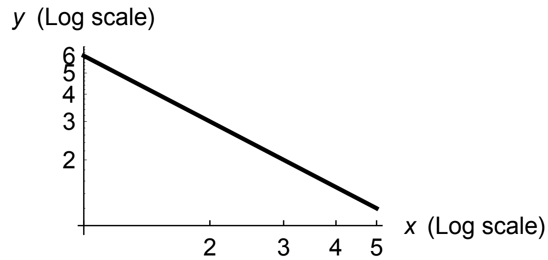

**62.** Taking logarithms on both sides gives $\log y = \log\left(2x^9\right) = \log 2 + \log x^9 = \log 2 + 9\log x$. Thus the linear relationship is $Y = 9X + \log 2$; a graph is below:



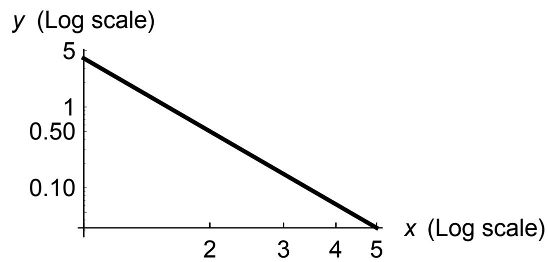

**63.** Taking logarithms on both sides gives $\log y = \log\left(2x^{-2}\right) = \log 2 + \log x^{-2} = \log 2 - 2\log x$. Thus the linear relationship is $Y = -2X + \log 2$; a graph is below:



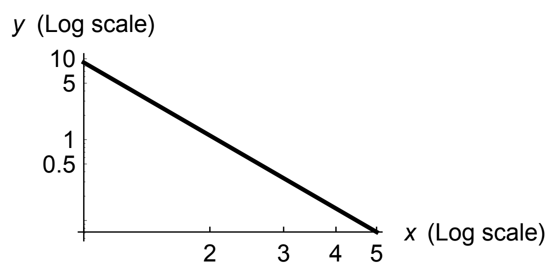
**64.** Taking logarithms on both sides gives $\log y = \log\left(6x^{-1}\right) = \log 6 + \log x^{-1} = \log 6 - \log x$. Thus the linear relationship is $Y = -X + \log 6$; a graph is below:




**65.** Taking logarithms on both sides gives $\log y = \log\left(4x^{-3}\right) = \log 4 + \log x^{-3} = \log 4 - 3\log x$. Thus the linear relationship is $Y = -3X + \log 4$; a graph is below:




**66.** Taking logarithms on both sides gives $\log y = \log\left(9x^{-3}\right) = \log 9 + \log x^{-3} = \log 9 - 3\log x$. Thus the linear relationship is $Y = -3X + \log 9$; a graph is below:




**67.** Starting with $f(x) = 3x^{1.7}$ we take logarithms on both sides, to get $\log f = \log 3x^{1.7} = \log 3 + 1.7\log x$. Thus the linear relationship is $Y = \log 3 + 1.7\text{X}$, and we should use a log-log transformation.

**68.** $g(s) = 9e^{-1.65}$ is already a horizontal line, so no transformation is necessary.

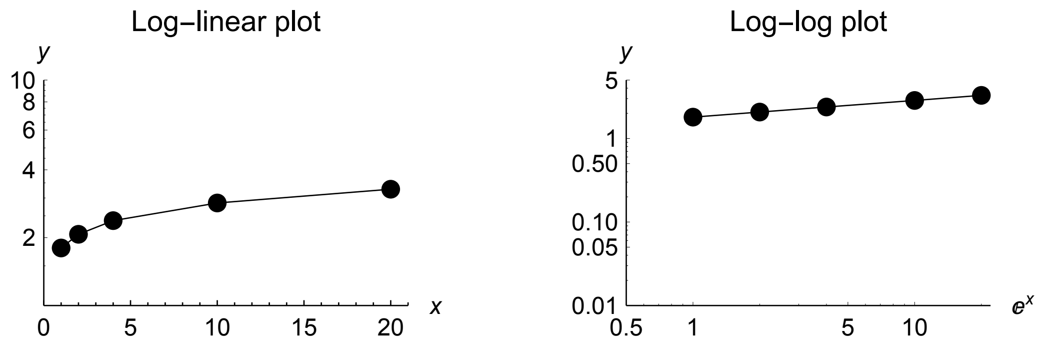
**69.** Starting with $N(t) = 130 \cdot 2^{1.2t}$ we take logarithms on both sides, to get $\log N(t) = \log(130 \cdot 2^{1.2t}) = \log 130 + 1.2t\log 2$. Thus the linear relationship is $Y = \log 130 + (1.2\log 2)t$, and we should use a log-linear plot.

**70.** Starting with $I(u) = 4.8u^{-0.89}$ we take logarithms on both sides, to get $\log I(u) = \log 4.8u^{-0.89} = \log 4.8 - 0.89\log u$. Thus the linear relationship is $Y = \log 4.8 - 0.89\text{X}$, and we should use a log-log transformation.

**71.** Starting with $R(t) = 3.6t^{1.2}$ we take logarithms on both sides, to get $\log R(t) = \log 3.6t^{1.2} = \log 3.6 + 1.2\log t$. Thus the linear relationship is $Y = \log 3.6 + 1.2\text{X}$, and we should use a log-log transformation.

**72.** Starting with $L(t) = 2^{-2.7t+1}$, take logarithms on both sides, giving $\log L(t) = \log\left(2^{-2.7t+1}\right) = (-2.7t+1)\log 2$. Simplifying gives $\log L(t) = -2.7t\log 2 - \log 2$. The linear relationship is $Y = -2.7t\log 2 - \log 2$, and we should use a log-linear plot.

**73.** The plots below are log-linear and log-log:




Clearly the data represents a power function since the log-log plot is a straight line. To find the functional relationship, use for example the first two points. The slope of the line is

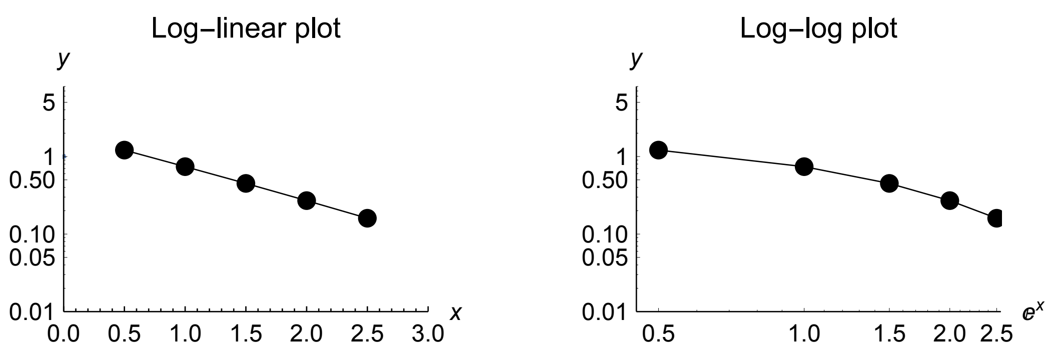
$$\frac{\log 2.07 - \log 1.8}{\log 2 - \log 1} \approx 0.20,$$

so that the relationship is

$$\log y - \log 1.8 = 0.2(\log x - \log 1) \quad \Rightarrow \quad \log\frac{y}{1.8} = 0.2\log x = \log x^{0.2}.$$

Exponentiating both sides gives $y = 1.8x^{0.2}$.

**74.** The plots below are log-linear and log-log:




Clearly the data represents an exponential function since the log-linear plot is a straight line. To find the functional relationship, use for example the first two points. The slope of the line is

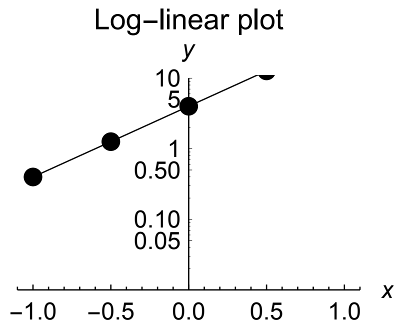
$$\frac{\log 1.21 - \log 0.74}{0.5 - 1} \approx -0.43,$$

so that the relationship is

$$\log y - \log 0.74 = -0.43(x-1) \quad \Rightarrow \quad \log\frac{y}{0.74} = -0.43(x-1) = -0.43x + 0.43.$$

Exponentiating both sides gives $y = 0.74 \times 10^{-0.43x+0.43} \approx 1.99 \times 10^{-0.43x}$.

**75.** Since some entries in the first column are nonpositive, we assume this is an exponential relationship. This is confirmed by a log-linear plot:




Clearly the data represents an exponential function since the log-linear plot is a straight line. To find the functional relationship, use for example the first two points. The slope of the line is

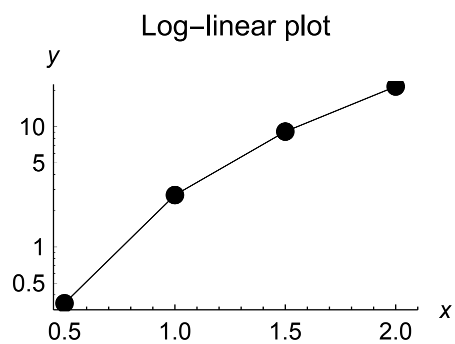
$$\frac{\log 1.26 - \log 0.398}{-0.5 - (-1)} \approx 1,$$

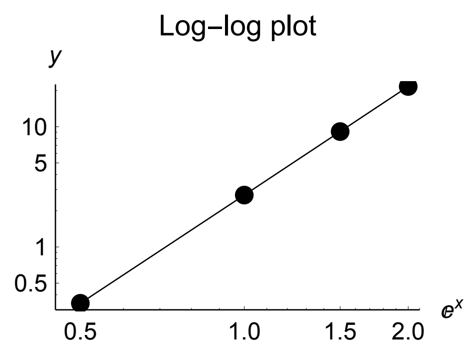
so that the relationship is

$$\log y - \log 4 = (x - 0) \quad \Rightarrow \quad \log \frac{y}{4} = x.$$

Exponentiating both sides gives $y = 4 \times 10^x$.

**76.** The data cannot fit an exponential model since $y(0) = 0$. So we hope it fits a power model. Ignoring the first data point for graphing purposes, the plots below are log-linear and log-log:




Clearly the data represents a power function since the log-log plot is a straight line. To find the functional relationship, use for example the points $(1, 2.7)$ and $(2, 21.60)$. The slope is

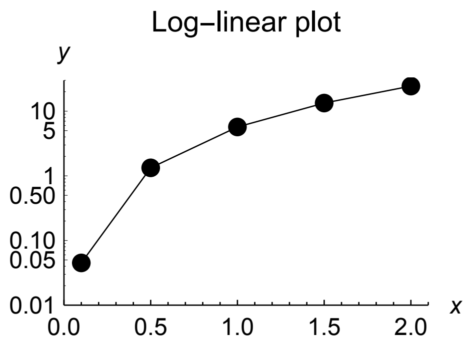
$$\frac{\log 21.60 - \log 2.7}{\log 2 - \log 1} = \frac{\log 8}{\log 2} = 3,$$

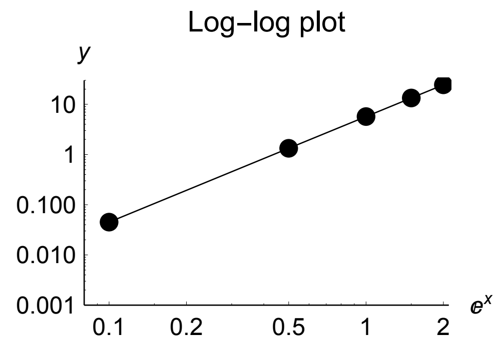
so that the relationship is

$$\log y - \log 2.7 = 3(\log x - \log 1) \quad \Rightarrow \quad \log \frac{y}{2.7} = 3 \log x.$$

Exponentiating both sides and simplifying gives $y = 2.7x^3$, which is easily seen to fit the other data points as well.

**77.** The plots below are log-linear and log-log:




Clearly the data represents an power function since the log-log plot is a straight line. To find the functional relationship, use for example the points $(1, 5.7)$ and $(2, 24.44)$. The slope is

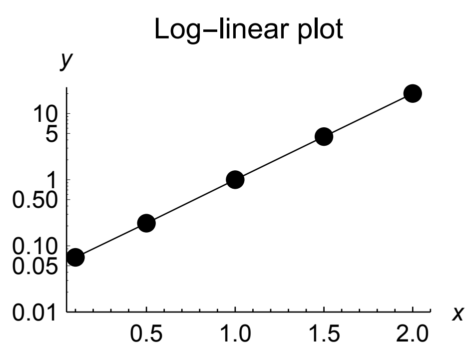
$$\frac{\log 24.44 - \log 5.7}{\log 2 - \log 1} \approx 2.1,$$

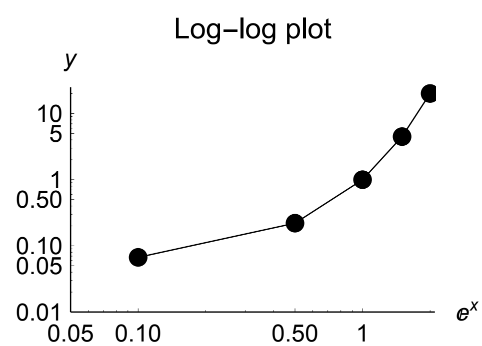
so that the relationship is

$$\log y - \log 5.7 = 2.1(\log x - \log 1) \quad \Rightarrow \quad \log \frac{y}{5.7} = 2.1 \log x.$$

Exponentiating both sides and simplifying gives $y = 5.7x^{2.1}$, which is easily seen to fit the other data points as well.

**78.** The plots below are log-linear and log-log:




Clearly the data represents an exponential function since the log-linear plot is a straight line. To find the functional relationship, use for example the points $(1, 1)$ and $(2, 20.09)$. The slope is

$$\frac{\log 20.09 - \log 1}{2 - 1} \approx 1.303,$$

so that the relationship is

$$\log y - \log 1 = 1.303(x - 1) \quad \Rightarrow \quad \log y = 1.303x - 1.303.$$

Exponentiating both sides gives $y = 10^{1.303x - 1.3} \approx 0.0498 \times 10^{1.303x}$. This is easily seen to fit the other data points as well.

**79.** Taking logarithms in base 2 we have $\log_2 y = x$, so the linear relationship is $Y = X$ on a $\log_2$ linear plot.

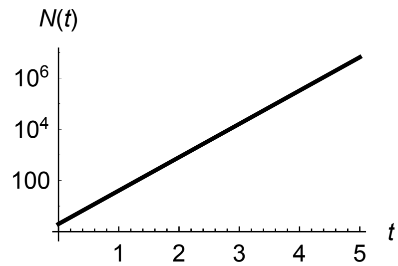
**80.** Taking logarithms in base 5 we have $\log_5 y = x \log_5 5 = x$, so the linear relationship is $Y = X$ on a $\log_5$-linear plot.

**81.** Taking logarithms in base 2 we have $\log_2 y = -x$, so the linear relationship is $Y = -x$ on a $\log_2$ linear plot.

**82.** Taking logarithms in base 3 we have $\log_3 y = \log_3 \left(3e^{-2x}\right) = \log_3 3 + \log_3 e^{-2x} = 1 - 2x \log_3 e$, so the linear relationship is $Y = 1 - 2x \log_3 e$ on a $\log_3$-linear plot.

**83.** **(a)** Taking logs, we have $\log N = \log(2e^{3t}) = \log 2 + 3t \log e$. Setting $Y = \log N$ and $x = t$ we have $Y = 3x \log e + \log 2$, which is a linear relationship.

**(b)** The slope of the line, as computed in part (a), is $3 \log e \approx 1.303$.




**84.** Since the population size is initially 20, the $y$-intercept on a semi-log plot would be $\log 20$. Thus the equation of the line on the semi-log plot is given by

$$Y = 0.03X + \log 20.$$

Converting variables we have

$$\log N(t) = 0.03\text{t} + \log 20$$

or solving for $N(t)$ , we have

$$N(t) = 10^{0.03t + \log 20} = 10^{\log 20} \cdot (10^{0.03})^t = 20(1.07)^t$$

**85.** The relationship $S = CA^z$ leads to $\log S = \log(CA^z) = \log C + z \log A$. Plotting $Y = \log S$ and $X = \log A$ on a log-log scale would lead to a straight line given by $Y = zX + \log C$. Thus the model $S = CA^z$ is appropriate. The value of $z$ is the slope of the straight line.

**86.** **(a)** Since $v = \frac{v_{\max} s}{s+K}$, we can invert both sides to obtain

$$\frac{1}{v} = \frac{s+K}{v_{\max} s} = \frac{s}{v_{\max} s} + \frac{K}{v_{\max} s} = \frac{K}{v_{\max}} \cdot \frac{1}{s} + \frac{1}{v_{\max}}$$

**(b)** From the linear relationship $\frac{1}{v} = \frac{K}{v_{\max}} \cdot \frac{1}{s} + \frac{1}{v_{\max}}$, we find that the vertical axis intercept is $\frac{1}{v_{\max}}$, and the horizontal axis intercept is $-\frac{1}{v_{\max}} \cdot \frac{v_{\max}}{K} = -\frac{1}{K}$.

**(c)** $\frac{1}{v_{\max}}$ will be the $y$-intercept of the graph, so we can obtain $v_{\max}$ by taking the reciprocal of the $y$-intercept. Similarly, we can obtain $K$ by taking the negative reciprocal of the $x$-intercept, which is $-\frac{1}{K}$. Alternatively, we could obtain $K$ by multiplying the slope of the line by the value of $v_{\max}$ obtained in the previous step.

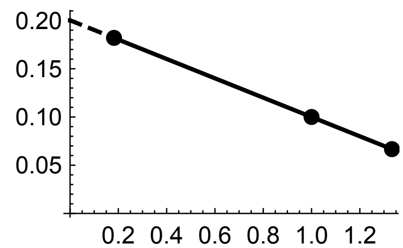
**87.** **(a)** Since $v = \frac{v_{\max} s}{s+K}$, we can multiply both sides by $s + K$ to obtain

$$vs + vK = v_{\max} s \quad \Rightarrow \quad \frac{vs + vK}{sK} = \frac{v_{\max} s}{sK} \quad \Rightarrow \quad \frac{v}{K} + \frac{v}{s} = \frac{v_{\max}}{K} \quad \Rightarrow \quad \frac{v}{s} = \frac{v_{\max}}{K} - \frac{1}{K} v$$

If we set $Y = \frac{v}{x}$ and $X = v$, this becomes $Y = \frac{v_{\max}}{K} - \frac{1}{K} X$, which is the equation of a straight line.

**(b)** In part (a), $K$ is the negative reciprocal of the slope, and $v_{\max}$ is the $Y$-intercept multiplied by $K$.

**(c)** A plot of $v$ against $v/s$ is below, with the line extended to the vertical axis:




From the data we estimate the slope of the line to be about $-0.1$, so that $K = -\frac{1}{-0.1} = 10$. The vertical intercept appears to be 0.2, so that $v_{\max} = 0.2K = 2$.

**88.** Let us denote the maximal oxygen consumption by $\mathcal{O}$, and the body mass by $M$. The equation in the log-log plot is given by

$$Y = 0.8X + 0.105$$

With $Y = \log \mathcal{O}$ and $X = \log M$, we can transform this to

$$\log \mathcal{O} = 0.8 \log M + 0.105$$

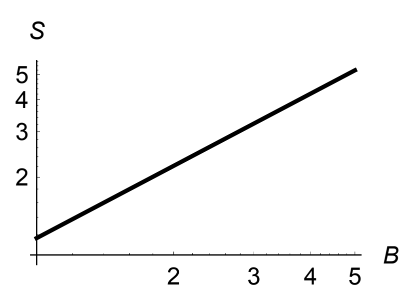
and exponentiating both sides we have

$$\mathcal{O} = 10^{(0.8 \log M + 0.105)}$$

$$= (10^{\log M})^{0.8} \cdot 10^{0.105}$$

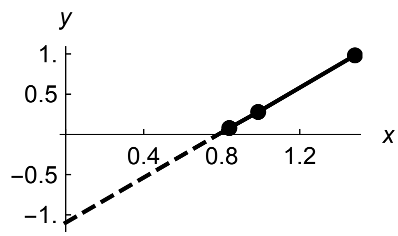
$$= 1.27(M)^{0.8}$$

**89.** **(a)** Taking logarithms of the relation $S = 1.162B^{0.93}$, we have $\log S = \log 1.162 + 0.93 \log B$. If we set $Y = \log S$ and $X = \log B$, we get the linear relationship $Y = \log 1.162 + 0.93\text{X}$. The slope is 0.93 and the $y$-intercept $\log 1.162$. A log-log plot of $S$ against $B$ showing the linear relationship is below:



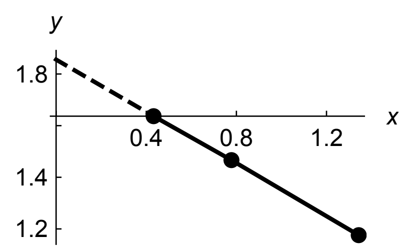
**(b)** When $B = 10$ cm, $S = 1.162(10)^{0.93} \approx 9.89$ so the ratio of $S$ to $B$ is $S/B = 0.989$. When $B = 100$ cm, $S = 1.162(100)^{0.93} \approx 84.2$ so the ratio of $S$ to $B$ is $S/B = 0.842$. When $B = 500$ cm, $S = 1.162(500)^{0.93} \approx 376$ so the ratio of $S$ to $B$ is $S/B = 0.752$. When $B$ is smaller, the ratio is larger. As $B$ increases (and the animal grows) the ratio decreases.

**90.** The plot below shows the line joining the logarithms (base 10) of the three given points; that line is extended as a dashed line until it meets the $y$-axis:



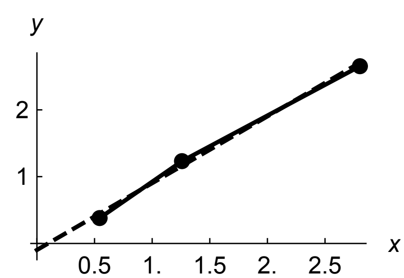

The $x$-intercept of the line is about 0.78, while the $y$-intercept is about $-1.1$, so the slope is $c \approx \frac{1.1}{0.78} \approx 1.41$. Since the $y$-intercept is $-1.1$, the value of the constant $a$ is $10^{-1.1} \approx 0.08$. The model is approximately $f = 0.08B^{1.41}$.

**91.** The plot below shows the line joining the logarithms (base 10) of the three given points; that line is extended as a dashed line until it meets the $y$-axis:




The slope of the line is about $a \approx \frac{1.635-1.86}{0.45} \approx -0.5$ Since the $y$-intercept is about 1.86, the value of the constant $b$ is $10^{1.86} \approx 72.44$. The model is approximately $S = 72.44B^{-0.5}$.

**92.** **(a)** The plot below shows the curve (which is not quite a straight line) joining the logarithms (base 10) of the three given points together with a line that closely fits the points.

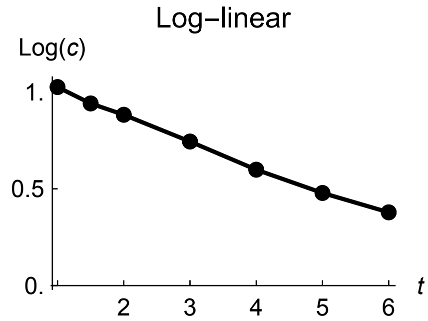


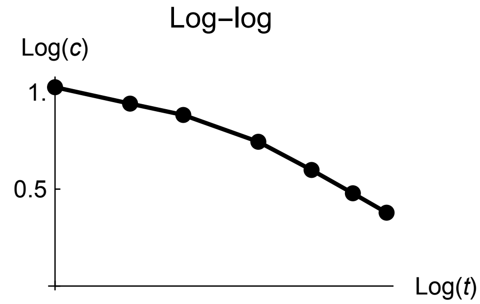

The slope of the line is about $\frac{\log_{10} 450-\log_{10} 2.4}{\log_{10} 635-\log_{10} 3.5} \approx 1.00$. This is the value of the constant $a$. Since the $y$-intercept is about $-0.1$, the value of the constant $b$ is $10^{-0.1} \approx 0.79$. The model is approximately $u = 0.79M$. This model does not fit the points perfectly, due to a combination of the fact that the data are not exactly log-linear and roundoff/estimation errors.

**(b)** With $M = 80$ we get $u = 0.79 \cdot 80 = 63.2\,\text{ml/s}$.

**93.** **(a)** Plot $\log c$ as a function of $t$ and also $\log c$ as a function of $\log t$. If the first plot is a straight line, then the relationship is an exponential law, while if the second is, the relationship is a power law.

**(b)** The two graphs are below:



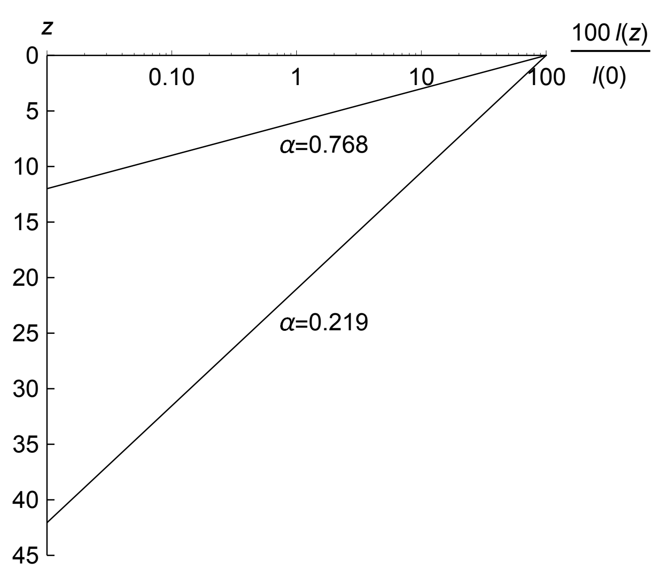

Since the log-linear plot more closely approximates a straight line, this is an exponential law $c = kd^t$.

**(c)** $\log_{10} d$ is the slope of the line, which is approximately $\frac{0.38-1}{6-1} = -0.124$. Thus $d \approx 10^{-0.124} \approx 0.751$.

**94.** **(a)** Since $\frac{I(z)}{I(0)} = e^{-\alpha z}$, we get $\ln \frac{I(z)}{I(0)} = -\alpha z$. To estimate the slope of the lines we must divide each horizontal coordinate by 100, since the horizontal axis is $100\frac{I(z)}{I(0)}$ (represented in log scale). For each line, we compute its slope using the two axis intercepts. We get:

$$\begin{aligned} \text{Lake 1}: \quad \alpha &= \frac{\ln 0.0001 - \ln 1}{15 - 0} \approx 0.614 \\ \text{Lake 2}: \quad \alpha &= \frac{\ln 0.0001 - \ln 1}{22.5 - 0} \approx 0.409 \\ \text{Lake 3}: \quad \alpha &= \frac{\ln 0.0001 - \ln 1}{30 - 0} \approx 0.307. \end{aligned}$$

**(b)** A graph is below:



Note that even though the slopes appear positive, they are really negative since the positive $z$-axis goes downwards.

**(c)** The graphs are straight lines since the light intensity is an exponential function of depth, so a log-linear plot produces a straight line.

**95.** **(a)** The given information implies that $I(1) = 0.9$, and so we have

$$0.9 = \frac{I(1)}{I(0)} = e^{-\alpha \cdot 1} = e^{-\alpha}.$$

Therefore $\alpha = -\ln 0.9\,\text{m}^{-1}$.

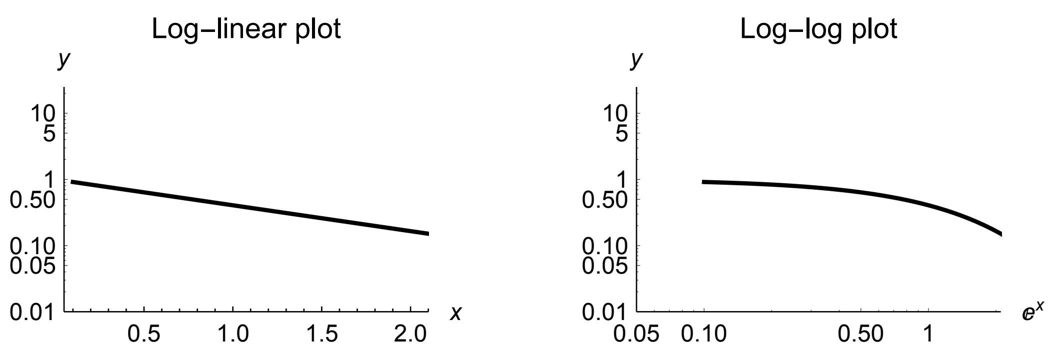
**(b)** With $\alpha = -\ln 0.9$, we have

$$\frac{I(2)}{I(1)} = \frac{e^{-2\alpha}}{e^{-\alpha}} = e^{-\alpha} = e^{\ln 0.9} = 0.9.$$

Thus 90% of the intensity at the first meter remains at the second meter. That is, 10% is absorbed. A similar calculation shows that 10% of the intensity at the second meter is absorbed in the third meter.

**(c)** When $z = 1\,\text{m}$, we have $\frac{I(1)}{I(0)} = e^{-\alpha} = e^{\ln 0.9} = 0.9$, and so 90% remains. When $z = 2\,\text{m}$, we have $\frac{I(2)}{I(0)} = e^{-2\alpha} = e^{2\ln 0.9} = 0.9^2 = 0.81$, and so 81% remains. When $z = 3\,\text{m}$, we have $\frac{I(3)}{I(0)} = e^{-3\alpha} = e^{3\ln 0.9} = 0.9^3 = 0.729$, and so 72.9% remains.

**(d)**




**(e)** Applying a logarithmic transformation to the equation $I(z) = I(0)e^{-\alpha z}$ we have

$$\log I(z) = \log(I(0)e^{-\alpha z}) = \log I(0) - \alpha z \log e = \log I(0) - (\alpha \log e)z$$

Setting $Y = \log I(z)$, this gives the linear relationship $Y = \log I(0) - (\alpha \log e)z$. The slope of the line is given by $m = -\alpha \log e = \ln 0.9 \log e = \ln 0.9 / \ln 10$.

**(f)** We want the ratio $I(z)/I(0)$ to equal 1%. That is, $0.01 = I(z)/I(0) = e^{-\alpha z} = e^{z \ln 0.9} = 0.9^z$ Thus $z = -\frac{\ln 0.01}{\alpha} = \frac{\ln(0.01)}{\ln(0.9)} \approx 43.7\text{m}$

**(g)** A clear lake would have small $\alpha$, and a milky lake would have large $\alpha$.

**96.** Since the straight line has slope $-\frac{3}{2}$, the equation of the line in the log-log plot is

$$\log w = -\frac{3}{2}\log d + c.$$

Since $d = 10^3/\text{m}^2$ when $w = 1\,\text{g}$, we have

$$\log 1 = -\frac{3}{2}\log 10^3 + c \quad \Rightarrow \quad 0 = -\frac{9}{2} + c \quad \Rightarrow \quad c = \frac{9}{2}.$$

The linear equation is therefore

$$\log w = -\frac{3}{2}\log d + \frac{9}{2}.$$

Exponentiating both sides gives

$$w = 10^{-(3/2)\log d + 9/2} = 10^{9/2} d^{-3/2} \approx 31622.8 d^{-3/2}.$$

**97.** The equation of the line is $\log y = \frac{\log 1000 - \log 100}{3-0}x + \log 100$, or $\log y = \frac{1}{3}x + \log 100$. Exponentiating both sides gives $y = 100 \cdot 10^{(1/3)x}$.

**98.** The equation of the line is $Y = -2X+$, or $\log y = -2\log x + 5 = 5 - \log x^2$. Collecting terms gives

$$\log y + \log x^2 = 5 \quad \Rightarrow \quad \log x^2 y = 5.$$

Now exponentiate both sides, giving $x^2 y = 10^5$, or $y = 10^5 x^{-2}$.

**99.** The $y$-axis is a log scale; using base 2 logs we get as the equation of the line $\log_2 y = \frac{\log_2 8 - \log_2 2}{4-1}(x - 1) + \log_2 2$, or $\log_2 y = \frac{2}{3}x + \frac{1}{3}$. Exponentiate both sides, giving

$$y = 2^{(2/3)x + 1/3} = 2^{1/3} \cdot \left(2^{2/3}\right)^x.$$

**100.** The $y$-axis is a log scale; using base 2 logs we get as the equation of the line $\log_2 y = \frac{\log_2 2 - \log_2 8}{2-1}(x - 1) + \log_2 8$, or $\log_2 y = -2x + 5$. Exponentiate both sides, giving

$$y = 2^{-2x+5} = 32\left(\frac{1}{4}\right)^x.$$

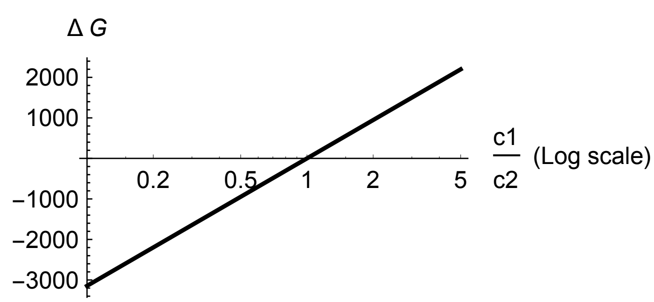
**101.** Since the $x$-axis is a log scale while the $y$-axis is linear, the equation of the line is

$$y = \frac{1-0}{\log 10 - \log 1}\log x = \log x.$$

**102.** Since the $x$-axis is a log scale while the $y$-axis is linear, the equation of the line is

$$y = \frac{1-3}{\log 1000 - \log 10}(\log x - \log 10) + 3 = 4 - \log x.$$

**103.**



**104.** **(a)** Invert both sides, obtaining

$$\frac{1}{f(x)} = 1 + e^{-(b+mx)}.$$

Subtract 1 from both sides, giving

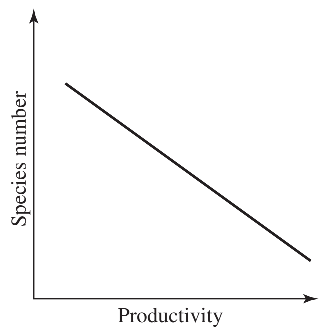
$$\frac{1 - f(x)}{f(x)} = e^{-(b+mx)}.$$

Now take natural logarithms of both sides, giving

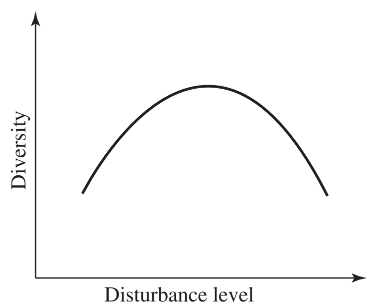
$$\ln \frac{1 - f(x)}{f(x)} = -(b + mx), \text{ or } \ln \frac{f(x)}{1 - f(x)} = b + mx.$$

## ■ 1.4.4

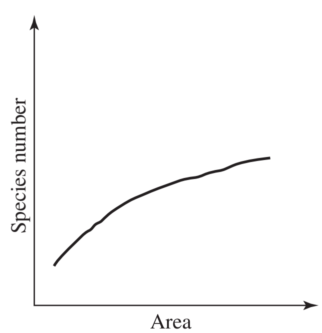
**105.**



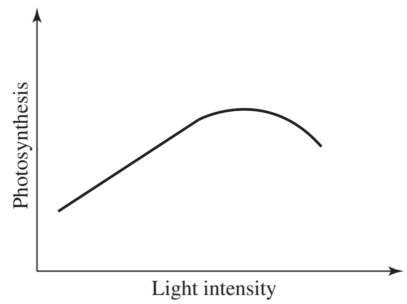

**106.**



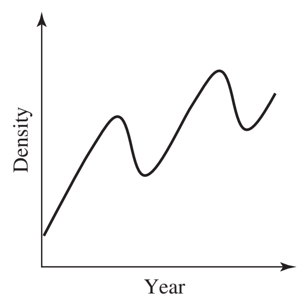

**107.**



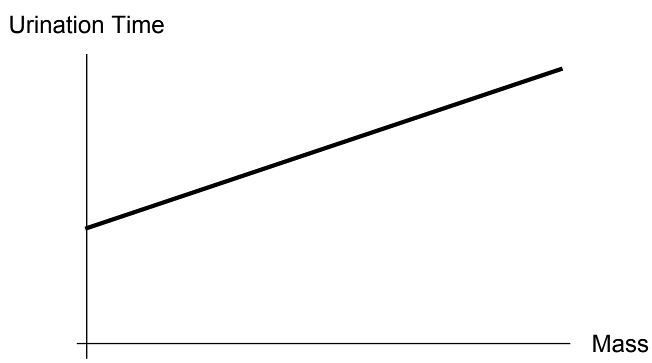
**108.**



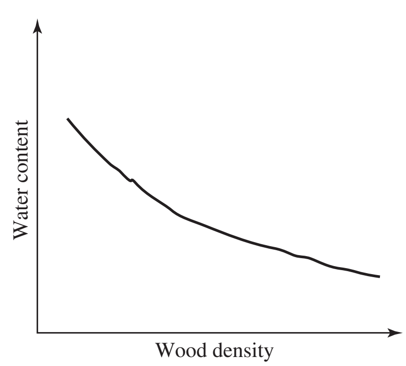

**109.**



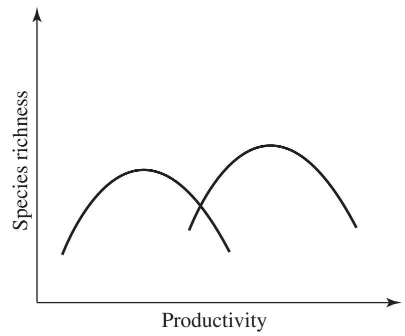

**110.**



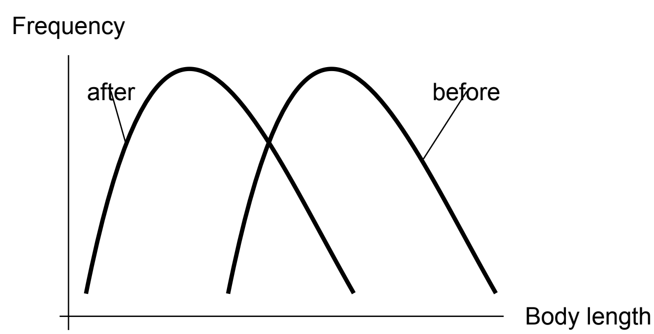

**111.**



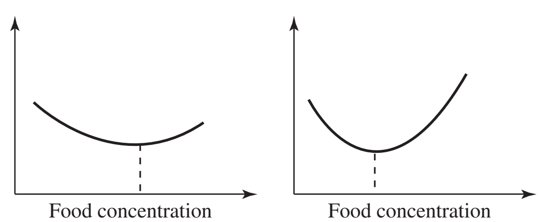
**112.**



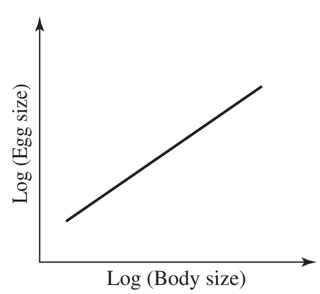

**113.**



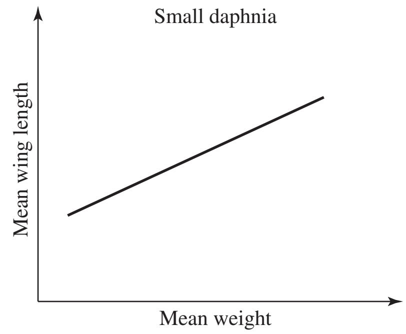

**114.**



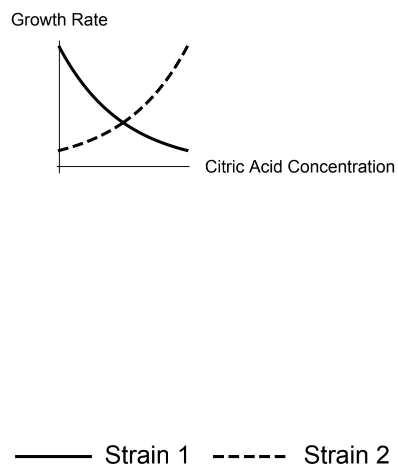

**115.**



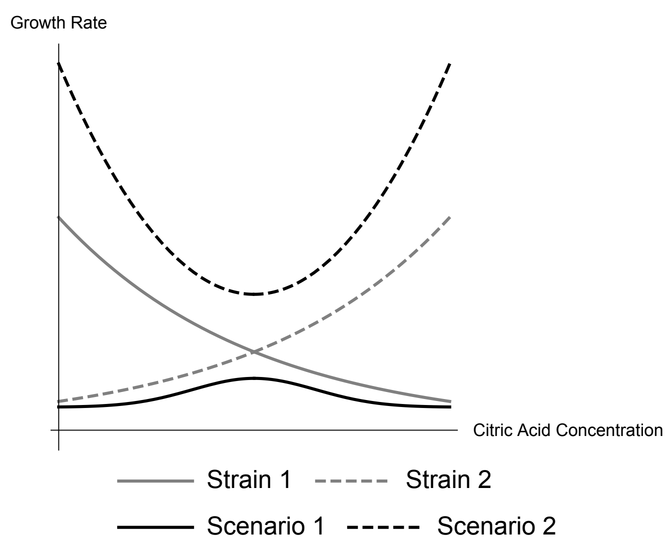
**116.**


Small daphnia
Mean wing length
Mean weight

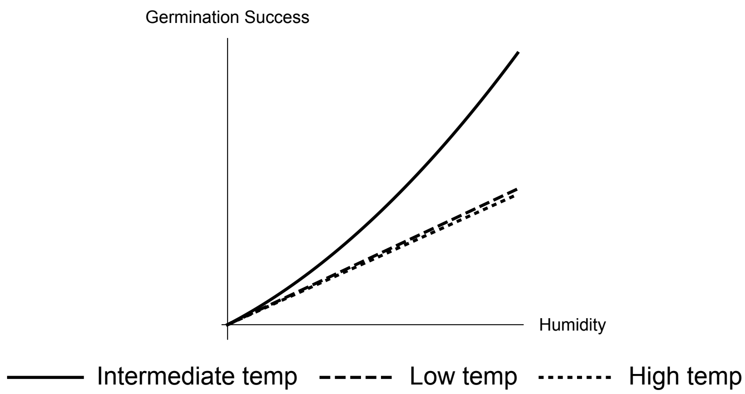
**117. (a)**


Growth Rate
Citric Acid Concentration
Strain 1
Strain 2

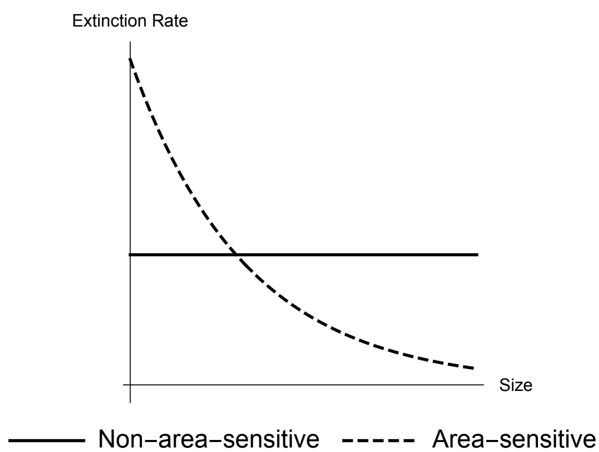
**(b)**


Growth Rate
Citric Acid Concentration
Strain 1
Strain 2
Scenario 1
Scenario 2

**118.** From Figure 1.84, at low temperature and at high temperature the response of growth rate to humidity changes is about the same, and roughly linear. At intermediate temperatures, germination success seems to respond more than linearly to humidity changes. This suggests graphs like these:




**119.**




## Chapter 1 Review

**1. (a)** $10^3$, $1.1 \times 10^3$, $1.22 \times 10^3$, $1.35 \times 10^3$, $1.49 \times 10^3$.

**(b)** Solving $100\,000 = 1000e^{0.1t}$ gives $\ln 100 = 0.1t$, so that $t = 10 \ln 100 \approx 46.05$ hour.

**2. (a)** After three hours, since $t$ is measured in hours the number of bacteria left is

$$B(3) = 25\,000e^{-1.5 \cdot 3} = 25\,000e^{-4.5} \approx 278.$$

**(b)** 1% of the bacteria are left when $B(3) = 0.01 \cdot 25\,000 = 250$. Solving $250 = 25\,000e^{-1.5t}$ for $t$, we get $0.01 = e^{-1.5t}$, so that $-1.5t = \ln 0.01 \approx -4.6$; then $t = \frac{4.6}{1.5} \approx 3.07$ hours.

**3. (a)** The reaction rate $R$ is proportional to the amount of the two reactants that are left. Denote by $k$ the constant of proportionality. If $y$ is the concentration of $C$, then the concentration of $D$ is $1 + y$. Since the reaction rate is proportional to the concentrations of $C$ and $D$, we can write $R(y) = ky(y+1)$. Here $0 \leq y \leq 1.5$, since there are 1.5 units of $C$, so that is the domain of $R(y)$. When $y = 0$, we have $R(y) = 0$, and when $y = 1.5$ we have $R(y) = k \cdot 1.5 \cdot 2.5 = 3.75k$. Since $R(y)$ is increasing, its range is $0 \leq R(y) \leq 3.75k$.

**(b)** The reaction rate $R$ is proportional to the amount of the two reactants that are left. Denote by $k$ the constant of proportionality. If $z$ is the concentration of $D$, then the concentration of $C$ is $z-1$. Since the reaction rate is proportional to the concentrations of $C$ and $D$, we can write $R(z)=k(z-1)z$. Here $0 \le z \le 2.5$, since there are 2.5 units of $D$, but in fact $z-1$ must be nonnegative also, since that is the number of units of $C$, so that also $z \ge 1$. Therefore the domain of $R(z)$ is $1 \le z \le 2.5$. When $z=1$, we have $R(z)=0$, and when $z=2.5$ we have $R(z)=k\cdot 1.5\cdot 2.5=3.75k$. Since $R(z)$ is increasing, its range is $0 \le R(z) \le 3.75k$.

**4. (a)** Using the given data values we get

$$\begin{aligned} 0 &= L(0) = L_\infty\left(1-e^{0k}\right) = 0 \\ 1 &= L(1) = L_\infty\left(1-e^{k}\right) \\ 1.5 &= L(2) = L_\infty\left(1-e^{2k}\right). \end{aligned}$$

Divide the third equation by the second, giving

$$1.5 = \frac{L_\infty\left(1-e^{2k}\right)}{L_\infty\left(1-e^{k}\right)} = \frac{1-e^{2k}}{1-e^{k}} = \frac{\left(1-e^{k}\right)\left(1+e^{k}\right)}{1-e^{k}} = 1+e^{k},$$

so that we must have $e^k=0.5$.

**(b)** Substituting $e^k=0.5$ into the second equation gives $1=L_\infty(1-0.5)$, so that $L_\infty=2$. Further, $k=\ln 0.5=-\ln 2$. This gives the equation $L(t)=2(1-e^{-t\ln 2})$.

**5. (a)** We have $L(1)=\ln 1+1=1$ and $E(1)=e^{1-1}=e^0=1$, so that the plants have the same height at $t=1$.

**(b)** Computing the two height equations gives

$$\begin{aligned} L(2) &= \ln(2)+1 \approx 1.693\,\text{ft} & E(2) &= e^{2-1} = e \approx 2.718\,\text{ft} \\ L(10) &= \ln(10)+1 \approx 3.303\,\text{ft} & E(10) &= e^{2-1} = e \approx 8103\,\text{ft} \\ L(100) &= \ln(100)+1 \approx 5.605\,\text{ft} & E(100) &= e^{2-1} = e \approx 9.9\times 10^{42}\,\text{ft}. \end{aligned}$$

**(c)** $L(t)=2L(1)=2$ when $\ln(t)+1=2$, or when $\ln t=1$. This happens at $t=e\approx 2.718$ years. So the time it takes the first plant to double its height is 2.718 years − 1 year = 1.718 years. On the other hand, $E(t)=2E(1)=2$ when $e^{t-1}=2$. This means that $t-1=\ln 2$ or $t=1+\ln 2\approx 1.693$ years, so the time it takes this plant to double its height is 1.693 years − 1 year = 0.693 years.

**(d)** $L(t)=4$ when $\ln(t)+1=4$, or when $\ln t=3$. This happens at $t=e^3\approx 20.086$ years. On the other hand, $E(t)=4$ when $e^{t-1}=4$. This means that $t-1=\ln 4=2\ln 2$ or $t=1+2\ln 2\approx 2.386$ years. So the time it takes the first plant to double again in height is $20.086-2.718\approx 17.368$ years, while the second plant doubles in height again in $2.386-1.693\approx 0.693$ years.

**(e)** $L(t)=3$ when $\ln(t)+1=3$, or when $\ln t=2$. This happens at $t=e^2\approx 7.389$ years. On the other hand, $E(t)=3$ when $e^{t-1}=3$. This means that $t-1=\ln 3$ or $t=1+\ln 3\approx 2.099$ years.

**6.** In the given model, the population (in pairs) is $2^N$, where $N$ is the number of doubling periods since the appearance of the first pair. Each doubling period is worth 25 years, so $N=40$ and 80 respectively. This gives $2^{40}\times 2=2.2\times 10^{12}$ individuals after 1000 years, and $2^{80}\times 2=2.4\times 10^{24}$ after 2000 years.

The area (in square feet) of the 29% of the earth's surface covered by land is $0.29\times 4\pi(5280\text{mi/ft}\times(7900/2)\text{mi})^2=1.58\times 10^{15}\text{ft}^2$ Therefore after 1000 years, with about 2 trillion people on earth, we would have (roughly) one person per 1000 square feet (the area of a modest house). After 2000 years, we would supposedly have over a billion people per square foot.

**7.** If the population grows by $q\%$ each year, then the population is $P_q(t) = A\left(1+\frac{q}{100}\right)^t$, where $A$ is the initial population. The doubling time is found by solving for $P_q(t) = 2A$:

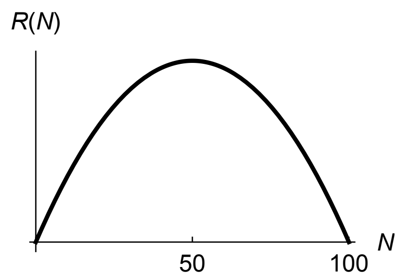
$$2A = A\left(1+\frac{q}{100}\right)^T \quad \Rightarrow \quad \ln 2 = T\ln\left(1+\frac{q}{100}\right) \quad \Rightarrow \quad T = \frac{\ln 2}{\ln\left(1+\frac{q}{100}\right)}.$$

A table showing doubling time, $T$, as a function of $q$ is below:

| $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | 69.7 | 35. | 23.4 | 17.7 | 14.2 | 11.9 | 10.2 | 9.0 | 8.0 | 7.3 |

As $q$ gets closer to zero, the denominator of the equation for $T$ gets closer to $\ln 1$, which is zero. Thus $T$ goes to infinity. This makes sense — if the population grows very slowly it will take a very long time for it to double.

**8. (a)**

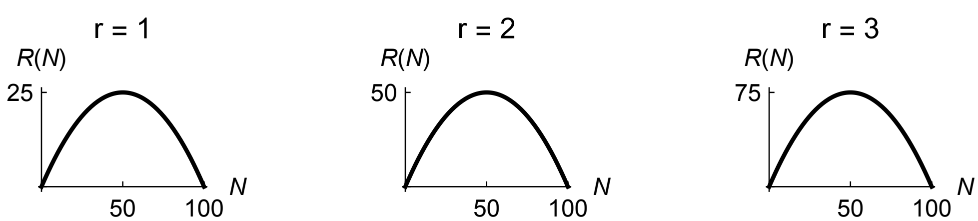



The function is

$$R(N) = rN\left(1-\frac{N}{100}\right).$$

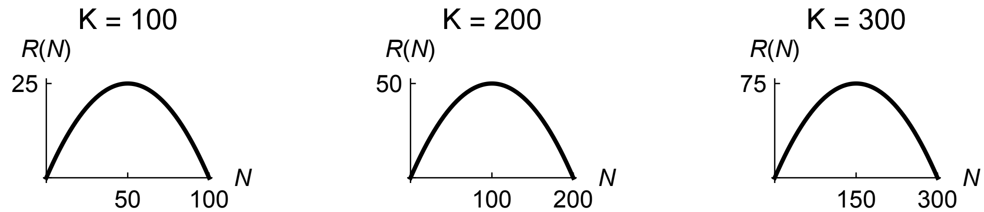
From the graph, it reaches its maximum value when $N = 50$, and that maximum is $R(50) = 50r\left(1-\frac{1}{2}\right) = 25r$, so the range of $R(N)A$ is $R(N) \le 25r$. (We do not show any points on the curve where the reproduction rate is negative, but certainly the death rate could exceed the birth rate. If you wish to exclude this possibility, then the range is $0 \le R(N) \le 25r$.) The domain consists of all nonnegative values of $N$, so $0 \le N$. (Again, if you wish to exclude the possibility of negative reproduction rates, the domain becomes $0 \le N \le 100$.)

**(b)**



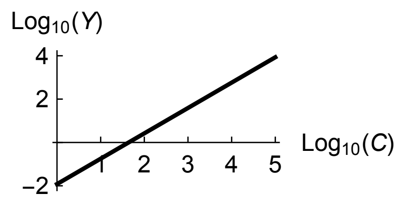

The reproduction rate appears to depend linearly on $r$; that is, the maximum for $r = 2$ is twice the maximum for $r = 1$, and similarly for $r = 3$.

**(c)**




The reproduction rate becomes zero when $N = K$, so that it the point at which the population stops increasing. This implies that the area cannot support more than $K$ individuals.

**9.** **(a)**




**(b)** $Y = C^{1.17}10^{-1.92}$

**(c)** $Y_p = 2.25Y_c$

**(d)** 8.5%

**10.** **(a)** Let $x$ be the ratio $C^{14}$ : $C^{12}$ At the time of death, $x = x_0 = 10^{-12}$ Subsequently, $x$ halves every 5730 years; more precisely, there is an exponential decay given by

$$x = x_0 2^{-t/5730\text{yr}},$$

where $t$ is time since death. We must invert this formula to find $t$. Multiplying by $1/x_0$ and taking logs in base 2 gives

$$t = -(5730\text{yr}) \log_2(x/x_0),$$

or, using the formula $\log_2 a = \log_{10} a / \log_{10} 2$,

$$t = -(5730\text{yr}/0.301) \log_{10}(x/x_0) = (19.0 \times 10^3\text{yr})(-12 - \log_{10} x).$$

**(b)** Here $x = 1.61 \times 10^{-13}$, whose $\log_{10}$ is $-12.793$. Therefore $t = (19.0 \times 10^3\text{yr}) \times 0.793$, or about 15,000 years.

**11.** **(a)** An hour is defined as 3600 s, so the length of an hour is constant over time and therefore the number of hours per year is constant. The number of seconds per year is

$$24\,\text{hours/day} \cdot 3600\,\text{s/hour} \cdot 365.25\,\text{days/year} \approx 3.155\,76 \times 10^7\,\text{s/year}.$$

Every million years the length of a day is decreased by 20 s, so in 380 million years the length of a day decreases by 7600 s. Thus the length of a day 380 million years ago was $24\,\text{hours/day} \cdot 3600\,\text{s/hour} - 7600\,\text{s/day} = 78\,800\,\text{s/day}$. It follows that the length of a year in days at that time was

$$\frac{3.155\,76 \times 10^7\,\text{s/year}}{78\,800\,\text{s/day}} = 400.477\,\text{days/year},$$

so the length of a year was about 400 days.

**(b)** From the above discussion, the number of days per year $t$ million years ago is

$$\frac{24 \cdot 3600 \cdot 365.25}{24 \cdot 3600 - 20t} \text{ days/year.}$$

The number of hours per year is constant, so we can compute it from today's value, which is $24 \cdot 365.25$ hours/year. So the number of hours per day $t$ million years ago was

$$\frac{24 \cdot 365.25 \text{ hours/year}}{\frac{24 \cdot 3600 \cdot 365.25}{24 \cdot 3600 - 20t} \text{ days/year}} = \frac{86\,400 - 20t}{3600} = 24 - \frac{1}{180}t.$$

**(c)** From the first equation in part (b), we know a formula for the number of days per year $t$ million years ago. So if we want to know when there were $d$ days per year, we must solve

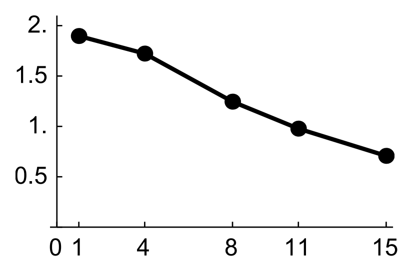
$$\frac{24 \cdot 3600 \cdot 365.25}{24 \cdot 3600 - 20t} = d.$$

Inverting both sides and simplifying gives $\frac{1}{365.25} - \frac{20}{24 \cdot 3600 \cdot 365.25}t = \frac{1}{d}$, so that

$$t = \left(\frac{1}{365.25} - \frac{1}{d}\right) \cdot \frac{24 \cdot 3600 \cdot 365.25}{20} = 4320 - \frac{1.577\,88 \times 10^6}{d}.$$

Evaluating for $d = 400$ and $d = 420$ gives $t = 375.3$ and $t = 563.1$, so the stromatolites were probably between 375 and 563 million years old.

**12.** **(a)** Plotting the time against the log of the virus concentration gives the following graph:




**(b)** The number of viruses is halved when the graph decreases by $\log 2 \approx 0.301$. The value at $x = 1$ is approximately 1.9, and the value at $x = 15$ is about 0.7, so the slope is $\frac{0.7-1.9}{14} \approx -0.086$. So the time it takes to halve the amount of virus is $\frac{0.301}{0.086} \approx 3.5$ days.

**(c)** No, assuming the model is accurate, since it is an exponential model and exponentials never reach zero. However, since the viral count is an integer, at some point the model will predict substantially less than one virus, so one could say that the viral load at that time was zero.

**13.** **(a)** For males, $S(t) = \exp\left(-(0.019t)^{3.41}\right)$; for females, $S(t) = \exp\left(-(0.022t)^{3.24}\right)$.

**(b)**

$$\begin{aligned}
\text{Males:}\quad & 0.5 = \exp\left(-(0.019t)^{3.41}\right) \quad\Rightarrow\quad \ln 0.5 = -(0.019t)^{3.41} \quad\Rightarrow\quad 0.693 = (0.019t)^{3.41} \\
& \Rightarrow\quad \ln 0.693 = 3.41 \ln(0.019t) \quad\Rightarrow\quad t \approx 47.27 \text{ days} \\
\text{Females:}\quad & 0.5 = \exp\left(-(0.022t)^{3.24}\right) \quad\Rightarrow\quad \ln 0.5 = -(0.022t)^{3.24} \quad\Rightarrow\quad 0.693 = (0.022t)^{3.24} \\
& \Rightarrow\quad \ln 0.693 = 3.24 \ln(0.022t) \quad\Rightarrow\quad t \approx 40.59 \text{ days.}
\end{aligned}$$

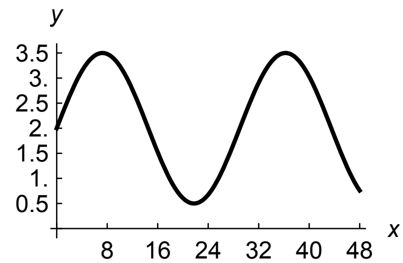
**(c)** Males should live longer.

**14.** **(a)** Since this is a circadian clock, the period should be 24 hours, and indeed, looking at the next time after $t = 0$ where the graph reaches 2.0 and is increasing, that is at $t = 24$.

**(b)** This looks like a sine curve that has been shifted up by 2 units. Its amplitude is $3.5 - 2 = 1.5$, and its period is $\frac{2\pi}{24}$. Therefore

$$E(t) = 2.0 + 1.5 \sin\left(\frac{2\pi}{24}t\right).$$

**(c)** The formula for the *frq*[7] expression level is $E(t) = 2.0 + 1.5 \sin\left(\frac{2\pi}{29}t\right)$, since it is the same as for *frq* except for the period. A graph of this function is




**15.** **(a)** Setting $x = k$ in the equation gives

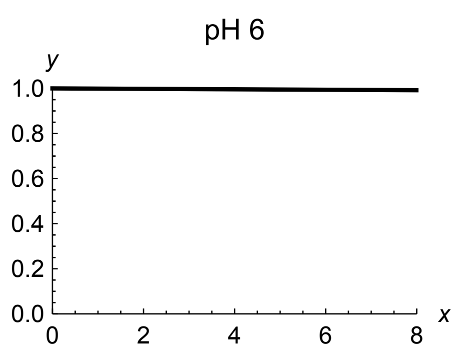
$$v = \frac{ak}{k+k} = \frac{ak}{2k} = \frac{a}{2}.$$

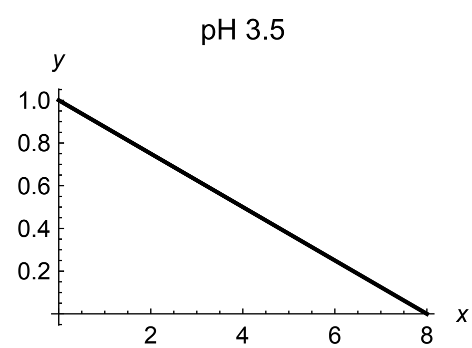
**(b)** Solving for $x$ in both instances gives

$$0.1a = \frac{ax}{k+x} \quad\Rightarrow\quad 0.1 = \frac{x}{k+x} \quad\Rightarrow\quad 0.1(k+x) = x \quad\Rightarrow\quad x = \frac{k}{9}$$
$$0.9a = \frac{ax}{k+x} \quad\Rightarrow\quad 0.9 = \frac{x}{k+x} \quad\Rightarrow\quad 0.9(k+x) = x \quad\Rightarrow\quad x = 9k.$$

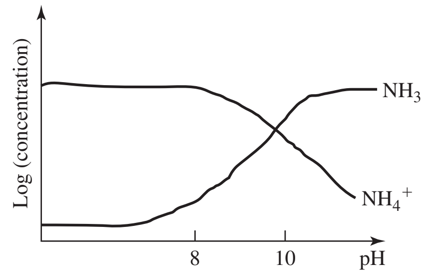
The ratio of these two values is indeed 81, so an 81-fold change in substrate concentration is required.

**16.** The problem does not give enough information to plot a precise curve of survivorship versus time. Qualitatively, what we can say is that the curve at pH6 starts at 100% and remains essentially horizontal for an unspecified period of time (days or weeks rather than hours). On the other hand at pH3.5 the curve, again starting at 100%, slopes down toward 0%, which it reaches after 8 hours. Whether it has a more or less constant negative slope during this period, or whether the dying off is concentrated at the beginning or at the end or follows some other pattern, we cannot

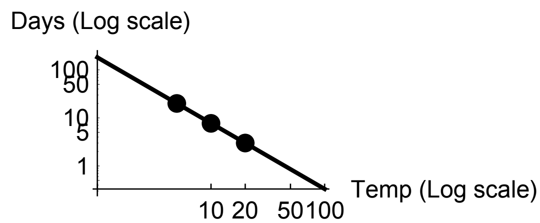


**17.**




**18.** **(a)** To make the log-log plot, we must first express the temperature $T$ and the development time $t$ as pure numbers, since we can only take logs of pure numbers. We therefore plot $x = \log_{10}(T/\circ\mathrm{C})$ on the horizontal axis and $y = \log_{10}(T/\mathrm{day})$ on the vertical axis. (One can equally well use natural logarithms or logarithms in any other base; this just corresponds to a change of scale.) Two data points are given, $(5\,°\mathrm{C}, 20\,\mathrm{days})$ and $(20\,°\mathrm{C}, 3\,\mathrm{days})$. The plot is therefore a straight line through these points on a log-log plot:




(The third point, corresponding to 10°, is the point computed in part (c)).

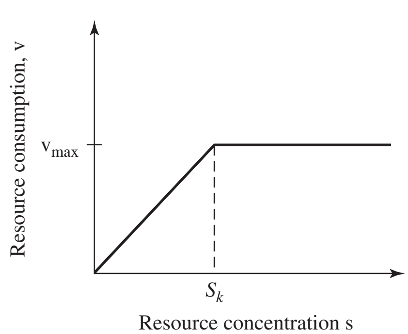
**(b)** The straight line has equation

$$Y = \log_{10} 20 + (X - \log_{10} 5) \cdot \frac{\log_{10} 3 - \log_{10} 20}{\log_{10} 20 - \log_{10} 5} = 1.301 + (X - 0.699) \frac{0.477 - 1.301}{1.301 - .699} = 2.26 - 1.37X.$$

**(c)** At $T = 10°$ we have $X = \log_{10}(10) = 1.00$, so $Y = 0.89$ and $t = 10^{0.89} \approx 7.8$ days.

**(d)** No. If the centigrade and Fahrenheit scales had their zeros at the same temperature, the two would be proportional, and their logarithms would differ by a constant, so the linear character of the plot would be preserved. But that is not the case, since $0\,°\mathrm{C} = 32\,°\mathrm{F}$. Therefore the log of a quantity measured in degrees centigrade and the log of the same quantity measured in degrees Fahrenheit are related by a nonlinear relation, and this means that a function that is linear on one of the logs is not linear on the other.

**19.** Since $g(0) = 0$, $g(S_k) = v_{\max}$, and the function is linear, it has slope $\frac{v_{\max}}{S_k}$. Since it passes through $(0, 0)$, its equation is $v = \frac{v_{\max}}{S_k} S$.



**20.** **(a)** For red light, $\frac{I(z)}{I(0)} = 0.35$ (that is, 100%-65%) at a depth of 1 m. Thus $0.35 = e^{-\alpha \cdot 1\text{m}}$. Taking natural logs we get $-1.05 = -\alpha \cdot 1\,\text{m}$, so $\alpha = 1.05\,\text{m}^{-1}$.

**(b)** For blue light, $0.95 = e^{-\alpha \cdot 1\,\text{m}}$. Taking natural logs we get $-0.05 = -\alpha \cdot 1\,\text{m}$, so $\alpha = 0.05\,\text{m}^{-1}$. The attenuation of red light (720 nm) is about 21 times that of blue light (475 nm).

**(c)** A few meters below the surface, the red component of sunlight has been almost entirely filtered out. (However if a diver carries her own white light source, she will be able to see red hues on objects very near her.)

**21.** **(a)** Since $N(0) = N_0 e^{r \cdot 0} = N_0$, we have $N_0 = 2000$. Now, $4000 = N(1) = 2000e^r$, so that $e^r = 2$ and then $r = \ln 2 \approx 0.693$. The model is $N(t) = 2000e^{t \ln 2} = 2000 \cdot 2^t$.

**(b)** The population size is 10 000 when $10\,000 = 2000 \cdot 2^t$, so when $2^t = 5$. This happens when $t = \frac{\ln 5}{\ln 2} \approx 2.322$ hours.

**(c)** The maximum value of $r$ corresponds to the highest growth rate, which occurs when $N(0) = 1800$ and $N(1) = 4000 + 0.1 \cdot 4000 = 4400$. In this case $N(0) = N_0 = 1800$, and $4400 = N(1) = 1800e^r$, so that $r = \ln \frac{4400}{1800} \approx 0.893$. The minimum value of $r$ corresponds to the lowest growth rate, which occurs when $N(0) = 2200$ and $N(1) = 4000 - 0.1 \cdot 4000 = 3600$. In this case $N(0) = N_0 = 2200$, and $3600 = N(1) = 2200e^r$, so that $r = \ln \frac{3600}{2200} \approx 0.492$. The maximum and minimum consistent values of $r$ are 0.893 and 0.492.

**22.** **(a)** Consider $\frac{N(t+1)}{N(t)} = \frac{N_0 e^{r(t+1)}}{N_0 e^{rt}} = e^{rt+r-rt} = e^r$. Hence, $r = \ln \frac{N(t+1)}{N(t)}$.

**(b)** Suppose that doubling time is 1 year, then:

**(i)** In that year it has increased 100%

**(ii)** Since $r = \ln \frac{N(t+1)}{N(t)} = \ln 2 \approx 0.693$, it is 69%.

**(c)** We have:

$$2N_0 = N_0 \cdot e^{0.013t} \quad \Rightarrow \quad 2 = e^{0.013t} \quad \Rightarrow \quad \ln 2 = 0.013t \quad \Rightarrow \quad t = \frac{\ln 2}{0.013} \approx 53.3\,\text{years}.$$

**(d)** We have $\frac{70}{1.3} \approx 53.8$ years. The rule of 70 is an approximation to $\ln 2 \approx 0.693$, in fact:

$$2 = e^{rt} \quad \Rightarrow \quad t = \frac{\ln 2}{r} \approx \frac{0.693}{r}.$$

Now if we regard $r$ as a percentage, we get $\frac{0.693}{r/100} = \frac{69.3}{r}$, which is close to $\frac{70}{r}$.