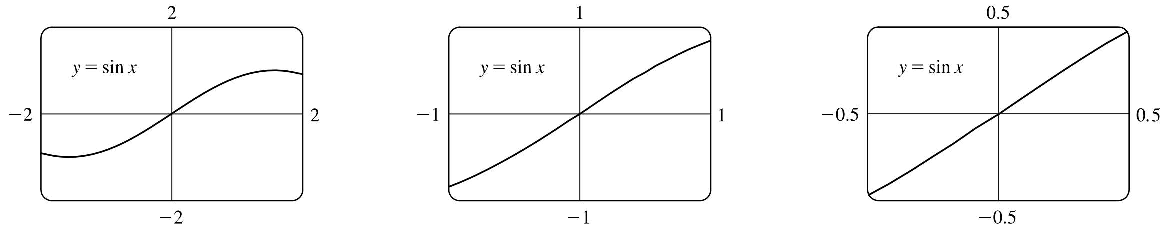
2 ☐ DERIVATIVES

2.1 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$.

2. The curve looks more like a line as the viewing rectangle gets smaller.



3. (a) (i) Using Definition 1 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{(4x - x^2) - 3}{x - 1} = \lim_{x \to 1} \frac{-(x^2 - 4x + 3)}{x - 1} = \lim_{x \to 1} \frac{-(x - 1)(x - 3)}{x - 1}$$

$$= \lim_{x \to 1} (3 - x) = 3 - 1 = 2$$

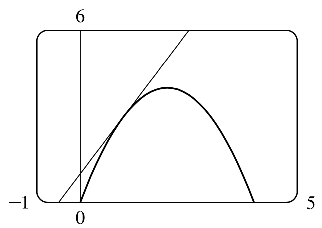
(ii) Using Equation 2 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[4(1 + h) - (1 + h)^2\right] - 3}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h \to 0} \frac{-h^2 + 2h}{h} = \lim_{h \to 0} \frac{h(-h + 2)}{h} = \lim_{h \to 0} (-h + 2) = 2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \quad \Rightarrow \quad y - f(1) = f'(1)(x - 1) \quad \Rightarrow \quad y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(c)



The graph of $y = 2x + 1$ is tangent to the graph of $y = 4x - x^2$ at the point $(1, 3)$. Now zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistiguishable.

4. (a) (i) Using Definition 1 with $f(x) = x - x^3$ and $P(1, 0)$,

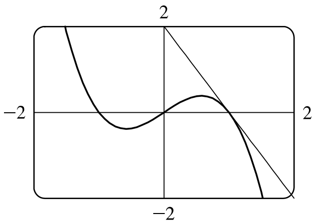
$$m = \lim_{x \to 1} \frac{f(x) - 0}{x - 1} = \lim_{x \to 1} \frac{x - x^3}{x - 1} = \lim_{x \to 1} \frac{x(1 - x^2)}{x - 1} = \lim_{x \to 1} \frac{x(1 + x)(1 - x)}{x - 1}$$

$$= \lim_{x \to 1} [-x(1 + x)] = -1(2) = -2$$

(ii) Using Equation 2 with $f(x) = x - x^3$ and $P(1, 0)$,

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\left[(1+h) - (1+h)^3\right] - 0}{h}$$
$$= \lim_{h\to 0} \frac{1 + h - (1 + 3h + 3h^2 + h^3)}{h} = \lim_{h\to 0} \frac{-h^3 - 3h^2 - 2h}{h} = \lim_{h\to 0} \frac{h(-h^2 - 3h - 2)}{h}$$
$$= \lim_{h\to 0}(-h^2 - 3h - 2) = -2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \quad \Rightarrow \quad y - f(1) = f'(1)(x - 1) \quad \Rightarrow \quad y - 0 = -2(x - 1)$, or $y = -2x + 2$.

(c) 

The graph of $y = -2x + 2$ is tangent to the graph of $y = x - x^3$ at the point $(1, 0)$. Now zoom in toward the point $(1, 0)$ until the cubic and the tangent line are indistinguishable.

5. Using (1) with $f(x) = 4x - 3x^2$ and $P(2, -4)$ [we could also use (2)],

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 2} \frac{(4x - 3x^2) - (-4)}{x - 2} = \lim_{x\to 2} \frac{-3x^2 + 4x + 4}{x - 2}$$
$$= \lim_{x\to 2} \frac{(-3x - 2)(x - 2)}{x - 2} = \lim_{x\to 2}(-3x - 2) = -3(2) - 2 = -8$$

Tangent line: $y - (-4) = -8(x - 2) \quad \Leftrightarrow \quad y + 4 = -8x + 16 \quad \Leftrightarrow \quad y = -8x + 12$.

6. Using (2) with $f(x) = x^3 - 3x + 1$ and $P(2, 3)$,

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f\,(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^3 - 3(2+h) + 1 - 3}{h}$$
$$= \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 6 - 3h - 2}{h} = \lim_{h\to 0} \frac{9h + 6h^2 + h^3}{h} = \lim_{h\to 0} \frac{h(9 + 6h + h^2)}{h}$$
$$= \lim_{h\to 0}(9 + 6h + h^2) = 9$$

Tangent line: $y - 3 = 9(x - 2) \quad \Leftrightarrow \quad y - 3 = 9x - 18 \quad \Leftrightarrow \quad y = 9x - 15$

7. Using (1),

$$m = \lim_{x\to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x\to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x\to 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x\to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$$

Tangent line: $y - 1 = \frac{1}{2}(x - 1) \quad \Leftrightarrow \quad y = \frac{1}{2}x + \frac{1}{2}$

8. Using (1) with $f(x) = \dfrac{2x+1}{x+2}$ and $P(1,1)$,

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 1} \frac{\dfrac{2x+1}{x+2} - 1}{x - 1} = \lim_{x\to 1} \frac{\dfrac{2x+1-(x+2)}{x+2}}{x-1} = \lim_{x\to 1} \frac{x-1}{(x-1)(x+2)}$$

$$= \lim_{x\to 1} \frac{1}{x+2} = \frac{1}{1+2} = \frac{1}{3}$$

Tangent line: $y - 1 = \frac{1}{3}(x-1) \quad\Leftrightarrow\quad y - 1 = \frac{1}{3}x - \frac{1}{3} \quad\Leftrightarrow\quad y = \frac{1}{3}x + \frac{2}{3}$

9. (a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$,

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{3 + 4(a+h)^2 - 2(a+h)^3 - (3 + 4a^2 - 2a^3)}{h}$$

$$= \lim_{h\to 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h}$$

$$= \lim_{h\to 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h}$$

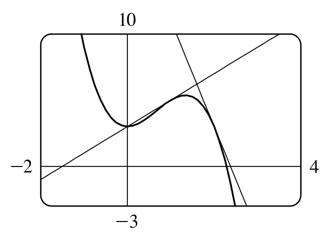
$$= \lim_{h\to 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h} = \lim_{h\to 0} \frac{h(8a + 4h - 6a^2 - 6ah - 2h^2)}{h}$$

$$= \lim_{h\to 0}(8a + 4h - 6a^2 - 6ah - 2h^2) = 8a - 6a^2$$

(b) At $(1,5)$: $m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is $y - 5 = 2(x-1) \quad\Leftrightarrow\quad y = 2x + 3$.

At $(2,3)$: $m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent line is $y - 3 = -8(x-2) \quad\Leftrightarrow\quad y = -8x + 19$.

(c)



10. (a) Using (1),

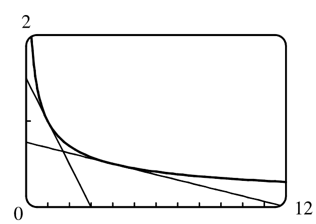
$$m = \lim_{x\to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} = \lim_{x\to a} \frac{\dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}}}{x - a} = \lim_{x\to a} \frac{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})}{\sqrt{ax}\,(x-a)(\sqrt{a} + \sqrt{x})} = \lim_{x\to a} \frac{a - x}{\sqrt{ax}\,(x-a)(\sqrt{a} + \sqrt{x})}$$

$$= \lim_{x\to a} \frac{-1}{\sqrt{ax}\,(\sqrt{a} + \sqrt{x})} = \frac{-1}{\sqrt{a^2}\,(2\sqrt{a})} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2} \quad [a > 0]$$

(b) At $(1,1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x-1) \quad\Leftrightarrow\quad y = -\frac{1}{2}x + \frac{3}{2}$.

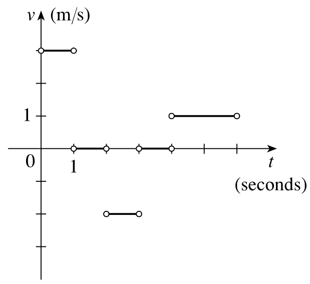
At $\left(4, \frac{1}{2}\right)$: $m = -\frac{1}{16}$, so an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x-4) \quad\Leftrightarrow\quad y = -\frac{1}{16}x + \frac{3}{4}$.

(c)



11. (a) The particle is moving to the right when s is increasing; that is, on the intervals $(0,1)$ and $(4,6)$. The particle is moving to the left when s is decreasing; that is, on the interval $(2,3)$. The particle is standing still when s is constant; that is, on the intervals $(1,2)$ and $(3,4)$.

(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval $(0, 1)$, the slope is $\frac{3-0}{1-0} = 3$. On the interval $(2, 3)$, the slope is $\frac{1-3}{3-2} = -2$. On the interval $(4, 6)$, the slope is $\frac{3-1}{6-4} = 1$.



12. (a) **Runner A** runs the entire 100-meter race at the same velocity since the slope of the position function is constant. **Runner B** starts the race at a slower velocity than runner A, but finishes the race at a faster velocity.

(b) The distance between the runners is the greatest at the time when the largest vertical line segment fits between the two graphs—this appears to be somewhere between 9 and 10 seconds.

(c) The runners had the same velocity when the slopes of their respective position functions are equal—this also appears to be at about 9.5 s. Note that the answers for parts (b) and (c) must be the same for these graphs because as soon as the velocity for runner B overtakes the velocity for runner A, the distance between the runners starts to decrease.

13. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \to 2} \frac{(40t - 16t^2) - 16}{t - 2} = \lim_{t \to 2} \frac{-16t^2 + 40t - 16}{t - 2} = \lim_{t \to 2} \frac{-8(2t^2 - 5t + 2)}{t - 2}$$

$$= \lim_{t \to 2} \frac{-8(t-2)(2t-1)}{t-2} = -8 \lim_{t \to 2} (2t - 1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

14. (a) Let $H(t) = 10t - 1.86t^2$.

$$v(1) = \lim_{h \to 0} \frac{H(1+h) - H(1)}{h} = \lim_{h \to 0} \frac{\left[10(1+h) - 1.86(1+h)^2\right] - (10 - 1.86)}{h}$$

$$= \lim_{h \to 0} \frac{10 + 10h - 1.86(1 + 2h + h^2) - 10 + 1.86}{h}$$

$$= \lim_{h \to 0} \frac{10 + 10h - 1.86 - 3.72h - 1.86h^2 - 10 + 1.86}{h}$$

$$= \lim_{h \to 0} \frac{6.28h - 1.86h^2}{h} = \lim_{h \to 0} (6.28 - 1.86h) = 6.28$$

The velocity of the rock after one second is 6.28 m/s.

(b) $$v(a) = \lim_{h \to 0} \frac{H(a+h) - H(a)}{h} = \lim_{h \to 0} \frac{\left[10(a+h) - 1.86(a+h)^2\right] - (10a - 1.86a^2)}{h}$$

$$= \lim_{h \to 0} \frac{10a + 10h - 1.86(a^2 + 2ah + h^2) - 10a + 1.86a^2}{h}$$

$$= \lim_{h \to 0} \frac{10a + 10h - 1.86a^2 - 3.72ah - 1.86h^2 - 10a + 1.86a^2}{h} = \lim_{h \to 0} \frac{10h - 3.72ah - 1.86h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(10 - 3.72a - 1.86h)}{h} = \lim_{h \to 0} (10 - 3.72a - 1.86h) = 10 - 3.72a$$

The velocity of the rock when $t = a$ is $(10 - 3.72a)$ m/s.

(c) The rock will hit the surface when $H = 0 \;\Leftrightarrow\; 10t - 1.86t^2 = 0 \;\Leftrightarrow\; t(10 - 1.86t) = 0 \;\Leftrightarrow\; t = 0$ or $1.86t = 10$. The rock hits the surface when $t = 10/1.86 \approx 5.4$ s.

(d) The velocity of the rock when it hits the surface is $v\left(\frac{10}{1.86}\right) = 10 - 3.72\left(\frac{10}{1.86}\right) = 10 - 20 = -10$ m/s.

15. $$v(a) = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\dfrac{a^2 - (a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2}$$
$$= \lim_{h\to 0} \frac{-(2ah + h^2)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-h(2a+h)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-(2a+h)}{a^2(a+h)^2} = \frac{-2a}{a^2 \cdot a^2} = \frac{-2}{a^3} \text{ m/s}$$

So $v(1) = \dfrac{-2}{1^3} = -2$ m/s, $v(2) = \dfrac{-2}{2^3} = -\dfrac{1}{4}$ m/s, and $v(3) = \dfrac{-2}{3^3} = -\dfrac{2}{27}$ m/s.

16. (a) The average velocity between times t and $t + h$ is
$$\frac{s(t+h) - s(t)}{(t+h) - t} = \frac{\frac{1}{2}(t+h)^2 - 6(t+h) + 23 - \left(\frac{1}{2}t^2 - 6t + 23\right)}{h}$$
$$= \frac{\frac{1}{2}t^2 + th + \frac{1}{2}h^2 - 6t - 6h + 23 - \frac{1}{2}t^2 + 6t - 23}{h}$$
$$= \frac{th + \frac{1}{2}h^2 - 6h}{h} = \frac{h\left(t + \frac{1}{2}h - 6\right)}{h} = \left(t + \tfrac{1}{2}h - 6\right) \text{ ft/s}$$

(i) $[4, 8]$: $t = 4$, $h = 8 - 4 = 4$, so the average velocity is $4 + \frac{1}{2}(4) - 6 = 0$ ft/s.

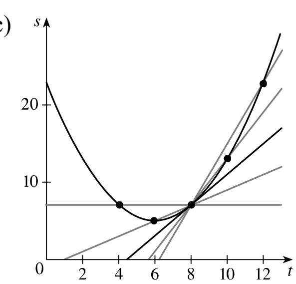
(ii) $[6, 8]$: $t = 6$, $h = 8 - 6 = 2$, so the average velocity is $6 + \frac{1}{2}(2) - 6 = 1$ ft/s.

(iii) $[8, 10]$: $t = 8$, $h = 10 - 8 = 2$, so the average velocity is $8 + \frac{1}{2}(2) - 6 = 3$ ft/s.

(iv) $[8, 12]$: $t = 8$, $h = 12 - 8 = 4$, so the average velocity is $8 + \frac{1}{2}(4) - 6 = 4$ ft/s.

(b) $v(t) = \lim_{h\to 0} \dfrac{s(t+h) - s(t)}{h} = \lim_{h\to 0} \left(t + \frac{1}{2}h - 6\right)$
$= t - 6$, so $v(8) = 2$ ft/s.

(c)



17. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

18. (a) On $[20, 60]$: $\dfrac{f(60) - f(20)}{60 - 20} = \dfrac{700 - 300}{40} = \dfrac{400}{40} = 10$

(b) Pick any interval that has the same y-value at its endpoints. $[0, 57]$ is such an interval since $f(0) = 600$ and $f(57) = 600$.

(c) On $[40, 60]$: $\dfrac{f(60) - f(40)}{60 - 40} = \dfrac{700 - 200}{20} = \dfrac{500}{20} = 25$

On $[40, 70]$: $\dfrac{f(70) - f(40)}{70 - 40} = \dfrac{900 - 200}{30} = \dfrac{700}{30} = 23\frac{1}{3}$

Since $25 > 23\frac{1}{3}$, the average rate of change on $[40, 60]$ is larger.

(d) $\dfrac{f(40) - f(10)}{40 - 10} = \dfrac{200 - 400}{30} = \dfrac{-200}{30} = -6\frac{2}{3}$

This value represents the slope of the line segment from $(10, f(10))$ to $(40, f(40))$.

19. (a) The tangent line at $x = 50$ appears to pass through the points $(43, 200)$ and $(60, 640)$, so

$f'(50) \approx \dfrac{640 - 200}{60 - 43} = \dfrac{440}{17} \approx 26.$

(b) The tangent line at $x = 10$ is steeper than the tangent line at $x = 30$, so it is larger in magnitude, but less in numerical value, that is, $f'(10) < f'(30)$.

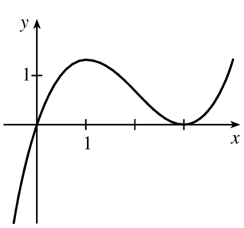
(c) The slope of the tangent line at $x = 60$, $f'(60)$, is greater than the slope of the line through $(40, f(40))$ and $(80, f(80))$. So yes, $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$.

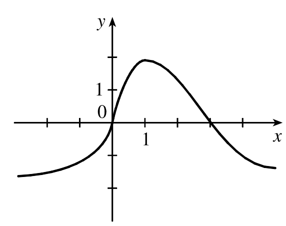
20. Since $g(5) = -3$, the point $(5, -3)$ is on the graph of g. Since $g'(5) = 4$, the slope of the tangent line at $x = 5$ is 4. Using the point-slope form of a line gives us $y - (-3) = 4(x - 5)$, or $y = 4x - 23$.

21. For the tangent line $y = 4x - 5$: when $x = 2$, $y = 4(2) - 5 = 3$ and its slope is 4 (the coefficient of x). At the point of tangency, these values are shared with the curve $y = f(x)$; that is, $f(2) = 3$ and $f'(2) = 4$.

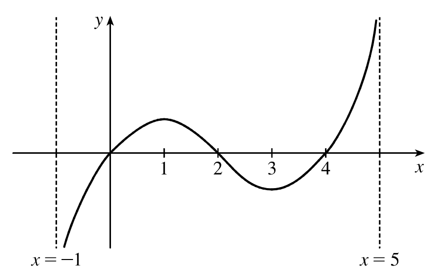
22. Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

23. We begin by drawing a curve through the origin with a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Last, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.

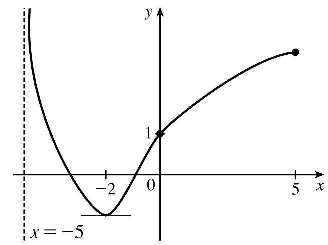




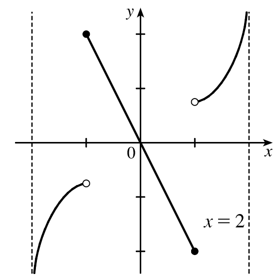
24. The condition $g(0) = g(2) = g(4) = 0$ means that the graph intersects the x-axis at $(0, 0)$, $(2, 0)$, and $(4, 0)$. The condition $g'(1) = g'(3) = 0$ means that the graph has horizontal tangents at $x = 1$ and $x = 3$. The conditions $g'(0) = g'(4) = 1$ and $g'(2) = -1$ mean that the tangents at $(0, 0)$ and $(4, 0)$ have slope 1, while the tangent at $(2, 0)$ has slope -1. Finally, the conditions $\lim_{x \to 5^-} g(x) = \infty$ and $\lim_{x \to -1^+} g(x) = -\infty$ imply that $x = -1$ and $x = 5$ are vertical asymptotes. A sample graph is shown. Note that the function shown has domain $(-1, 5)$. That domain could easily be extended by drawing additional graph segments in $(-\infty, -1]$ and $[5, \infty)$ that satisfy the vertical line test.



25. We begin by drawing a curve through $(0, 1)$ with a slope of 1 to satisfy $g(0) = 1$ and $g'(0) = 1$. We round off our figure at $x = -2$ to satisfy $g'(-2) = 0$. As $x \to -5^+$, $y \to \infty$, so we draw a vertical asymptote at $x = -5$. As $x \to 5^-$, $y \to 3$, so we draw a dot at $(5, 3)$ [the dot could be open or closed].



26. We begin by drawing an odd function (symmetric with respect to the origin) through the origin with slope -2 to satisfy $f'(0) = -2$. Now draw a curve starting at $x = 1$ and increasing without bound as $x \to 2^-$ since $\lim_{x \to 2^-} f(x) = \infty$. Lastly, reflect the last curve through the origin (rotate 180°) since f is an odd function.



27. Using (4) with $f(x) = 3x^2 - x^3$ and $a = 1$,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[3(1+h)^2 - (1+h)^3] - 2}{h} \\ &= \lim_{h \to 0} \frac{(3 + 6h + 3h^2) - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h \to 0} \frac{3h - h^3}{h} = \lim_{h \to 0} \frac{h(3 - h^2)}{h} \\ &= \lim_{h \to 0} (3 - h^2) = 3 - 0 = 3 \end{aligned}$$

Tangent line: $y - 2 = 3(x - 1) \quad \Leftrightarrow \quad y - 2 = 3x - 3 \quad \Leftrightarrow \quad y = 3x - 1$

28. Using (5) with $g(x) = x^4 - 2$ and $a = 1$,

$$\begin{aligned} g'(1) &= \lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1} \frac{(x^4 - 2) - (-1)}{x - 1} = \lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to 1} \frac{(x^2 + 1)(x^2 - 1)}{x - 1} \\ &= \lim_{x \to 1} \frac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = \lim_{x \to 1} [(x^2 + 1)(x + 1)] = 2(2) = 4 \end{aligned}$$

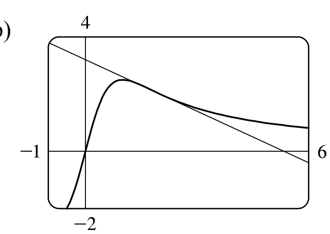
Tangent line: $y - (-1) = 4(x - 1) \quad \Leftrightarrow \quad y + 1 = 4x - 4 \quad \Leftrightarrow \quad y = 4x - 5$

29. (a) Using (4) with $F(x) = 5x/(1 + x^2)$ and the point $(2, 2)$, we have

$$\begin{aligned} F'(2) &= \lim_{h \to 0} \frac{F(2+h) - F(2)}{h} = \lim_{h \to 0} \frac{\dfrac{5(2+h)}{1 + (2+h)^2} - 2}{h} \\ &= \lim_{h \to 0} \frac{\dfrac{5h + 10}{h^2 + 4h + 5} - 2}{h} = \lim_{h \to 0} \frac{\dfrac{5h + 10 - 2(h^2 + 4h + 5)}{h^2 + 4h + 5}}{h} \\ &= \lim_{h \to 0} \frac{-2h^2 - 3h}{h(h^2 + 4h + 5)} = \lim_{h \to 0} \frac{h(-2h - 3)}{h(h^2 + 4h + 5)} = \lim_{h \to 0} \frac{-2h - 3}{h^2 + 4h + 5} = \frac{-3}{5} \end{aligned}$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = -\frac{3}{5}(x - 2)$ or $y = -\frac{3}{5}x + \frac{16}{5}$.

(b)

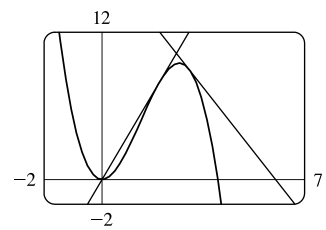


30. (a) Using (4) with $G(x) = 4x^2 - x^3$, we have

$$\begin{aligned} G'(a) &= \lim_{h\to 0}\frac{G(a+h)-G(a)}{h} = \lim_{h\to 0}\frac{[4(a+h)^2-(a+h)^3]-(4a^2-a^3)}{h} \\ &= \lim_{h\to 0}\frac{4a^2+8ah+4h^2-(a^3+3a^2h+3ah^2+h^3)-4a^2+a^3}{h} \\ &= \lim_{h\to 0}\frac{8ah+4h^2-3a^2h-3ah^2-h^3}{h} = \lim_{h\to 0}\frac{h(8a+4h-3a^2-3ah-h^2)}{h} \\ &= \lim_{h\to 0}(8a+4h-3a^2-3ah-h^2) = 8a-3a^2 \end{aligned}$$

At the point $(2, 8)$, $G'(2) = 16 - 12 = 4$, and an equation of the tangent line is $y - 8 = 4(x - 2)$, or $y = 4x$. At the point $(3, 9)$, $G'(3) = 24 - 27 = -3$, and an equation of the tangent line is $y - 9 = -3(x - 3)$, or $y = -3x + 18$.

(b)



31. Use (4) with $f(x) = 3x^2 - 4x + 1$.

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{[3(a+h)^2-4(a+h)+1]-(3a^2-4a+1)}{h} \\ &= \lim_{h\to 0}\frac{3a^2+6ah+3h^2-4a-4h+1-3a^2+4a-1}{h} = \lim_{h\to 0}\frac{6ah+3h^2-4h}{h} \\ &= \lim_{h\to 0}\frac{h(6a+3h-4)}{h} = \lim_{h\to 0}(6a+3h-4) = 6a-4 \end{aligned}$$

32. Use (4) with $f(t) = 2t^3 + t$.

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{[2(a+h)^3+(a+h)]-(2a^3+a)}{h} \\ &= \lim_{h\to 0}\frac{2a^3+6a^2h+6ah^2+2h^3+a+h-2a^3-a}{h} = \lim_{h\to 0}\frac{6a^2h+6ah^2+2h^3+h}{h} \\ &= \lim_{h\to 0}\frac{h(6a^2+6ah+2h^2+1)}{h} = \lim_{h\to 0}(6a^2+6ah+2h^2+1) = 6a^2+1 \end{aligned}$$

33. Use (4) with $f(t) = (2t + 1)/(t + 3)$.

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\dfrac{2(a+h)+1}{(a+h)+3}-\dfrac{2a+1}{a+3}}{h} \\ &= \lim_{h\to 0}\frac{(2a+2h+1)(a+3)-(2a+1)(a+h+3)}{h(a+h+3)(a+3)} \\ &= \lim_{h\to 0}\frac{(2a^2+6a+2ah+6h+a+3)-(2a^2+2ah+6a+a+h+3)}{h(a+h+3)(a+3)} \\ &= \lim_{h\to 0}\frac{5h}{h(a+h+3)(a+3)} = \lim_{h\to 0}\frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2} \end{aligned}$$

34. Use (4) with $f(x) = x^{-2} = 1/x^2$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h \to 0} \frac{\dfrac{a^2 - (a+h)^2}{a^2(a+h)^2}}{h}$$

$$= \lim_{h \to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2} = \lim_{h \to 0} \frac{-2ah - h^2}{ha^2(a+h)^2} = \lim_{h \to 0} \frac{h(-2a-h)}{ha^2(a+h)^2}$$

$$= \lim_{h \to 0} \frac{-2a-h}{a^2(a+h)^2} = \frac{-2a}{a^2(a^2)} = \frac{-2}{a^3}$$

35. Use (4) with $f(x) = \sqrt{1-2x}$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h} \cdot \frac{\sqrt{1-2(a+h)} + \sqrt{1-2a}}{\sqrt{1-2(a+h)} + \sqrt{1-2a}}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{1-2(a+h)}\right)^2 - \left(\sqrt{1-2a}\right)^2}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} = \lim_{h \to 0} \frac{(1-2a-2h) - (1-2a)}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)}$$

$$= \lim_{h \to 0} \frac{-2h}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} = \lim_{h \to 0} \frac{-2}{\sqrt{1-2(a+h)} + \sqrt{1-2a}}$$

$$= \frac{-2}{\sqrt{1-2a} + \sqrt{1-2a}} = \frac{-2}{2\sqrt{1-2a}} = \frac{-1}{\sqrt{1-2a}}$$

36. Use (4) with $f(x) = \dfrac{4}{\sqrt{1-x}}$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\dfrac{4}{\sqrt{1-(a+h)}} - \dfrac{4}{\sqrt{1-a}}}{h}$$

$$= 4 \lim_{h \to 0} \frac{\dfrac{\sqrt{1-a} - \sqrt{1-a-h}}{\sqrt{1-a-h}\,\sqrt{1-a}}}{h} = 4 \lim_{h \to 0} \frac{\sqrt{1-a} - \sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}}$$

$$= 4 \lim_{h \to 0} \frac{\sqrt{1-a} - \sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}} \cdot \frac{\sqrt{1-a} + \sqrt{1-a-h}}{\sqrt{1-a} + \sqrt{1-a-h}} = 4 \lim_{h \to 0} \frac{(\sqrt{1-a})^2 - (\sqrt{1-a-h})^2}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a} + \sqrt{1-a-h})}$$

$$= 4 \lim_{h \to 0} \frac{(1-a) - (1-a-h)}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a} + \sqrt{1-a-h})} = 4 \lim_{h \to 0} \frac{h}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a} + \sqrt{1-a-h})}$$

$$= 4 \lim_{h \to 0} \frac{1}{\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a} + \sqrt{1-a-h})} = 4 \cdot \frac{1}{\sqrt{1-a}\,\sqrt{1-a}(\sqrt{1-a} + \sqrt{1-a})}$$

$$= \frac{4}{(1-a)(2\sqrt{1-a})} = \frac{2}{(1-a)^1(1-a)^{1/2}} = \frac{2}{(1-a)^{3/2}}$$

37. By (4), $\displaystyle\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} = f'(9)$, where $f(x) = \sqrt{x}$ and $a = 9$.

38. By (4), $\lim\limits_{h\to 0}\dfrac{2^{3+h}-8}{h}=f'(3)$, where $f(x)=2^x$ and $a=3$.

39. By Equation 5, $\lim\limits_{x\to 2}\dfrac{x^6-64}{x-2}=f'(2)$, where $f(x)=x^6$ and $a=2$.

40. By Equation 5, $\lim\limits_{x\to 1/4}\dfrac{\frac{1}{x}-4}{x-\frac{1}{4}}=f'(4)$, where $f(x)=\dfrac{1}{x}$ and $a=\dfrac{1}{4}$.

41. By (4), $\lim\limits_{h\to 0}\dfrac{\cos(\pi+h)+1}{h}=f'(\pi)$, where $f(x)=\cos x$ and $a=\pi$.

Or: By (4), $\lim\limits_{h\to 0}\dfrac{\cos(\pi+h)+1}{h}=f'(0)$, where $f(x)=\cos(\pi+x)$ and $a=0$.

42. By Equation 5, $\lim\limits_{\theta\to \pi/6}\dfrac{\sin\theta-\frac{1}{2}}{\theta-\frac{\pi}{6}}=f'\left(\dfrac{\pi}{6}\right)$, where $f(\theta)=\sin\theta$ and $a=\dfrac{\pi}{6}$.

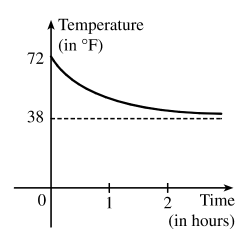
43.
$$v(4)=f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0}\frac{\left[80(4+h)-6(4+h)^2\right]-\left[80(4)-6(4)^2\right]}{h}$$
$$=\lim_{h\to 0}\frac{(320+80h-96-48h-6h^2)-(320-96)}{h}=\lim_{h\to 0}\frac{32h-6h^2}{h}$$
$$=\lim_{h\to 0}\frac{h(32-6h)}{h}=\lim_{h\to 0}(32-6h)=32\text{ m/s}$$

The speed when $t=4$ is $|32|=32$ m/s.

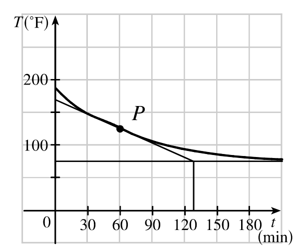
44.
$$v(4)=f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}=\lim_{h\to 0}\frac{\left(10+\frac{45}{4+h+1}\right)-\left(10+\frac{45}{4+1}\right)}{h}=\lim_{h\to 0}\frac{\frac{45}{5+h}-9}{h}$$
$$=\lim_{h\to 0}\frac{45-9(5+h)}{h(5+h)}=\lim_{h\to 0}\frac{-9h}{h(5+h)}=\lim_{h\to 0}\frac{-9}{5+h}=-\frac{9}{5}\text{ m/s.}$$

The speed when $t=4$ is $\left|-\frac{9}{5}\right|=\frac{9}{5}$ m/s.

45. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



46. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t=1$ h seems to be about $\dfrac{75-168}{132-0}\approx -0.7\,°\text{F/min}$.



47. (a) (i) $[1.0, 2.0]$: $\dfrac{C(2)-C(1)}{2-1} = \dfrac{0.018-0.033}{1} = -0.015\ \dfrac{\text{g/dL}}{\text{h}}$

(ii) $[1.5, 2.0]$: $\dfrac{C(2)-C(1.5)}{2-1.5} = \dfrac{0.018-0.024}{0.5} = \dfrac{-0.006}{0.5} = -0.012\ \dfrac{\text{g/dL}}{\text{h}}$

(iii) $[2.0, 2.5]$: $\dfrac{C(2.5)-C(2)}{2.5-2} = \dfrac{0.012-0.018}{0.5} = \dfrac{-0.006}{0.5} = -0.012\ \dfrac{\text{g/dL}}{\text{h}}$

(iv) $[2.0, 3.0]$: $\dfrac{C(3)-C(2)}{3-2} = \dfrac{0.007-0.018}{1} = -0.011\ \dfrac{\text{g/dL}}{\text{h}}$

(b) We estimate the instantaneous rate of change at $t = 2$ by averaging the average rates of change for $[1.5, 2.0]$ and $[2.0, 2.5]$: $\dfrac{-0.012+(-0.012)}{2} = -0.012\ \dfrac{\text{g/dL}}{\text{h}}$. After 2 hours, the BAC is decreasing at a rate of $0.012\ (\text{g/dL})/\text{h}$.

48. (a) (i) $[2006, 2008]$: $\dfrac{N(2008)-N(2006)}{2008-2006} = \dfrac{16{,}680-12{,}440}{2} = \dfrac{4240}{2} = 2120$ locations/year

(ii) $[2008, 2010]$: $\dfrac{N(2010)-N(2008)}{2010-2008} = \dfrac{16{,}858-16{,}680}{2} = \dfrac{178}{2} = 89$ locations/year.

The rate of growth decreased over the period from 2006 to 2010.

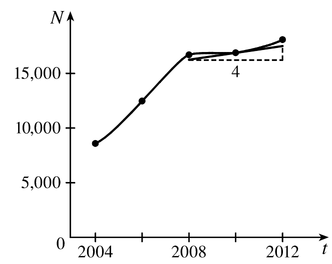
(b) $[2010, 2012]$: $\dfrac{N(2012)-N(2010)}{2012-2010} = \dfrac{18{,}066-16{,}858}{2} = \dfrac{1208}{2} = 604$ locations/year.

Using that value and the value from part (a)(ii), we have $\dfrac{89+604}{2} = \dfrac{693}{2} = 346.5$ locations/year.

(c) The tangent segment has endpoints $(2008, 16{,}250)$ and $(2012, 17{,}500)$.

An estimate of the instantaneous rate of growth in 2010 is

$\dfrac{17{,}500-16{,}250}{2012-2008} = \dfrac{1250}{4} = 312.5$ locations/year.



49. (a) $[1990, 2005]$: $\dfrac{84{,}077-66{,}533}{2005-1990} = \dfrac{17{,}544}{15} = 1169.6$ thousands of barrels per day per year. This means that oil consumption rose by an average of 1169.6 thousands of barrels per day each year from 1990 to 2005.

(b) $[1995, 2000]$: $\dfrac{76{,}784-70{,}099}{2000-1995} = \dfrac{6685}{5} = 1337$

$[2000, 2005]$: $\dfrac{84{,}077-76{,}784}{2005-2000} = \dfrac{7293}{5} = 1458.6$

An estimate of the instantaneous rate of change in 2000 is $\frac{1}{2}(1337+1458.6) = 1397.8$ thousands of barrels per day per year.

50. (a) (i) $[4, 11]$: $\dfrac{V(11)-V(4)}{11-4} = \dfrac{9.4-53}{7} = \dfrac{-43.6}{7} \approx -6.23\ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(ii) $[8, 11]$: $\dfrac{V(11)-V(8)}{11-8} = \dfrac{9.4-18}{3} = \dfrac{-8.6}{3} \approx -2.87\ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(iii) $[11, 15]$: $\dfrac{V(15) - V(11)}{15 - 11} = \dfrac{5.2 - 9.4}{4} = \dfrac{-4.2}{4} = -1.05\ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(iv) $[11, 22]$: $\dfrac{V(22) - V(11)}{22 - 11} = \dfrac{3.6 - 9.4}{11} = \dfrac{-5.8}{11} \approx -0.53\ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(b) An estimate of $V'(11)$ is the average of the answers from part (a)(ii) and (iii).

$V'(11) \approx \frac{1}{2}\left[-2.87 + (-1.05)\right] = -1.96\ \dfrac{\text{RNA copies/mL}}{\text{day}}$.

$V'(11)$ measures the instantaneous rate of change of patient 303's viral load 11 days after ABT-538 treatment began.

51. (a) (i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105) - C(100)}{105 - 100} = \dfrac{6601.25 - 6500}{5} = \$20.25/\text{unit}.$

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101) - C(100)}{101 - 100} = \dfrac{6520.05 - 6500}{1} = \$20.05/\text{unit}.$

(b) $\dfrac{C(100 + h) - C(100)}{h} = \dfrac{\left[5000 + 10(100 + h) + 0.05(100 + h)^2\right] - 6500}{h} = \dfrac{20h + 0.05h^2}{h}$

$= 20 + 0.05h,\ h \neq 0$

So the instantaneous rate of change is $\displaystyle\lim_{h \to 0} \frac{C(100 + h) - C(100)}{h} = \lim_{h \to 0}(20 + 0.05h) = \$20/\text{unit}.$

52. $\Delta V = V(t + h) - V(t) = 100{,}000\left(1 - \dfrac{t + h}{60}\right)^2 - 100{,}000\left(1 - \dfrac{t}{60}\right)^2$

$= 100{,}000\left[\left(1 - \dfrac{t + h}{30} + \dfrac{(t + h)^2}{3600}\right) - \left(1 - \dfrac{t}{30} + \dfrac{t^2}{3600}\right)\right] = 100{,}000\left(-\dfrac{h}{30} + \dfrac{2th}{3600} + \dfrac{h^2}{3600}\right)$

$= \dfrac{100{,}000}{3600}h\,(-120 + 2t + h) = \dfrac{250}{9}h\,(-120 + 2t + h)$

Dividing ΔV by h and then letting $h \to 0$, we see that the instantaneous rate of change is $\frac{500}{9}\,(t - 60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\overline{3}$	$100,000$
10	$-2777.\overline{7}$	$69,444.\overline{4}$
20	$-2222.\overline{2}$	$44,444.\overline{4}$
30	$-1666.\overline{6}$	$25,000$
40	$-1111.\overline{1}$	$11,111.\overline{1}$
50	$-555.\overline{5}$	$2,777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

53. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

54. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

55. (a) $H'(58)$ is the rate at which the daily heating cost changes with respect to temperature when the outside temperature is $58\,^\circ\mathrm{F}$. The units are dollars/ $^\circ\mathrm{F}$.

(b) If the outside temperature increases, the building should require less heating, so we would expect $H'(58)$ to be negative.

56. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).

(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

57. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\mathrm{mg/L})/^\circ\mathrm{C}$.

(b) For $T = 16^\circ\mathrm{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \dfrac{6-14}{32-0} = -\dfrac{8}{32} = -0.25\ (\mathrm{mg/L})/^\circ\mathrm{C}$. This means that as the temperature increases past $16^\circ\mathrm{C}$, the oxygen solubility is decreasing at a rate of $0.25\ (\mathrm{mg/L})/^\circ\mathrm{C}$.

58. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are $(\mathrm{cm/s})/^\circ\mathrm{C}$.

(b) For $T = 15^\circ\mathrm{C}$, it appears the tangent line to the curve goes through the points $(10, 25)$ and $(20, 32)$. So $S'(15) \approx \dfrac{32-25}{20-10} = 0.7\ (\mathrm{cm/s})/^\circ\mathrm{C}$. This tells us that at $T = 15^\circ\mathrm{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of $0.7\ (\mathrm{cm/s})/^\circ\mathrm{C}$. In a similar fashion for $T = 25^\circ\mathrm{C}$, we can use the points $(20, 35)$ and $(25, 25)$ to obtain $S'(25) \approx \dfrac{25-35}{25-20} = -2\ (\mathrm{cm/s})/^\circ\mathrm{C}$. As it gets warmer than $20^\circ\mathrm{C}$, the maximum sustainable speed decreases rapidly.

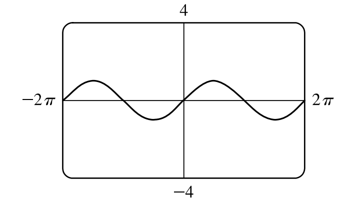
59. Since $f(x) = x\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h\sin(1/h)-0}{h} = \lim\limits_{h\to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 1.5.4.)

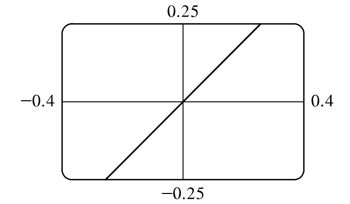
60. Since $f(x) = x^2\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^2\sin(1/h)-0}{h} = \lim\limits_{h\to 0} h\sin(1/h)$. Since $-1 \leq \sin\dfrac{1}{h} \leq 1$, we have $-|h| \leq |h|\sin\dfrac{1}{h} \leq |h| \quad\Rightarrow\quad -|h| \leq h\sin\dfrac{1}{h} \leq |h|$. Because $\lim\limits_{h\to 0}(-|h|) = 0$ and $\lim\limits_{h\to 0}|h| = 0$, we know that $\lim\limits_{h\to 0}\left(h\sin\dfrac{1}{h}\right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

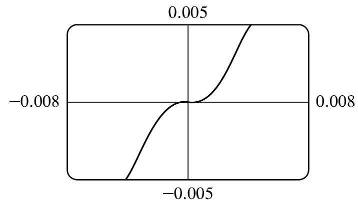
61. (a) The slope at the origin appears to be 1.



(b) The slope at the origin still appears to be 1.



(c) Yes, the slope at the origin now appears to be 0.



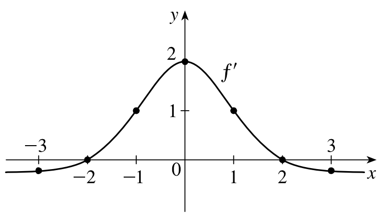
2.2 The Derivative as a Function

1. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

(a) $f'(-3) \approx -0.2$

(b) $f'(-2) \approx 0$ (c) $f'(-1) \approx 1$ (d) $f'(0) \approx 2$

(e) $f'(1) \approx 1$ (f) $f'(2) \approx 0$ (g) $f'(3) \approx -0.2$



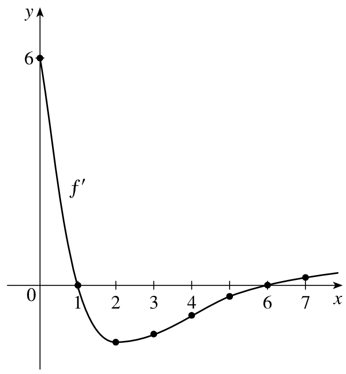
2. Your answers may vary depending on your estimates.

(a) *Note:* By estimating the slopes of tangent lines on the graph of f, it appears that $f'(0) \approx 6$.

(b) $f'(1) \approx 0$

(c) $f'(2) \approx -1.5$ (d) $f'(3) \approx -1.3$ (e) $f'(4) \approx -0.8$

(f) $f'(5) \approx -0.3$ (g) $f'(6) \approx 0$ (h) $f'(7) \approx 0.2$



3. (a)$'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

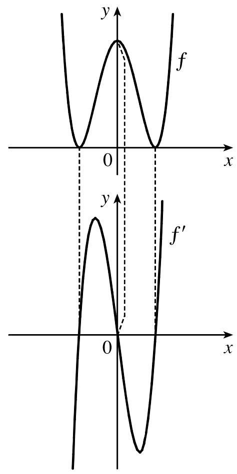
(b)$'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$'$ = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)$'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 4–11: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f—there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

4.



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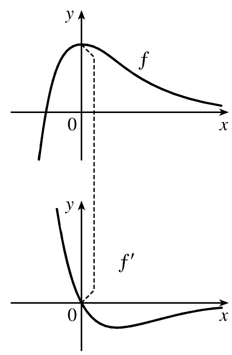

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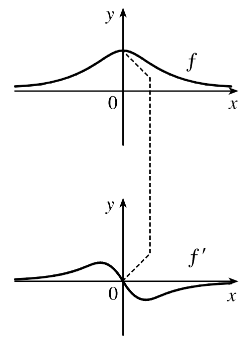

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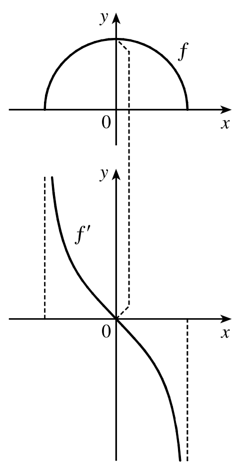

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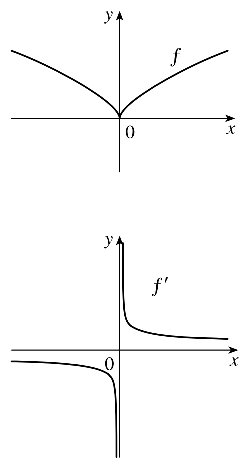

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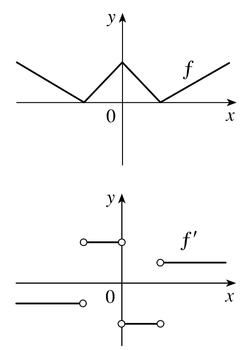

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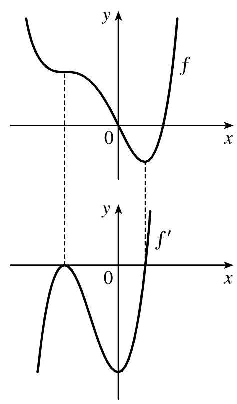

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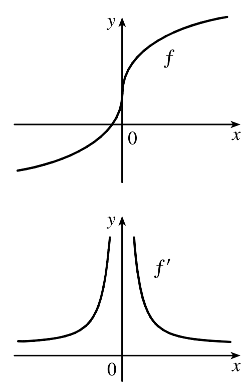
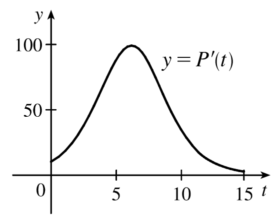
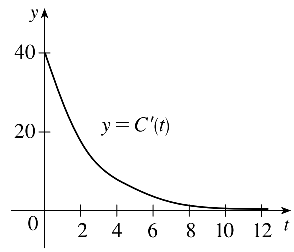


12. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y-values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y-values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



13. (a) $C'(t)$ is the instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours.

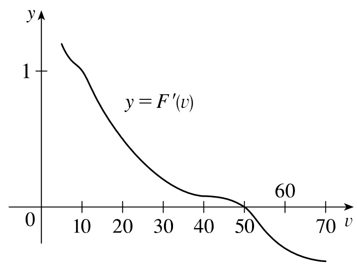
(b) The graph of $C'(t)$ tells us that the rate of change of percentage of full capacity is decreasing and approaching 0.



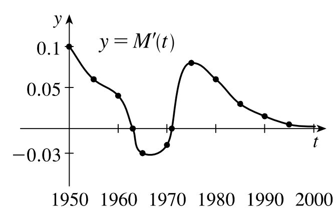
14. (a) $F'(v)$ is the instantaneous rate of change of fuel economy with respect to speed.

(b) Graphs will vary depending on estimates of F', but will change from positive to negative at about $v = 50$.

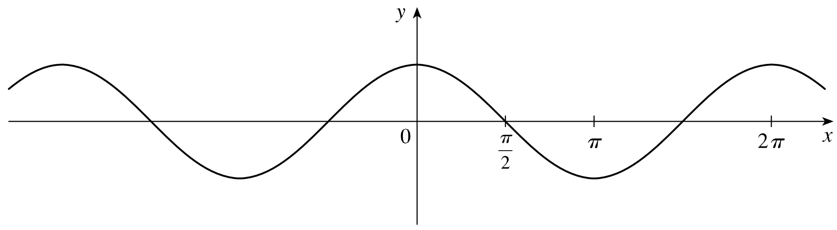
(c) To save on gas, drive at the speed where F is a maximum and F' is 0, which is about 50 mi/h.



15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.



16.

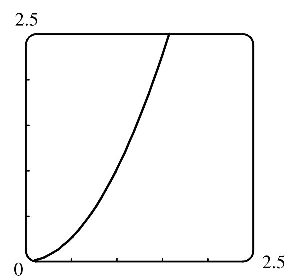


The graph of the derivative looks like the graph of the cosine function.

17. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.

(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.



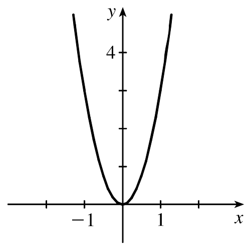
(d) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^2 - x^2}{h}$

$$= \lim_{h\to 0} \frac{\left(x^2+2hx+h^2\right)-x^2}{h} = \lim_{h\to 0} \frac{2hx+h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h) = 2x$$

18. (a) By zooming in, we estimate that $f'(0)=0$, $f'\left(\frac{1}{2}\right)\approx 0.75$, $f'(1)\approx 3$, $f'(2)\approx 12$, and $f'(3)\approx 27$.

(b) By symmetry, $f'(-x)=f'(x)$. So $f'\left(-\frac{1}{2}\right)\approx 0.75$, $f'(-1)\approx 3$, $f'(-2)\approx 12$, and $f'(-3)\approx 27$.

(c)



(d) Since $f'(0)=0$, it appears that f' may have the form $f'(x)=ax^2$. Using $f'(1)=3$, we have $a=3$, so $f'(x)=3x^2$.

(e) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^3-x^3}{h} = \lim_{h\to 0} \dfrac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}$

$$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2$$

19. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[3(x+h)-8]-(3x-8)}{h} = \lim_{h\to 0} \dfrac{3x+3h-8-3x+8}{h}$

$$= \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3$$

Domain of f = domain of $f' = \mathbb{R}$.

20. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[m(x+h)+b]-(mx+b)}{h} = \lim_{h\to 0} \dfrac{mx+mh+b-mx-b}{h}$

$$= \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m$$

Domain of f = domain of $f' = \mathbb{R}$.

21. $f'(t) = \lim_{h\to 0} \dfrac{f(t+h)-f(t)}{h} = \lim_{h\to 0} \dfrac{\left[2.5(t+h)^2+6(t+h)\right]-\left(2.5t^2+6t\right)}{h}$

$$= \lim_{h\to 0} \frac{2.5(t^2+2th+h^2)+6t+6h-2.5t^2-6t}{h} = \lim_{h\to 0} \frac{2.5t^2+5th+2.5h^2+6h-2.5t^2}{h}$$

$$= \lim_{h\to 0} \frac{5th+2.5h^2+6h}{h} = \lim_{h\to 0} \frac{h\,(5t+2.5h+6)}{h} = \lim_{h\to 0}\,(5t+2.5h+6)$$

$$= 5t+6$$

Domain of f = domain of $f' = \mathbb{R}$.

22. $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left[4+8(x+h)-5(x+h)^2\right]-(4+8x-5x^2)}{h}$$
$$=\lim_{h\to 0}\frac{4+8x+8h-5(x^2+2xh+h^2)-4-8x+5x^2}{h}=\lim_{h\to 0}\frac{8h-5x^2-10xh-5h^2+5x^2}{h}$$
$$=\lim_{h\to 0}\frac{8h-10xh-5h^2}{h}=\lim_{h\to 0}\frac{h(8-10x-5h)}{h}=\lim_{h\to 0}(8-10x-5h)$$
$$=8-10x$$

Domain of f = domain of $f'=\mathbb{R}$.

23. $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[(x+h)^2-2(x+h)^3]-(x^2-2x^3)}{h}$$
$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-2x^3-6x^2h-6xh^2-2h^3-x^2+2x^3}{h}$$
$$=\lim_{h\to 0}\frac{2xh+h^2-6x^2h-6xh^2-2h^3}{h}=\lim_{h\to 0}\frac{h(2x+h-6x^2-6xh-2h^2)}{h}$$
$$=\lim_{h\to 0}(2x+h-6x^2-6xh-2h^2)=2x-6x^2$$

Domain of f = domain of $f'=\mathbb{R}$.

24. $$g'(t)=\lim_{h\to 0}\frac{g(t+h)-g(t)}{h}=\lim_{h\to 0}\frac{\dfrac{1}{\sqrt{t+h}}-\dfrac{1}{\sqrt{t}}}{h}=\lim_{h\to 0}\frac{\dfrac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\,\sqrt{t}}}{h}=\lim_{h\to 0}\left(\frac{\sqrt{t}-\sqrt{t+h}}{h\sqrt{t+h}\,\sqrt{t}}\cdot\frac{\sqrt{t}+\sqrt{t+h}}{\sqrt{t}+\sqrt{t+h}}\right)$$
$$=\lim_{h\to 0}\frac{t-(t+h)}{h\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)}=\lim_{h\to 0}\frac{-h}{h\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)}=\lim_{h\to 0}\frac{-1}{\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)}$$
$$=\frac{-1}{\sqrt{t}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t}\right)}=\frac{-1}{t\left(2\sqrt{t}\right)}=-\frac{1}{2t^{3/2}}$$

Domain of g = domain of $g'=(0,\infty)$.

25. $$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h}\left[\frac{\sqrt{9-(x+h)}+\sqrt{9-x}}{\sqrt{9-(x+h)}+\sqrt{9-x}}\right]$$
$$=\lim_{h\to 0}\frac{[9-(x+h)]-(9-x)}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]}=\lim_{h\to 0}\frac{-h}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]}$$
$$=\lim_{h\to 0}\frac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}}=\frac{-1}{2\sqrt{9-x}}$$

Domain of $g=(-\infty,9]$, domain of $g'=(-\infty,9)$.

26. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{(x+h)^2-1}{2(x+h)-3} - \dfrac{x^2-1}{2x-3}}{h}$

$= \lim_{h\to 0} \dfrac{\dfrac{[(x+h)^2-1](2x-3)-[2(x+h)-3](x^2-1)}{[2(x+h)-3](2x-3)}}{h}$

$= \lim_{h\to 0} \dfrac{(x^2+2xh+h^2-1)(2x-3)-(2x+2h-3)(x^2-1)}{h[2(x+h)-3](2x-3)}$

$= \lim_{h\to 0} \dfrac{(2x^3+4x^2h+2xh^2-2x-3x^2-6xh-3h^2+3)-(2x^3+2x^2h-3x^2-2x-2h+3)}{h(2x+2h-3)(2x-3)}$

$= \lim_{h\to 0} \dfrac{4x^2h+2xh^2-6xh-3h^2-2x^2h+2h}{h(2x+2h-3)(2x-3)} = \lim_{h\to 0} \dfrac{h(2x^2+2xh-6x-3h+2)}{h(2x+2h-3)(2x-3)}$

$= \lim_{h\to 0} \dfrac{2x^2+2xh-6x-3h+2}{(2x+2h-3)(2x-3)} = \dfrac{2x^2-6x+2}{(2x-3)^2}$

Domain of f = domain of $f' = (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$.

27. $G'(t) = \lim_{h\to 0} \dfrac{G(t+h)-G(t)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1-2(t+h)}{3+(t+h)} - \dfrac{1-2t}{3+t}}{h}$

$= \lim_{h\to 0} \dfrac{\dfrac{[1-2(t+h)](3+t)-[3+(t+h)](1-2t)}{[3+(t+h)](3+t)}}{h}$

$= \lim_{h\to 0} \dfrac{3+t-6t-2t^2-6h-2ht-(3-6t+t-2t^2+h-2ht)}{h[3+(t+h)](3+t)} = \lim_{h\to 0} \dfrac{-6h-h}{h(3+t+h)(3+t)}$

$= \lim_{h\to 0} \dfrac{-7h}{h(3+t+h)(3+t)} = \lim_{h\to 0} \dfrac{-7}{(3+t+h)(3+t)} = \dfrac{-7}{(3+t)^2}$

Domain of G = domain of $G' = (-\infty, -3) \cup (-3, \infty)$.

28. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^{3/2}-x^{3/2}}{h} = \lim_{h\to 0} \dfrac{[(x+h)^{3/2}-x^{3/2}][(x+h)^{3/2}+x^{3/2}]}{h\,[(x+h)^{3/2}+x^{3/2}]}$

$= \lim_{h\to 0} \dfrac{(x+h)^3-x^3}{h[(x+h)^{3/2}+x^{3/2}]} = \lim_{h\to 0} \dfrac{x^3+3x^2h+3xh^2+h^3-x^3}{h[(x+h)^{3/2}+x^{3/2}]} = \lim_{h\to 0} \dfrac{h\left(3x^2+3xh+h^2\right)}{h[(x+h)^{3/2}+x^{3/2}]}$

$= \lim_{h\to 0} \dfrac{3x^2+3xh+h^2}{(x+h)^{3/2}+x^{3/2}} = \dfrac{3x^2}{2x^{3/2}} = \frac{3}{2}x^{1/2}$

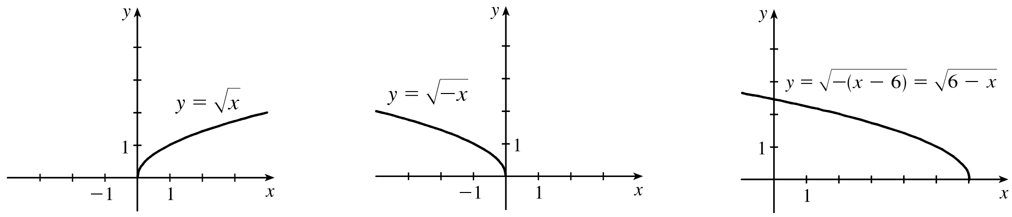
Domain of f = domain of $f' = [0, \infty)$. Strictly speaking, the domain of f' is $(0, \infty)$ because the limit that defines $f'(0)$ does not exist (as a two-sided limit). But the right-hand derivative (in the sense of Exercise 62) does exist at 0, so in that sense one could regard the domain of f' to be $[0, \infty)$.

29. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^4-x^4}{h} = \lim_{h\to 0} \dfrac{\left(x^4+4x^3h+6x^2h^2+4xh^3+h^4\right)-x^4}{h}$

$= \lim_{h\to 0} \dfrac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0} \left(4x^3+6x^2h+4xh^2+h^3\right) = 4x^3$

Domain of f = domain of $f' = \mathbb{R}$.

30. (a)

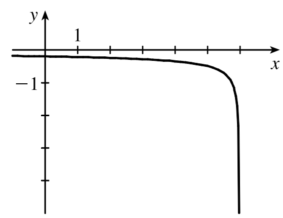


(b) Note that the third graph in part (a) has small negative values for its slope, f'; but as $x \to 6^-$, $f' \to -\infty$. See the graph in part (d).

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{6-(x+h)}-\sqrt{6-x}}{h}\left[\frac{\sqrt{6-(x+h)}+\sqrt{6-x}}{\sqrt{6-(x+h)}+\sqrt{6-x}}\right]$$
$$= \lim_{h\to 0} \frac{[6-(x+h)]-(6-x)}{h\left[\sqrt{6-(x+h)}+\sqrt{6-x}\right]} = \lim_{h\to 0} \frac{-h}{h\left(\sqrt{6-x-h}+\sqrt{6-x}\right)}$$
$$= \lim_{h\to 0} \frac{-1}{\sqrt{6-x-h}+\sqrt{6-x}} = \frac{-1}{2\sqrt{6-x}}$$

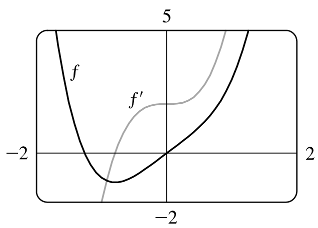
Domain of $f = (-\infty, 6]$, domain of $f' = (-\infty, 6)$.

(d)



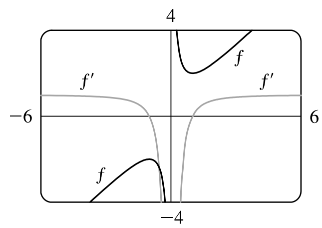
31. (a) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^4+2(x+h)]-(x^4+2x)}{h}$$
$$= \lim_{h\to 0} \frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4+2x+2h-x^4-2x}{h}$$
$$= \lim_{h\to 0} \frac{4x^3h+6x^2h^2+4xh^3+h^4+2h}{h} = \lim_{h\to 0} \frac{h(4x^3+6x^2h+4xh^2+h^3+2)}{h}$$
$$= \lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3+2) = 4x^3+2$$

(b) Notice that $f'(x)=0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope.



32. (a) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)+1/(x+h)]-(x+1/x)}{h} = \lim_{h\to 0} \frac{\dfrac{(x+h)^2+1}{x+h} - \dfrac{x^2+1}{x}}{h}$$
$$= \lim_{h\to 0} \frac{x[(x+h)^2+1]-(x+h)(x^2+1)}{h(x+h)x} = \lim_{h\to 0} \frac{(x^3+2hx^2+xh^2+x)-(x^3+x+hx^2+h)}{h(x+h)x}$$
$$= \lim_{h\to 0} \frac{hx^2+xh^2-h}{h(x+h)x} = \lim_{h\to 0} \frac{h(x^2+xh-1)}{h(x+h)x} = \lim_{h\to 0} \frac{x^2+xh-1}{(x+h)x} = \frac{x^2-1}{x^2}, \text{ or } 1-\frac{1}{x^2}$$

(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope. Both functions are discontinuous at $x = 0$.



33. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent unemployed per year.

(b) To find $U'(t)$, we use $\lim\limits_{h\to 0} \dfrac{U(t+h)-U(t)}{h} \approx \dfrac{U(t+h)-U(t)}{h}$ for small values of h.

For 2003: $U'(2003) \approx \dfrac{U(2004)-U(2003)}{2004-2003} = \dfrac{5.5-6.0}{1} = -0.5$

For 2004: We estimate $U'(2004)$ by using $h=-1$ and $h=1$, and then average the two results to obtain a final estimate.

$h=-1 \quad\Rightarrow\quad U'(2004) \approx \dfrac{U(2003)-U(2004)}{2003-2004} = \dfrac{6.0-5.5}{-1} = -0.5;$

$h=1 \quad\Rightarrow\quad U'(2004) \approx \dfrac{U(2005)-U(2004)}{2005-2004} = \dfrac{5.1-5.5}{1} = -0.4.$

So we estimate that $U'(2004) \approx \frac{1}{2}[-0.5+(-0.4)] = -0.45$. Other values for $U'(t)$ are calculated in a similar fashion.

t	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
$U'(t)$	−0.50	−0.45	−0.45	−0.25	0.60	2.35	1.90	−0.20	−0.75	−0.80

34. (a) $N'(t)$ is the rate at which the number of minimally invasive cosmetic surgery procedures performed in the United States is changing with respect to time. Its units are thousands of surgeries per year.

(b) To find $N'(t)$, we use $\lim\limits_{h\to 0} \dfrac{N(t+h)-N(t)}{h} \approx \dfrac{N(t+h)-N(t)}{h}$ for small values of h.

For 2000: $N'(2000) \approx \dfrac{N(2002)-N(2000)}{2002-2000} = \dfrac{4897-5500}{2} = -301.5$

For 2002: We estimate $N'(2002)$ by using $h=-2$ and $h=2$, and then average the two results to obtain a final estimate.

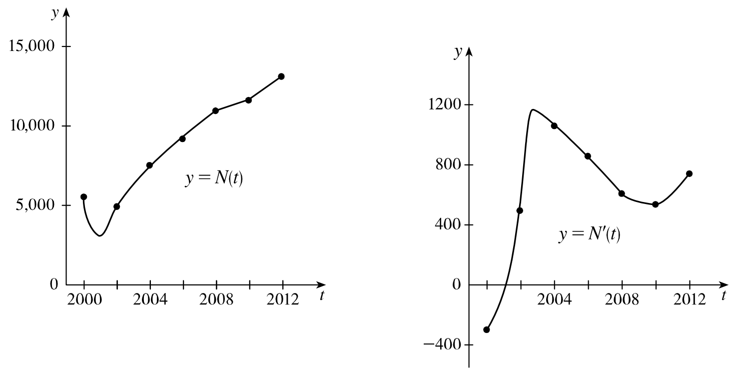
$h=-2 \quad\Rightarrow\quad N'(2002) \approx \dfrac{N(2000)-N(2002)}{2000-2002} = \dfrac{5500-4897}{-2} = -301.5$

$h=2 \quad\Rightarrow\quad N'(2002) \approx \dfrac{N(2004)-N(2002)}{2004-2002} = \dfrac{7470-4897}{2} = 1286.5$

So we estimate that $N'(2002) \approx \frac{1}{2}[-301.5+1286.5] = 492.5$.

t	2000	2002	2004	2006	2008	2010	2012
$N'(t)$	−301.5	492.5	1060.25	856.75	605.75	534.5	737

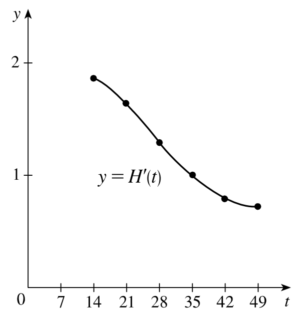
(c)



(d) We could get more accurate values for $N'(t)$ by obtaining data for more values of t.

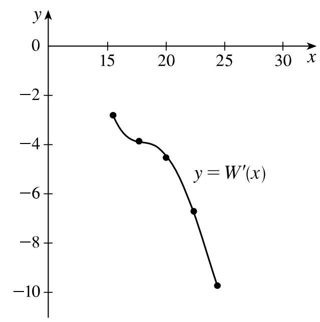
35. As in Exercise 33, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values.

t	14	21	28	35	42	49
$H(t)$	41	54	64	72	78	83
$H'(t)$	$\frac{13}{7}$	$\frac{23}{14}$	$\frac{18}{14}$	$\frac{14}{14}$	$\frac{11}{14}$	$\frac{5}{7}$



36. As in Exercise 33, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values. The units for $W'(x)$ are grams per degree (g/°C).

x	15.5	17.7	20.0	22.4	24.4
$W(x)$	37.2	31.0	19.8	9.7	−9.8
$W'(x)$	−2.82	−3.87	−4.53	−6.73	−9.75



37. (a) dP/dt is the rate at which the percentage of the city's electrical power produced by solar panels changes with respect to time t, measured in percentage points per year.

(b) 2 years after January 1, 2000 (January 1, 2002), the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.

38. dN/dp is the rate at which the number of people who travel by car to another state for a vacation changes with respect to the price of gasoline. If the price of gasoline goes up, we would expect fewer people to travel, so we would expect dN/dp to be negative.

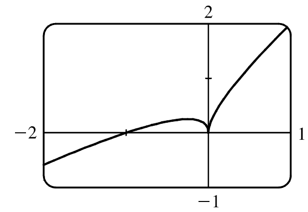
39. f is not differentiable at $x = -4$, because the graph has a corner there, and at $x = 0$, because there is a discontinuity there.

40. f is not differentiable at $x = -1$, because there is a discontinuity there, and at $x = 2$, because the graph has a corner there.

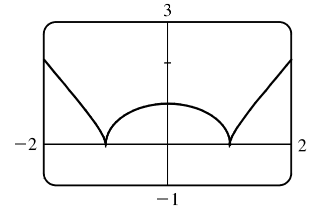
41. f is not differentiable at $x = 1$, because f is not defined there, and at $x = 5$, because the graph has a vertical tangent there.

42. f is not differentiable at $x = -2$ and $x = 3$, because the graph has corners there, and at $x = 1$, because there is a discontinuity there.

43. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



44. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so $g(x) = (x^2 - 1)^{2/3}$ is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So g is not differentiable at $x = \pm 1$.



45. Call the curve with the positive y-intercept g and the other curve h. Notice that g has a maximum (horizontal tangent) at $x = 0$, but $h \neq 0$, so h cannot be the derivative of g. Also notice that where g is positive, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is negative since f' is below the x-axis there and $f''(1)$ is positive since f is concave upward at $x = 1$. Therefore, $f''(1)$ is greater than $f'(-1)$.

46. Call the curve with the smallest positive x-intercept g and the other curve h. Notice that where g is positive in the first quadrant, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is positive since f' is above the x-axis there and $f''(1)$ appears to be zero since f has an inflection point at $x = 1$. Therefore, $f'(1)$ is greater than $f''(-1)$.

47. $a = f, b = f', c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

48. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a, so $b' = a$. We conclude that $d = f$, $c = f'$, $b = f''$, and $a = f'''$.

49. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

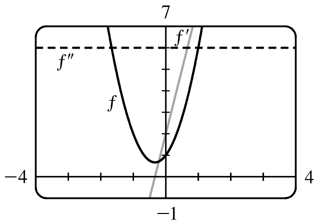
50. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

51.
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 + 2(x+h) + 1] - (3x^2 + 2x + 1)}{h}$$
$$= \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 + 2x + 2h + 1) - (3x^2 + 2x + 1)}{h} = \lim_{h \to 0} \frac{6xh + 3h^2 + 2h}{h}$$
$$= \lim_{h \to 0} \frac{h(6x + 3h + 2)}{h} = \lim_{h \to 0} (6x + 3h + 2) = 6x + 2$$

[continued]

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[6(x+h)+2] - (6x+2)}{h} = \lim_{h \to 0} \frac{(6x+6h+2) - (6x+2)}{h}$$

$$= \lim_{h \to 0} \frac{6h}{h} = \lim_{h \to 0} 6 = 6$$



We see from the graph that our answers are reasonable because the graph of f' is that of a linear function and the graph of f'' is that of a constant function.

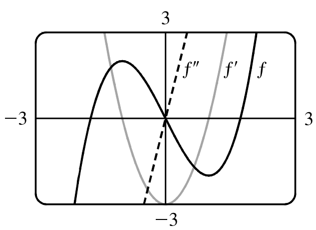
52. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - 3(x+h)] - (x^3 - 3x)}{h}$$

$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h) - (x^3 - 3x)}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 - 3] - (3x^2 - 3)}{h} = \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 - 3) - (3x^2 - 3)}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} \frac{h(6x + 3h)}{h} = \lim_{h \to 0} (6x + 3h) = 6x$$



We see from the graph that our answers are reasonable because the graph of f' is that of an even function (f is an odd function) and the graph of f'' is that of an odd function. Furthermore, $f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent.

53. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[2(x+h)^2 - (x+h)^3\right] - (2x^2 - x^3)}{h}$$

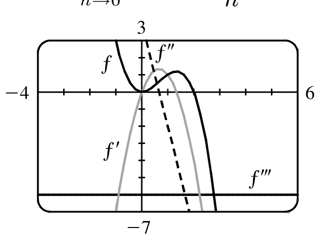
$$= \lim_{h \to 0} \frac{h(4x + 2h - 3x^2 - 3xh - h^2)}{h} = \lim_{h \to 0} (4x + 2h - 3x^2 - 3xh - h^2) = 4x - 3x^2$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{\left[4(x+h) - 3(x+h)^2\right] - (4x - 3x^2)}{h} = \lim_{h \to 0} \frac{h(4 - 6x - 3h)}{h}$$

$$= \lim_{h \to 0} (4 - 6x - 3h) = 4 - 6x$$

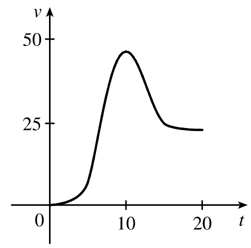
$$f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h \to 0} \frac{[4 - 6(x+h)] - (4 - 6x)}{h} = \lim_{h \to 0} \frac{-6h}{h} = \lim_{h \to 0} (-6) = -6$$

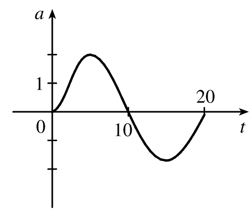
$$f^{(4)}(x) = \lim_{h \to 0} \frac{f'''(x+h) - f'''(x)}{h} = \lim_{h \to 0} \frac{-6 - (-6)}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} (0) = 0$$



The graphs are consistent with the geometric interpretations of the derivatives because f' has zeros where f has a local minimum and a local maximum, f'' has a zero where f' has a local maximum, and f''' is a constant function equal to the slope of f''.

54. (a) Since we estimate the velocity to be a maximum at $t = 10$, the acceleration is 0 at $t = 10$.





(b) Drawing a tangent line at $t = 10$ on the graph of a, a appears to decrease by 10 ft/s^2 over a period of 20 s. So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5\ (\text{ft/s}^2)/\text{s}$ or ft/s^3.

55. (a) Note that we have factored $x - a$ as the difference of two cubes in the third step.

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{x-a} = \lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$
$$= \lim_{x\to a}\frac{1}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \tfrac{1}{3}a^{-2/3}$$

(b) $f'(0) = \lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0}\dfrac{\sqrt[3]{h}-0}{h} = \lim\limits_{h\to 0}\dfrac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

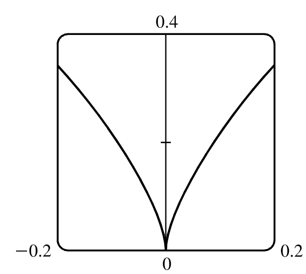
(c) $\lim\limits_{x\to 0}|f'(x)| = \lim\limits_{x\to 0}\dfrac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

56. (a) $g'(0) = \lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{x^{2/3}-0}{x} = \lim\limits_{x\to 0}\dfrac{1}{x^{1/3}}$, which does not exist.

(b) $$g'(a) = \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = \lim_{x\to a}\frac{x^{2/3}-a^{2/3}}{x-a} = \lim_{x\to a}\frac{(x^{1/3}-a^{1/3})(x^{1/3}+a^{1/3})}{(x^{1/3}-a^{1/3})(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$
$$= \lim_{x\to a}\frac{x^{1/3}+a^{1/3}}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \quad\text{or}\quad \tfrac{2}{3}a^{-1/3}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and $\lim\limits_{x\to 0}|g'(x)| = \lim\limits_{x\to 0}\dfrac{2}{3\,|x|^{1/3}} = \infty$. This shows that g has a vertical tangent line at $x = 0$.

(d)



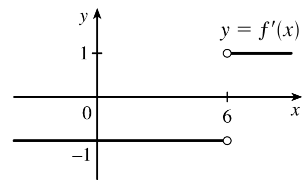
57. $f(x) = |x-6| = \begin{cases} x-6 & \text{if } x-6 \geq 6 \\ -(x-6) & \text{if } x-6 < 0 \end{cases} = \begin{cases} x-6 & \text{if } x \geq 6 \\ 6-x & \text{if } x < 6 \end{cases}$

So the right-hand limit is $\lim\limits_{x\to 6^+}\dfrac{f(x)-f(6)}{x-6} = \lim\limits_{x\to 6^+}\dfrac{|x-6|-0}{x-6} = \lim\limits_{x\to 6^+}\dfrac{x-6}{x-6} = \lim\limits_{x\to 6^+} 1 = 1$, and the left-hand limit is $\lim\limits_{x\to 6^-}\dfrac{f(x)-f(6)}{x-6} = \lim\limits_{x\to 6^-}\dfrac{|x-6|-0}{x-6} = \lim\limits_{x\to 6^-}\dfrac{6-x}{x-6} = \lim\limits_{x\to 6^-}(-1) = -1$. Since these limits are not equal,

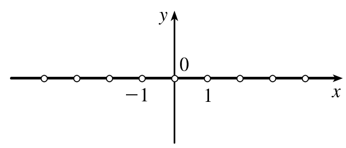
$f'(6) = \lim_{x \to 6} \dfrac{f(x) - f(6)}{x - 6}$ does not exist and f is not differentiable at 6.

However, a formula for f' is $f'(x) = \begin{cases} 1 & \text{if } x > 6 \\ -1 & \text{if } x < 6 \end{cases}$

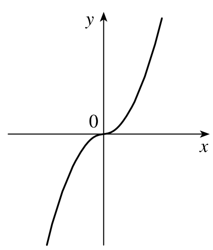
Another way of writing the formula is $f'(x) = \dfrac{x - 6}{|x - 6|}$.



58. $f(x) = [\![x]\!]$ is not continuous at any integer n, so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a, so $f'(a) = 0$. Thus, $f'(x) = 0$, x not an integer.



59. (a) $f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



(b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$. [See Exercise 17(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x\,|x|}{x} = \lim_{x \to 0} |x| = 0.$$

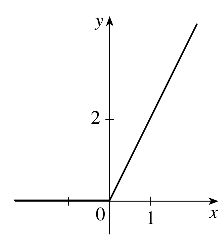
So f is differentiable at 0. Thus, f is differentiable for all x.

(c) From part (b), we have $f'(x) = \left\{ \begin{array}{rl} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{array} \right\} = 2\,|x|$.

60. (a) $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

so $g(x) = x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$.

Graph the line $y = 2x$ for $x \geq 0$ and graph $y = 0$ (the x-axis) for $x < 0$.



(b) g is not differentiable at $x = 0$ because the graph has a corner there, but is differentiable at all other values; that is, g is differentiable on $(-\infty, 0) \cup (0, \infty)$.

(c) $g(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \Rightarrow \quad g'(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

Another way of writing the formula is $g'(x) = 1 + \operatorname{sgn} x$ for $x \neq 0$.

61. (a) If f is even, then

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f[-(x-h)] - f(-x)}{h} \\ &= \lim_{h \to 0} \frac{f(x-h) - f(x)}{h} = -\lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= -\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = -f'(x) \end{aligned}$$

Therefore, f' is odd.

(b) If f is odd, then

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{f[-(x-h)]-f(-x)}{h} \\ &= \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) \end{aligned}$$

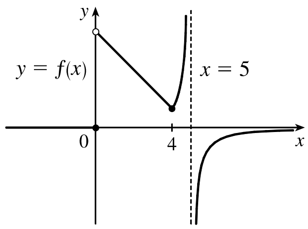
Therefore, f' is even.

62. (a) $f'_-(4) = \lim_{h\to 0^-}\dfrac{f(4+h)-f(4)}{h} = \lim_{h\to 0^-}\dfrac{5-(4+h)-1}{h} = \lim_{h\to 0^-}\dfrac{-h}{h} = -1$

and

$$f'_+(4) = \lim_{h\to 0^+}\frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^+}\frac{\dfrac{1}{5-(4+h)}-1}{h} = \lim_{h\to 0^+}\frac{1-(1-h)}{h(1-h)} = \lim_{h\to 0^+}\frac{1}{1-h} = 1$$

(b)



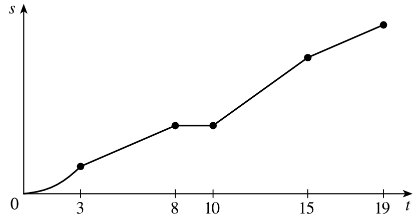
(c) $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5-x & \text{if } 0 < x < 4 \\ 1/(5-x) & \text{if } x \ge 4 \end{cases}$

At 4 we have $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(5-x) = 1$ and $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\dfrac{1}{5-x} = 1$, so $\lim_{x\to 4} f(x) = 1 = f(4)$ and f is continuous at 4. Since $f(5)$ is not defined, f is discontinuous at 5. These expressions show that f is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ and $(5, \infty)$. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(5-x) = 5 \neq 0 = \lim_{x\to 0^-} f(x)$, $\lim_{x\to 0} f(x)$ does not exist, so f is discontinuous (and therefore not differentiable) at 0.

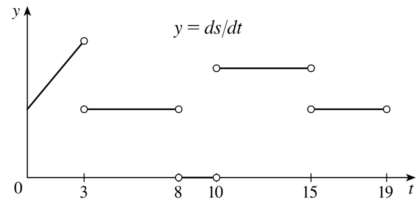
(d) From (a), f is not differentiable at 4 since $f'_-(4) \neq f'_+(4)$, and from (c), f is not differentiable at 0 or 5.

63. These graphs are idealizations conveying the spirit of the problem. In reality, changes in speed are not instantaneous, so the graph in (a) would not have corners and the graph in (b) would be continuous.

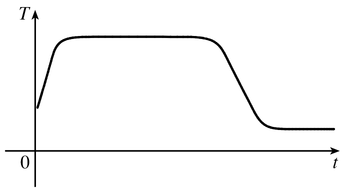
(a)



(b)

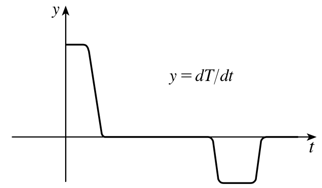


64. (a)

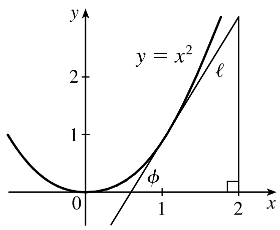


(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

(c)



65.



In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent to angle ϕ. Then the slope of the tangent line ℓ is $m = \Delta y/\Delta x = \tan\phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 17) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63^\circ$.

2.3 Differentiation Formulas

1. $f(x) = 2^{40}$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

2. $f(x) = \pi^2$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

3. $f(x) = 5.2x + 2.3 \quad\Rightarrow\quad f'(x) = 5.2(1) + 0 = 5.2$

4. $g(x) = \frac{7}{4}x^2 - 3x + 12 \quad\Rightarrow\quad g'(x) = \frac{7}{4}(2x) - 3(1) + 0 = \frac{7}{2}x - 3$

5. $f(t) = 2t^3 - 3t^2 - 4t \quad\Rightarrow\quad f'(t) = 2(3t^2) - 3(2t) - 4(1) = 6t^2 - 6t - 4$

6. $f(t) = 1.4t^5 - 2.5t^2 + 6.7 \quad\Rightarrow\quad f'(t) = 1.4(5t^4) - 2.5(2t) + 0 = 7t^4 - 5t$

7. $g(x) = x^2(1 - 2x) = x^2 - 2x^3 \quad\Rightarrow\quad g'(x) = 2x - 2(3x^2) = 2x - 6x^2$

8. $H(u) = (3u - 1)(u + 2) = 3u^2 + 5u - 2 \quad\Rightarrow\quad H'(u) = 3(2u) + 5(1) - 0 = 6u + 5$

9. $g(t) = 2t^{-3/4} \quad\Rightarrow\quad g'(t) = 2\left(-\frac{3}{4}t^{-7/4}\right) = -\frac{3}{2}t^{-7/4}$

10. $B(y) = cy^{-6} \quad\Rightarrow\quad B'(y) = c(-6y^{-7}) = -6cy^{-7}$

11. $F(r) = \dfrac{5}{r^3} = 5r^{-3} \quad\Rightarrow\quad F'(r) = 5(-3r^{-4}) = -15r^{-4} = -\dfrac{15}{r^4}$

12. $y = x^{5/3} - x^{2/3} \quad\Rightarrow\quad y' = \frac{5}{3}x^{2/3} - \frac{2}{3}x^{-1/3}$

13. $S(p) = \sqrt{p} - p = p^{1/2} - p \quad \Rightarrow \quad S'(p) = \frac{1}{2}p^{-1/2} - 1$ or $\dfrac{1}{2\sqrt{p}} - 1$

14. $y = \sqrt[3]{x}\,(2 + x) = 2x^{1/3} + x^{4/3} \quad \Rightarrow \quad y' = 2\left(\frac{1}{3}x^{-2/3}\right) + \frac{4}{3}x^{1/3} = \frac{2}{3}x^{-2/3} + \frac{4}{3}x^{1/3}$ or $\dfrac{2}{3\sqrt[3]{x^2}} + \dfrac{4}{3}\sqrt[3]{x}$

15. $R(a) = (3a + 1)^2 = 9a^2 + 6a + 1 \quad \Rightarrow \quad R'(a) = 9(2a) + 6(1) + 0 = 18a + 6$

16. $S(R) = 4\pi R^2 \quad \Rightarrow \quad S'(R) = 4\pi(2R) = 8\pi R$

17. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2} \quad \Rightarrow$

$y' = \frac{3}{2}x^{1/2} + 4\left(\frac{1}{2}\right)x^{-1/2} + 3\left(-\frac{1}{2}\right)x^{-3/2} = \frac{3}{2}\sqrt{x} + \dfrac{2}{\sqrt{x}} - \dfrac{3}{2x\sqrt{x}} \quad \left[\text{note that } x^{3/2} = x^{2/2} \cdot x^{1/2} = x\sqrt{x}\right]$

The last expression can be written as $\dfrac{3x^2}{2x\sqrt{x}} + \dfrac{4x}{2x\sqrt{x}} - \dfrac{3}{2x\sqrt{x}} = \dfrac{3x^2 + 4x - 3}{2x\sqrt{x}}$.

18. $y = \dfrac{\sqrt{x} + x}{x^2} = \dfrac{\sqrt{x}}{x^2} + \dfrac{x}{x^2} = x^{1/2-2} + x^{1-2} = x^{-3/2} + x^{-1} \quad \Rightarrow \quad y' = -\frac{3}{2}x^{-5/2} + (-1x^{-2}) = -\frac{3}{2}x^{-5/2} - x^{-2}$

19. $G(q) = \left(1 + q^{-1}\right)^2 = 1 + 2q^{-1} + q^{-2} \quad \Rightarrow \quad G'(q) = 0 + 2(-1q^{-2}) + (-2q^{-3}) = -2q^{-2} - 2q^{-3}$

20. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t} = \sqrt{5}\,t^{1/2} + \sqrt{7}\,t^{-1} \quad \Rightarrow \quad G'(t) = \sqrt{5}\left(\frac{1}{2}t^{-1/2}\right) + \sqrt{7}\left(-1t^{-2}\right) = \dfrac{\sqrt{5}}{2\sqrt{t}} - \dfrac{\sqrt{7}}{t^2}$

21. $u = \left(\dfrac{1}{t} - \dfrac{1}{\sqrt{t}}\right)^2 = \dfrac{1}{t^2} - \dfrac{2}{t^{3/2}} + \dfrac{1}{t} = t^{-2} - 2t^{-3/2} + t^{-1} \quad \Rightarrow$

$u' = -2t^{-3} - 2\left(-\frac{3}{2}\right)t^{-5/2} + (-1)t^{-2} = -\dfrac{2}{t^3} + \dfrac{3}{t^{5/2}} - \dfrac{1}{t^2} = -\dfrac{2}{t^3} + \dfrac{3}{t^2\sqrt{t}} - \dfrac{1}{t^2}$

22. $D(t) = \dfrac{1 + 16t^2}{(4t)^3} = \dfrac{1 + 16t^2}{64t^3} = \frac{1}{64}t^{-3} + \frac{1}{4}t^{-1} \quad \Rightarrow$

$D'(t) = \frac{1}{64}(-3t^{-4}) + \frac{1}{4}(-1t^{-2}) = -\frac{3}{64}t^{-4} - \frac{1}{4}t^{-2}$ or $-\dfrac{3}{64t^4} - \dfrac{1}{4t^2}$

23. Product Rule: $f(x) = (1 + 2x^2)(x - x^2) \quad \Rightarrow$

$$f'(x) = (1 + 2x^2)(1 - 2x) + (x - x^2)(4x) = 1 - 2x + 2x^2 - 4x^3 + 4x^2 - 4x^3 = 1 - 2x + 6x^2 - 8x^3.$$

Multiplying first: $f(x) = (1 + 2x^2)(x - x^2) = x - x^2 + 2x^3 - 2x^4 \quad \Rightarrow \quad f'(x) = 1 - 2x + 6x^2 - 8x^3$ (equivalent).

24. Quotient Rule: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = \dfrac{x^4 - 5x^3 + x^{1/2}}{x^2} \quad \Rightarrow$

$$F'(x) = \frac{x^2(4x^3 - 15x^2 + \frac{1}{2}x^{-1/2}) - (x^4 - 5x^3 + x^{1/2})(2x)}{(x^2)^2} = \frac{4x^5 - 15x^4 + \frac{1}{2}x^{3/2} - 2x^5 + 10x^4 - 2x^{3/2}}{x^4}$$

$$= \frac{2x^5 - 5x^4 - \frac{3}{2}x^{3/2}}{x^4} = 2x - 5 - \tfrac{3}{2}x^{-5/2}$$

Simplifying first: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = x^2 - 5x + x^{-3/2} \quad \Rightarrow \quad F'(x) = 2x - 5 - \frac{3}{2}x^{-5/2}$ (equivalent).

For this problem, simplifying first seems to be the better method.

25. $f(x) = (5x^2 - 2)(x^3 + 3x) \overset{\text{PR}}{\Rightarrow}$

$f'(x) = (5x^2 - 2)(3x^2 + 3) + (x^3 + 3x)(10x) = 15x^4 + 9x^2 - 6 + 10x^4 + 30x^2 = 25x^4 + 39x^2 - 6$

26. $B(u) = (u^3 + 1)(2u^2 - 4u - 1) \overset{\text{PR}}{\Rightarrow}$

$$\begin{aligned} B'(u) &= (u^3 + 1)(4u - 4) + (2u^2 - 4u - 1)(3u^2) \\ &= 4u^4 - 4u^3 + 4u - 4 + 6u^4 - 12u^3 - 3u^2 = 10u^4 - 16u^3 - 3u^2 + 4u - 4 \end{aligned}$$

27. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3) = (y^{-2} - 3y^{-4})(y + 5y^3) \overset{\text{PR}}{\Rightarrow}$

$$\begin{aligned} F'(y) &= (y^{-2} - 3y^{-4})(1 + 15y^2) + (y + 5y^3)(-2y^{-3} + 12y^{-5}) \\ &= (y^{-2} + 15 - 3y^{-4} - 45y^{-2}) + (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2}) \\ &= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4 \end{aligned}$$

28. $J(v) = (v^3 - 2v)(v^{-4} + v^{-2}) \overset{\text{PR}}{\Rightarrow}$

$$\begin{aligned} J'(v) &= (v^3 - 2v)(-4v^{-5} - 2v^{-3}) + (v^{-4} + v^{-2})(3v^2 - 2) \\ &= -4v^{-2} - 2v^0 + 8v^{-4} + 4v^{-2} + 3v^{-2} - 2v^{-4} + 3v^0 - 2v^{-2} = 1 + v^{-2} + 6v^{-4} \end{aligned}$$

29. $g(x) = \dfrac{1 + 2x}{3 - 4x} \overset{\text{QR}}{\Rightarrow} g'(x) = \dfrac{(3 - 4x)(2) - (1 + 2x)(-4)}{(3 - 4x)^2} = \dfrac{6 - 8x + 4 + 8x}{(3 - 4x)^2} = \dfrac{10}{(3 - 4x)^2}$

30. $h(t) = \dfrac{6t + 1}{6t - 1} \overset{\text{QR}}{\Rightarrow} h'(t) = \dfrac{(6t - 1)(6) - (6t + 1)(6)}{(6t - 1)^2} = \dfrac{36t - 6 - 36t - 6}{(6t - 1)^2} = -\dfrac{12}{(6t - 1)^2}$

31. $y = \dfrac{x^2 + 1}{x^3 - 1} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(x^3 - 1)(2x) - (x^2 + 1)(3x^2)}{(x^3 - 1)^2} = \frac{x\left[(x^3 - 1)(2) - (x^2 + 1)(3x)\right]}{(x^3 - 1)^2} = \frac{x(2x^3 - 2 - 3x^3 - 3x)}{(x^3 - 1)^2} = \frac{x(-x^3 - 3x - 2)}{(x^3 - 1)^2}$$

32. $y = \dfrac{1}{t^3 + 2t^2 - 1} \overset{\text{QR}}{\Rightarrow} y' = \dfrac{(t^3 + 2t^2 - 1)(0) - 1(3t^2 + 4t)}{(t^3 + 2t^2 - 1)^2} = -\dfrac{3t^2 + 4t}{(t^3 + 2t^2 - 1)^2}$

33. $y = \dfrac{t^3 + 3t}{t^2 - 4t + 3} \overset{\text{QR}}{\Rightarrow}$

$$\begin{aligned} y' &= \frac{(t^2 - 4t + 3)(3t^2 + 3) - (t^3 + 3t)(2t - 4)}{(t^2 - 4t + 3)^2} \\ &= \frac{3t^4 + 3t^2 - 12t^3 - 12t + 9t^2 + 9 - (2t^4 - 4t^3 + 6t^2 - 12t)}{(t^2 - 4t + 3)^2} = \frac{t^4 - 8t^3 + 6t^2 + 9}{(t^2 - 4t + 3)^2} \end{aligned}$$

34. $y = \dfrac{(u + 2)^2}{1 - u} = \dfrac{u^2 + 4u + 4}{1 - u} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(1 - u)(2u + 4) - (u^2 + 4u + 4)(-1)}{(1 - u)^2} = \frac{2u + 4 - 2u^2 - 4u + u^2 + 4u + 4}{(1 - u)^2} = \frac{-u^2 + 2u + 8}{(1 - u)^2}$$

35. $y = \dfrac{s - \sqrt{s}}{s^2} = \dfrac{s}{s^2} - \dfrac{\sqrt{s}}{s^2} = s^{-1} - s^{-3/2} \quad \Rightarrow \quad y' = -s^{-2} + \frac{3}{2}s^{-5/2} = \dfrac{-1}{s^2} + \dfrac{3}{2s^{5/2}} = \dfrac{3 - 2\sqrt{s}}{2s^{5/2}}$

36. $y = \dfrac{\sqrt{x}}{2 + x} \quad \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(2+x)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(2+x)^2} = \frac{\dfrac{1}{\sqrt{x}} + \dfrac{\sqrt{x}}{2} - \sqrt{x}}{(2+x)^2} = \frac{\dfrac{2 + x - 2x}{2\sqrt{x}}}{(2+x)^2} = \frac{2 - x}{2\sqrt{x}(2+x)^2}$$

37. $f(t) = \dfrac{\sqrt[3]{t}}{t - 3} \quad \overset{\text{QR}}{\Rightarrow}$

$$f'(t) = \frac{(t-3)\left(\frac{1}{3}t^{-2/3}\right) - t^{1/3}(1)}{(t-3)^2} = \frac{\frac{1}{3}t^{1/3} - t^{-2/3} - t^{1/3}}{(t-3)^2} = \frac{-\frac{2}{3}t^{1/3} - t^{-2/3}}{(t-3)^2} = \frac{\dfrac{-2t}{3t^{2/3}} - \dfrac{3}{3t^{2/3}}}{(t-3)^2} = \frac{-2t - 3}{3t^{2/3}(t-3)^2}$$

38. $y = \dfrac{cx}{1 + cx} \quad \Rightarrow \quad y' = \dfrac{(1 + cx)(c) - (cx)(c)}{(1 + cx)^2} = \dfrac{c + c^2x - c^2x}{(1 + cx)^2} = \dfrac{c}{(1 + cx)^2}$

39. $F(x) = \dfrac{2x^5 + x^4 - 6x}{x^3} = 2x^2 + x - 6x^{-2} \quad \Rightarrow \quad F'(x) = 4x + 1 + 12x^{-3} = 4x + 1 + \dfrac{12}{x^3}$ or $\dfrac{4x^4 + x^3 + 12}{x^3}$

40. $A(v) = v^{2/3}(2v^2 + 1 - v^{-2}) = 2v^{8/3} + v^{2/3} - v^{-4/3} \quad \Rightarrow$

$$A'(v) = \tfrac{16}{3}v^{5/3} + \tfrac{2}{3}v^{-1/3} + \tfrac{4}{3}v^{-7/3} = \tfrac{2}{3}v^{-7/3}(8v^{12/3} + v^{6/3} + 2) = \frac{2(8v^4 + v^2 + 2)}{3v^{7/3}}$$

41. $G(y) = \dfrac{B}{Ay^3 + B} \quad \overset{\text{QR}}{\Rightarrow} \quad G'(y) = \dfrac{(Ay^3 + B)(0) - B(3Ay^2)}{(Ay^3 + B)^2} = -\dfrac{3ABy^2}{(Ay^3 + B)^2}$

42. $F(t) = \dfrac{At}{Bt^2 + Ct^3} = \dfrac{A}{Bt + Ct^2} \quad \overset{\text{QR}}{\Rightarrow}$

$$F'(t) = \frac{(Bt + Ct^2)(0) - A(B + 2Ct)}{(Bt + Ct^2)^2} = \frac{-A(B + 2Ct)}{(t)^2(B + Ct)^2} = -\frac{A(B + 2Ct)}{t^2(B + Ct)^2}$$

43. $f(x) = \dfrac{x}{x + c/x} \quad \Rightarrow \quad f'(x) = \dfrac{(x + c/x)(1) - x(1 - c/x^2)}{\left(x + \dfrac{c}{x}\right)^2} = \dfrac{x + c/x - x + c/x}{\left(\dfrac{x^2 + c}{x}\right)^2} = \dfrac{2c/x}{\dfrac{(x^2 + c)^2}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{2cx}{(x^2 + c)^2}$

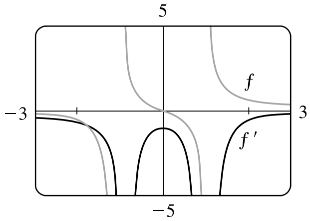
44. $f(x) = \dfrac{ax + b}{cx + d} \quad \Rightarrow \quad f'(x) = \dfrac{(cx + d)(a) - (ax + b)(c)}{(cx + d)^2} = \dfrac{acx + ad - acx - bc}{(cx + d)^2} = \dfrac{ad - bc}{(cx + d)^2}$

45. $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \quad \Rightarrow \quad P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$

46. $f(x) = \dfrac{x}{x^2 - 1} \quad \Rightarrow$

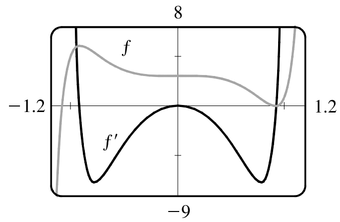
$$f'(x) = \frac{(x^2 - 1)1 - x(2x)}{(x^2 - 1)^2} = \frac{-x^2 - 1}{(x^2 - 1)^2} = -\frac{x^2 + 1}{(x^2 - 1)^2}$$

Notice that the slopes of all tangents to f are negative and $f'(x) < 0$ always.



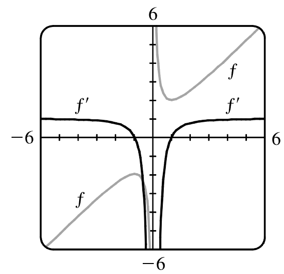
47. $f(x) = 3x^{15} - 5x^3 + 3 \quad \Rightarrow \quad f'(x) = 45x^{14} - 15x^2$.

Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

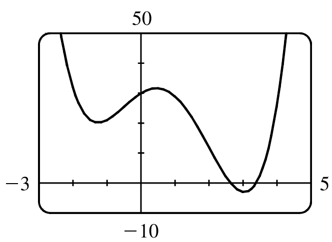


48. $f(x) = x + 1/x = x + x^{-1} \quad \Rightarrow \quad f'(x) = 1 - x^{-2} = 1 - 1/x^2$.

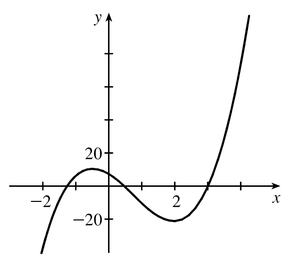
Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



49. (a)

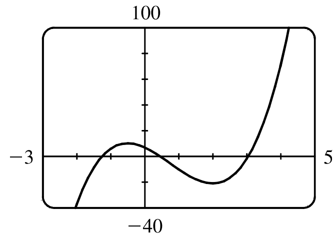


(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

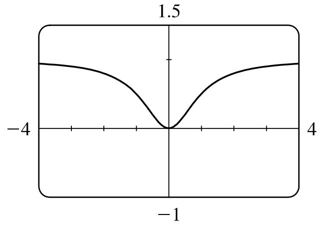


(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \quad \Rightarrow$

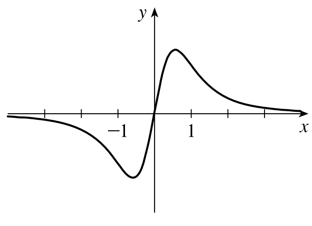
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



50. (a)



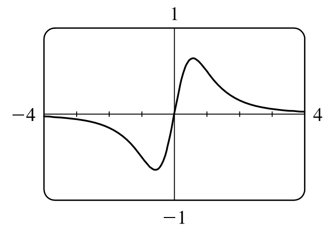
(b)



From the graph in part (a), it appears that g' is zero at $x = 0$. The slopes are negative (so g' is negative) on $(-\infty, 0)$. The slopes are positive (so g' is positive) on $(0, \infty)$.

(c) $g(x) = \dfrac{x^2}{x^2+1} \quad \Rightarrow$

$$g'(x) = \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$



51. $y = \dfrac{2x}{x+1} \quad \Rightarrow \quad y' = \dfrac{(x+1)(2) - (2x)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$.

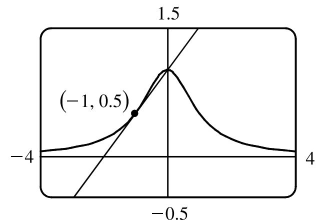
At $(1,1)$, $y' = \frac{1}{2}$, and an equation of the tangent line is $y - 1 = \frac{1}{2}(x-1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

52. $y = 2x^3 - x^2 + 2 \quad \Rightarrow \quad y' = 6x^2 - 2x$. At $(1,3)$, $y' = 6(1)^2 - 2(1) = 4$ and an equation of the tangent line is $y - 3 = 4(x-1)$ or $y = 4x - 1$.

53. (a) $y = f(x) = \dfrac{1}{1+x^2} \quad \Rightarrow$

$f'(x) = \dfrac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x + 1$.

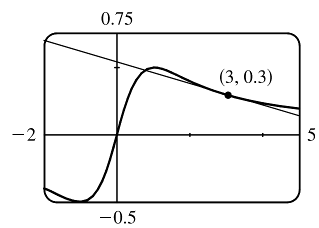
(b)



54. (a) $y = f(x) = \dfrac{x}{1+x^2} \quad \Rightarrow$

$f'(x) = \dfrac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. So the slope of the tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100}$ and its equation is $y - 0.3 = -0.08(x-3)$ or $y = -0.08x + 0.54$.

(b)



55. $y = x + \sqrt{x} \quad \Rightarrow \quad y' = 1 + \frac{1}{2}x^{-1/2} = 1 + 1/(2\sqrt{x})$. At $(1,2)$, $y' = \frac{3}{2}$, and an equation of the tangent line is $y - 2 = \frac{3}{2}(x-1)$, or $y = \frac{3}{2}x + \frac{1}{2}$. The slope of the normal line is $-\frac{2}{3}$, so an equation of the normal line is $y - 2 = -\frac{2}{3}(x-1)$, or $y = -\frac{2}{3}x + \frac{8}{3}$.

56. $y^2 = x^3 \quad \Rightarrow \quad y = x^{3/2}$ [since x and y are positive at $(1,1)$] $\Rightarrow \quad y' = \frac{3}{2}x^{1/2}$. At $(1,1)$, $y' = \frac{3}{2}$ and an equation of the tangent line is $y - 1 = \frac{3}{2}(x-1)$ or $y = \frac{3}{2}x - \frac{1}{2}$. The slope of the normal line is $-\frac{2}{3}$ $\left(\text{the negative reciprocal of } \frac{3}{2}\right)$ and an equation of the normal line is $y - 1 = -\frac{2}{3}(x-1)$ or $y = -\frac{2}{3}x + \frac{5}{3}$.

57. $y = \dfrac{3x+1}{x^2+1} \quad \Rightarrow \quad y' = \dfrac{(x^2+1)(3) - (3x+1)(2x)}{(x^2+1)^2}$. At $(1,2)$, $y' = \dfrac{6-8}{2^2} = -\dfrac{1}{2}$, and an equation of the tangent line is $y - 2 = -\frac{1}{2}(x-1)$, or $y = -\frac{1}{2}x + \frac{5}{2}$. The slope of the normal line is 2, so an equation of the normal line is $y - 2 = 2(x-1)$, or $y = 2x$.

58. $y = \dfrac{\sqrt{x}}{x+1} \quad\Rightarrow\quad y' = \dfrac{(x+1)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(x+1)^2} = \dfrac{(x+1)-(2x)}{2\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{2\sqrt{x}\,(x+1)^2}.$

At $(4, 0.4)$, $y' = \frac{-3}{100} = -0.03$, and an equation of the tangent line is $y - 0.4 = -0.03(x-4)$, or $y = -0.03x + 0.52$. The slope of the normal line is $\frac{100}{3}$, so an equation of the normal line is $y - 0.4 = \frac{100}{3}(x-4) \quad\Leftrightarrow\quad y = \frac{100}{3}x - \frac{400}{3} + \frac{2}{5} \quad\Leftrightarrow$ $y = \frac{100}{3}x - \frac{1994}{15}$.

59. $f(x) = 0.001x^5 - 0.02x^3 \quad\Rightarrow\quad f'(x) = 0.005x^4 - 0.06x^2 \quad\Rightarrow\quad f''(x) = 0.02x^3 - 0.12x$

60. $G(r) = \sqrt{r} + \sqrt[3]{r} \quad\Rightarrow\quad G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3} \quad\Rightarrow\quad G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$

61. $f(x) = \dfrac{x^2}{1+2x} \quad\Rightarrow\quad f'(x) = \dfrac{(1+2x)(2x) - x^2(2)}{(1+2x)^2} = \dfrac{2x + 4x^2 - 2x^2}{(1+2x)^2} = \dfrac{2x^2+2x}{(1+2x)^2} \quad\Rightarrow$

$$f''(x) = \frac{(1+2x)^2(4x+2) - (2x^2+2x)(1+4x+4x^2)'}{[(1+2x)^2]^2} = \frac{2(1+2x)^2(2x+1) - 2x(x+1)(4+8x)}{(1+2x)^4}$$

$$= \frac{2(1+2x)[(1+2x)^2 - 4x(x+1)]}{(1+2x)^4} = \frac{2(1+4x+4x^2-4x^2-4x)}{(1+2x)^3} = \frac{2}{(1+2x)^3}$$

62. Using the Reciprocal Rule, $f(x) = \dfrac{1}{3-x} \quad\Rightarrow\quad f'(x) = -\dfrac{(3-x)'}{(3-x)^2} = -\dfrac{-1}{(3-x)^2} = \dfrac{1}{(3-x)^2} \quad\Rightarrow$

$$f''(x) = -\frac{[(3-x)^2]'}{[(3-x)^2]^2} = -\frac{(9-6x+x^2)'}{(3-x)^4} = -\frac{-6+2x}{(3-x)^4} = -\frac{-2(3-x)}{(3-x)^4} = \frac{2}{(3-x)^3}.$$

63. (a) $s = t^3 - 3t \quad\Rightarrow\quad v(t) = s'(t) = 3t^2 - 3 \quad\Rightarrow\quad a(t) = v'(t) = 6t$

(b) $a(2) = 6(2) = 12\ \text{m/s}^2$

(c) $v(t) = 3t^2 - 3 = 0$ when $t^2 = 1$, that is, $t = 1$ $[t \geq 0]$ and $a(1) = 6\ \text{m/s}^2$.

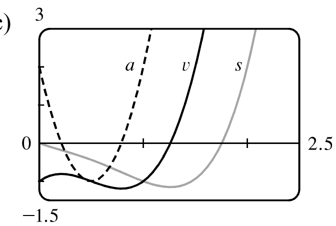
64. (a) $s = t^4 - 2t^3 + t^2 - t \quad\Rightarrow$

$v(t) = s'(t) = 4t^3 - 6t^2 + 2t - 1 \quad\Rightarrow$

$a(t) = v'(t) = 12t^2 - 12t + 2$

(b) $a(1) = 12(1)^2 - 12(1) + 2 = 2\ \text{m/s}^2$

(c)



65. $L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21 \quad\Rightarrow\quad \dfrac{dL}{dA} = 0.0465A^2 - 0.744A + 3.95$, so

$\left.\dfrac{dL}{dA}\right|_{A=12} = 0.0465(12)^2 - 0.744(12) + 3.95 = 1.718$. The derivative is the instantaneous rate of change of the length of an Alaskan rockfish with respect to its age when its age is 12 years.

66. $S(A) = 0.882A^{0.842} \quad\Rightarrow\quad S'(A) = 0.882(0.842A^{-0.158}) = 0.742644A^{-0.158}$, so

$S'(100) = 0.742644(100)^{-0.158} \approx 0.36$. The derivative is the instantaneous rate of change of the number of tree species with respect to area. Its units are number of species per square meter.

67. (a) $P = \dfrac{k}{V}$ and $P = 50$ when $V = 0.106$, so $k = PV = 50(0.106) = 5.3$. Thus, $P = \dfrac{5.3}{V}$ and $V = \dfrac{5.3}{P}$.

(b) $V = 5.3P^{-1} \;\Rightarrow\; \dfrac{dV}{dP} = 5.3(-1P^{-2}) = -\dfrac{5.3}{P^2}$. When $P = 50$, $\dfrac{dV}{dP} = -\dfrac{5.3}{50^2} = -0.00212$. The derivative is the instantaneous rate of change of the volume with respect to the pressure at $25\,^\circ\mathrm{C}$. Its units are $\mathrm{m}^3/\mathrm{kPa}$.

68. (a) $L = aP^2 + bP + c$, where $a \approx -0.275428$, $b \approx 19.74853$, and $c \approx -273.55234$.

(b) $\dfrac{dL}{dP} = 2aP + b$. When $P = 30$, $\dfrac{dL}{dP} \approx 3.2$, and when $P = 40$, $\dfrac{dL}{dP} \approx -2.3$. The derivative is the instantaneous rate of change of tire life with respect to pressure. Its units are (thousands of miles)$/(\mathrm{lb/in}^2)$. When $\dfrac{dL}{dP}$ is positive, tire life is increasing, and when $\dfrac{dL}{dP} < 0$, tire life is decreasing.

69. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

70. We are given that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$.

(a) $h(x) = 3f(x) + 8g(x) \;\Rightarrow\; h'(x) = 3f'(x) + 8g'(x)$, so

$h'(4) = 3f'(4) + 8g'(4) = 3(6) + 8(-3) = 18 - 24 = -6$.

(b) $h(x) = f(x)\,g(x) \;\Rightarrow\; h'(x) = f(x)\,g'(x) + g(x)\,f'(x)$, so

$h'(4) = f(4)\,g'(4) + g(4)\,f'(4) = 2(-3) + 5(6) = -6 + 30 = 24$.

(c) $h(x) = \dfrac{f(x)}{g(x)} \;\Rightarrow\; h'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$, so

$$h'(4) = \frac{g(4)\,f'(4) - f(4)\,g'(4)}{[g(4)]^2} = \frac{5(6) - 2(-3)}{5^2} = \frac{30 + 6}{25} = \frac{36}{25}.$$

(d) $h(x) = \dfrac{g(x)}{f(x) + g(x)} \;\Rightarrow$

$$h'(4) = \frac{[f(4) + g(4)]\,g'(4) - g(4)\,[f'(4) + g'(4)]}{[f(4) + g(4)]^2} = \frac{(2+5)\,(-3) - 5\,[6 + (-3)]}{(2+5)^2} = \frac{-21 - 15}{7^2} = -\frac{36}{49}$$

71. $f(x) = \sqrt{x}\,g(x) \;\Rightarrow\; f'(x) = \sqrt{x}\,g'(x) + g(x)\cdot\frac{1}{2}x^{-1/2}$, so $f'(4) = \sqrt{4}\,g'(4) + g(4)\cdot\dfrac{1}{2\sqrt{4}} = 2\cdot 7 + 8\cdot\dfrac{1}{4} = 16$.

72. $\dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right] = \dfrac{xh'(x) - h(x)\cdot 1}{x^2} \;\Rightarrow\; \dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right]_{x=2} = \dfrac{2h'(2) - h(2)}{2^2} = \dfrac{2(-3) - (4)}{4} = \dfrac{-10}{4} = -2.5$

73. (a) From the graphs of f and g, we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f; $g(1) = 1$ since the point $(1, 1)$ is on the graph of g; $f'(1) = 2$ since the slope of the line segment between $(0, 0)$ and $(2, 4)$ is $\dfrac{4-0}{2-0} = 2$; $g'(1) = -1$ since the slope of the line segment between $(-2, 4)$ and $(2, 0)$ is $\dfrac{0-4}{2-(-2)} = -1$.

Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)\,f'(1) = 2 \cdot (-1) + 1 \cdot 2 = 0$.

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{2\left(-\frac{1}{3}\right) - 3 \cdot \frac{2}{3}}{2^2} = \dfrac{-\frac{8}{3}}{4} = -\dfrac{2}{3}$

74. (a) $P(x) = F(x)\,G(x)$, so $P'(2) = F(2)\,G'(2) + G(2)\,F'(2) = 3 \cdot \frac{2}{4} + 2 \cdot 0 = \frac{3}{2}$.

(b) $Q(x) = F(x)/G(x)$, so $Q'(7) = \dfrac{G(7)\,F'(7) - F(7)\,G'(7)}{[G(7)]^2} = \dfrac{1 \cdot \frac{1}{4} - 5 \cdot \left(-\frac{2}{3}\right)}{1^2} = \dfrac{1}{4} + \dfrac{10}{3} = \dfrac{43}{12}$

75. (a) $y = xg(x) \quad\Rightarrow\quad y' = xg'(x) + g(x) \cdot 1 = xg'(x) + g(x)$

(b) $y = \dfrac{x}{g(x)} \quad\Rightarrow\quad y' = \dfrac{g(x) \cdot 1 - xg'(x)}{[g(x)]^2} = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \dfrac{g(x)}{x} \quad\Rightarrow\quad y' = \dfrac{xg'(x) - g(x) \cdot 1}{(x)^2} = \dfrac{xg'(x) - g(x)}{x^2}$

76. (a) $y = x^2 f(x) \quad\Rightarrow\quad y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \quad\Rightarrow\quad y' = \dfrac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

(c) $y = \dfrac{x^2}{f(x)} \quad\Rightarrow\quad y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}} \quad\Rightarrow$

$$y' = \frac{\sqrt{x}\,[xf'(x) + f(x)] - [1 + xf(x)]\,\dfrac{1}{2\sqrt{x}}}{(\sqrt{x}\,)^2}$$
$$= \frac{x^{3/2}f'(x) + x^{1/2}f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

77. The curve $y = 2x^3 + 3x^2 - 12x + 1$ has a horizontal tangent when $y' = 6x^2 + 6x - 12 = 0 \quad\Leftrightarrow\quad 6(x^2 + x - 2) = 0 \quad\Leftrightarrow\quad 6(x + 2)(x - 1) = 0 \quad\Leftrightarrow\quad x = -2$ or $x = 1$. The points on the curve are $(-2, 21)$ and $(1, -6)$.

78. $f(x) = x^3 + 3x^2 + x + 3$ has a horizontal tangent when $f'(x) = 3x^2 + 6x + 1 = 0 \quad\Leftrightarrow$

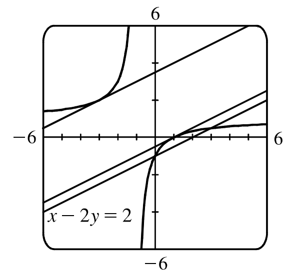
$x = \dfrac{-6 \pm \sqrt{36 - 12}}{6} = -1 \pm \frac{1}{3}\sqrt{6}$.

79. $y = 6x^3 + 5x - 3 \quad\Rightarrow\quad m = y' = 18x^2 + 5$, but $x^2 \geq 0$ for all x, so $m \geq 5$ for all x.

80. $y = x^4 + 1 \quad\Rightarrow\quad y' = 4x^3$. The slope of the line $32x - y = 15$ (or $y = 32x - 15$) is 32, so the slope of any line parallel to it is also 32. Thus, $y' = 32 \quad\Leftrightarrow\quad 4x^3 = 32 \quad\Leftrightarrow\quad x^3 = 8 \quad\Leftrightarrow\quad x = 2$, which is the x-coordinate of the point on the curve at which the slope is 32. The y-coordinate is $2^4 + 1 = 17$, so an equation of the tangent line is $y - 17 = 32(x - 2)$ or $y = 32x - 47$.

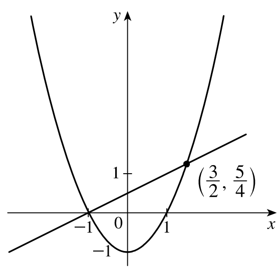
81. The slope of the line $3x - y = 15$ (or $y = 3x - 15$) is 3, so the slope of both tangent lines to the curve is 3. $y = x^3 - 3x^2 + 3x - 3 \;\Rightarrow\; y' = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$. Thus, $3(x - 1)^2 = 3 \;\Rightarrow\; (x - 1)^2 = 1 \;\Rightarrow\; x - 1 = \pm 1 \;\Rightarrow\; x = 0$ or 2, which are the x-coordinates at which the tangent lines have slope 3. The points on the curve are $(0, -3)$ and $(2, -1)$, so the tangent line equations are $y - (-3) = 3(x - 0)$ or $y = 3x - 3$ and $y - (-1) = 3(x - 2)$ or $y = 3x - 7$.

82. $y = \dfrac{x - 1}{x + 1} \;\Rightarrow\; y' = \dfrac{(x + 1)(1) - (x - 1)(1)}{(x + 1)^2} = \dfrac{2}{(x + 1)^2}$. If the tangent intersects the curve when $x = a$, then its slope is $2/(a + 1)^2$. But if the tangent is parallel to $x - 2y = 2$, that is, $y = \frac{1}{2}x - 1$, then its slope is $\frac{1}{2}$. Thus, $\dfrac{2}{(a + 1)^2} = \dfrac{1}{2} \;\Rightarrow\; (a + 1)^2 = 4 \;\Rightarrow\; a + 1 = \pm 2 \;\Rightarrow\; a = 1$ or -3. When $a = 1$, $y = 0$ and the equation of the tangent is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$. When $a = -3$, $y = 2$ and the equation of the tangent is $y - 2 = \frac{1}{2}(x + 3)$ or $y = \frac{1}{2}x + \frac{7}{2}$.

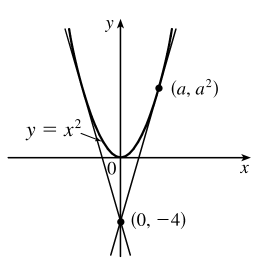


83. The slope of $y = \sqrt{x}$ is given by $y = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$. The slope of $2x + y = 1$ (or $y = -2x + 1$) is -2, so the desired normal line must have slope -2, and hence, the tangent line to the curve must have slope $\frac{1}{2}$. This occurs if $\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2} \;\Rightarrow\; \sqrt{x} = 1 \;\Rightarrow\; x = 1$. When $x = 1$, $y = \sqrt{1} = 1$, and an equation of the normal line is $y - 1 = -2(x - 1)$ or $y = -2x + 3$.

84. $y = f(x) = x^2 - 1 \;\Rightarrow\; f'(x) = 2x$. So $f'(-1) = -2$, and the slope of the normal line is $\frac{1}{2}$. The equation of the normal line at $(-1, 0)$ is $y - 0 = \frac{1}{2}[x - (-1)]$ or $y = \frac{1}{2}x + \frac{1}{2}$. Substituting this into the equation of the parabola, we obtain $\frac{1}{2}x + \frac{1}{2} = x^2 - 1 \;\Leftrightarrow\; x + 1 = 2x^2 - 2 \;\Leftrightarrow\; 2x^2 - x - 3 = 0 \;\Leftrightarrow\; (2x - 3)(x + 1) = 0 \;\Leftrightarrow\; x = \frac{3}{2}$ or -1. Substituting $\frac{3}{2}$ into the equation of the normal line gives us $y = \frac{5}{4}$. Thus, the second point of intersection is $\left(\frac{3}{2}, \frac{5}{4}\right)$, as shown in the sketch.



85.



Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \;\Leftrightarrow\; y = 2ax - 4$. Since (a, a^2) also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

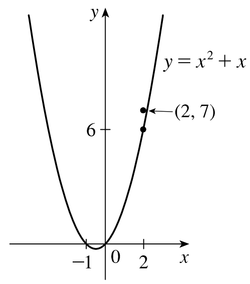
86. (a) If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $(a, a^2 + a)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be $\dfrac{\Delta y}{\Delta x} = \dfrac{a^2 + a + 3}{a - 2}$.

Therefore, $2a + 1 = \dfrac{a^2 + a + 3}{a - 2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \quad \Leftrightarrow$ $a^2 - 4a - 5 = (a - 5)(a + 1) = 0 \quad \Leftrightarrow \quad a = 5$ or -1. If $a = -1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y - 0 = (-1)(x + 1)$ or $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y - 30 = 11(x - 5)$ or $y = 11x - 25$.

(b) As in part (a), but using the point $(2, 7)$, we get the equation

$$2a + 1 = \frac{a^2 + a - 7}{a - 2} \quad \Rightarrow \quad 2a^2 - 3a - 2 = a^2 + a - 7 \quad \Leftrightarrow \quad a^2 - 4a + 5 = 0.$$

The last equation has no real solution (discriminant $= -16 < 0$), so there is no line through the point $(2, 7)$ that is tangent to the parabola. The diagram shows that the point $(2, 7)$ is "inside" the parabola, but tangent lines to the parabola do not pass through points inside the parabola.



87. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\dfrac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x)$.

(c) $y = (x^4 + 3x^3 + 17x + 82)^3 \quad \Rightarrow \quad y' = 3(x^4 + 3x^3 + 17x + 82)^2(4x^3 + 9x^2 + 17)$

88. (a) $f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1} \quad \Rightarrow \quad f''(x) = n(n-1)x^{n-2} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = n(n-1)(n-2)\cdots 2 \cdot 1x^{n-n} = n!$

(b) $f(x) = x^{-1} \quad \Rightarrow \quad f'(x) = (-1)x^{-2} \quad \Rightarrow \quad f''(x) = (-1)(-2)x^{-3} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = (-1)(-2)(-3)\cdots(-n)x^{-(n+1)} = (-1)^n n! x^{-(n+1)}$ or $\dfrac{(-1)^n\, n!}{x^{n+1}}$

89. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \quad \Rightarrow \quad 2a = 2 \quad \Rightarrow \quad a = 1$.
$P'(2) = 3 \quad \Rightarrow \quad 2(1)(2) + b = 3 \quad \Rightarrow \quad 4 + b = 3 \quad \Rightarrow \quad b = -1$.
$P(2) = 5 \quad \Rightarrow \quad 1(2)^2 + (-1)(2) + c = 5 \quad \Rightarrow \quad 2 + c = 5 \quad \Rightarrow \quad c = 3$. So $P(x) = x^2 - x + 3$.

90. $y = Ax^2 + Bx + C \quad \Rightarrow \quad y' = 2Ax + B \quad \Rightarrow \quad y'' = 2A$. We substitute these expressions into the equation $y'' + y' - 2y = x^2$ to get

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$
$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$
$$(-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = (1)x^2 + (0)x + (0)$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \quad \Rightarrow \quad A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \quad \Rightarrow$ $A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \quad \Rightarrow \quad -1 - \frac{1}{2} - 2C = 0 \quad \Rightarrow \quad C = -\frac{3}{4}$.

91. $y = f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f, so $f(-2) = 6 \quad \Rightarrow$ $-8a + 4b - 2c + d = 6$ **(1)**. The point $(2, 0)$ is on f, so $f(2) = 0 \quad \Rightarrow \quad 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \quad \Rightarrow \quad 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \quad \Rightarrow$ $12a + 4b + c = 0$ **(4)**. Subtracting equation **(3)** from **(4)** gives $8b = 0 \quad \Rightarrow \quad b = 0$. Adding **(1)** and **(2)** gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From **(3)** we have $c = -12a$, so **(2)** becomes $8a + 4(0) + 2(-12a) + 3 = 0 \quad \Rightarrow \quad 3 = 16a \quad \Rightarrow$ $a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

92. $y = ax^2 + bx + c \quad \Rightarrow \quad y'(x) = 2ax + b$. The parabola has slope 4 at $x = 1$ and slope -8 at $x = -1$, so $y'(1) = 4 \quad \Rightarrow$ $2a + b = 4$ **(1)** and $y'(-1) = -8 \quad \Rightarrow \quad -2a + b = -8$ **(2)**. Adding **(1)** and **(2)** gives us $2b = -4 \quad \Leftrightarrow \quad b = -2$. From **(1)**, $2a - 2 = 4 \quad \Leftrightarrow \quad a = 3$. Thus, the equation of the parabola is $y = 3x^2 - 2x + c$. Since it passes through the point $(2, 15)$, we have $15 = 3(2)^2 - 2(2) + c \quad \Rightarrow \quad c = 7$, so the equation is $y = 3x^2 - 2x + 7$.

93. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \quad \Rightarrow$

$$\begin{aligned} T'(1999) &= P(1999)A'(1999) + A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr}) + (\$30{,}593)(9200/\text{yr}) \\ &= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr} \end{aligned}$$

So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

94. (a) $f(20) = 10{,}000$ means that when the price of the fabric is \$20/yard, 10,000 yards will be sold.

$f'(20) = -350$ means that as the price of the fabric increases past \$20/yard, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

(b) $R(p) = pf(p) \quad \Rightarrow \quad R'(p) = pf'(p) + f(p) \cdot 1 \quad \Rightarrow \quad R'(20) = 20f'(20) + f(20) \cdot 1 = 20(-350) + 10{,}000 = 3000$.

This means that as the price of the fabric increases past \$20/yard, the total revenue is increasing at \$3000/(\$/yard). Note that the Product Rule indicates that we will lose \$7000/(\$/yard) due to selling less fabric, but this loss is more than made up for by the additional revenue due to the increase in price.

95. $v = \dfrac{0.14[\text{S}]}{0.015 + [\text{S}]} \quad \Rightarrow \quad \dfrac{dv}{d[\text{S}]} = \dfrac{(0.015 + [\text{S}])(0.14) - (0.14[\text{S}])(1)}{(0.015 + [\text{S}])^2} = \dfrac{0.0021}{(0.015 + [\text{S}])^2}$.

$dv/d[\text{S}]$ represents the rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S.

96. $B(t) = N(t)\,M(t) \quad \Rightarrow \quad B'(t) = N(t)\,M'(t) + M(t)\,N'(t)$, so

$B'(4) = N(4)\,M'(4) + M(4)\,N'(4) = 820(0.14) + 1.2(50) = 174.8$ g/week.

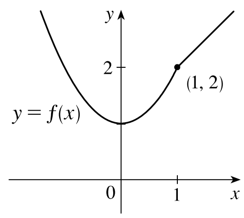
97. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$

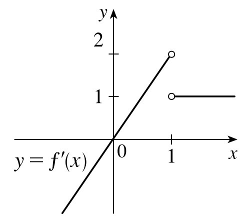
Calculate the left- and right-hand derivatives as defined in Exercise 2.2.62:

$$f'_-(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[(1+h)^2 + 1] - (1+1)}{h} = \lim_{h \to 0^-} \frac{h^2 + 2h}{h} = \lim_{h \to 0^-} (h+2) = 2 \text{ and}$$

$$f'_+(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[(1+h) + 1] - (1+1)}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

Since the left and right limits are different, $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ does not exist, that is, $f'(1)$ does not exist. Therefore, f is not differentiable at 1.





98. $g(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \geq 2 \end{cases}$

Investigate the left- and right-hand derivatives at $x = 0$ and $x = 2$:

$$g'_-(0) = \lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{2h - 2(0)}{h} = 2 \text{ and}$$

$$g'_+(0) = \lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^+} \frac{(2h - h^2) - 2(0)}{h} = \lim_{h \to 0^+} (2 - h) = 2, \text{ so } g \text{ is differentiable at } x = 0.$$

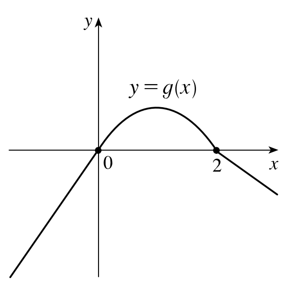
$$g'_-(2) = \lim_{h \to 0^-} \frac{g(2+h) - g(2)}{h} = \lim_{h \to 0^-} \frac{2(2+h) - (2+h)^2 - (2-2)}{h} = \lim_{h \to 0^-} \frac{-2h - h^2}{h} = \lim_{h \to 0^-} (-2 - h) = -2$$

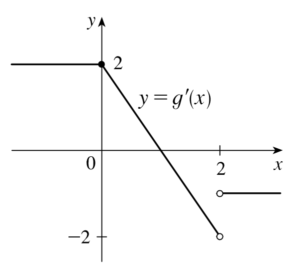
and

$$g'_+(2) = \lim_{h \to 0^+} \frac{g(2+h) - g(2)}{h} = \lim_{h \to 0^+} \frac{[2 - (2+h)] - (2-2)}{h} = \lim_{h \to 0^+} \frac{-h}{h} = \lim_{h \to 0^+} (-1) = -1,$$

so g is not differentiable at $x = 2$. Thus, a formula for g' is

$$g'(x) = \begin{cases} 2 & \text{if } x \leq 0 \\ 2 - 2x & \text{if } 0 < x < 2 \\ -1 & \text{if } x > 2 \end{cases}$$





99. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \iff |x| < 3 \iff -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \leq -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \geq 3 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

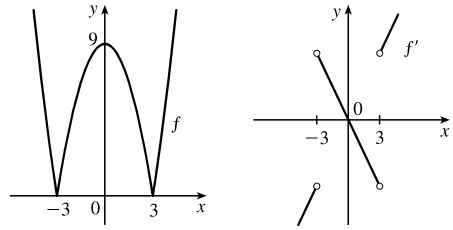
To show that $f'(3)$ does not exist we investigate $\lim\limits_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 2.2.62.

$$f'_-(3)=\lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^-}\frac{[-(3+h)^2+9]-0}{h}=\lim_{h\to 0^-}(-6-h)=-6 \quad \text{and}$$

$$f'_+(3)=\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^+}\frac{\left[(3+h)^2-9\right]-0}{h}=\lim_{h\to 0^+}\frac{6h+h^2}{h}=\lim_{h\to 0^+}(6+h)=6.$$

Since the left and right limits are different, $\lim\limits_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

(b)



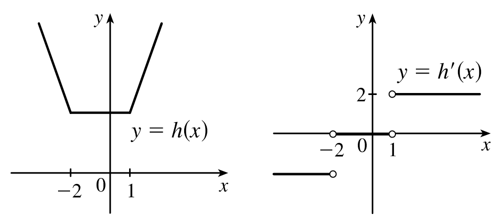
100. If $x\geq 1$, then $h(x)=|x-1|+|x+2|=x-1+x+2=2x+1$.

If $-2<x<1$, then $h(x)=-(x-1)+x+2=3$.

If $x\leq -2$, then $h(x)=-(x-1)-(x+2)=-2x-1$. Therefore,

$$h(x)=\begin{cases}-2x-1 & \text{if } x\leq -2\\ 3 & \text{if } -2<x<1\\ 2x+1 & \text{if } x\geq 1\end{cases} \quad\Rightarrow\quad h'(x)=\begin{cases}-2 & \text{if } x<-2\\ 0 & \text{if } -2<x<1\\ 2 & \text{if } x>1\end{cases}$$

To see that $h'(1)=\lim\limits_{x\to 1}\dfrac{h(x)-h(1)}{x-1}$ does not exist, observe that $\lim\limits_{x\to 1^-}\dfrac{h(x)-h(1)}{x-1}=\lim\limits_{x\to 1^-}\dfrac{3-3}{3-1}=0$ but $\lim\limits_{x\to 1^+}\dfrac{h(x)-h(1)}{x-1}=\lim\limits_{x\to 1^+}\dfrac{2x-2}{x-1}=2$. Similarly, $h'(-2)$ does not exist.



101. $y=f(x)=ax^2 \quad\Rightarrow\quad f'(x)=2ax$. So the slope of the tangent to the parabola at $x=2$ is $m=2a(2)=4a$. The slope of the given line, $2x+y=b \quad\Leftrightarrow\quad y=-2x+b$, is seen to be -2, so we must have $4a=-2 \quad\Leftrightarrow\quad a=-\frac{1}{2}$. So when $x=2$, the point in question has y-coordinate $-\frac{1}{2}\cdot 2^2=-2$. Now we simply require that the given line, whose equation is $2x+y=b$, pass through the point $(2,-2)$: $2(2)+(-2)=b \quad\Leftrightarrow\quad b=2$. So we must have $a=-\frac{1}{2}$ and $b=2$.

102. (a) We use the Product Rule repeatedly: $F=fg \quad\Rightarrow\quad F'=f'g+fg' \quad\Rightarrow$

$$F''=(f''g+f'g')+(f'g'+fg'')=f''g+2f'g'+fg''.$$

(b) $F'''=f'''g+f''g'+2\,(f''g'+f'g'')+f'g''+fg'''=f'''g+3f''g'+3f'g''+fg''' \quad\Rightarrow$

$$\begin{aligned}F^{(4)}&=f^{(4)}g+f'''g'+3\,(f'''g'+f''g'')+3\,(f''g''+f'g''')+f'g'''+fg^{(4)}\\&=f^{(4)}g+4f'''g'+6f''g''+4f'g'''+fg^{(4)}\end{aligned}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g'' + \cdots + \binom{n}{k}f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)},$$

where $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

103. The slope of the curve $y = c\sqrt{x}$ is $y' = \dfrac{c}{2\sqrt{x}}$ and the slope of the tangent line $y = \frac{3}{2}x + 6$ is $\frac{3}{2}$. These must be equal at the point of tangency $\left(a, c\sqrt{a}\right)$, so $\dfrac{c}{2\sqrt{a}} = \dfrac{3}{2} \;\Rightarrow\; c = 3\sqrt{a}$. The y-coordinates must be equal at $x = a$, so $c\sqrt{a} = \frac{3}{2}a + 6 \;\Rightarrow\; \left(3\sqrt{a}\right)\sqrt{a} = \frac{3}{2}a + 6 \;\Rightarrow\; 3a = \frac{3}{2}a + 6 \;\Rightarrow\; \frac{3}{2}a = 6 \;\Rightarrow\; a = 4$. Since $c = 3\sqrt{a}$, we have $c = 3\sqrt{4} = 6$.

104. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}(4x + b) = 8 + b$. Hence, $b = -4$.

105. $F = f/g \;\Rightarrow\; f = Fg \;\Rightarrow\; f' = F'g + Fg' \;\Rightarrow\; F' = \dfrac{f' - Fg'}{g} = \dfrac{f' - (f/g)g'}{g} = \dfrac{f'g - fg'}{g^2}$

106. (a) $xy = c \;\Rightarrow\; y = \dfrac{c}{x}$. Let $P = \left(a, \dfrac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\dfrac{c}{a^2}$. Its equation is $y - \dfrac{c}{a} = -\dfrac{c}{a^2}(x - a)$ or $y = -\dfrac{c}{a^2}x + \dfrac{2c}{a}$, so its y-intercept is $\dfrac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \dfrac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \dfrac{c}{a}\right) = P$.

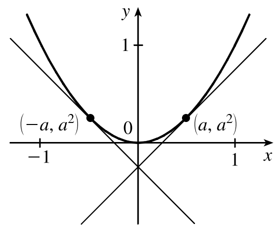
(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

107. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative, $f'(1) = \lim\limits_{x\to 1}\dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x\to 1}\dfrac{x^{1000} - 1}{x - 1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim\limits_{x\to 1}\dfrac{x^{1000} - 1}{x - 1} = 1000$.

Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

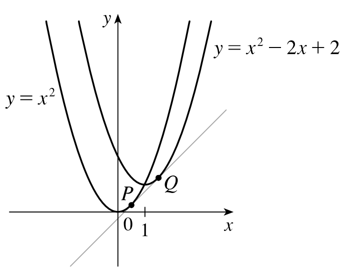
$$\lim_{x\to 1}\frac{x^{1000} - 1}{x - 1} = \lim_{x\to 1}\frac{(x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1} = \lim_{x\to 1}(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$$
$$= \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{1000 \text{ ones}} = 1000, \text{ as above.}$$

108. In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y = x^2$ is symmetric about the y-axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $dy/dx = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x + a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation $y - a^2 = 2a(x - a)$, or $y = 2ax - a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a) = -1 \;\Leftrightarrow\; a^2 = \frac{1}{4} \;\Leftrightarrow\; a = \frac{1}{2}$. So the lines intersect at $\left(0, -\frac{1}{4}\right)$.



109. $y = x^2 \;\Rightarrow\; y' = 2x$, so the slope of a tangent line at the point (a, a^2) is $y' = 2a$ and the slope of a normal line is $-1/(2a)$, for $a \neq 0$. The slope of the normal line through the points (a, a^2) and $(0, c)$ is $\dfrac{a^2 - c}{a - 0}$, so $\dfrac{a^2 - c}{a} = -\dfrac{1}{2a} \;\Rightarrow$ $a^2 - c = -\frac{1}{2} \;\Rightarrow\; a^2 = c - \frac{1}{2}$. The last equation has two solutions if $c > \frac{1}{2}$, one solution if $c = \frac{1}{2}$, and no solution if $c < \frac{1}{2}$. Since the y-axis is normal to $y = x^2$ regardless of the value of c (this is the case for $a = 0$), we have three normal lines if $c > \frac{1}{2}$ and one normal line if $c \leq \frac{1}{2}$.

110.



From the sketch, it appears that there may be a line that is tangent to both curves. The slope of the line through the points $P(a, a^2)$ and $Q(b, b^2 - 2b + 2)$ is $\dfrac{b^2 - 2b + 2 - a^2}{b - a}$. The slope of the tangent line at P is $2a$ $\;[y' = 2x]\;$ and at Q is $2b - 2$ $\;[y' = 2x - 2]$. All three slopes are equal, so $2a = 2b - 2 \;\Leftrightarrow\; a = b - 1$.

Also, $2b - 2 = \dfrac{b^2 - 2b + 2 - a^2}{b - a} \;\Rightarrow\; 2b - 2 = \dfrac{b^2 - 2b + 2 - (b - 1)^2}{b - (b - 1)} \;\Rightarrow\; 2b - 2 = b^2 - 2b + 2 - b^2 + 2b - 1 \;\Rightarrow$ $2b = 3 \;\Rightarrow\; b = \frac{3}{2}$ and $a = \frac{3}{2} - 1 = \frac{1}{2}$. Thus, an equation of the tangent line at P is $y - \left(\frac{1}{2}\right)^2 = 2\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)$ or $y = x - \frac{1}{4}$.

APPLIED PROJECT Building a Better Roller Coaster

1. (a) $f(x) = ax^2 + bx + c \;\Rightarrow\; f'(x) = 2ax + b$.

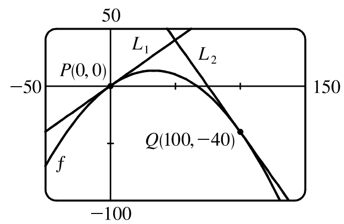
The origin is at P:	$f(0) = 0$	$\Rightarrow$	$c = 0$
The slope of the ascent is 0.8:	$f'(0) = 0.8$	$\Rightarrow$	$b = 0.8$
The slope of the drop is -1.6:	$f'(100) = -1.6$	$\Rightarrow$	$200a + b = -1.6$

(b) $b = 0.8$, so $200a + b = -1.6 \;\Rightarrow\; 200a + 0.8 = -1.6 \;\Rightarrow\; 200a = -2.4 \;\Rightarrow\; a = -\dfrac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

(c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$.

The horizontal distance between P and Q is 100, so the y-coordinate at Q is $f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the point $(100, -40)$ and has slope -1.6, it has equation $y + 40 = -1.6(x - 100)$ or $y = -1.6x + 120$.



(d) The difference in elevation between $P(0, 0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L_1'(x) = 0.8$	$L_1''(x) = 0$
$[0, 10)$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L_2'(x) = -1.6$	$L_2''(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition — these are found after solving the system in part (b).

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L_1'(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L_1''(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729{,}000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24{,}300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1{,}000{,}000p + 10{,}000q + 100r + s = -40$
	$h'(100) = L_2'(100)$	$-\frac{8}{5}$	$30{,}000p + 200q + r = -1.6$
	$h''(100) = L_2''(100)$	0	$600p + 2q = 0$

(b) We can arrange our work in a 12×12 matrix as follows.

a	b	c	k	l	m	n	p	q	r	s	constant
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0.8
0	0	0	0	2	0	0	0	0	0	0	0
−100	−10	−1	1000	100	10	1	0	0	0	0	0
−20	−1	0	300	20	1	0	0	0	0	0	0
−2	0	0	60	2	0	0	0	0	0	0	0
−8100	−90	−1	0	0	0	0	729,000	8100	90	1	0
−180	−1	0	0	0	0	0	24,300	180	1	0	0
−2	0	0	0	0	0	0	540	2	0	0	0
0	0	0	0	0	0	0	1,000,000	10,000	100	1	−40
0	0	0	0	0	0	0	30,000	200	1	0	−1.6
0	0	0	0	0	0	0	600	2	0	0	0

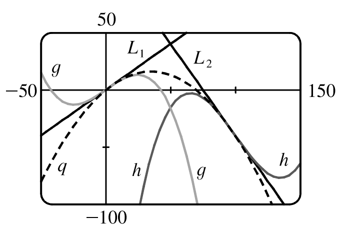
Solving the system gives us the formulas for q, g, and h.

$$\left.\begin{aligned} a &= -0.01\overline{3} = -\tfrac{1}{75} \\ b &= 0.9\overline{3} = \tfrac{14}{15} \\ c &= -0.\overline{4} = -\tfrac{4}{9} \end{aligned}\right\} q(x) = -\tfrac{1}{75}x^2 + \tfrac{14}{15}x - \tfrac{4}{9}$$

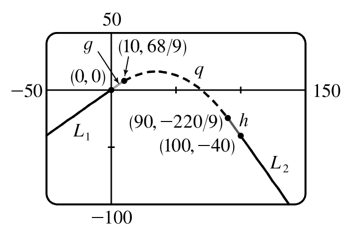
$$\left.\begin{aligned} k &= -0.000\overline{4} = -\tfrac{1}{2250} \\ l &= 0 \\ m &= 0.8 = \tfrac{4}{5} \\ n &= 0 \end{aligned}\right\} g(x) = -\tfrac{1}{2250}x^3 + \tfrac{4}{5}x$$

$$\left.\begin{aligned} p &= 0.000\overline{4} = \tfrac{1}{2250} \\ q &= -0.1\overline{3} = -\tfrac{2}{15} \\ r &= 11.7\overline{3} = \tfrac{176}{15} \\ s &= -324.\overline{4} = -\tfrac{2920}{9} \end{aligned}\right\} h(x) = \tfrac{1}{2250}x^3 - \tfrac{2}{15}x^2 + \tfrac{176}{15}x - \tfrac{2920}{9}$$

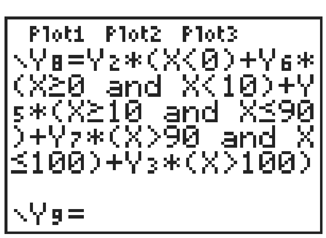
(c) Graph of L_1, q, g, h, and L_2:



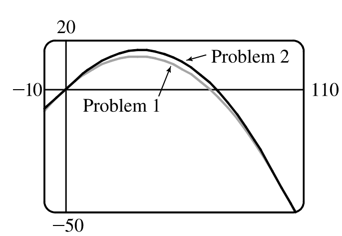
The graph of the five functions as a piecewise-defined function:



This is the piecewise-defined function assignment on a TI-83/4 Plus calculator, where $Y_2 = L_1$, $Y_6 = g$, $Y_5 = q$, $Y_7 = h$, and $Y_3 = L_2$.



A comparison of the graphs in part 1(c) and part 2(c):



2.4 Derivatives of Trigonometric Functions

1. $f(x) = x^2 \sin x \overset{\text{PR}}{\Rightarrow} f'(x) = x^2 \cos x + (\sin x)(2x) = x^2 \cos x + 2x \sin x$

2. $f(x) = x \cos x + 2 \tan x \Rightarrow f'(x) = x(-\sin x) + (\cos x)(1) + 2 \sec^2 x = \cos x - x \sin x + 2 \sec^2 x$

3. $f(x) = 3 \cot x - 2 \cos x \Rightarrow f'(x) = 3(-\csc^2 x) - 2(-\sin x) = -3 \csc^2 x + 2 \sin x$

4. $y = 2 \sec x - \csc x \Rightarrow y' = 2(\sec x \tan x) - (-\csc x \cot x) = 2 \sec x \tan x + \csc x \cot x$

5. $y = \sec\theta \tan\theta \Rightarrow y' = \sec\theta(\sec^2\theta) + \tan\theta(\sec\theta\tan\theta) = \sec\theta(\sec^2\theta + \tan^2\theta)$. Using the identity $1 + \tan^2\theta = \sec^2\theta$, we can write alternative forms of the answer as $\sec\theta(1 + 2\tan^2\theta)$ or $\sec\theta(2\sec^2\theta - 1)$.

6. $g(t) = 4 \sec t + \tan t \Rightarrow g'(t) = 4 \sec t \tan t + \sec^2 t$

7. $y = c \cos t + t^2 \sin t \Rightarrow y' = c(-\sin t) + t^2(\cos t) + \sin t\,(2t) = -c \sin t + t(t \cos t + 2 \sin t)$

8. $y = u(a \cos u + b \cot u) \Rightarrow$

$y' = u(-a \sin u - b \csc^2 u) + (a \cos u + b \cot u) \cdot 1 = a \cos u + b \cot u - au \sin u - bu \csc^2 u$

9. $y = \dfrac{x}{2 - \tan x} \Rightarrow y' = \dfrac{(2 - \tan x)(1) - x(-\sec^2 x)}{(2 - \tan x)^2} = \dfrac{2 - \tan x + x \sec^2 x}{(2 - \tan x)^2}$

10. $y = \sin\theta \cos\theta \Rightarrow y' = \sin\theta(-\sin\theta) + \cos\theta(\cos\theta) = \cos^2\theta - \sin^2\theta$ [or $\cos 2\theta$]

11. $f(\theta) = \dfrac{\sin\theta}{1 + \cos\theta} \Rightarrow$

$$f'(\theta) = \frac{(1 + \cos\theta)\cos\theta - (\sin\theta)(-\sin\theta)}{(1 + \cos\theta)^2} = \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1 + \cos\theta)^2} = \frac{\cos\theta + 1}{(1 + \cos\theta)^2} = \frac{1}{1 + \cos\theta}$$

12. $y = \dfrac{\cos x}{1 - \sin x} \Rightarrow$

$$y' = \frac{(1 - \sin x)(-\sin x) - \cos x(-\cos x)}{(1 - \sin x)^2} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} = \frac{-\sin x + 1}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

13. $y = \dfrac{t \sin t}{1 + t} \Rightarrow$

$$y' = \frac{(1 + t)(t \cos t + \sin t) - t \sin t(1)}{(1 + t)^2} = \frac{t \cos t + \sin t + t^2 \cos t + t \sin t - t \sin t}{(1 + t)^2} = \frac{(t^2 + t)\cos t + \sin t}{(1 + t)^2}$$

14. $y = \dfrac{\sin t}{1 + \tan t} \Rightarrow$

$$y' = \frac{(1 + \tan t)\cos t - (\sin t)\sec^2 t}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \dfrac{\sin t}{\cos^2 t}}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \tan t \sec t}{(1 + \tan t)^2}$$

15. Using Exercise 2.3.87(a), $f(\theta) = \theta \cos\theta \sin\theta \Rightarrow$

$$\begin{aligned} f'(\theta) &= 1 \cos\theta \sin\theta + \theta(-\sin\theta)\sin\theta + \theta\cos\theta(\cos\theta) = \cos\theta \sin\theta - \theta\sin^2\theta + \theta\cos^2\theta \\ &= \sin\theta \cos\theta + \theta(\cos^2\theta - \sin^2\theta) = \tfrac{1}{2}\sin 2\theta + \theta\cos 2\theta \quad \text{[using double-angle formulas]} \end{aligned}$$

16. Using Exercise 2.3.87(a), $f(x) = x^2 \sin x \tan x \Rightarrow$

$$f'(x) = (x^2)' \sin x \tan x + x^2 (\sin x)' \tan x + x^2 \sin x (\tan x)' = 2x \sin x \tan x + x^2 \cos x \tan x + x^2 \sin x \sec^2 x$$
$$= 2x \sin x \tan x + x^2 \sin x + x^2 \sin x \sec^2 x = x \sin x (2 \tan x + x + x \sec^2 x).$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x} \cdot \dfrac{\cos x}{\sin x} = -\csc x \cot x$

18. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x} = \sec x \tan x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

20. $f(x) = \cos x \Rightarrow$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$
$$= \lim_{h \to 0} \left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}\right) = \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h}$$
$$= (\cos x)(0) - (\sin x)(1) = -\sin x$$

21. $y = \sin x + \cos x \Rightarrow y' = \cos x - \sin x$, so $y'(0) = \cos 0 - \sin 0 = 1 - 0 = 1$. An equation of the tangent line to the curve $y = \sin x + \cos x$ at the point $(0, 1)$ is $y - 1 = 1(x - 0)$ or $y = x + 1$.

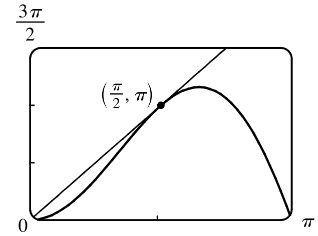
22. $y = (1 + x)\cos x \Rightarrow y' = (1 + x)(-\sin x) + \cos x \cdot 1$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$ or $y = x + 1$.

23. $y = \cos x - \sin x \Rightarrow y' = -\sin x - \cos x$, so $y'(\pi) = -\sin \pi - \cos \pi = 0 - (-1) = 1$. An equation of the tangent line to the curve $y = \cos x - \sin x$ at the point $(\pi, -1)$ is $y - (-1) = 1(x - \pi)$ or $y = x - \pi - 1$.

24. $y = x + \tan x \Rightarrow y' = 1 + \sec^2 x$, so $y'(\pi) = 1 + (-1)^2 = 2$. An equation of the tangent line to the curve $y = x + \tan x$ at the point (π, π) is $y - \pi = 2(x - \pi)$ or $y = 2x - \pi$.

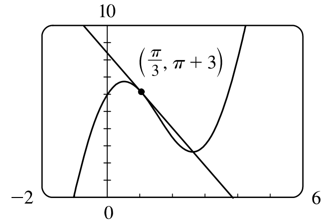
25. (a) $y = 2x \sin x \Rightarrow y' = 2(x \cos x + \sin x \cdot 1)$. At $\left(\frac{\pi}{2}, \pi\right)$, $y' = 2\left(\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}\right) = 2(0 + 1) = 2$, and an equation of the tangent line is $y - \pi = 2\left(x - \frac{\pi}{2}\right)$, or $y = 2x$.

(b)



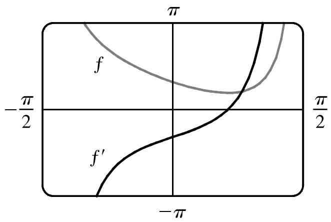
26. (a) $y = 3x + 6 \cos x \Rightarrow y' = 3 - 6 \sin x$. At $\left(\frac{\pi}{3}, \pi + 3\right)$, $y' = 3 - 6 \sin \frac{\pi}{3} = 3 - 6\frac{\sqrt{3}}{2} = 3 - 3\sqrt{3}$, and an equation of the tangent line is $y - (\pi + 3) = \left(3 - 3\sqrt{3}\right)\left(x - \frac{\pi}{3}\right)$, or $y = \left(3 - 3\sqrt{3}\right)x + 3 + \pi\sqrt{3}$.

(b)



27. (a) $f(x) = \sec x - x \quad\Rightarrow\quad f'(x) = \sec x\,\tan x - 1$

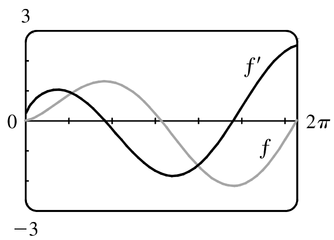
(b)



Note that $f' = 0$ where f has a minimum. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

28. (a) $f(x) = \sqrt{x}\sin x \quad\Rightarrow\quad f'(x) = \sqrt{x}\cos x + (\sin x)\left(\frac{1}{2}x^{-1/2}\right) = \sqrt{x}\cos x + \dfrac{\sin x}{2\sqrt{x}}$

(b)



Notice that $f'(x) = 0$ when f has a horizontal tangent.
f' is positive when f is increasing and f' is negative when f is decreasing.

29. $H(\theta) = \theta\sin\theta \quad\Rightarrow\quad H'(\theta) = \theta\,(\cos\theta) + (\sin\theta)\cdot 1 = \theta\cos\theta + \sin\theta \quad\Rightarrow$

$H''(\theta) = \theta\,(-\sin\theta) + (\cos\theta)\cdot 1 + \cos\theta = -\theta\sin\theta + 2\cos\theta$

30. $f(t) = \sec t \quad\Rightarrow\quad f'(t) = \sec t\,\tan t \quad\Rightarrow\quad f''(t) = (\sec t)\sec^2 t + (\tan t)\sec t\,\tan t = \sec^3 t + \sec t\,\tan^2 t$, so

$f''\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^3 + \sqrt{2}(1)^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$.

31. (a) $f(x) = \dfrac{\tan x - 1}{\sec x} \quad\Rightarrow$

$$f'(x) = \frac{\sec x(\sec^2 x) - (\tan x - 1)(\sec x\tan x)}{(\sec x)^2} = \frac{\sec x(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

(b) $f(x) = \dfrac{\tan x - 1}{\sec x} = \dfrac{\dfrac{\sin x}{\cos x} - 1}{\dfrac{1}{\cos x}} = \dfrac{\dfrac{\sin x - \cos x}{\cos x}}{\dfrac{1}{\cos x}} = \sin x - \cos x \quad\Rightarrow\quad f'(x) = \cos x - (-\sin x) = \cos x + \sin x$

(c) From part (a), $f'(x) = \dfrac{1 + \tan x}{\sec x} = \dfrac{1}{\sec x} + \dfrac{\tan x}{\sec x} = \cos x + \sin x$, which is the expression for $f'(x)$ in part (b).

32. (a) $g(x) = f(x)\sin x \quad\Rightarrow\quad g'(x) = f(x)\cos x + \sin x \cdot f'(x)$, so

$g'\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right)\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right) = 4\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot(-2) = 2 - \sqrt{3}$

(b) $h(x) = \dfrac{\cos x}{f(x)} \quad\Rightarrow\quad h'(x) = \dfrac{f(x)\cdot(-\sin x) - \cos x\cdot f'(x)}{[f(x)]^2}$, so

$$h'\left(\tfrac{\pi}{3}\right) = \frac{f\left(\frac{\pi}{3}\right)\cdot\left(-\sin\frac{\pi}{3}\right) - \cos\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right)}{\left[f\left(\frac{\pi}{3}\right)\right]^2} = \frac{4\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)(-2)}{4^2} = \frac{-2\sqrt{3} + 1}{16} = \frac{1 - 2\sqrt{3}}{16}$$

33. $f(x) = x + 2\sin x$ has a horizontal tangent when $f'(x) = 0 \quad\Leftrightarrow\quad 1 + 2\cos x = 0 \quad\Leftrightarrow\quad \cos x = -\frac{1}{2} \quad\Leftrightarrow$

$x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n + 1)\pi \pm \frac{\pi}{3}$, n an integer.

34. $y = \dfrac{\cos x}{2 + \sin x} \quad \Rightarrow \quad y' = \dfrac{(2 + \sin x)(-\sin x) - \cos x \, \cos x}{(2 + \sin x)^2} = \dfrac{-2\sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2} = \dfrac{-2\sin x - 1}{(2 + \sin x)^2} = 0$ when

$-2\sin x - 1 = 0 \quad \Leftrightarrow \quad \sin x = -\frac{1}{2} \quad \Leftrightarrow \quad x = \frac{11\pi}{6} + 2\pi n$ or $x = \frac{7\pi}{6} + 2\pi n$, n an integer. So $y = \frac{1}{\sqrt{3}}$ or $y = -\frac{1}{\sqrt{3}}$ and

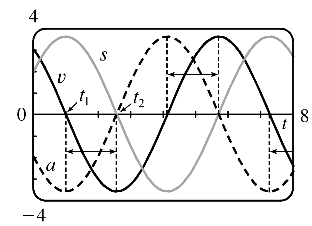
the points on the curve with horizontal tangents are: $\left(\frac{11\pi}{6} + 2\pi n, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{7\pi}{6} + 2\pi n, -\frac{1}{\sqrt{3}}\right)$, n an integer.

35. (a) $x(t) = 8\sin t \quad \Rightarrow \quad v(t) = x'(t) = 8\cos t \quad \Rightarrow \quad a(t) = x''(t) = -8\sin t$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8\sin\frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$, velocity $v\left(\frac{2\pi}{3}\right) = 8\cos\frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$, and acceleration $a\left(\frac{2\pi}{3}\right) = -8\sin\frac{2\pi}{3} = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

36. (a) $s(t) = 2\cos t + 3\sin t \quad \Rightarrow \quad v(t) = -2\sin t + 3\cos t \quad \Rightarrow$
$a(t) = -2\cos t - 3\sin t$

(b)

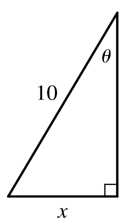


(c) $s = 0 \quad \Rightarrow \quad t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \quad \Rightarrow \quad t_1 \approx 0.98$, $s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$, that is, when $t = t_2 + n\pi$, n a positive integer.

37.

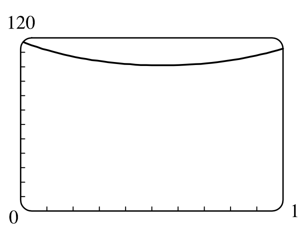


From the diagram we can see that $\sin\theta = x/10 \quad \Leftrightarrow \quad x = 10\sin\theta$. We want to find the rate of change of x with respect to θ, that is, $dx/d\theta$. Taking the derivative of $x = 10\sin\theta$, we get $dx/d\theta = 10(\cos\theta)$. So when $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 10\cos\frac{\pi}{3} = 10\left(\frac{1}{2}\right) = 5$ ft/rad.

38. (a) $F = \dfrac{\mu W}{\mu\sin\theta + \cos\theta} \quad \Rightarrow \quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}$

(b) $\dfrac{dF}{d\theta} = 0 \quad \Leftrightarrow \quad \mu W(\sin\theta - \mu\cos\theta) = 0 \quad \Leftrightarrow \quad \sin\theta = \mu\cos\theta \quad \Leftrightarrow \quad \tan\theta = \mu \quad \Leftrightarrow \quad \theta = \tan^{-1}\mu$

(c)



From the graph of $F = \dfrac{0.6(50)}{0.6\sin\theta + \cos\theta}$ for $0 \le \theta \le 1$, we see that $\dfrac{dF}{d\theta} = 0 \quad \Rightarrow \quad \theta \approx 0.54$. Checking this with part (b) and $\mu = 0.6$, we calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the value from the graph is consistent with the value in part (b).

39. $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{3x} = \lim_{x\to 0} \frac{5}{3}\left(\frac{\sin 5x}{5x}\right) = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{3}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad [\theta = 5x] \quad = \frac{5}{3}\cdot 1 = \frac{5}{3}$

40. $\lim_{x\to 0} \frac{\sin x}{\sin \pi x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\pi x}{\sin \pi x} \cdot \frac{1}{\pi} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{\theta\to 0} \frac{\theta}{\sin\theta} \cdot \frac{1}{\pi} \quad [\theta = \pi x]$

$$= 1 \cdot \lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} \cdot \frac{1}{\pi} = 1 \cdot 1 \cdot \frac{1}{\pi} = \frac{1}{\pi}$$

41. $\lim_{t\to 0} \frac{\tan 6t}{\sin 2t} = \lim_{t\to 0} \left(\frac{\sin 6t}{t} \cdot \frac{1}{\cos 6t} \cdot \frac{t}{\sin 2t}\right) = \lim_{t\to 0} \frac{6\sin 6t}{6t} \cdot \lim_{t\to 0} \frac{1}{\cos 6t} \cdot \lim_{t\to 0} \frac{2t}{2\sin 2t}$

$$= 6 \lim_{t\to 0} \frac{\sin 6t}{6t} \cdot \lim_{t\to 0} \frac{1}{\cos 6t} \cdot \frac{1}{2} \lim_{t\to 0} \frac{2t}{\sin 2t} = 6(1) \cdot \frac{1}{1} \cdot \frac{1}{2}(1) = 3$$

42. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0} \frac{\frac{\cos\theta - 1}{\theta}}{\frac{\sin\theta}{\theta}} = \frac{\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta}}{\lim_{\theta\to 0} \frac{\sin\theta}{\theta}} = \frac{0}{1} = 0$

43. $\lim_{x\to 0} \frac{\sin 3x}{5x^3 - 4x} = \lim_{x\to 0} \left(\frac{\sin 3x}{3x} \cdot \frac{3}{5x^2 - 4}\right) = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \lim_{x\to 0} \frac{3}{5x^2 - 4} = 1 \cdot \left(\frac{3}{-4}\right) = -\frac{3}{4}$

44. $\lim_{x\to 0} \frac{\sin 3x \sin 5x}{x^2} = \lim_{x\to 0} \left(\frac{3\sin 3x}{3x} \cdot \frac{5\sin 5x}{5x}\right) = \lim_{x\to 0} \frac{3\sin 3x}{3x} \cdot \lim_{x\to 0} \frac{5\sin 5x}{5x}$

$$= 3 \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot 5 \lim_{x\to 0} \frac{\sin 5x}{5x} = 3(1) \cdot 5(1) = 15$$

45. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta + \tan\theta} = \lim_{\theta\to 0} \frac{\frac{\sin\theta}{\theta}}{1 + \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta}} = \frac{\lim_{\theta\to 0} \frac{\sin\theta}{\theta}}{1 + \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \lim_{\theta\to 0} \frac{1}{\cos\theta}} = \frac{1}{1 + 1 \cdot 1} = \frac{1}{2}$$

46. $\lim_{x\to 0} \csc x \, \sin(\sin x) = \lim_{x\to 0} \frac{\sin(\sin x)}{\sin x} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta}$ [As $x \to 0$, $\theta = \sin x \to 0$.] $= 1$

47. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{2\theta^2} = \lim_{\theta\to 0} \frac{\cos\theta - 1}{2\theta^2} \cdot \frac{\cos\theta + 1}{\cos\theta + 1} = \lim_{\theta\to 0} \frac{\cos^2\theta - 1}{2\theta^2(\cos\theta + 1)} = \lim_{\theta\to 0} \frac{-\sin^2\theta}{2\theta^2(\cos\theta + 1)}$

$$= -\frac{1}{2} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\theta} \cdot \frac{1}{\cos\theta + 1} = -\frac{1}{2} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{1}{\cos\theta + 1}$$

$$= -\frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{1}{1 + 1} = -\frac{1}{4}$$

48. $\lim_{x\to 0} \frac{\sin(x^2)}{x} = \lim_{x\to 0} \left[x \cdot \frac{\sin(x^2)}{x \cdot x}\right] = \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{\sin(x^2)}{x^2} = 0 \cdot \lim_{y\to 0^+} \frac{\sin y}{y}$ [where $y = x^2$] $= 0 \cdot 1 = 0$

49. $\lim_{x\to \pi/4} \frac{1 - \tan x}{\sin x - \cos x} = \lim_{x\to \pi/4} \frac{\left(1 - \frac{\sin x}{\cos x}\right) \cdot \cos x}{(\sin x - \cos x) \cdot \cos x} = \lim_{x\to \pi/4} \frac{\cos x - \sin x}{(\sin x - \cos x)\cos x} = \lim_{x\to \pi/4} \frac{-1}{\cos x} = \frac{-1}{1/\sqrt{2}} = -\sqrt{2}$

50. $\lim_{x\to 1} \frac{\sin(x-1)}{x^2 + x - 2} = \lim_{x\to 1} \frac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x\to 1} \frac{1}{x+2} \lim_{x\to 1} \frac{\sin(x-1)}{x-1} = \frac{1}{3} \cdot 1 = \frac{1}{3}$

51. $\dfrac{d}{dx}(\sin x) = \cos x \quad\Rightarrow\quad \dfrac{d^2}{dx^2}(\sin x) = -\sin x \quad\Rightarrow\quad \dfrac{d^3}{dx^3}(\sin x) = -\cos x \quad\Rightarrow\quad \dfrac{d^4}{dx^4}(\sin x) = \sin x.$

The derivatives of $\sin x$ occur in a cycle of four. Since $99 = 4(24) + 3$, we have $\dfrac{d^{99}}{dx^{99}}(\sin x) = \dfrac{d^3}{dx^3}(\sin x) = -\cos x$.

52. Let $f(x) = x\sin x$ and $h(x) = \sin x$, so $f(x) = xh(x)$. Then $f'(x) = h(x) + xh'(x)$,

$f''(x) = h'(x) + h'(x) + xh''(x) = 2h'(x) + xh''(x)$,

$f'''(x) = 2h''(x) + h''(x) + xh'''(x) = 3h''(x) + xh'''(x), \cdots, f^{(n)}(x) = nh^{(n-1)}(x) + xh^{(n)}(x)$.

Since $34 = 4(8) + 2$, we have $h^{(34)}(x) = h^{(2)}(x) = \dfrac{d^2}{dx^2}(\sin x) = -\sin x$ and $h^{(35)}(x) = -\cos x$.

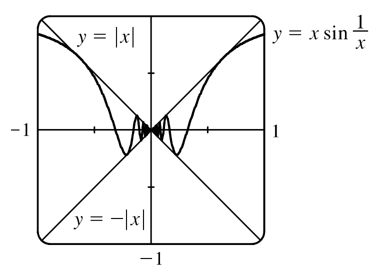
Thus, $\dfrac{d^{35}}{dx^{35}}(x\sin x) = 35h^{(34)}(x) + xh^{(35)}(x) = -35\sin x - x\cos x$.

53. $y = A\sin x + B\cos x \quad\Rightarrow\quad y' = A\cos x - B\sin x \quad\Rightarrow\quad y'' = -A\sin x - B\cos x$. Substituting these expressions for y, y', and y'' into the given differential equation $y'' + y' - 2y = \sin x$ gives us

$(-A\sin x - B\cos x) + (A\cos x - B\sin x) - 2(A\sin x + B\cos x) = \sin x \quad\Leftrightarrow$

$-3A\sin x - B\sin x + A\cos x - 3B\cos x = \sin x \quad\Leftrightarrow\quad (-3A - B)\sin x + (A - 3B)\cos x = 1\sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$ (since 0 is the coefficient of $\cos x$ on the right side). Solving for A and B, we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

54. Since $-1 \le \sin(1/x) \le 1$, we have (as illustrated in the figure) $-|x| \le x\sin(1/x) \le |x|$. We know that $\lim\limits_{x\to 0}(|x|) = 0$ and $\lim\limits_{x\to 0}(-|x|) = 0$; so by the Squeeze Theorem, $\lim\limits_{x\to 0} x\sin(1/x) = 0$.



55. (a) $\dfrac{d}{dx}\tan x = \dfrac{d}{dx}\dfrac{\sin x}{\cos x} \quad\Rightarrow\quad \sec^2 x = \dfrac{\cos x\,\cos x - \sin x\,(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \dfrac{1}{\cos^2 x}$.

(b) $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\dfrac{1}{\cos x} \quad\Rightarrow\quad \sec x\,\tan x = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x\,\tan x = \dfrac{\sin x}{\cos^2 x}$.

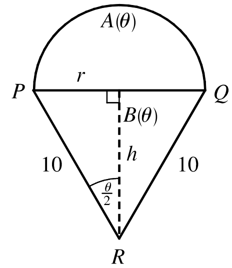
(c) $\dfrac{d}{dx}(\sin x + \cos x) = \dfrac{d}{dx}\dfrac{1 + \cot x}{\csc x} \quad\Rightarrow$

$$\cos x - \sin x = \frac{\csc x\,(-\csc^2 x) - (1 + \cot x)(-\csc x\,\cot x)}{\csc^2 x} = \frac{\csc x\,[-\csc^2 x + (1 + \cot x)\,\cot x]}{\csc^2 x}$$
$$= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1 + \cot x}{\csc x}$$

So $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$.

56. We get the following formulas for r and h in terms of θ:

$$\sin\frac{\theta}{2} = \frac{r}{10} \quad\Rightarrow\quad r = 10\sin\frac{\theta}{2} \quad\text{and}\quad \cos\frac{\theta}{2} = \frac{h}{10} \quad\Rightarrow\quad h = 10\cos\frac{\theta}{2}$$



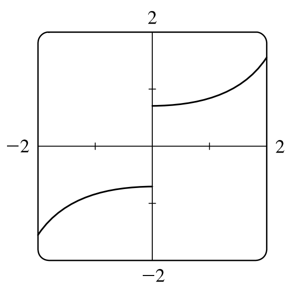
Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h} = \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\frac{10\sin(\theta/2)}{10\cos(\theta/2)}$$
$$= \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2) = 0$$

57. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle. By drawing the bisector of the angle θ, we can see that $\sin\frac{\theta}{2} = \frac{d/2}{r} \;\Rightarrow\; d = 2r\sin\frac{\theta}{2}$. So $\lim\limits_{\theta\to 0^+}\frac{s}{d} = \lim\limits_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim\limits_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim\limits_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1$.

[This is just the reciprocal of the limit $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta\to 0$, $\frac{\theta}{2}\to 0$ also.]

58. (a)



It appears that $f(x) = \dfrac{x}{\sqrt{1-\cos 2x}}$ has a jump discontinuity at $x = 0$.

(b) Using the identity $\cos 2x = 1 - \sin^2 x$, we have $\dfrac{x}{\sqrt{1-\cos 2x}} = \dfrac{x}{\sqrt{1-(1-2\sin^2 x)}} = \dfrac{x}{\sqrt{2\sin^2 x}} = \dfrac{x}{\sqrt{2}\,|\sin x|}$.

Thus,

$$\lim_{x\to 0^-}\frac{x}{\sqrt{1-\cos 2x}} = \lim_{x\to 0^-}\frac{x}{\sqrt{2}\,|\sin x|} = \frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{x}{-(\sin x)}$$
$$= -\frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{1}{\sin x/x} = -\frac{1}{\sqrt{2}}\cdot\frac{1}{1} = -\frac{\sqrt{2}}{2}$$

Evaluating $\lim\limits_{x\to 0^+} f(x)$ is similar, but $|\sin x| = +\sin x$, so we get $\frac{1}{2}\sqrt{2}$. These values appear to be reasonable values for the graph, so they confirm our answer to part (a).

Another method: Multiply numerator and denominator by $\sqrt{1+\cos 2x}$.

2.5 The Chain Rule

1. Let $u = g(x) = 1 + 4x$ and $y = f(u) = \sqrt[3]{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\frac{1}{3}u^{-2/3})(4) = \dfrac{4}{3\sqrt[3]{(1+4x)^2}}$.

2. Let $u = g(x) = 2x^3 + 5$ and $y = f(u) = u^4$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (4u^3)(6x^2) = 24x^2(2x^3+5)^3$.

3. Let $u = g(x) = \pi x$ and $y = f(u) = \tan u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\sec^2 u)(\pi) = \pi\sec^2\pi x$.

4. Let $u = g(x) = \cot x$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)(-\csc^2 x) = -\cos(\cot x)\csc^2 x$.

5. Let $u = g(x) = \sin x$ and $y = f(u) = \sqrt{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \frac{1}{2}u^{-1/2}\cos x = \dfrac{\cos x}{2\sqrt{u}} = \dfrac{\cos x}{2\sqrt{\sin x}}$.

6. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)\left(\frac{1}{2}x^{-1/2}\right) = \dfrac{\cos u}{2\sqrt{x}} = \dfrac{\cos\sqrt{x}}{2\sqrt{x}}$.

7. $F(x) = (5x^6 + 2x^3)^4 \overset{\text{CR}}{\Rightarrow} F'(x) = 4(5x^6 + 2x^3)^3 \cdot \dfrac{d}{dx}(5x^6 + 2x^3) = 4(5x^6 + 2x^3)^3(30x^5 + 6x^2)$.

We can factor as follows: $4(x^3)^3(5x^3 + 2)^3 6x^2(5x^3 + 1) = 24x^{11}(5x^3 + 2)^3(5x^3 + 1)$

8. $F(x) = (1 + x + x^2)^{99} \overset{\text{CR}}{\Rightarrow} F'(x) = 99(1 + x + x^2)^{98} \cdot \dfrac{d}{dx}\left(1 + x + x^2\right) = 99(1 + x + x^2)^{98}(1 + 2x)$

9. $f(x) = \sqrt{5x + 1} = (5x + 1)^{1/2} \overset{\text{CR}}{\Rightarrow} f'(x) = \frac{1}{2}(5x + 1)^{-1/2} \cdot \dfrac{d}{dx}(5x + 1) = \dfrac{5}{2\sqrt{5x + 1}}$

10. $g(x) = (2 - \sin x)^{3/2} \overset{\text{CR}}{\Rightarrow}$

$g'(x) = \frac{3}{2}(2 - \sin x)^{1/2} \cdot \dfrac{d}{dx}(2 - \sin x) = \frac{3}{2}(2 - \sin x)^{1/2}(-\cos x) = -\frac{3}{2}\cos x\,(2 - \sin x)^{1/2}$

11. $A(t) = \dfrac{1}{(\cos t + \tan t)^2} = (\cos t + \tan t)^{-2} \overset{\text{CR}}{\Rightarrow} A'(t) = -2(\cos t + \tan t)^{-3}(-\sin t + \sec^2 t) = \dfrac{2(\sin t - \sec^2 t)}{(\cos t + \tan t)^3}$

12. $f(x) = \dfrac{1}{\sqrt[3]{x^2 - 1}} = (x^2 - 1)^{-1/3} \Rightarrow f'(x) = -\dfrac{1}{3}(x^2 - 1)^{-4/3}(2x) = \dfrac{-2x}{3(x^2 - 1)^{4/3}}$

13. $f(\theta) = \cos(\theta^2) \Rightarrow f'(\theta) = -\sin(\theta^2) \cdot \dfrac{d}{d\theta}(\theta^2) = -\sin(\theta^2) \cdot (2\theta) = -2\theta\sin(\theta^2)$

14. $g(\theta) = \cos^2\theta = (\cos\theta)^2 \Rightarrow g'(\theta) = 2\,(\cos\theta)^1\,(-\sin\theta) = -2\sin\theta\,\cos\theta = -\sin 2\theta$

15. $h(v) = v\sqrt[3]{1 + v^2} = v(1 + v^2)^{1/3} \overset{\text{PR}}{\Rightarrow}$

$h'(v) = v \cdot \frac{1}{3}(1 + v^2)^{-2/3}(2v) + (1 + v^2)^{1/3} \cdot 1 = \frac{1}{3}(1 + v^2)^{-2/3}[2v^2 + 3(1 + v^2)] = \dfrac{5v^2 + 3}{3(\sqrt[3]{1 + v^2}\,)^2}$

16. $f(t) = t\sin\pi t \Rightarrow f'(t) = t(\cos\pi t) \cdot \pi + (\sin\pi t) \cdot 1 = \pi t\cos\pi t + \sin\pi t$

17. $f(x) = (2x - 3)^4(x^2 + x + 1)^5 \Rightarrow$

$$\begin{aligned} f'(x) &= (2x - 3)^4 \cdot 5(x^2 + x + 1)^4(2x + 1) + (x^2 + x + 1)^5 \cdot 4(2x - 3)^3 \cdot 2 \\ &= (2x - 3)^3(x^2 + x + 1)^4[(2x - 3) \cdot 5(2x + 1) + (x^2 + x + 1) \cdot 8] \\ &= (2x - 3)^3(x^2 + x + 1)^4(20x^2 - 20x - 15 + 8x^2 + 8x + 8) = (2x - 3)^3(x^2 + x + 1)^4(28x^2 - 12x - 7) \end{aligned}$$

18. $g(x) = (x^2+1)^3(x^2+2)^6 \quad \Rightarrow$

$$g'(x) = (x^2+1)^3 \cdot 6(x^2+2)^5 \cdot 2x + (x^2+2)^6 \cdot 3(x^2+1)^2 \cdot 2x$$
$$= 6x(x^2+1)^2(x^2+2)^5[2(x^2+1)+(x^2+2)] = 6x(x^2+1)^2(x^2+2)^5(3x^2+4)$$

19. $h(t) = (t+1)^{2/3}(2t^2-1)^3 \quad \Rightarrow$

$$h'(t) = (t+1)^{2/3} \cdot 3(2t^2-1)^2 \cdot 4t + (2t^2-1)^3 \cdot \tfrac{2}{3}(t+1)^{-1/3} = \tfrac{2}{3}(t+1)^{-1/3}(2t^2-1)^2[18t(t+1)+(2t^2-1)]$$
$$= \tfrac{2}{3}(t+1)^{-1/3}(2t^2-1)^2(20t^2+18t-1)$$

20. $F(t) = (3t-1)^4(2t+1)^{-3} \quad \Rightarrow$

$$F'(t) = (3t-1)^4(-3)(2t+1)^{-4}(2) + (2t+1)^{-3} \cdot 4(3t-1)^3(3)$$
$$= 6(3t-1)^3(2t+1)^{-4}[-(3t-1)+2(2t+1)] = 6(3t-1)^3(2t+1)^{-4}(t+3)$$

21. $g(u) = \left(\dfrac{u^3-1}{u^3+1}\right)^8 \quad \Rightarrow$

$$g'(u) = 8\left(\frac{u^3-1}{u^3+1}\right)^7 \frac{d}{du}\frac{u^3-1}{u^3+1} = 8\,\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{(u^3+1)(3u^2)-(u^3-1)(3u^2)}{(u^3+1)^2}$$
$$= 8\,\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{3u^2[(u^3+1)-(u^3-1)]}{(u^3+1)^2} = 8\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{3u^2(2)}{(u^3+1)^2} = \frac{48u^2(u^3-1)^7}{(u^3+1)^9}$$

22. $y = \left(x+\dfrac{1}{x}\right)^5 \quad \Rightarrow \quad y' = 5\left(x+\dfrac{1}{x}\right)^4 \dfrac{d}{dx}\left(x+\dfrac{1}{x}\right) = 5\left(x+\dfrac{1}{x}\right)^4\left(1-\dfrac{1}{x^2}\right).$

Another form of the answer is $\dfrac{5(x^2+1)^4(x^2-1)}{x^6}$.

23. $y = \sqrt{\dfrac{x}{x+1}} = \left(\dfrac{x}{x+1}\right)^{1/2} \quad \Rightarrow$

$$y' = \frac{1}{2}\left(\frac{x}{x+1}\right)^{-1/2}\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{1}{2}\,\frac{x^{-1/2}}{(x+1)^{-1/2}}\,\frac{(x+1)(1)-x(1)}{(x+1)^2}$$
$$= \frac{1}{2}\,\frac{(x+1)^{1/2}}{x^{1/2}}\,\frac{1}{(x+1)^2} = \frac{1}{2\sqrt{x}(x+1)^{3/2}}$$

24. $U(y) = \left(\dfrac{y^4+1}{y^2+1}\right)^5 \quad \Rightarrow$

$$U'(y) = 5\left(\frac{y^4+1}{y^2+1}\right)^4\frac{(y^2+1)(4y^3)-(y^4+1)(2y)}{(y^2+1)^2} = \frac{5(y^4+1)^4 2y[2y^2(y^2+1)-(y^4+1)]}{(y^2+1)^4(y^2+1)^2}$$
$$= \frac{10y(y^4+1)^4(y^4+2y^2-1)}{(y^2+1)^6}$$

25. $h(\theta) = \tan(\theta^2\sin\theta) \quad \overset{\text{CR}}{\Rightarrow}$

$$h'(\theta) = \sec^2(\theta^2\sin\theta)\cdot\frac{d}{d\theta}(\theta^2\sin\theta) = \sec^2(\theta^2\sin\theta)\cdot[\theta^2\cos\theta+(\sin\theta)(2\theta)] = \theta\sec^2(\theta^2\sin\theta)(\theta\cos\theta+2\sin\theta)$$

26. $f(t) = \sqrt{\dfrac{t}{t^2+4}} = \left(\dfrac{t}{t^2+4}\right)^{1/2} \Rightarrow$

$$f'(t) = \frac{1}{2}\left(\frac{t}{t^2+4}\right)^{-1/2} \cdot \frac{d}{dt}\left(\frac{t}{t^2+4}\right) = \frac{1}{2}\left(\frac{t^2+4}{t}\right)^{1/2} \cdot \frac{(t^2+4)(1)-t(2t)}{(t^2+4)^2}$$

$$= \frac{(t^2+4)^{1/2}}{2t^{1/2}} \cdot \frac{t^2+4-2t^2}{(t^2+4)^2} = \frac{4-t^2}{2t^{1/2}(t^2+4)^{3/2}}$$

27. $y = \dfrac{\cos x}{\sqrt{1+\sin x}} = (\cos x)(1+\sin x)^{-1/2} \Rightarrow$

$$y' = (\cos x)\cdot\left(-\tfrac{1}{2}\right)(1+\sin x)^{-3/2}\cos x + (1+\sin x)^{-1/2}(-\sin x)$$

$$= -\tfrac{1}{2}(1+\sin x)^{-3/2}[\cos^2 x + 2(1+\sin x)\,\sin x] = -\tfrac{1}{2}(1+\sin x)^{-3/2}(\cos^2 x + 2\sin x + 2\sin^2 x)$$

$$= -\tfrac{1}{2}(1+\sin x)^{-3/2}(1+2\sin x+\sin^2 x) = -\tfrac{1}{2}(1+\sin x)^{-3/2}(1+\sin x)^2$$

$$= -\tfrac{1}{2}(1+\sin x)^{1/2} \text{ or } -\tfrac{1}{2}\sqrt{1+\sin x}$$

28. $F(t) = \dfrac{t^2}{\sqrt{t^3+1}} \Rightarrow$

$$F'(t) = \frac{(t^3+1)^{1/2}(2t) - t^2\cdot\frac{1}{2}(t^3+1)^{-1/2}(3t^2)}{\left(\sqrt{t^3+1}\right)^2} = \frac{t(t^3+1)^{-1/2}\left[2(t^3+1)-\frac{3}{2}t^3\right]}{(t^3+1)^1}$$

$$= \frac{t\left(\frac{1}{2}t^3+2\right)}{(t^3+1)^{3/2}} = \frac{t(t^3+4)}{2(t^3+1)^{3/2}}$$

29. $H(r) = \dfrac{(r^2-1)^3}{(2r+1)^5} \Rightarrow$

$$H'(r) = \frac{(2r+1)^5\cdot 3(r^2-1)^2(2r) - (r^2-1)^3\cdot 5(2r+1)^4(2)}{[(2r+1)^5]^2} = \frac{2(2r+1)^4(r^2-1)^2[3r(2r+1)-5(r^2-1)]}{(2r+1)^{10}}$$

$$= \frac{2(r^2-1)^2(6r^2+3r-5r^2+5)}{(2r+1)^6} = \frac{2(r^2-1)^2(r^2+3r+5)}{(2r+1)^6}$$

30. $s(t) = \sqrt{\dfrac{1+\sin t}{1+\cos t}} = \left(\dfrac{1+\sin t}{1+\cos t}\right)^{1/2} \Rightarrow$

$$s'(t) = \frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-1/2} \frac{(1+\cos t)\cos t - (1+\sin t)(-\sin t)}{(1+\cos t)^2}$$

$$= \frac{1}{2}\frac{(1+\sin t)^{-1/2}}{(1+\cos t)^{-1/2}}\frac{\cos t+\cos^2 t+\sin t+\sin^2 t}{(1+\cos t)^2} = \frac{\cos t+\sin t+1}{2\sqrt{1+\sin t}\,(1+\cos t)^{3/2}}$$

31. $y = \cos(\sec 4x) \Rightarrow$

$$y' = -\sin(\sec 4x)\frac{d}{dx}\sec 4x = -\sin(\sec 4x)\cdot\sec 4x\tan 4x\cdot 4 = -4\sin(\sec 4x)\sec 4x\tan 4x$$

32. $J(\theta) = \tan^2(n\theta) = [\tan(n\theta)]^2 \Rightarrow$

$$J'(\theta) = 2\,[\tan(n\theta)]^1\frac{d}{d\theta}\tan(n\theta) = 2\tan(n\theta)\,\sec^2(n\theta)\cdot n = 2n\tan(n\theta)\,\sec^2(n\theta)$$

33. $y = \sin\sqrt{1+x^2} \quad\Rightarrow\quad y' = \cos\sqrt{1+x^2}\cdot\frac{1}{2}(1+x^2)^{-1/2}\cdot 2x = \left(x\,\cos\sqrt{1+x^2}\,\right)/\sqrt{1+x^2}$

34. $y = \sqrt{\sin(1+x^2)} = [\sin(1+x^2)]^{1/2} \quad\Rightarrow\quad y' = \frac{1}{2}[\sin(1+x^2)]^{-1/2}\cdot\cos(1+x^2)\cdot 2x = \dfrac{x\cos(1+x^2)}{\sqrt{\sin(1+x^2)}}$

35. $y = \left(\dfrac{1-\cos 2x}{1+\cos 2x}\right)^4 \quad\Rightarrow$

$$y' = 4\left(\frac{1-\cos 2x}{1+\cos 2x}\right)^3\cdot\frac{(1+\cos 2x)(2\sin 2x)+(1-\cos 2x)(-2\sin 2x)}{(1+\cos 2x)^2}$$

$$= 4\left(\frac{1-\cos 2x}{1+\cos 2x}\right)^3\cdot\frac{2\sin 2x\,(1+\cos 2x+1-\cos 2x)}{(1+\cos 2x)^2} = \frac{4(1-\cos 2x)^3}{(1+\cos 2x)^3}\,\frac{2\sin 2x\,(2)}{(1+\cos 2x)^2} = \frac{16\sin 2x\,(1-\cos 2x)^3}{(1+\cos 2x)^5}$$

36. $y = x\sin\dfrac{1}{x} \quad\Rightarrow\quad y' = \sin\dfrac{1}{x} + x\cos\dfrac{1}{x}\left(-\dfrac{1}{x^2}\right) = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}$

37. $y = \cot^2(\sin\theta) = [\cot(\sin\theta)]^2 \quad\Rightarrow$

$$y' = 2[\cot(\sin\theta)]\cdot\frac{d}{d\theta}[\cot(\sin\theta)] = 2\cot(\sin\theta)\cdot[-\csc^2(\sin\theta)\cdot\cos\theta] = -2\cos\theta\,\cot(\sin\theta)\,\csc^2(\sin\theta)$$

38. $y = \sin(t+\cos\sqrt{t}\,) \quad\Rightarrow$

$$y' = \cos(t+\cos\sqrt{t}\,)\cdot\frac{d}{dt}(t+\cos\sqrt{t}\,) = \cos(t+\cos\sqrt{t}\,)\cdot\left(1-\sin\sqrt{t}\cdot\frac{1}{2\sqrt{t}}\right) = \cos(t+\cos\sqrt{t}\,)\,\frac{2\sqrt{t}-\sin\sqrt{t}}{2\sqrt{t}}$$

39. $f(t) = \tan(\sec(\cos t)) \quad\Rightarrow$

$$f'(t) = \sec^2(\sec(\cos t))\cdot\frac{d}{dt}\,\sec(\cos t) = \sec^2(\sec(\cos t))\cdot\sec(\cos t)\tan(\cos t)\cdot\frac{d}{dt}\,\cos t$$

$$= -\sin t\,\sec^2(\sec(\cos t))\sec(\cos t)\tan(\cos t)$$

40. $g(u) = [(u^2-1)^6-3u]^4 \quad\Rightarrow$

$$g'(u) = 4[(u^2-1)^6-3u]^3\cdot\frac{d}{du}[(u^2-1)^6-3u] = 4[(u^2-1)^6-3u]^3\cdot[6(u^2-1)^5\cdot 2u-3]$$

$$= 12[(u^2-1)^6-3u]^3[4u(u^2-1)^5-1]$$

41. $y = \sqrt{x+\sqrt{x}} \quad\Rightarrow\quad y' = \frac{1}{2}\,(x+\sqrt{x}\,)^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

42. $y = \sqrt{x+\sqrt{x+\sqrt{x}}} \quad\Rightarrow\quad y' = \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left[1+\frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right)\right]$

43. $g(x) = (2r\sin rx+n)^p \quad\Rightarrow\quad g'(x) = p(2r\sin rx+n)^{p-1}(2r\cos rx\cdot r) = p(2r\sin rx+n)^{p-1}(2r^2\cos rx)$

44. $y = \cos^4(\sin^3 x) = [\cos(\sin^3 x)]^4 \quad\Rightarrow$

$$y' = 4[\cos(\sin^3 x)]^3(-\sin(\sin^3 x))\,3\sin^2 x\,\cos x = -12\sin^2 x\,\cos x\,\cos^3(\sin^3 x)\,\sin(\sin^3 x)$$

45. $y = \cos\sqrt{\sin(\tan\pi x)} = \cos(\sin(\tan\pi x))^{1/2} \quad\Rightarrow$

$$y' = -\sin(\sin(\tan\pi x))^{1/2}\cdot\frac{d}{dx}(\sin(\tan\pi x))^{1/2} = -\sin(\sin(\tan\pi x))^{1/2}\cdot\tfrac{1}{2}(\sin(\tan\pi x))^{-1/2}\cdot\frac{d}{dx}(\sin(\tan\pi x))$$

$$= \frac{-\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}}\cdot\cos(\tan\pi x)\cdot\frac{d}{dx}\tan\pi x = \frac{-\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}}\cdot\cos(\tan\pi x)\cdot\sec^2(\pi x)\cdot\pi$$

$$= \frac{-\pi\cos(\tan\pi x)\sec^2(\pi x)\sin\sqrt{\sin(\tan\pi x)}}{2\sqrt{\sin(\tan\pi x)}}$$

46. $y = \left[x + (x+\sin^2 x)^3\right]^4 \quad\Rightarrow\quad y' = 4\left[x+(x+\sin^2 x)^3\right]^3\cdot\left[1 + 3(x+\sin^2 x)^2\cdot(1+2\sin x\,\cos x)\right]$

47. $y = \cos(\sin 3\theta) \quad\Rightarrow\quad y' = -\sin(\sin 3\theta)\cdot(\cos 3\theta)\cdot 3 = -3\cos 3\theta\,\sin(\sin 3\theta) \quad\Rightarrow$

$y'' = -3\left[(\cos 3\theta)\cos(\sin 3\theta)(\cos 3\theta)\cdot 3 + \sin(\sin 3\theta)(-\sin 3\theta)\cdot 3\right] = -9\cos^2(3\theta)\cos(\sin 3\theta) + 9(\sin 3\theta)\sin(\sin 3\theta)$

48. $y = \dfrac{1}{(1+\tan x)^2} = (1+\tan x)^{-2} \quad\Rightarrow\quad y' = -2(1+\tan x)^{-3}\sec^2 x = \dfrac{-2\sec^2 x}{(1+\tan x)^3}.$

Using the Product Rule with $y' = \left[-2(1+\tan x)^{-3}\right](\sec x)^2$, we get

$$y'' = -2(1+\tan x)^{-3}\cdot 2(\sec x)(\sec x\,\tan x) + (\sec x)^2\cdot 6(1+\tan x)^{-4}\sec^2 x$$

$$= 2\sec^2 x\,(1+\tan x)^{-4}\left[-2(1+\tan x)\tan x + 3\sec^2 x\right] \qquad \left[\begin{array}{c}\text{2 is the lesser exponent for } \sec x\\ \text{and } -4 \text{ for } (1+\tan x)\end{array}\right]$$

$$= 2\sec^2 x\,(1+\tan x)^{-4}\left[-2\tan x - 2\tan^2 x + 3(\tan^2 x + 1)\right]$$

$$= \frac{2\sec^2 x\left(\tan^2 x - 2\tan x + 3\right)}{(1+\tan x)^4}$$

49. $y = \sqrt{1-\sec t} \quad\Rightarrow\quad y' = \tfrac{1}{2}(1-\sec t)^{-1/2}(-\sec t\,\tan t) = \dfrac{-\sec t\,\tan t}{2\sqrt{1-\sec t}}.$

Using the Product Rule with $y' = \left(-\tfrac{1}{2}\sec t\,\tan t\right)(1-\sec t)^{-1/2}$, we get

$y'' = \left(-\tfrac{1}{2}\sec t\,\tan t\right)\left[-\tfrac{1}{2}(1-\sec t)^{-3/2}(-\sec t\,\tan t)\right] + (1-\sec t)^{-1/2}\left(-\tfrac{1}{2}\right)[\sec t\,\sec^2 t + \tan t\,\sec t\,\tan t].$

Now factor out $-\tfrac{1}{2}\sec t(1-\sec t)^{-3/2}$. Note that $-\tfrac{3}{2}$ is the lesser exponent on $(1-\sec t)$. Continuing,

$$y'' = -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left[\tfrac{1}{2}\sec t\,\tan^2 t + (1-\sec t)(\sec^2 t + \tan^2 t)\right]$$

$$= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left(\tfrac{1}{2}\sec t\,\tan^2 t + \sec^2 t + \tan^2 t - \sec^3 t - \sec t\,\tan^2 t\right)$$

$$= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left[-\tfrac{1}{2}\sec t\,(\sec^2 t - 1) + \sec^2 t + (\sec^2 t - 1) - \sec^3 t\right]$$

$$= -\tfrac{1}{2}\sec t\,(1-\sec t)^{-3/2}\left(-\tfrac{3}{2}\sec^3 t + 2\sec^2 t + \tfrac{1}{2}\sec t - 1\right)$$

$$= \sec t\,(1-\sec t)^{-3/2}\left(\tfrac{3}{4}\sec^3 t - \sec^2 t - \tfrac{1}{4}\sec t + \tfrac{1}{2}\right)$$

$$= \frac{\sec t\,(3\sec^3 t - 4\sec^2 t - \sec t + 2)}{4(1-\sec t)^{3/2}}$$

There are many other correct forms of y'', such as $y'' = \dfrac{\sec t\,(3\sec t + 2)\sqrt{1-\sec t}}{4}$. We chose to find a factored form with only secants in the final form.

50. $y = \dfrac{4x}{\sqrt{x+1}} \quad \Rightarrow$

$$y' = \frac{\sqrt{x+1}\cdot 4 - 4x\cdot\frac{1}{2}(x+1)^{-1/2}}{\left(\sqrt{x+1}\right)^2} = \frac{4\sqrt{x+1} - 2x/\sqrt{x+1}}{x+1} = \frac{4(x+1)-2x}{(x+1)^{3/2}} = \frac{2x+4}{(x+1)^{3/2}} \quad \Rightarrow$$

$$y'' = \frac{(x+1)^{3/2}\cdot 2 - (2x+4)\cdot\frac{3}{2}(x+1)^{1/2}}{[(x+1)^{3/2}]^2} = \frac{(x+1)^{1/2}[2(x+1)-3(x+2)]}{(x+1)^3} = \frac{2x+2-3x-6}{(x+1)^{5/2}} = \frac{-x-4}{(x+1)^{5/2}}$$

51. $y = (3x-1)^{-6} \quad \Rightarrow \quad y' = -6(3x-1)^{-7}\cdot 3 = -18(3x-1)^{-7}$. At $(0,1)$, $y' = -18(-1)^{-7} = -18(-1) = 18$, and an equation of the tangent line is $y - 1 = 18(x-0)$, or $y = 18x+1$.

52. $y = \sqrt{1+x^3} = (1+x^3)^{1/2} \quad \Rightarrow \quad y' = \frac{1}{2}(1+x^3)^{-1/2}\cdot 3x^2 = \dfrac{3x^2}{2\sqrt{1+x^3}}$. At $(2,3)$, $y' = \dfrac{3\cdot 4}{2\sqrt{9}} = 2$, and an equation of the tangent line is $y - 3 = 2(x-2)$, or $y = 2x-1$.

53. $y = \sin(\sin x) \quad \Rightarrow \quad y' = \cos(\sin x)\cdot\cos x$. At $(\pi, 0)$, $y' = \cos(\sin\pi)\cdot\cos\pi = \cos(0)\cdot(-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x-\pi)$, or $y = -x+\pi$.

54. $y = \sin^2 x\cos x \quad \Rightarrow \quad y' = \sin^2 x(-\sin x) + \cos x(2\sin x\cos x)$. At $(\pi/2, 0)$, $y' = 1(-1)+0 = -1$, and an equation of the tangent line is $y - 0 = -1\left(x - \frac{\pi}{2}\right)$, or $y = -x + \frac{\pi}{2}$.

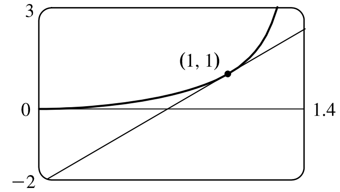
55. (a) $y = f(x) = \tan\left(\frac{\pi}{4}x^2\right) \quad \Rightarrow \quad f'(x) = \sec^2\left(\frac{\pi}{4}x^2\right)\left(2\cdot\frac{\pi}{4}x\right)$.

The slope of the tangent at $(1,1)$ is thus

$f'(1) = \sec^2\frac{\pi}{4}\left(\frac{\pi}{2}\right) = 2\cdot\frac{\pi}{2} = \pi$, and its equation

is $y - 1 = \pi(x-1)$ or $y = \pi x - \pi + 1$.

(b)



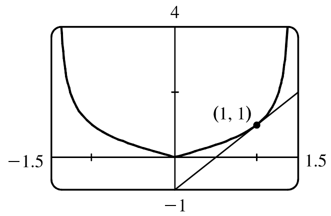
56. (a) For $x > 0$, $|x| = x$, and $y = f(x) = \dfrac{x}{\sqrt{2-x^2}} \quad \Rightarrow$

$$f'(x) = \frac{\sqrt{2-x^2}\,(1) - x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2}\cdot\frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}}$$

$$= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}}$$

So at $(1,1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y - 1 = 2(x-1)$ or $y = 2x-1$.

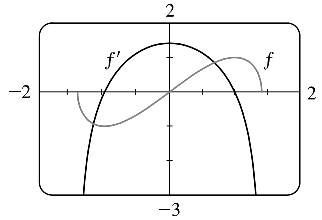
(b)



57. (a) $f(x) = x\sqrt{2-x^2} = x(2-x^2)^{1/2} \quad \Rightarrow$

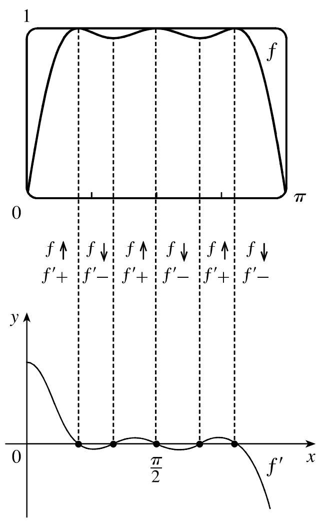
$$f'(x) = x\cdot\tfrac{1}{2}(2-x^2)^{-1/2}(-2x) + (2-x^2)^{1/2}\cdot 1 = (2-x^2)^{-1/2}\left[-x^2 + (2-x^2)\right] = \frac{2-2x^2}{\sqrt{2-x^2}}$$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

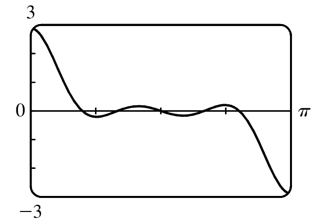
58. (a)



From the graph of f, we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f'. From the symmetry of the graph of f, we must have the graph of f' as high at $x = 0$ as it is low at $x = \pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

(b) $f(x) = \sin(x + \sin 2x) \quad \Rightarrow$

$$f'(x) = \cos(x + \sin 2x) \cdot \frac{d}{dx}\,(x + \sin 2x) = \cos(x + \sin 2x)(1 + 2\cos 2x)$$



59. For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2\sin x + \sin^2 x \quad \Rightarrow \quad f'(x) = 2\cos x + 2\sin x\,\cos x = 0 \quad \Leftrightarrow$ $2\cos x(1 + \sin x) = 0 \quad \Leftrightarrow \quad \cos x = 0$ or $\sin x = -1$, so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right) = 3$ and $f\left(\frac{3\pi}{2}\right) = -1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2} + 2n\pi, 3\right)$ and $\left(\frac{3\pi}{2} + 2n\pi, -1\right)$, where n is any integer.

60. $y = \sqrt{1 + 2x} \quad \Rightarrow \quad y' = \frac{1}{2}(1 + 2x)^{-1/2} \cdot 2 = \dfrac{1}{\sqrt{1 + 2x}}$. The line $6x + 2y = 1$ $\left(\text{or } y = -3x + \frac{1}{2}\right)$ has slope -3, so the tangent line perpendicular to it must have slope $\frac{1}{3}$. Thus, $\dfrac{1}{3} = \dfrac{1}{\sqrt{1 + 2x}} \quad \Leftrightarrow \quad \sqrt{1 + 2x} = 3 \quad \Rightarrow \quad 1 + 2x = 9 \quad \Leftrightarrow$ $2x = 8 \quad \Leftrightarrow \quad x = 4$. When $x = 4$, $y = \sqrt{1 + 2(4)} = 3$, so the point is $(4, 3)$.

61. $F(x) = f(g(x)) \quad \Rightarrow \quad F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(5) = f'(g(5)) \cdot g'(5) = f'(-2) \cdot 6 = 4 \cdot 6 = 24$.

62. $h(x) = \sqrt{4 + 3f(x)} \quad \Rightarrow \quad h'(x) = \frac{1}{2}(4 + 3f(x))^{-1/2} \cdot 3f'(x)$, so

$h'(1) = \frac{1}{2}(4 + 3f(1))^{-1/2} \cdot 3f'(1) = \frac{1}{2}(4 + 3 \cdot 7)^{-1/2} \cdot 3 \cdot 4 = \frac{6}{\sqrt{25}} = \frac{6}{5}$.

63. (a) $h(x) = f(g(x)) \quad \Rightarrow \quad h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \quad \Rightarrow \quad H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

64. (a) $F(x) = f(f(x)) \quad \Rightarrow \quad F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

(b) $G(x) = g(g(x)) \quad \Rightarrow \quad G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.

65. (a) $u(x) = f(g(x)) \Rightarrow u'(x) = f'(g(x))g'(x)$. So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2, 4)$ to $(6, 3)$, so its slope is $\frac{3-4}{6-2} = -\frac{1}{4}$. To find $g'(1)$, note that g is linear from $(0, 6)$ to $(2, 0)$, so its slope is $\frac{0-6}{2-0} = -3$. Thus, $f'(3)g'(1) = \left(-\frac{1}{4}\right)(-3) = \frac{3}{4}$.

(b) $v(x) = g(f(x)) \Rightarrow v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.

(c) $w(x) = g(g(x)) \Rightarrow w'(x) = g'(g(x))g'(x)$. So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2, 0)$ to $(5, 2)$, so its slope is $\frac{2-0}{5-2} = \frac{2}{3}$. Thus, $g'(3)g'(1) = \left(\frac{2}{3}\right)(-3) = -2$.

66. (a) $h(x) = f(f(x)) \Rightarrow h'(x) = f'(f(x))f'(x)$. So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

(b) $g(x) = f(x^2) \Rightarrow g'(x) = f'(x^2) \cdot \frac{d}{dx}(x^2) = f'(x^2)(2x)$. So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(2) = 8$.

67. The point $(3, 2)$ is on the graph of f, so $f(3) = 2$. The tangent line at $(3, 2)$ has slope $\frac{\Delta y}{\Delta x} = \frac{-4}{6} = -\frac{2}{3}$.

$g(x) = \sqrt{f(x)} \Rightarrow g'(x) = \frac{1}{2}[f(x)]^{-1/2} \cdot f'(x) \Rightarrow$

$g'(3) = \frac{1}{2}[f(3)]^{-1/2} \cdot f'(3) = \frac{1}{2}(2)^{-1/2}(-\frac{2}{3}) = -\frac{1}{3\sqrt{2}}$ or $-\frac{1}{6}\sqrt{2}$.

68. (a) $F(x) = f(x^\alpha) \Rightarrow F'(x) = f'(x^\alpha)\frac{d}{dx}(x^\alpha) = f'(x^\alpha)\alpha x^{\alpha-1}$

(b) $G(x) = [f(x)]^\alpha \Rightarrow G'(x) = \alpha[f(x)]^{\alpha-1} f'(x)$

69. $r(x) = f(g(h(x))) \Rightarrow r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$, so
$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1) = f'(g(2)) \cdot g'(2) \cdot 4 = f'(3) \cdot 5 \cdot 4 = 6 \cdot 5 \cdot 4 = 120$

70. $f(x) = xg(x^2) \Rightarrow f'(x) = xg'(x^2)\,2x + g(x^2) \cdot 1 = 2x^2g'(x^2) + g(x^2) \Rightarrow$
$f''(x) = 2x^2g''(x^2)\,2x + g'(x^2)\,4x + g'(x^2)\,2x = 4x^3g''(x^2) + 4xg'(x^2) + 2xg'(x^2) = 6xg'(x^2) + 4x^3g''(x^2)$

71. $F(x) = f(3f(4f(x))) \Rightarrow$

$$F'(x) = f'(3f(4f(x))) \cdot \frac{d}{dx}(3f(4f(x))) = f'(3f(4f(x))) \cdot 3f'(4f(x)) \cdot \frac{d}{dx}(4f(x))$$
$$= f'(3f(4f(x))) \cdot 3f'(4f(x)) \cdot 4f'(x), \quad \text{so}$$

$F'(0) = f'(3f(4f(0))) \cdot 3f'(4f(0)) \cdot 4f'(0) = f'(3f(4 \cdot 0)) \cdot 3f'(4 \cdot 0) \cdot 4 \cdot 2 = f'(3 \cdot 0) \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 2 \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 96$.

72. $F(x) = f(xf(xf(x))) \Rightarrow$

$$F'(x) = f'(xf(xf(x))) \cdot \frac{d}{dx}(xf(xf(x))) = f'(xf(xf(x))) \cdot \left[x \cdot f'(xf(x)) \cdot \frac{d}{dx}(xf(x)) + f(xf(x)) \cdot 1\right]$$
$$= f'(xf(xf(x))) \cdot [xf'(xf(x)) \cdot (xf'(x) + f(x) \cdot 1) + f(xf(x))], \quad \text{so}$$

$$F'(1) = f'(f(f(1))) \cdot [f'(f(1)) \cdot (f'(1) + f(1)) + f(f(1))] = f'(f(2)) \cdot [f'(2) \cdot (4 + 2) + f(2)]$$
$$= f'(3) \cdot [5 \cdot 6 + 3] = 6 \cdot 33 = 198.$$

73. Let $f(x) = \cos x$. Then $Df(2x) = 2f'(2x)$, $D^2 f(2x) = 2^2 f''(2x)$, $D^3 f(2x) = 2^3 f'''(2x)$, ..., $D^{(n)} f(2x) = 2^n f^{(n)}(2x)$. Since the derivatives of $\cos x$ occur in a cycle of four, and since $103 = 4(25) + 3$, we have $f^{(103)}(x) = f^{(3)}(x) = \sin x$ and $D^{103} \cos 2x = 2^{103} f^{(103)}(2x) = 2^{103} \sin 2x$.

74. Let $f(x) = x \sin \pi x$ and $h(x) = \sin \pi x$, so $f(x) = xh(x)$. Then $Df(x) = xh'(x) + h(x)$, $D^2 f(x) = xh''(x) + h'(x) + h'(x) = xh''(x) + 2h'(x)$, $D^3 f(x) = xh'''(x) + h''(x) + 2h''(x) = xh'''(x) + 3h''(x)$, ..., $D^n f(x) = xh^{(n)}(x) + nh^{(n-1)}(x)$. We now find a pattern for the derivatives of h: $h'(x) = \pi \cos \pi x$, $h''(x) = -\pi^2 \sin \pi x$, $h'''(x) = -\pi^3 \cos \pi x$, $h^4(x) = \pi^4 \sin \pi x$, and so on. Since $34 = 4(8) + 2$, we have $h^{(34)}(x) = -\pi^{34} \sin \pi x$ and $h^{(35)}(x) = -\pi^{35} \cos \pi x$. Thus,

$D^{35} f(x) = xh^{(35)}(x) + 35h^{(34)}(x) = x(-\pi^{35} \cos \pi x) + 35(-\pi^{34} \sin \pi x) = -\pi^{35} x \cos \pi x - 35\pi^{34} \sin \pi x$.

75. $s(t) = 10 + \frac{1}{4} \sin(10\pi t) \quad \Rightarrow \quad$ the velocity after t seconds is $v(t) = s'(t) = \frac{1}{4} \cos(10\pi t)(10\pi) = \frac{5\pi}{2} \cos(10\pi t)$ cm/s.

76. (a) $s = A\cos(\omega t + \delta) \quad \Rightarrow \quad \text{velocity} = s' = -\omega A \sin(\omega t + \delta)$.

(b) If $A \neq 0$ and $\omega \neq 0$, then $s' = 0 \quad \Leftrightarrow \quad \sin(\omega t + \delta) = 0 \quad \Leftrightarrow \quad \omega t + \delta = n\pi \quad \Leftrightarrow \quad t = \dfrac{n\pi - \delta}{\omega}$, n an integer.

77. (a) $B(t) = 4.0 + 0.35 \sin \dfrac{2\pi t}{5.4} \quad \Rightarrow \quad \dfrac{dB}{dt} = \left(0.35 \cos \dfrac{2\pi t}{5.4}\right)\left(\dfrac{2\pi}{5.4}\right) = \dfrac{0.7\pi}{5.4} \cos \dfrac{2\pi t}{5.4} = \dfrac{7\pi}{54} \cos \dfrac{2\pi t}{5.4}$

(b) At $t = 1$, $\dfrac{dB}{dt} = \dfrac{7\pi}{54} \cos \dfrac{2\pi}{5.4} \approx 0.16$.

78. $L(t) = 12 + 2.8 \sin\left(\frac{2\pi}{365}(t - 80)\right) \quad \Rightarrow \quad L'(t) = 2.8 \cos\left(\frac{2\pi}{365}(t - 80)\right)\left(\frac{2\pi}{365}\right)$.

On March 21, $t = 80$, and $L'(80) \approx 0.0482$ hours per day. On May 21, $t = 141$, and $L'(141) \approx 0.02398$, which is approximately one-half of $L'(80)$.

79. By the Chain Rule, $a(t) = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \dfrac{dv}{ds} v(t) = v(t) \dfrac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

80. (a) The derivative dV/dr represents the rate of change of the volume with respect to the radius and the derivative dV/dt represents the rate of change of the volume with respect to time.

(b) Since $V = \dfrac{4}{3}\pi r^3$, $\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.

81. (a) Derive gives $g'(t) = \dfrac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get

$g'(t) = 9\dfrac{(t-2)^8}{(2t+1)^9} - 18\dfrac{(t-2)^9}{(2t+1)^{10}}$, and the simplification command results in the expression given by Derive.

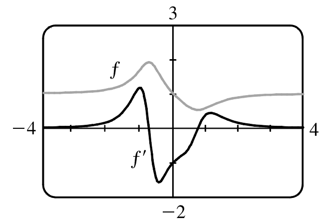
(b) Derive gives $y' = 2(x^3 - x + 1)^3(2x + 1)^4(17x^3 + 6x^2 - 9x + 3)$ without simplifying. With either Maple or Mathematica, we first get $y' = 10(2x + 1)^4(x^3 - x + 1)^4 + 4(2x + 1)^5(x^3 - x + 1)^3(3x^2 - 1)$. If we use

Mathematica's `Factor` or `Simplify`, or Maple's `factor`, we get the above expression, but Maple's `simplify` gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

82. (a) $f(x) = \left(\dfrac{x^4 - x + 1}{x^4 + x + 1}\right)^{1/2}$. Derive gives $f'(x) = \dfrac{(3x^4 - 1)\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}}{(x^4 + x + 1)(x^4 - x + 1)}$ whereas either Maple or Mathematica give $f'(x) = \dfrac{3x^4 - 1}{\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}\,(x^4 + x + 1)^2}$ after simplification.

(b) $f'(x) = 0 \quad \Leftrightarrow \quad 3x^4 - 1 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt[4]{\frac{1}{3}} \approx \pm 0.7598$.

(c) Yes. $f'(x) = 0$ where f has horizontal tangents. f' has two maxima and one minimum where f has inflection points.



83. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get

$f'(x) = f'(-x)\,\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

(b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

84.
$$\left[\frac{f(x)}{g(x)}\right]' = \left\{f(x)\,[g(x)]^{-1}\right\}' = f'(x)\,[g(x)]^{-1} + (-1)\,[g(x)]^{-2}\,g'(x)f(x)$$
$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

This is an alternative derivation of the *formula* in the Quotient Rule. But part of the purpose of the Quotient Rule is to show that if f and g are differentiable, so is f/g. The proof in Section 2.3 does that; this one doesn't.

85. (a)
$$\begin{aligned}\frac{d}{dx}(\sin^n x\,\cos nx) &= n\sin^{n-1} x\,\cos x\,\cos nx + \sin^n x\,(-n\sin nx) && \text{[Product Rule]}\\ &= n\sin^{n-1} x\,(\cos nx\,\cos x - \sin nx\,\sin x) && \text{[factor out } n\sin^{n-1} x\text{]}\\ &= n\sin^{n-1} x\cos(nx + x) && \text{[Addition Formula for cosine]}\\ &= n\sin^{n-1} x\cos[(n+1)x] && \text{[factor out } x\text{]}\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dx}(\cos^n x\,\cos nx) &= n\cos^{n-1} x\,(-\sin x)\cos nx + \cos^n x\,(-n\sin nx) && \text{[Product Rule]}\\ &= -n\cos^{n-1} x\,(\cos nx\,\sin x + \sin nx\,\cos x) && \text{[factor out } -n\cos^{n-1} x\text{]}\\ &= -n\cos^{n-1} x\sin(nx + x) && \text{[Addition Formula for sine]}\\ &= -n\cos^{n-1} x\sin[(n+1)x] && \text{[factor out } x\text{]}\end{aligned}$$

86. "The rate of change of y^5 with respect to x is eighty times the rate of change of y with respect to x" $\Leftrightarrow$

$\dfrac{d}{dx}y^5 = 80\dfrac{dy}{dx} \quad \Leftrightarrow \quad 5y^4\dfrac{dy}{dx} = 80\dfrac{dy}{dx} \quad \Leftrightarrow \quad 5y^4 = 80$ (Note that $dy/dx \neq 0$ since the curve never has a horizontal tangent) $\Leftrightarrow \quad y^4 = 16 \quad \Leftrightarrow \quad y = 2$ (since $y > 0$ for all x)

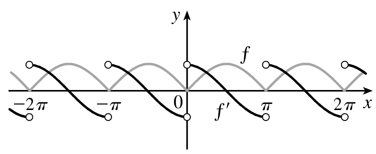
87. Since $\theta^\circ = \left(\frac{\pi}{180}\right)\theta$ rad, we have $\dfrac{d}{d\theta}(\sin\theta^\circ) = \dfrac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta^\circ$.

88. (a) $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \quad\Rightarrow\quad f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = x/\sqrt{x^2} = x/|x|$ for $x \neq 0$.

f is not differentiable at $x = 0$.

(b) $f(x) = |\sin x| = \sqrt{\sin^2 x} \quad\Rightarrow$

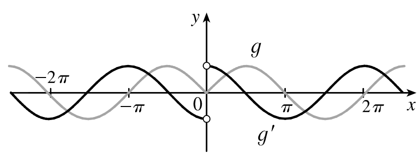
$$f'(x) = \tfrac{1}{2}(\sin^2 x)^{-1/2} 2\sin x\,\cos x = \frac{\sin x}{|\sin x|}\cos x = \begin{cases} \cos x & \text{if } \sin x > 0 \\ -\cos x & \text{if } \sin x < 0 \end{cases}$$



f is not differentiable when $x = n\pi$, n an integer.

(c) $g(x) = \sin|x| = \sin\sqrt{x^2} \quad\Rightarrow$

$$g'(x) = \cos|x| \cdot \frac{x}{|x|} = \frac{x}{|x|}\cos x = \begin{cases} \cos x & \text{if } x > 0 \\ -\cos x & \text{if } x < 0 \end{cases}$$



g is not differentiable at 0.

89. The Chain Rule says that $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \quad \text{[Product Rule]}$$

$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

90. From Exercise 89, $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{dy}{du}\dfrac{d^2u}{dx^2} \quad\Rightarrow$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2\right] + \frac{d}{dx}\left[\frac{dy}{du}\frac{d^2u}{dx^2}\right]$$

$$= \left[\frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\right]\left(\frac{du}{dx}\right)^2 + \left[\frac{d}{dx}\left(\frac{du}{dx}\right)^2\right]\frac{d^2y}{du^2} + \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{d^2u}{dx^2} + \left[\frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\right]\frac{dy}{du}$$

$$= \left[\frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\right]\left(\frac{du}{dx}\right)^2 + 2\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\left(\frac{d^2u}{dx^2}\right) + \frac{d^3u}{dx^3}\frac{dy}{du}$$

$$= \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 3\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d^3u}{dx^3}$$

APPLIED PROJECT Where Should a Pilot Start Descent?

1. Condition (i) will hold if and only if all of the following four conditions hold:

(α) $P(0) = 0$

(β) $P'(0) = 0$ (for a smooth landing)

(γ) $P'(\ell) = 0$ (since the plane is cruising horizontally when it begins its descent)

(δ) $P(\ell) = h$.

[continued]

First of all, condition α implies that $P(0) = d = 0$, so $P(x) = ax^3 + bx^2 + cx \Rightarrow P'(x) = 3ax^2 + 2bx + c$. But $P'(0) = c = 0$ by condition β. So $P'(\ell) = 3a\ell^2 + 2b\ell = \ell\,(3a\ell + 2b)$. Now by condition γ, $3a\ell + 2b = 0 \Rightarrow a = -\dfrac{2b}{3\ell}$.

Therefore, $P(x) = -\dfrac{2b}{3\ell}x^3 + bx^2$. Setting $P(\ell) = h$ for condition δ, we get $P(\ell) = -\dfrac{2b}{3\ell}\ell^3 + b\ell^2 = h \Rightarrow$
$-\dfrac{2}{3}b\ell^2 + b\ell^2 = h \Rightarrow \dfrac{1}{3}b\ell^2 = h \Rightarrow b = \dfrac{3h}{\ell^2} \Rightarrow a = -\dfrac{2h}{\ell^3}$. So $y = P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$.

2. By condition (ii), $\dfrac{dx}{dt} = -v$ for all t, so $x(t) = \ell - vt$. Condition (iii) states that $\left|\dfrac{d^2y}{dt^2}\right| \le k$. By the Chain Rule,

we have $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = -\dfrac{2h}{\ell^3}\left(3x^2\right)\dfrac{dx}{dt} + \dfrac{3h}{\ell^2}(2x)\dfrac{dx}{dt} = \dfrac{6hx^2v}{\ell^3} - \dfrac{6hxv}{\ell^2}$ (for $x \le \ell$) $\Rightarrow$

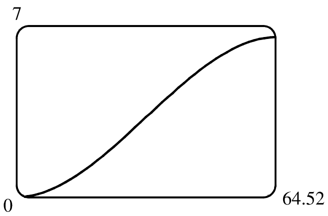
$\dfrac{d^2y}{dt^2} = \dfrac{6hv}{\ell^3}(2x)\dfrac{dx}{dt} - \dfrac{6hv}{\ell^2}\dfrac{dx}{dt} = -\dfrac{12hv^2}{\ell^3}x + \dfrac{6hv^2}{\ell^2}$. In particular, when $t = 0$, $x = \ell$ and so

$\left.\dfrac{d^2y}{dt^2}\right|_{t=0} = -\dfrac{12hv^2}{\ell^3}\ell + \dfrac{6hv^2}{\ell^2} = -\dfrac{6hv^2}{\ell^2}$. Thus, $\left|\dfrac{d^2y}{dt^2}\right|_{t=0} = \dfrac{6hv^2}{\ell^2} \le k$. (This condition also follows from taking $x = 0$.)

3. We substitute $k = 860\ \text{mi/h}^2$, $h = 35{,}000\ \text{ft} \times \dfrac{1\ \text{mi}}{5280\ \text{ft}}$, and $v = 300\ \text{mi/h}$ into the result of part (b):

$$\frac{6\left(35{,}000 \cdot \frac{1}{5280}\right)(300)^2}{\ell^2} \le 860 \quad\Rightarrow\quad \ell \ge 300\sqrt{6 \cdot \frac{35{,}000}{5280 \cdot 860}} \approx 64.5 \text{ miles.}$$

4. Substituting the values of h and ℓ in Problem 3 into $P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$ gives us $P(x) = ax^3 + bx^2$, where $a \approx -4.937 \times 10^{-5}$ and $b \approx 4.78 \times 10^{-3}$.



2.6 Implicit Differentiation

1. (a) $\dfrac{d}{dx}(9x^2 - y^2) = \dfrac{d}{dx}(1) \Rightarrow 18x - 2y\,y' = 0 \Rightarrow 2y\,y' = 18x \Rightarrow y' = \dfrac{9x}{y}$

(b) $9x^2 - y^2 = 1 \Rightarrow y^2 = 9x^2 - 1 \Rightarrow y = \pm\sqrt{9x^2 - 1}$, so $y' = \pm\frac{1}{2}(9x^2 - 1)^{-1/2}(18x) = \pm\dfrac{9x}{\sqrt{9x^2 - 1}}$.

(c) From part (a), $y' = \dfrac{9x}{y} = \dfrac{9x}{\pm\sqrt{9x^2 - 1}}$, which agrees with part (b).

2. (a) $\dfrac{d}{dx}(2x^2 + x + xy) = \dfrac{d}{dx}(1) \Rightarrow 4x + 1 + xy' + y \cdot 1 = 0 \Rightarrow xy' = -4x - y - 1 \Rightarrow y' = -\dfrac{4x + y + 1}{x}$

(b) $2x^2 + x + xy = 1 \Rightarrow xy = 1 - 2x^2 - x \Rightarrow y = \dfrac{1}{x} - 2x - 1$, so $y' = -\dfrac{1}{x^2} - 2$

(c) From part (a),

$$y' = -\frac{4x + y + 1}{x} = -4 - \frac{1}{x}y - \frac{1}{x} = -4 - \frac{1}{x}\left(\frac{1}{x} - 2x - 1 - \frac{1}{x}\right) = -4 - \frac{1}{x^2} + 2 + \frac{1}{x} - \frac{1}{x} = -\frac{1}{x^2} - 2, \text{ which}$$

agrees with part (b).

3. (a) $\frac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=\frac{d}{dx}(1) \quad\Rightarrow\quad \frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}y'=0 \quad\Rightarrow\quad \frac{1}{2\sqrt{y}}\,y'=-\frac{1}{2\sqrt{x}} \quad\Rightarrow\quad y'=-\frac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{x}+\sqrt{y}=1 \quad\Rightarrow\quad \sqrt{y}=1-\sqrt{x} \quad\Rightarrow\quad y=\left(1-\sqrt{x}\right)^2 \quad\Rightarrow\quad y=1-2\sqrt{x}+x$, so

$y'=-2\cdot\frac{1}{2}x^{-1/2}+1=1-\frac{1}{\sqrt{x}}$.

(c) From part (a), $y'=-\frac{\sqrt{y}}{\sqrt{x}}=-\frac{1-\sqrt{x}}{\sqrt{x}}$ [from part (b)] $=-\frac{1}{\sqrt{x}}+1$, which agrees with part (b).

4. (a) $\frac{d}{dx}\left(\frac{2}{x}-\frac{1}{y}\right)=\frac{d}{dx}(4) \quad\Rightarrow\quad -2x^{-2}+y^{-2}y'=0 \quad\Rightarrow\quad \frac{1}{y^2}\,y'=\frac{2}{x^2} \quad\Rightarrow\quad y'=\frac{2y^2}{x^2}$

(b) $\frac{2}{x}-\frac{1}{y}=4 \quad\Rightarrow\quad \frac{1}{y}=\frac{2}{x}-4 \quad\Rightarrow\quad \frac{1}{y}=\frac{2-4x}{x} \quad\Rightarrow\quad y=\frac{x}{2-4x}$, so

$y'=\frac{(2-4x)(1)-x(-4)}{(2-4x)^2}=\frac{2}{(2-4x)^2}\ \left[\text{or } \frac{1}{2(1-2x)^2}\right]$.

(c) From part (a), $y'=\frac{2y^2}{x^2}=\frac{2\left(\frac{x}{2-4x}\right)^2}{x^2}$ [from part (b)] $=\frac{2x^2}{x^2(2-4x)^2}=\frac{2}{(2-4x)^2}$, which agrees with part (b).

5. $\frac{d}{dx}\left(x^2-4xy+y^2\right)=\frac{d}{dx}(4) \quad\Rightarrow\quad 2x-4[xy'+y(1)]+2y\,y'=0 \quad\Rightarrow\quad 2y\,y'-4xy'=4y-2x \quad\Rightarrow$

$y'(y-2x)=2y-x \quad\Rightarrow\quad y'=\frac{2y-x}{y-2x}$

6. $\frac{d}{dx}\left(2x^2+xy-y^2\right)=\frac{d}{dx}(2) \quad\Rightarrow\quad 4x+xy'+y(1)-2y\,y'=0 \quad\Rightarrow\quad xy'-2y\,y'=-4x-y \quad\Rightarrow$

$(x-2y)y'=-4x-y \quad\Rightarrow\quad y'=\frac{-4x-y}{x-2y}$

7. $\frac{d}{dx}\left(x^4+x^2y^2+y^3\right)=\frac{d}{dx}(5) \quad\Rightarrow\quad 4x^3+x^2\cdot 2y\,y'+y^2\cdot 2x+3y^2y'=0 \quad\Rightarrow\quad 2x^2y\,y'+3y^2y'=-4x^3-2xy^2 \quad\Rightarrow$

$(2x^2y+3y^2)y'=-4x^3-2xy^2 \quad\Rightarrow\quad y'=\frac{-4x^3-2xy^2}{2x^2y+3y^2}=-\frac{2x(2x^2+y^2)}{y(2x^2+3y)}$

8. $\frac{d}{dx}\left(x^3-xy^2+y^3\right)=\frac{d}{dx}(1) \quad\Rightarrow\quad 3x^2-x\cdot 2y\,y'-y^2\cdot 1+3y^2y'=0 \quad\Rightarrow\quad 3y^2y'-2x\,y\,y'=y^2-3x^2 \quad\Rightarrow$

$(3y^2-2xy)\,y'=y^2-3x^2 \quad\Rightarrow\quad y'=\frac{y^2-3x^2}{3y^2-2xy}=\frac{y^2-3x^2}{y(3y-2x)}$

9. $\frac{d}{dx}\left(\frac{x^2}{x+y}\right)=\frac{d}{dx}(y^2+1) \quad\Rightarrow\quad \frac{(x+y)(2x)-x^2(1+y')}{(x+y)^2}=2y\,y' \quad\Rightarrow$

$2x^2+2xy-x^2-x^2\,y'=2y(x+y)^2\,y' \quad\Rightarrow\quad x^2+2xy=2y(x+y)^2\,y'+x^2\,y' \quad\Rightarrow$

$x(x+2y)=\left[2y(x^2+2xy+y^2)+x^2\right]y' \quad\Rightarrow\quad y'=\frac{x(x+2y)}{2x^2y+4xy^2+2y^3+x^2}$

Or: Start by clearing fractions and then differentiate implicitly.

10. $\dfrac{d}{dx}\left(y^5+x^2y^3\right)=\dfrac{d}{dx}\left(1+x^4y\right) \quad\Rightarrow\quad 5y^4y'+x^2\cdot 3y^2y'+y^3\cdot 2x=0+x^4y'+y\cdot 4x^3 \quad\Rightarrow$

$$y'\left(5y^4+3x^2y^2-x^4\right)=4x^3y-2xy^3 \quad\Rightarrow\quad y'=\frac{4x^3y-2xy^3}{5y^4+3x^2y^2-x^4}$$

11. $\dfrac{d}{dx}(y\cos x)=\dfrac{d}{dx}(x^2+y^2) \quad\Rightarrow\quad y(-\sin x)+\cos x\cdot y'=2x+2y\,y' \quad\Rightarrow\quad \cos x\cdot y'-2y\,y'=2x+y\sin x \quad\Rightarrow$

$$y'(\cos x-2y)=2x+y\sin x \quad\Rightarrow\quad y'=\frac{2x+y\sin x}{\cos x-2y}$$

12. $\dfrac{d}{dx}\cos(xy)=\dfrac{d}{dx}(1+\sin y) \quad\Rightarrow\quad -\sin(xy)(xy'+y\cdot 1)=\cos y\cdot y' \quad\Rightarrow\quad -xy'\sin(xy)-\cos y\cdot y'=y\sin(xy) \quad\Rightarrow$

$$y'[-x\sin(xy)-\cos y]=y\sin(xy) \quad\Rightarrow\quad y'=\frac{y\sin(xy)}{-x\sin(xy)-\cos y}=-\frac{y\sin(xy)}{x\sin(xy)+\cos y}$$

13. $\dfrac{d}{dx}\sqrt{x+y}=\dfrac{d}{dx}\left(x^4+y^4\right) \quad\Rightarrow\quad \tfrac{1}{2}(x+y)^{-1/2}(1+y')=4x^3+4y^3y' \quad\Rightarrow$

$$\frac{1}{2\sqrt{x+y}}+\frac{1}{2\sqrt{x+y}}\,y'=4x^3+4y^3y' \quad\Rightarrow\quad \frac{1}{2\sqrt{x+y}}-4x^3=4y^3y'-\frac{1}{2\sqrt{x+y}}y' \quad\Rightarrow$$

$$\frac{1-8x^3\sqrt{x+y}}{2\sqrt{x+y}}=\frac{8y^3\sqrt{x+y}-1}{2\sqrt{x+y}}\,y' \quad\Rightarrow\quad y'=\frac{1-8x^3\sqrt{x+y}}{8y^3\sqrt{x+y}-1}$$

14. $\dfrac{d}{dx}\left[y\sin(x^2)\right]=\dfrac{d}{dx}\left[x\sin(y^2)\right] \quad\Rightarrow\quad y\cos(x^2)\cdot 2x+\sin(x^2)\cdot y'=x\cos(y^2)\cdot 2y\,y'+\sin(y^2)\cdot 1 \quad\Rightarrow$

$$y'\left[\sin(x^2)-2xy\cos(y^2)\right]=\sin(y^2)-2xy\cos(x^2) \quad\Rightarrow\quad y'=\frac{\sin(y^2)-2xy\cos(x^2)}{\sin(x^2)-2xy\cos(y^2)}$$

15. $\dfrac{d}{dx}\tan(x/y)=\dfrac{d}{dx}(x+y) \quad\Rightarrow\quad \sec^2(x/y)\cdot\dfrac{y\cdot 1-x\cdot y'}{y^2}=1+y' \quad\Rightarrow$

$$y\sec^2(x/y)-x\sec^2(x/y)\cdot y'=y^2+y^2y' \quad\Rightarrow\quad y\sec^2(x/y)-y^2=y^2y'+x\sec^2(x/y) \quad\Rightarrow$$

$$y\sec^2(x/y)-y^2=\left[y^2+x\sec^2(x/y)\right]\cdot y' \quad\Rightarrow\quad y'=\frac{y\sec^2(x/y)-y^2}{y^2+x\sec^2(x/y)}$$

16. $\dfrac{d}{dx}(xy)=\dfrac{d}{dx}\sqrt{x^2+y^2} \quad\Rightarrow\quad xy'+y(1)=\tfrac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x+2y\,y') \quad\Rightarrow$

$$xy'+y=\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}\,y' \quad\Rightarrow\quad xy'-\frac{y}{\sqrt{x^2+y^2}}\,y'=\frac{x}{\sqrt{x^2+y^2}}-y \quad\Rightarrow$$

$$\frac{x\sqrt{x^2+y^2}-y}{\sqrt{x^2+y^2}}\,y'=\frac{x-y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \quad\Rightarrow\quad y'=\frac{x-y\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2}-y}$$

17. $\dfrac{d}{dx}\sqrt{xy}=\dfrac{d}{dx}(1+x^2y) \quad\Rightarrow\quad \tfrac{1}{2}(xy)^{-1/2}(xy'+y\cdot 1)=0+x^2y'+y\cdot 2x \quad\Rightarrow$

$$\frac{x}{2\sqrt{xy}}\,y'+\frac{y}{2\sqrt{xy}}=x^2y'+2xy \quad\Rightarrow\quad y'\left(\frac{x}{2\sqrt{xy}}-x^2\right)=2xy-\frac{y}{2\sqrt{xy}} \quad\Rightarrow$$

$$y'\left(\frac{x-2x^2\sqrt{xy}}{2\sqrt{xy}}\right)=\frac{4xy\sqrt{xy}-y}{2\sqrt{xy}} \quad\Rightarrow\quad y'=\frac{4xy\sqrt{xy}-y}{x-2x^2\sqrt{xy}}$$

18. $\dfrac{d}{dx}(x\sin y + y\sin x) = \dfrac{d}{dx}(1) \quad\Rightarrow\quad x\cos y\cdot y' + \sin y\cdot 1 + y\cos x + \sin x\cdot y' = 0 \quad\Rightarrow$

$x\cos y\cdot y' + \sin x\cdot y' = -\sin y - y\cos x \quad\Rightarrow\quad y'(x\cos y + \sin x) = -\sin y - y\cos x \quad\Rightarrow\quad y' = \dfrac{-\sin y - y\cos x}{x\cos y + \sin x}$

19. $\dfrac{d}{dx}\sin(xy) = \dfrac{d}{dx}\cos(x+y) \quad\Rightarrow\quad \cos(xy)\cdot(xy' + y\cdot 1) = -\sin(x+y)\cdot(1+y') \quad\Rightarrow$

$x\cos(xy)\,y' + y\cos(xy) = -\sin(x+y) - y'\sin(x+y) \quad\Rightarrow$

$x\cos(xy)\,y' + y'\sin(x+y) = -y\cos(xy) - \sin(x+y) \quad\Rightarrow$

$[x\cos(xy) + \sin(x+y)]\,y' = -1\,[y\cos(xy) + \sin(x+y)] \quad\Rightarrow\quad y' = -\dfrac{y\cos(xy) + \sin(x+y)}{x\cos(xy) + \sin(x+y)}$

20. $\dfrac{d}{dx}\tan(x-y) = \dfrac{d}{dx}\left(\dfrac{y}{1+x^2}\right) \quad\Rightarrow\quad (1+x^2)\tan(x-y) = y \quad\Rightarrow$

$(1+x^2)\sec^2(x-y)\cdot(1-y') + \tan(x-y)\cdot 2x = y' \quad\Rightarrow$

$(1+x^2)\sec^2(x-y) - (1+x^2)\sec^2(x-y)\cdot y' + 2x\tan(x-y) = y' \quad\Rightarrow$

$(1+x^2)\sec^2(x-y) + 2x\tan(x-y) = \left[1 + (1+x^2)\sec^2(x-y)\right]\cdot y' \quad\Rightarrow$

$y' = \dfrac{(1+x^2)\sec^2(x-y) + 2x\tan(x-y)}{1 + (1+x^2)\sec^2(x-y)}$

21. $\dfrac{d}{dx}\left\{f(x) + x^2[f(x)]^3\right\} = \dfrac{d}{dx}(10) \quad\Rightarrow\quad f'(x) + x^2\cdot 3[f(x)]^2\cdot f'(x) + [f(x)]^3\cdot 2x = 0.$ If $x = 1$, we have

$f'(1) + 1^2\cdot 3[f(1)]^2\cdot f'(1) + [f(1)]^3\cdot 2(1) = 0 \quad\Rightarrow\quad f'(1) + 1\cdot 3\cdot 2^2\cdot f'(1) + 2^3\cdot 2 = 0 \quad\Rightarrow$

$f'(1) + 12f'(1) = -16 \quad\Rightarrow\quad 13f'(1) = -16 \quad\Rightarrow\quad f'(1) = -\frac{16}{13}.$

22. $\dfrac{d}{dx}[g(x) + x\sin g(x)] = \dfrac{d}{dx}(x^2) \quad\Rightarrow\quad g'(x) + x\cos g(x)\cdot g'(x) + \sin g(x)\cdot 1 = 2x.$ If $x = 0$, we have

$g'(0) + 0 + \sin g(0) = 2(0) \quad\Rightarrow\quad g'(0) + \sin 0 = 0 \quad\Rightarrow\quad g'(0) + 0 = 0 \quad\Rightarrow\quad g'(0) = 0.$

23. $\dfrac{d}{dy}(x^4y^2 - x^3y + 2xy^3) = \dfrac{d}{dy}(0) \quad\Rightarrow\quad x^4\cdot 2y + y^2\cdot 4x^3\,x' - (x^3\cdot 1 + y\cdot 3x^2\,x') + 2(x\cdot 3y^2 + y^3\cdot x') = 0 \quad\Rightarrow$

$4x^3y^2\,x' - 3x^2y\,x' + 2y^3\,x' = -2x^4y + x^3 - 6xy^2 \quad\Rightarrow\quad (4x^3y^2 - 3x^2y + 2y^3)\,x' = -2x^4y + x^3 - 6xy^2 \quad\Rightarrow$

$x' = \dfrac{dx}{dy} = \dfrac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$

24. $\dfrac{d}{dy}(y\sec x) = \dfrac{d}{dy}(x\tan y) \quad\Rightarrow\quad y\cdot\sec x\,\tan x\cdot x' + \sec x\cdot 1 = x\cdot\sec^2 y + \tan y\cdot x' \quad\Rightarrow$

$y\sec x\,\tan x\cdot x' - \tan y\cdot x' = x\sec^2 y - \sec x \quad\Rightarrow\quad (y\sec x\,\tan x - \tan y)\,x' = x\sec^2 y - \sec x \quad\Rightarrow$

$x' = \dfrac{dx}{dy} = \dfrac{x\sec^2 y - \sec x}{y\sec x\,\tan x - \tan y}$

25. $y\sin 2x = x\cos 2y \quad\Rightarrow\quad y\cdot\cos 2x\cdot 2 + \sin 2x\cdot y' = x(-\sin 2y\cdot 2y') + \cos(2y)\cdot 1 \quad\Rightarrow$

$\sin 2x\cdot y' + 2x\sin 2y\cdot y' = -2y\cos 2x + \cos 2y \quad\Rightarrow\quad y'(\sin 2x + 2x\sin 2y) = -2y\cos 2x + \cos 2y \quad\Rightarrow$

$y' = \dfrac{-2y\cos 2x + \cos 2y}{\sin 2x + 2x\sin 2y}$. When $x = \frac{\pi}{2}$ and $y = \frac{\pi}{4}$, we have $y' = \dfrac{(-\pi/2)(-1)+0}{0+\pi\cdot 1} = \dfrac{\pi/2}{\pi} = \dfrac{1}{2}$, so an equation of the tangent line is $y - \frac{\pi}{4} = \frac{1}{2}(x - \frac{\pi}{2})$, or $y = \frac{1}{2}x$.

26. $\sin(x+y) = 2x - 2y \quad\Rightarrow\quad \cos(x+y)\cdot(1+y') = 2 - 2y' \quad\Rightarrow\quad \cos(x+y)\cdot y' + 2y' = 2 - \cos(x+y) \quad\Rightarrow$

$y'[\cos(x+y)+2] = 2 - \cos(x+y) \quad\Rightarrow\quad y' = \dfrac{2-\cos(x+y)}{\cos(x+y)+2}$. When $x = \pi$ and $y = \pi$, we have $y' = \dfrac{2-1}{1+2} = \dfrac{1}{3}$, so an equation of the tangent line is $y - \pi = \frac{1}{3}(x - \pi)$, or $y = \frac{1}{3}x + \frac{2\pi}{3}$.

27. $x^2 - xy - y^2 = 1 \quad\Rightarrow\quad 2x - (xy' + y\cdot 1) - 2y\,y' = 0 \quad\Rightarrow\quad 2x - xy' - y - 2y\,y' = 0 \quad\Rightarrow\quad 2x - y = xy' + 2y\,y' \quad\Rightarrow$

$2x - y = (x+2y)\,y' \quad\Rightarrow\quad y' = \dfrac{2x-y}{x+2y}$. When $x = 2$ and $y = 1$, we have $y' = \dfrac{4-1}{2+2} = \dfrac{3}{4}$, so an equation of the tangent line is $y - 1 = \frac{3}{4}(x-2)$, or $y = \frac{3}{4}x - \frac{1}{2}$.

28. $x^2 + 2xy + 4y^2 = 12 \quad\Rightarrow\quad 2x + 2x\,y' + 2y + 8y\,y' = 0 \quad\Rightarrow\quad 2x\,y' + 8y\,y' = -2x - 2y \quad\Rightarrow$

$(x+4y)\,y' = -x - y \quad\Rightarrow\quad y' = -\dfrac{x+y}{x+4y}$. When $x = 2$ and $y = 1$, we have $y' = -\dfrac{2+1}{2+4} = -\dfrac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x-2)$ or $y = -\frac{1}{2}x + 2$.

29. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \quad\Rightarrow\quad 2x + 2y\,y' = 2(2x^2+2y^2-x)(4x+4y\,y'-1)$. When $x = 0$ and $y = \frac{1}{2}$, we have $0 + y' = 2(\frac{1}{2})(2y'-1) \quad\Rightarrow\quad y' = 2y' - 1 \quad\Rightarrow\quad y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x-0)$ or $y = x + \frac{1}{2}$.

30. $x^{2/3} + y^{2/3} = 4 \quad\Rightarrow\quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \quad\Rightarrow\quad \dfrac{1}{\sqrt[3]{x}} + \dfrac{y'}{\sqrt[3]{y}} = 0 \quad\Rightarrow\quad y' = -\dfrac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x = -3\sqrt{3}$ and $y = 1$, we have $y' = -\dfrac{1}{\left(-3\sqrt{3}\right)^{1/3}} = -\dfrac{\left(-3\sqrt{3}\right)^{2/3}}{-3\sqrt{3}} = \dfrac{3}{3\sqrt{3}} = \dfrac{1}{\sqrt{3}}$, so an equation of the tangent line is $y - 1 = \frac{1}{\sqrt{3}}\left(x + 3\sqrt{3}\right)$ or $y = \frac{1}{\sqrt{3}}x + 4$.

31. $2(x^2+y^2)^2 = 25(x^2-y^2) \quad\Rightarrow\quad 4(x^2+y^2)(2x+2y\,y') = 25(2x - 2y\,y') \quad\Rightarrow$

$4(x+y\,y')(x^2+y^2) = 25(x - y\,y') \quad\Rightarrow\quad 4y\,y'(x^2+y^2) + 25yy' = 25x - 4x(x^2+y^2) \quad\Rightarrow$

$y' = \dfrac{25x - 4x(x^2+y^2)}{25y + 4y(x^2+y^2)}$. When $x = 3$ and $y = 1$, we have $y' = \frac{75-120}{25+40} = -\frac{45}{65} = -\frac{9}{13}$, so an equation of the tangent line is $y - 1 = -\frac{9}{13}(x-3)$ or $y = -\frac{9}{13}x + \frac{40}{13}$.

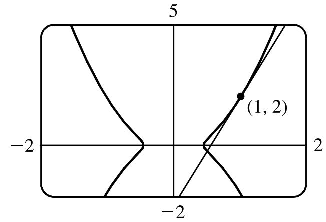
32. $y^2(y^2-4)=x^2(x^2-5) \Rightarrow y^4-4y^2=x^4-5x^2 \Rightarrow 4y^3\,y'-8y\,y'=4x^3-10x$.

When $x=0$ and $y=-2$, we have $-32y'+16y'=0 \Rightarrow -16y'=0 \Rightarrow y'=0$, so an equation of the tangent line is $y+2=0(x-0)$ or $y=-2$.

33. (a) $y^2=5x^4-x^2 \Rightarrow 2y\,y'=5(4x^3)-2x \Rightarrow y'=\dfrac{10x^3-x}{y}$.

So at the point $(1,2)$ we have $y'=\dfrac{10(1)^3-1}{2}=\dfrac{9}{2}$, and an equation of the tangent line is $y-2=\frac{9}{2}(x-1)$ or $y=\frac{9}{2}x-\frac{5}{2}$.

(b)



34. (a) $y^2=x^3+3x^2 \Rightarrow 2y\,y'=3x^2+3(2x) \Rightarrow y'=\dfrac{3x^2+6x}{2y}$. So at the point $(1,-2)$ we have $y'=\dfrac{3(1)^2+6(1)}{2(-2)}=-\dfrac{9}{4}$, and an equation of the tangent line is $y+2=-\frac{9}{4}(x-1)$ or $y=-\frac{9}{4}x+\frac{1}{4}$.

(b) The curve has a horizontal tangent where $y'=0 \Leftrightarrow$

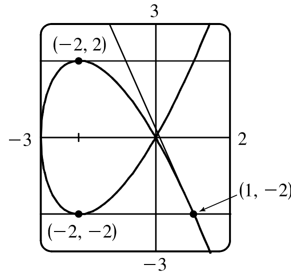
$3x^2+6x=0 \Leftrightarrow 3x(x+2)=0 \Leftrightarrow x=0$ or $x=-2$.

But note that at $x=0$, $y=0$ also, so the derivative does not exist.

At $x=-2$, $y^2=(-2)^3+3(-2)^2=-8+12=4$, so $y=\pm2$.

So the two points at which the curve has a horizontal tangent are $(-2,-2)$ and $(-2,2)$.

(c)



35. $x^2+4y^2=4 \Rightarrow 2x+8y\,y'=0 \Rightarrow y'=-x/(4y) \Rightarrow$

$$y''=-\frac{1}{4}\frac{y\cdot1-x\cdot y'}{y^2}=-\frac{1}{4}\frac{y-x[-x/(4y)]}{y^2}=-\frac{1}{4}\frac{4y^2+x^2}{4y^3}=-\frac{1}{4}\frac{4}{4y^3}\quad\left[\begin{array}{c}\text{since } x \text{ and } y \text{ must satisfy the}\\ \text{original equation } x^2+4y^2=4\end{array}\right]$$

Thus, $y''=-\dfrac{1}{4y^3}$.

36. $x^2+xy+y^2=3 \Rightarrow 2x+xy'+y+2y\,y'=0 \Rightarrow (x+2y)y'=-2x-y \Rightarrow y'=\dfrac{-2x-y}{x+2y}$.

Differentiating $2x+xy'+y+2y\,y'=0$ to find y'' gives $2+xy''+y'+y'+2y\,y''+2y'y'=0 \Rightarrow$

$$(x+2y)\,y''=-2-2y'-2(y')^2=-2\left[1-\frac{2x+y}{x+2y}+\left(\frac{2x+y}{x+2y}\right)^2\right] \Rightarrow$$

$$\begin{aligned}y''&=-\frac{2}{x+2y}\left[\frac{(x+2y)^2-(2x+y)(x+2y)+(2x+y)^2}{(x+2y)^2}\right]\\&=-\frac{2}{(x+2y)^3}\,(x^2+4xy+4y^2-2x^2-4xy-xy-2y^2+4x^2+4xy+y^2)\\&=-\frac{2}{(x+2y)^3}\,(3x^2+3xy+3y^2)=-\frac{2}{(x+2y)^3}\,(9)\quad\left[\begin{array}{c}\text{since } x \text{ and } y \text{ must satisfy the}\\ \text{original equation } x^2+xy+y^2=3\end{array}\right]\end{aligned}$$

Thus, $y''=-\dfrac{18}{(x+2y)^3}$.

37. $\sin y + \cos x = 1 \quad \Rightarrow \quad \cos y \cdot y' - \sin x = 0 \quad \Rightarrow \quad y' = \dfrac{\sin x}{\cos y} \quad \Rightarrow$

$$y'' = \frac{\cos y\, \cos x - \sin x(-\sin y)\, y'}{(\cos y)^2} = \frac{\cos y\, \cos x + \sin x\, \sin y(\sin x/\cos y)}{\cos^2 y}$$
$$= \frac{\cos^2 y\, \cos x + \sin^2 x\, \sin y}{\cos^2 y\, \cos y} = \frac{\cos^2 y\, \cos x + \sin^2 x\, \sin y}{\cos^3 y}$$

Using $\sin y + \cos x = 1$, the expression for y'' can be simplified to $y'' = (\cos^2 x + \sin y)/\cos^3 y$.

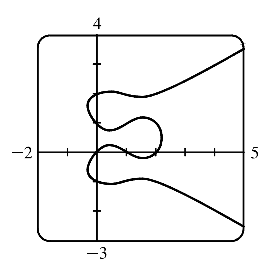
38. $x^3 - y^3 = 7 \quad \Rightarrow \quad 3x^2 - 3y^2 y' = 0 \quad \Rightarrow \quad y' = \dfrac{x^2}{y^2} \quad \Rightarrow$

$$y'' = \frac{y^2(2x) - x^2(2y\, y')}{(y^2)^2} = \frac{2xy[y - x(x^2/y^2)]}{y^4} = \frac{2x(y - x^3/y^2)}{y^3} = \frac{2x(y^3 - x^3)}{y^3 y^2} = \frac{2x(-7)}{y^5} = \frac{-14x}{y^5}$$

39. If $x = 0$ in $xy + y^3 = 1$, then we get $y^3 = 1 \quad \Rightarrow \quad y = 1$, so the point where $x = 0$ is $(0, 1)$. Differentiating implicitly with respect to x gives us $xy' + y \cdot 1 + 3y^2\, y' = 0$. Substituting 0 for x and 1 for y gives us $1 + 3y' = 0 \quad \Rightarrow \quad y' = -\frac{1}{3}$. Differentiating $xy' + y + 3y^2\, y' = 0$ implicitly with respect to x gives us $xy'' + y' + y' + 3(y^2 y'' + y' \cdot 2y\, y') = 0$. Now substitute 0 for x, 1 for y, and $-\frac{1}{3}$ for y'. $\quad 0 - \frac{1}{3} - \frac{1}{3} + 3\left[y'' + \left(-\frac{1}{3}\right) \cdot 2\left(-\frac{1}{3}\right)\right] = 0 \quad \Rightarrow \quad 3\left(y'' + \frac{2}{9}\right) = \frac{2}{3} \quad \Rightarrow$
$y'' + \frac{2}{9} = \frac{2}{9} \quad \Rightarrow \quad y'' = 0$.

40. If $x = 1$ in $x^2 + xy + y^3 = 1$, then we get $1 + y + y^3 = 1 \quad \Rightarrow \quad y^3 + y = 0 \quad \Rightarrow \quad y(y^2 + 1) \quad \Rightarrow \quad y = 0$, so the point where $x = 1$ is $(1, 0)$. Differentiating implicitly with respect to x gives us $2x + xy' + y \cdot 1 + 3y^2 \cdot y' = 0$. Substituting 1 for x and 0 for y gives us $2 + y' + 0 + 0 = 0 \quad \Rightarrow \quad y' = -2$. Differentiating $2x + xy' + y + 3y^2 y' = 0$ implicitly with respect to x gives us $2 + xy'' + y' \cdot 1 + y' + 3(y^2 y'' + y' \cdot 2yy') = 0$. Now substitute 1 for x, 0 for y, and -2 for y'.
$2 + y'' + (-2) + (-2) + 3(0 + 0) = 0 \quad \Rightarrow \quad y'' = 2$. Differentiating $2 + xy'' + 2y' + 3y^2 y'' + 6y(y')^2 = 0$ implicitly with respect to x gives us $xy''' + y'' \cdot 1 + 2y'' + 3(y^2 y''' + y'' \cdot 2yy') + 6[y \cdot 2y'y'' + (y')^2 y'] = 0$. Now substitute 1 for x, 0 for y, -2 for y', and 2 for y''. $\; y''' + 2 + 4 + 3(0 + 0) + 6[0 + (-8)] = 0 \quad \Rightarrow \quad y''' = -2 - 4 + 48 = 42$.

41. (a) There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

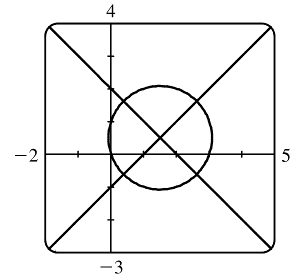


(b) $y' = \dfrac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \quad \Rightarrow \quad y' = -1$ at $(0, 1)$ and $y' = \frac{1}{3}$ at $(0, 2)$.

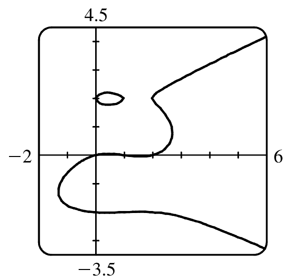
Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

(c) $y' = 0 \quad \Rightarrow \quad 3x^2 - 6x + 2 = 0 \quad \Rightarrow \quad x = 1 \pm \frac{1}{3}\sqrt{3}$

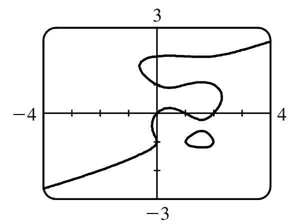
(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph. By modifying the equation in other ways, we can generate the other graphs.



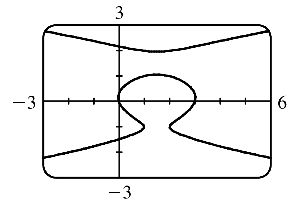
$y(y^2 - 1)(y - 2)$
$= x(x-1)(x-2)(x-3)$



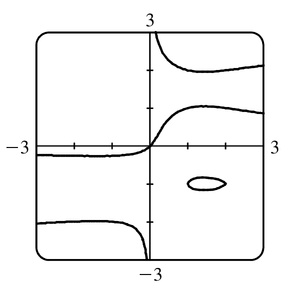
$y(y^2 - 4)(y - 2)$
$= x(x-1)(x-2)$



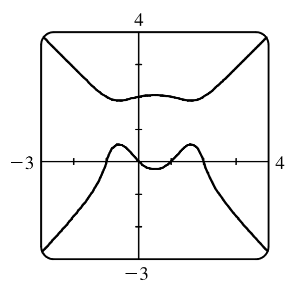
$y(y+1)(y^2 - 1)(y - 2)$
$= x(x-1)(x-2)$



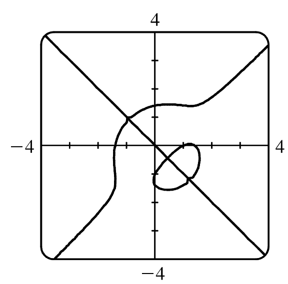
$(y+1)(y^2 - 1)(y - 2)$
$= (x-1)(x-2)$



$x(y+1)(y^2 - 1)(y - 2)$
$= y(x-1)(x-2)$

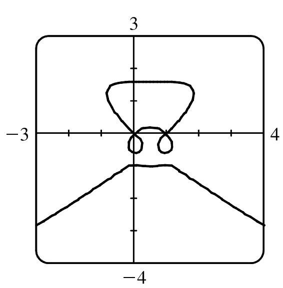


$y(y^2 + 1)(y - 2)$
$= x(x^2-1)(x-2)$



$y(y+1)(y^2 - 2)$
$= x(x-1)(x^2-2)$

42. (a)



(b) $\dfrac{d}{dx}(2y^3 + y^2 - y^5) = \dfrac{d}{dx}(x^4 - 2x^3 + x^2) \quad\Rightarrow$

$6y^2y' + 2y\,y' - 5y^4y' = 4x^3 - 6x^2 + 2x \quad\Rightarrow$

$y' = \dfrac{2x(2x^2 - 3x + 1)}{6y^2 + 2y - 5y^4} = \dfrac{2x(2x-1)(x-1)}{y(6y + 2 - 5y^3)}$. From the graph and the values for which $y' = 0$, we speculate that there are 9 points with horizontal tangents: 3 at $x = 0$, 3 at $x = \frac{1}{2}$, and 3 at $x = 1$. The three horizontal tangents along the top of the wagon are hard to find, but by limiting the y-range of the graph (to $[1.6, 1.7]$, for example) they are distinguishable.

43. From Exercise 31, a tangent to the lemniscate will be horizontal if $y' = 0 \Rightarrow 25x - 4x(x^2 + y^2) = 0 \Rightarrow$ $x[25 - 4(x^2 + y^2)] = 0 \Rightarrow x^2 + y^2 = \frac{25}{4}$ (**1**). (Note that when x is 0, y is also 0, and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for $x^2 + y^2$ in the equation of the lemniscate, $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, we get $x^2 - y^2 = \frac{25}{8}$ (**2**). Solving (**1**) and (**2**), we have $x^2 = \frac{75}{16}$ and $y^2 = \frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4}, \pm\frac{5}{4}\right)$.

44. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = -\dfrac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{-b^2x_0}{a^2y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = -\dfrac{x_0x}{a^2} + \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the ellipse, we have $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$.

45. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{b^2x_0}{a^2y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = \dfrac{x_0x}{a^2} - \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the hyperbola, we have $\dfrac{x_0x}{a^2} - \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$.

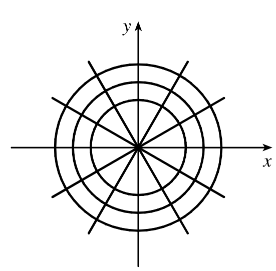
46. $\sqrt{x} + \sqrt{y} = \sqrt{c} \Rightarrow \dfrac{1}{2\sqrt{x}} + \dfrac{y'}{2\sqrt{y}} = 0 \Rightarrow y' = -\dfrac{\sqrt{y}}{\sqrt{x}} \Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$. Now $x = 0 \Rightarrow y = y_0 - \dfrac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0) = y_0 + \sqrt{x_0}\sqrt{y_0}$, so the y-intercept is $y_0 + \sqrt{x_0}\sqrt{y_0}$. And $y = 0 \Rightarrow -y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \Rightarrow x - x_0 = \dfrac{y_0\sqrt{x_0}}{\sqrt{y_0}} \Rightarrow$ $x = x_0 + \sqrt{x_0}\sqrt{y_0}$, so the x-intercept is $x_0 + \sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is

$$\left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c.$$

47. If the circle has radius r, its equation is $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\dfrac{x}{y}$, so the slope of the tangent line at $P(x_0, y_0)$ is $-\dfrac{x_0}{y_0}$. The negative reciprocal of that slope is $\dfrac{-1}{-x_0/y_0} = \dfrac{y_0}{x_0}$, which is the slope of OP, so the tangent line at P is perpendicular to the radius OP.

48. $y^q = x^p \Rightarrow qy^{q-1}y' = px^{p-1} \Rightarrow y' = \dfrac{px^{p-1}}{qy^{q-1}} = \dfrac{px^{p-1}y}{qy^q} = \dfrac{px^{p-1}x^{p/q}}{qx^p} = \dfrac{p}{q}x^{(p/q)-1}$

49. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O [assume a and b are not both zero]. $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -x/y$, so the slope of the tangent line at $P_0\,(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0, which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.



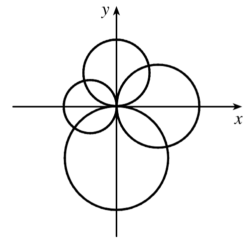
50. The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ intersect at the origin where the tangents are vertical and horizontal [assume a and b are both nonzero]. If (x_0, y_0) is the other point of intersection, then $x_0^2 + y_0^2 = ax_0$ (**1**) and $x_0^2 + y_0^2 = by_0$ (**2**).

Now $x^2 + y^2 = ax \quad\Rightarrow\quad 2x + 2yy' = a \quad\Rightarrow\quad y' = \dfrac{a - 2x}{2y}$ and $x^2 + y^2 = by \quad\Rightarrow$

$2x + 2yy' = by' \quad\Rightarrow\quad y' = \dfrac{2x}{b - 2y}$. Thus, the curves are orthogonal at $(x_0, y_0) \quad\Leftrightarrow$

$\dfrac{a - 2x_0}{2y_0} = -\dfrac{b - 2y_0}{2x_0} \quad\Leftrightarrow\quad 2ax_0 - 4x_0^2 = 4y_0^2 - 2by_0 \quad\Leftrightarrow\quad ax_0 + by_0 = 2(x_0^2 + y_0^2)$,

which is true by (**1**) and (**2**).

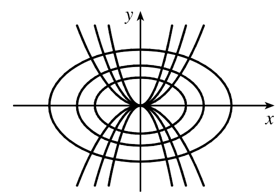


51. $y = cx^2 \quad\Rightarrow\quad y' = 2cx$ and $x^2 + 2y^2 = k$ [assume $k > 0$] $\quad\Rightarrow\quad 2x + 4yy' = 0 \quad\Rightarrow$

$2yy' = -x \quad\Rightarrow\quad y' = -\dfrac{x}{2(y)} = -\dfrac{x}{2(cx^2)} = -\dfrac{1}{2cx}$, so the curves are orthogonal if

$c \neq 0$. If $c = 0$, then the horizontal line $y = cx^2 = 0$ intersects $x^2 + 2y^2 = k$ orthogonally

at $\left(\pm\sqrt{k}, 0\right)$, since the ellipse $x^2 + 2y^2 = k$ has vertical tangents at those two points.

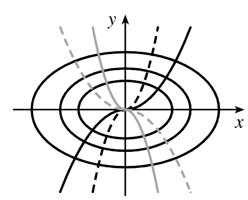


52. $y = ax^3 \quad\Rightarrow\quad y' = 3ax^2$ and $x^2 + 3y^2 = b$ [assume $b > 0$] $\quad\Rightarrow\quad 2x + 6yy' = 0 \quad\Rightarrow$

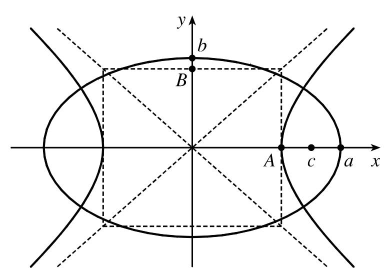
$3yy' = -x \quad\Rightarrow\quad y' = -\dfrac{x}{3(y)} = -\dfrac{x}{3(ax^3)} = -\dfrac{1}{3ax^2}$, so the curves are orthogonal if

$a \neq 0$. If $a = 0$, then the horizontal line $y = ax^3 = 0$ intesects $x^2 + 3y^2 = b$ orthogonally

at $\left(\pm\sqrt{b}, 0\right)$, since the ellipse $x^2 + 3y^2 = b$ has vertical tangents at those two points.



53. Since $A^2 < a^2$, we are assured that there are four points of intersection.



(1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad\Rightarrow\quad \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \quad\Rightarrow\quad \dfrac{yy'}{b^2} = -\dfrac{x}{a^2} \quad\Rightarrow$

$y' = m_1 = -\dfrac{xb^2}{ya^2}$.

(2) $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1 \quad\Rightarrow\quad \dfrac{2x}{A^2} - \dfrac{2yy'}{B^2} = 0 \quad\Rightarrow\quad \dfrac{yy'}{B^2} = \dfrac{x}{A^2} \quad\Rightarrow$

$y' = m_2 = \dfrac{xB^2}{yA^2}$.

Now $m_1 m_2 = -\dfrac{xb^2}{ya^2} \cdot \dfrac{xB^2}{yA^2} = -\dfrac{b^2B^2}{a^2A^2} \cdot \dfrac{x^2}{y^2}$ (**3**). Subtracting equations, (**1**) − (**2**), gives us $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 0 \quad\Rightarrow$

$\dfrac{y^2}{b^2} + \dfrac{y^2}{B^2} = \dfrac{x^2}{A^2} - \dfrac{x^2}{a^2} \quad\Rightarrow\quad \dfrac{y^2B^2 + y^2b^2}{b^2B^2} = \dfrac{x^2a^2 - x^2A^2}{A^2a^2} \quad\Rightarrow\quad \dfrac{y^2(b^2 + B^2)}{b^2B^2} = \dfrac{x^2(a^2 - A^2)}{a^2A^2}$ (**4**). Since

$a^2 - b^2 = A^2 + B^2$, we have $a^2 - A^2 = b^2 + B^2$. Thus, equation (**4**) becomes $\dfrac{y^2}{b^2B^2} = \dfrac{x^2}{A^2a^2} \quad\Rightarrow\quad \dfrac{x^2}{y^2} = \dfrac{A^2a^2}{b^2B^2}$, and

substituting for $\dfrac{x^2}{y^2}$ in equation (**3**) gives us $m_1 m_2 = -\dfrac{b^2B^2}{a^2A^2} \cdot \dfrac{a^2A^2}{b^2B^2} = -1$. Hence, the ellipse and hyperbola are orthogonal trajectories.

54. $y = (x+c)^{-1} \;\Rightarrow\; y' = -(x+c)^{-2}$ and $y = a(x+k)^{1/3} \;\Rightarrow\; y' = \frac{1}{3}a(x+k)^{-2/3}$, so the curves are othogonal if the product of the slopes is -1, that is, $\dfrac{-1}{(x+c)^2} \cdot \dfrac{a}{3(x+k)^{2/3}} = -1 \;\Rightarrow\; a = 3(x+c)^2(x+k)^{2/3} \;\Rightarrow$

$a = 3\left(\dfrac{1}{y}\right)^2\left(\dfrac{y}{a}\right)^2$ [since $y^2 = (x+c)^{-2}$ and $y^2 = a^2(x+k)^{2/3}$] $\;\Rightarrow\; a = 3\left(\dfrac{1}{a^2}\right) \;\Rightarrow\; a^3 = 3 \;\Rightarrow\; a = \sqrt[3]{3}$.

55. (a) $\left(P + \dfrac{n^2 a}{V^2}\right)(V - nb) = nRT \;\Rightarrow\; PV - Pnb + \dfrac{n^2 a}{V} - \dfrac{n^3 ab}{V^2} = nRT \;\Rightarrow$

$$\frac{d}{dP}(PV - Pnb + n^2 aV^{-1} - n^3 abV^{-2}) = \frac{d}{dP}(nRT) \;\Rightarrow$$

$$PV' + V \cdot 1 - nb - n^2 aV^{-2} \cdot V' + 2n^3 abV^{-3} \cdot V' = 0 \;\Rightarrow\; V'(P - n^2 aV^{-2} + 2n^3 abV^{-3}) = nb - V \;\Rightarrow$$

$$V' = \frac{nb - V}{P - n^2 aV^{-2} + 2n^3 abV^{-3}} \text{ or } \frac{dV}{dP} = \frac{V^3(nb - V)}{PV^3 - n^2 aV + 2n^3 ab}$$

(b) Using the last expression for dV/dP from part (a), we get

$$\frac{dV}{dP} = \frac{(10\text{ L})^3[(1\text{ mole})(0.04267\text{ L/mole}) - 10\text{ L}]}{\left[\begin{array}{l}(2.5\text{ atm})(10\text{ L})^3 - (1\text{ mole})^2(3.592\text{ L}^2\text{-atm/ mole}^2)(10\text{ L}) \\ \qquad + 2(1\text{ mole})^3(3.592\text{ L}^2\text{-atm/ mole}^2)(0.04267\text{ L/ mole})\end{array}\right]}$$

$$= \frac{-9957.33\text{ L}^4}{2464.386541\text{ L}^3\text{-atm}} \approx -4.04\text{ L/ atm}.$$

56. (a) $x^2 + xy + y^2 + 1 = 0 \;\Rightarrow\; 2x + xy' + y \cdot 1 + 2yy' + 0 = 0 \;\Rightarrow\; y'(x + 2y) = -2x - y \;\Rightarrow\; y' = \dfrac{-2x - y}{x + 2y}$

(b) Plotting the curve in part (a) gives us an empty graph, that is, there are no points that satisfy the equation. If there were any points that satisfied the equation, then x and y would have opposite signs; otherwise, all the terms are positive and their sum can not equal 0. $x^2 + xy + y^2 + 1 = 0 \;\Rightarrow\; x^2 + 2xy + y^2 - xy + 1 = 0 \;\Rightarrow\; (x+y)^2 = xy - 1$. The left side of the last equation is nonnegative, but the right side is at most -1, so that proves there are no points that satisfy the equation.

Another solution: $x^2 + xy + y^2 + 1 = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + 1 = \frac{1}{2}(x^2 + 2xy + y^2) + \frac{1}{2}(x^2 + y^2) + 1$

$$= \tfrac{1}{2}(x+y)^2 + \tfrac{1}{2}(x^2 + y^2) + 1 \geq 1$$

Another solution: Regarding $x^2 + xy + y^2 + 1 = 0$ as a quadratic in x, the discriminant is $y^2 - 4(y^2 + 1) = -3y^2 - 4$. This is negative, so there are no real solutions.

(c) The expression for y' in part (a) is meaningless; that is, since the equation in part (a) has no solution, it does not implicitly define a function y of x, and therefore it is meaningless to consider y'.

57. To find the points at which the ellipse $x^2 - xy + y^2 = 3$ crosses the x-axis, let $y = 0$ and solve for x.

$y = 0 \;\Rightarrow\; x^2 - x(0) + 0^2 = 3 \;\Leftrightarrow\; x = \pm\sqrt{3}$. So the graph of the ellipse crosses the x-axis at the points $(\pm\sqrt{3}, 0)$.

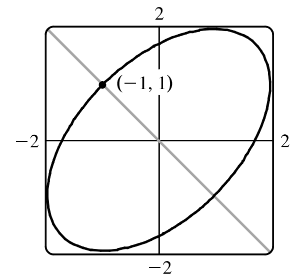
[continued]

Using implicit differentiation to find y', we get $2x - xy' - y + 2yy' = 0 \;\Rightarrow\; y'(2y - x) = y - 2x \;\Leftrightarrow\; y' = \dfrac{y - 2x}{2y - x}$.

So y' at $(\sqrt{3}, 0)$ is $\dfrac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$ and y' at $(-\sqrt{3}, 0)$ is $\dfrac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$. Thus, the tangent lines at these points are parallel.

58. (a) We use implicit differentiation to find $y' = \dfrac{y - 2x}{2y - x}$ as in Exercise 57. The slope of the tangent line at $(-1, 1)$ is $m = \dfrac{1 - 2(-1)}{2(1) - (-1)} = \dfrac{3}{3} = 1$, so the slope of the normal line is $-\dfrac{1}{m} = -1$, and its equation is $y - 1 = -1(x + 1) \;\Leftrightarrow\; y = -x$. Substituting $-x$ for y in the equation of the ellipse, we get $x^2 - x(-x) + (-x)^2 = 3 \;\Rightarrow\; 3x^2 = 3 \;\Leftrightarrow\; x = \pm 1$. So the normal line must intersect the ellipse again at $x = 1$, and since the equation of the line is $y = -x$, the other point of intersection must be $(1, -1)$.

(b)



59. $x^2y^2 + xy = 2 \;\Rightarrow\; x^2 \cdot 2yy' + y^2 \cdot 2x + x \cdot y' + y \cdot 1 = 0 \;\Leftrightarrow\; y'(2x^2y + x) = -2xy^2 - y \;\Leftrightarrow$

$y' = -\dfrac{2xy^2 + y}{2x^2y + x}$. So $-\dfrac{2xy^2 + y}{2x^2y + x} = -1 \;\Leftrightarrow\; 2xy^2 + y = 2x^2y + x \;\Leftrightarrow\; y(2xy + 1) = x(2xy + 1) \;\Leftrightarrow$

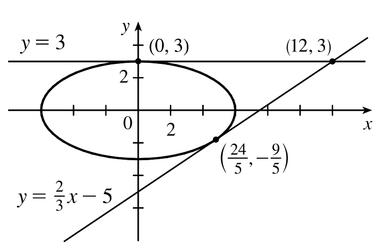
$y(2xy + 1) - x(2xy + 1) = 0 \;\Leftrightarrow\; (2xy + 1)(y - x) = 0 \;\Leftrightarrow\; xy = -\frac{1}{2}$ or $y = x$. But $xy = -\frac{1}{2} \;\Rightarrow$

$x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2$, so we must have $x = y$. Then $x^2y^2 + xy = 2 \;\Rightarrow\; x^4 + x^2 = 2 \;\Leftrightarrow\; x^4 + x^2 - 2 = 0 \;\Leftrightarrow$

$(x^2 + 2)(x^2 - 1) = 0$. So $x^2 = -2$, which is impossible, or $x^2 = 1 \;\Leftrightarrow\; x = \pm 1$. Since $x = y$, the points on the curve where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

60. $x^2 + 4y^2 = 36 \;\Rightarrow\; 2x + 8yy' = 0 \;\Rightarrow\; y' = -\dfrac{x}{4y}$. Let (a, b) be a point on $x^2 + 4y^2 = 36$ whose tangent line passes through $(12, 3)$. The tangent line is then $y - 3 = -\dfrac{a}{4b}(x - 12)$, so $b - 3 = -\dfrac{a}{4b}(a - 12)$. Multiplying both sides by $4b$ gives $4b^2 - 12b = -a^2 + 12a$, so $4b^2 + a^2 = 12(a + b)$. But $4b^2 + a^2 = 36$, so $36 = 12(a + b) \;\Rightarrow\; a + b = 3 \;\Rightarrow$ $b = 3 - a$. Substituting $3 - a$ for b into $a^2 + 4b^2 = 36$ gives $a^2 + 4(3 - a)^2 = 36 \;\Leftrightarrow\; a^2 + 36 - 24a + 4a^2 = 36 \;\Leftrightarrow$ $5a^2 - 24a = 0 \;\Leftrightarrow\; a(5a - 24) = 0$, so $a = 0$ or $a = \frac{24}{5}$. If $a = 0$, $b = 3 - 0 = 3$, and if $a = \frac{24}{5}$, $b = 3 - \frac{24}{5} = -\frac{9}{5}$.

So the two points on the ellipse are $(0, 3)$ and $\left(\frac{24}{5}, -\frac{9}{5}\right)$. Using $y - 3 = -\dfrac{a}{4b}(x - 12)$ with $(a, b) = (0, 3)$ gives us the tangent line $y - 3 = 0$ or $y = 3$. With $(a, b) = \left(\frac{24}{5}, -\frac{9}{5}\right)$, we have $y - 3 = -\frac{24/5}{4(-9/5)}(x - 12) \;\Leftrightarrow\; y - 3 = \frac{2}{3}(x - 12) \;\Leftrightarrow\; y = \frac{2}{3}x - 5$. A graph of the ellipse and the tangent lines confirms our results.



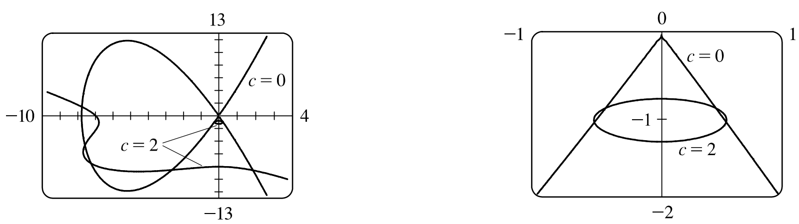
61. (a) $y = J(x)$ and $xy'' + y' + xy = 0 \quad \Rightarrow \quad xJ''(x) + J'(x) + xJ(x) = 0$. If $x = 0$, we have $0 + J'(0) + 0 = 0$, so $J'(0) = 0$.

(b) Differentiating $xy'' + y' + xy = 0$ implicitly, we get $xy''' + y'' \cdot 1 + y'' + xy' + y \cdot 1 = 0 \quad \Rightarrow$ $xy''' + 2y'' + xy' + y = 0$, so $xJ'''(x) + 2J''(x) + xJ'(x) + J(x) = 0$. If $x = 0$, we have $0 + 2J''(0) + 0 + 1 \quad [J(0) = 1 \text{ is given}] \quad = 0 \quad \Rightarrow \quad 2J''(0) = -1 \quad \Rightarrow \quad J''(0) = -\frac{1}{2}$.

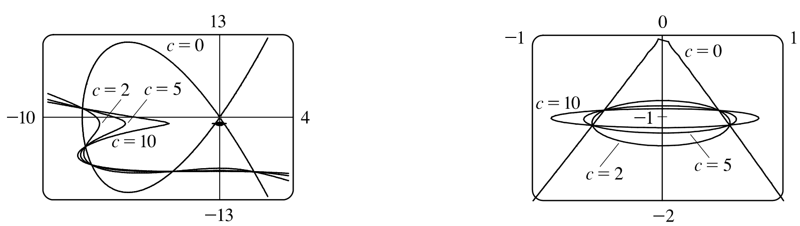
62. $x^2 + 4y^2 = 5 \quad \Rightarrow \quad 2x + 4(2yy') = 0 \quad \Rightarrow \quad y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$, or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a + 5} \quad \Leftrightarrow \quad 4b^2 = -a^2 - 5a \quad \Leftrightarrow \quad a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$ [since (a, b) is on the ellipse], so $5 = -5a \quad \Leftrightarrow \quad a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \quad \Rightarrow \quad b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a + 5} = \dfrac{1}{-1 + 5} = \dfrac{1}{4} \quad \Rightarrow \quad h = 2$. So the lamp is located 2 units above the x-axis.

LABORATORY PROJECT Families of Implicit Curves

1. (a) There appear to be nine points of intersection. The "inner four" near the origin are about $(\pm 0.2, -0.9)$ and $(\pm 0.3, -1.1)$. The "outer five" are about $(2.0, -8.9)$, $(-2.8, -8.8)$, $(-7.5, -7.7)$, $(-7.8, -4.7)$, and $(-8.0, 1.5)$.



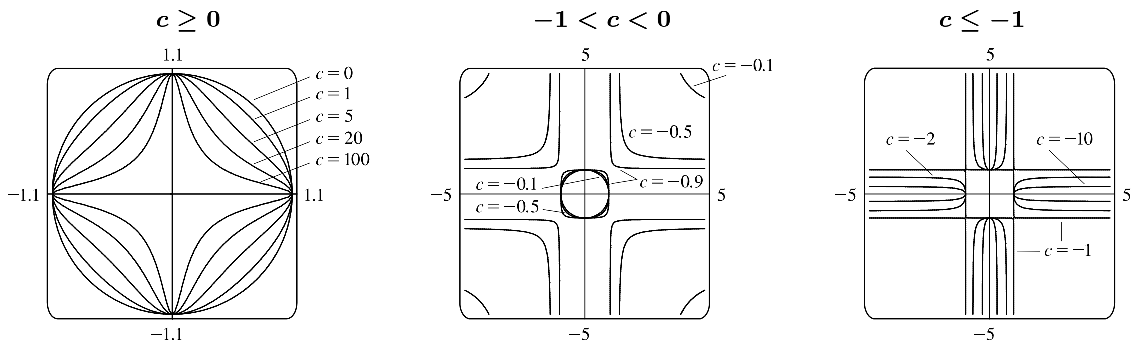
(b) We see from the graphs with $c = 5$ and $c = 10$, and for other values of c, that the curves change shape but the nine points of intersection are the same.



2. (a) If $c = 0$, the graph is the unit circle. As c increases, the graph looks more diamondlike and then more crosslike (see the graph for $c \geq 0$).

For $-1 < c < 0$ (see the graph), there are four hyperboliclike branches as well as an ellipticlike curve bounded by $|x| \le 1$ and $|y| \le 1$ for values of c close to 0. As c gets closer to -1, the branches and the curve become more rectangular, approaching the lines $|x| = 1$ and $|y| = 1$.

For $c = -1$, we get the lines $x = \pm 1$ and $y = \pm 1$. As c decreases, we get four test-tubelike curves (see the graph) that are bounded by $|x| = 1$ and $|y| = 1$, and get thinner as $|c|$ gets larger.



(b) The curve for $c = -1$ is described in part (a). When $c = -1$, we get $x^2 + y^2 - x^2y^2 = 1 \quad \Leftrightarrow$ $0 = x^2y^2 - x^2 - y^2 + 1 \quad \Leftrightarrow \quad 0 = (x^2 - 1)(y^2 - 1) \quad \Leftrightarrow \quad x = \pm 1$ or $y = \pm 1$, which algebraically proves that the graph consists of the stated lines.

(c) $\dfrac{d}{dx}(x^2 + y^2 + cx^2y^2) = \dfrac{d}{dx}(1) \quad \Rightarrow \quad 2x + 2y\,y' + c(x^2 \cdot 2y\,y' + y^2 \cdot 2x) = 0 \quad \Rightarrow$

$2y\,y' + 2cx^2y\,y' = -2x - 2cxy^2 \quad \Rightarrow \quad 2y(1 + cx^2)y' = -2x(1 + cy^2) \quad \Rightarrow \quad y' = -\dfrac{x(1 + cy^2)}{y(1 + cx^2)}.$

For $c = -1$, $y' = -\dfrac{x(1 - y^2)}{y(1 - x^2)} = -\dfrac{x(1 + y)(1 - y)}{y(1 + x)(1 - x)}$, so $y' = 0$ when $y = \pm 1$ or $x = 0$ (which leads to $y = \pm 1$) and y' is undefined when $x = \pm 1$ or $y = 0$ (which leads to $x = \pm 1$). Since the graph consists of the lines $x = \pm 1$ and $y = \pm 1$, the slope at any point on the graph is undefined or 0, which is consistent with the expression found for y'.

2.7 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^3 - 9t^2 + 24t$ (in feet) $\quad \Rightarrow \quad v(t) = f'(t) = 3t^2 - 18t + 24$ (in ft/s)

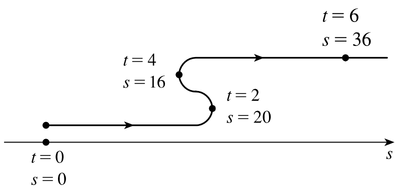
(b) $v(1) = 3(1)^2 - 18(1) + 24 = 9$ ft/s

(c) The particle is at rest when $v(t) = 0$. $3t^2 - 18t + 24 = 0 \quad \Leftrightarrow \quad 3(t^2 - 6t + 8) = 0 \quad \Leftrightarrow \quad 3(t - 2)(t - 4) = 0 \quad \Rightarrow$ $t = 2$ s or $t = 4$ s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $3(t - 2)(t - 4) > 0 \quad \Leftrightarrow \quad 0 \le t < 2$ or $t > 4$.

(e) v changes sign at $t = 2$ and 4 in the interval $[0, 6]$. The total distance traveled during the first 6 seconds is $|f(2) - f(0)| + |f(4) - f(2)| + |f(6) - f(4)| = |20 - 0| + |16 - 20| + |36 - 16| = 20 + 4 + 20 = 44$ ft.

(f)

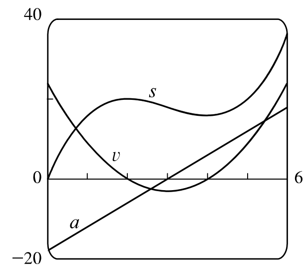


(g) $v(t) = 3t^2 - 18t + 24 \quad \Rightarrow$

$a(t) = v'(t) = 6t - 18$ [in (ft/s)/s or ft/s^2].

$a(1) = 6 - 18 = -12$ ft/s^2.

(h)



(i) $a(t) > 0 \quad \Leftrightarrow \quad 6t - 18 > 0 \quad \Leftrightarrow \quad t > 3$. The particle is speeding up when v and a have the same sign. From the figure in part (h), we see that v and a are both positive when $4 < t \leq 6$ and both negative when $2 < t < 3$. Thus, the particle is speeding up when $2 < t < 3$ and $4 < t \leq 6$. The particle is slowing down when v and a have opposite signs; that is, when $0 \leq t < 2$ and $3 < t < 4$.

2. (a) $s = f(t) = 0.01t^4 - 0.04t^3$ (in feet) $\quad \Rightarrow \quad v(t) = f'(t) = 0.04t^3 - 0.12t^2$ (in ft/s)

(b) $v(3) = 0.04(3)^3 - 0.12(3)^2 = 0$ ft/s

(c) The particle is at rest when $v(t) = 0$. $\quad 0.04t^3 - 0.12t^2 = 0 \quad \Leftrightarrow \quad 0.04t^2(t - 3) = 0 \quad \Leftrightarrow \quad t = 0$ s or 3 s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $\quad 0.04t^2(t - 3) > 0 \quad \Leftrightarrow \quad t > 3$.

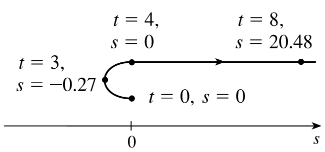
(e) See Exercise 1(e).

$|f(3) - f(0)| = |-0.27 - 0| = 0.27$.

$|f(8) - f(3)| = |20.48 - (-0.27)| = 20.75$.

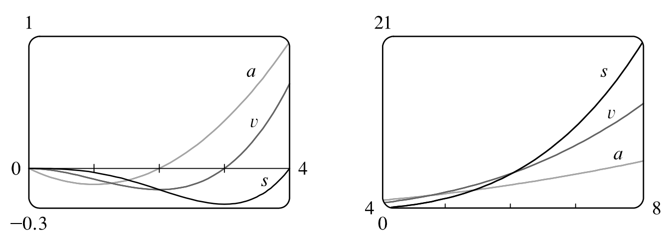
The total distance is $0.27 + 20.75 = 21.02$ ft.

(f)



(g) $v(t) = 0.04t^3 - 0.12t^2 \quad \Rightarrow \quad a(t) = v'(t) = 0.12t^2 - 0.24t$. $\quad a(3) = 0.12(3)^2 - 0.24(3) = 0.36$ (ft/s)/s or ft/s^2.

(h) Here we show the graph of s, v, and a for $0 \leq t \leq 4$ and $4 \leq t \leq 8$.



(i) The particle is speeding up when v and a have the same sign. This occurs when $0 < t < 2$ [v and a are both negative] and when $t > 3$ [v and a are both positive]. It is slowing down when v and a have opposite signs; that is, when $2 < t < 3$.

3. (a) $s = f(t) = \sin(\pi t/2)$ (in feet) $\Rightarrow$ $v(t) = f'(t) = \cos(\pi t/2) \cdot (\pi/2) = \frac{\pi}{2}\cos(\pi t/2)$ (in ft/s)

(b) $v(1) = \frac{\pi}{2}\cos\frac{\pi}{2} = \frac{\pi}{2}(0) = 0$ ft/s

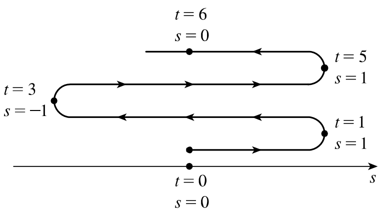
(c) The particle is at rest when $v(t) = 0$. $\frac{\pi}{2}\cos\frac{\pi}{2}t = 0 \;\Leftrightarrow\; \cos\frac{\pi}{2}t = 0 \;\Leftrightarrow\; \frac{\pi}{2}t = \frac{\pi}{2} + n\pi \;\Leftrightarrow\; t = 1 + 2n$, where n is a nonnegative integer since $t \geq 0$.

(d) The particle is moving in the positive direction when $v(t) > 0$. From part (c), we see that v changes sign at every positive odd integer. v is positive when $0 < t < 1$, $3 < t < 5$, $7 < t < 9$, and so on.

(e) v changes sign at $t = 1$, 3, and 5 in the interval $[0, 6]$. The total distance traveled during the first 6 seconds is

$$\begin{aligned}|f(1) - f(0)| + |f(3) - f(1)| + |f(5) - f(3)| + |f(6) - f(5)| &= |1 - 0| + |-1 - 1| + |1 - (-1)| + |0 - 1| \\ &= 1 + 2 + 2 + 1 = 6 \text{ ft}\end{aligned}$$

(f)

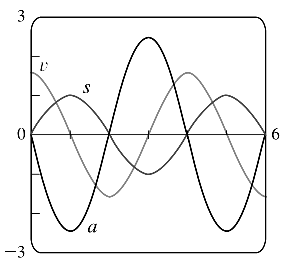


(g) $v(t) = \frac{\pi}{2}\cos(\pi t/2) \quad\Rightarrow$

$$\begin{aligned}a(t) = v'(t) &= \tfrac{\pi}{2}\left[-\sin(\pi t/2) \cdot (\pi/2)\right] \\ &= (-\pi^2/4)\sin(\pi t/2) \text{ ft/s}^2\end{aligned}$$

$$a(1) = (-\pi^2/4)\sin(\pi/2) = -\pi^2/4 \text{ ft/s}^2$$

(h)



(i) The particle is speeding up when v and a have the same sign. From the figure in part (h), we see that v and a are both positive when $3 < t < 4$ and both negative when $1 < t < 2$ and $5 < t < 6$. Thus, the particle is speeding up when $1 < t < 2$, $3 < t < 4$, and $5 < t < 6$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 1$, $2 < t < 3$, and $4 < t < 5$.

4. (a) $s = f(t) = \dfrac{9t}{t^2 + 9}$ (in feet) $\Rightarrow$ $v(t) = f'(t) = \dfrac{(t^2+9)(9) - 9t(2t)}{(t^2+9)^2} = \dfrac{-9t^2 + 81}{(t^2+9)^2} = \dfrac{-9(t^2 - 9)}{(t^2+9)^2}$ (in ft/s)

(b) $v(1) = \dfrac{-9(1-9)}{(1+9)^2} = \dfrac{72}{100} = 0.72$ ft/s

(c) The particle is at rest when $v(t) = 0$. $\dfrac{-9(t^2-9)}{(t^2+9)^2} = 0 \;\Leftrightarrow\; t^2 - 9 = 0 \;\Rightarrow\; t = 3$ s [since $t \geq 0$].

(d) The particle is moving in the positive direction when $v(t) > 0$.

$$\frac{-9(t^2-9)}{(t^2+9)^2} > 0 \;\Rightarrow\; -9(t^2 - 9) > 0 \;\Rightarrow\; t^2 - 9 < 0 \;\Rightarrow\; t^2 < 9 \;\Rightarrow\; 0 \leq t < 3.$$

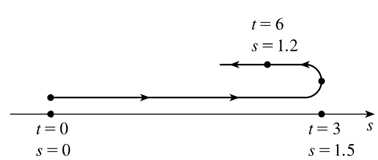
(e) Since the particle is moving in the positve direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 3]$ and$[3, 6]$, respectively.

$|f(3) - f(0)| = \left|\frac{27}{18} - 0\right| = \frac{3}{2}$

$|f(6) - f(3)| = \left|\frac{54}{45} - \frac{27}{18}\right| = \frac{3}{10}$

The total distance is $\frac{3}{2} + \frac{3}{10} = \frac{9}{5}$ or 1.8 ft.

(f)

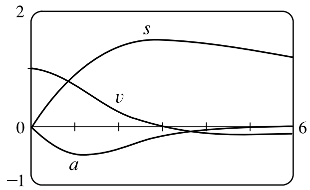


(g) $v(t) = -9\,\dfrac{t^2-9}{(t^2+9)^2} \quad \Rightarrow$

$$a(t) = v'(t) = -9\,\frac{(t^2+9)^2(2t)-(t^2-9)2(t^2+9)(2t)}{[(t^2+9)^2]^2} = -9\,\frac{2t(t^2+9)[(t^2+9)-2(t^2-9)]}{(t^2+9)^4} = \frac{18t(t^2-27)}{(t^2+9)^3}.$$

$$a(1) = \frac{18(-26)}{10^3} = -0.468 \text{ ft/s}^2$$

(h)



(i) The particle is speeding up when v and a have the same sign. a is negative for $0 < t < \sqrt{27}$ $[\approx 5.2]$, so from the figure in part (h), we see that v and a are both negative for $3 < t < 3\sqrt{3}$. The particle is slowing down when v and a have opposite signs. This occurs when $0 < t < 3$ and when $t > 3\sqrt{3}$.

5. (a) From the figure, the velocity v is positive on the interval $(0, 2)$ and negative on the interval $(2, 3)$. The acceleration a is positive (negative) when the slope of the tangent line is positive (negative), so the acceleration is positive on the interval $(0, 1)$, and negative on the interval $(1, 3)$. The particle is speeding up when v and a have the same sign, that is, on the interval $(0, 1)$ when $v > 0$ and $a > 0$, and on the interval $(2, 3)$ when $v < 0$ and $a < 0$. The particle is slowing down when v and a have opposite signs, that is, on the interval $(1, 2)$ when $v > 0$ and $a < 0$.

(b) $v > 0$ on $(0, 3)$ and $v < 0$ on $(3, 4)$. $a > 0$ on $(1, 2)$ and $a < 0$ on $(0, 1)$ and $(2, 4)$. The particle is speeding up on $(1, 2)$ $[v > 0, a > 0]$ and on $(3, 4)$ $[v < 0, a < 0]$. The particle is slowing down on $(0, 1)$ and $(2, 3)$ $[v > 0, a < 0]$.

6. (a) The velocity v is positive when s is increasing, that is, on the intervals $(0, 1)$ and $(3, 4)$; and it is negative when s is decreasing, that is, on the interval $(1, 3)$. The acceleration a is positive when the graph of s is concave upward (v is increasing), that is, on the interval $(2, 4)$; and it is negative when the graph of s is concave downward (v is decreasing), that is, on the interval $(0, 2)$. The particle is speeding up on the interval $(1, 2)$ $[v < 0, a < 0]$ and on $(3, 4)$ $[v > 0, a > 0]$. The particle is slowing down on the interval $(0, 1)$ $[v > 0, a < 0]$ and on $(2, 3)$ $[v < 0, a > 0]$.

(b) The velocity v is positive on $(3, 4)$ and negative on $(0, 3)$. The acceleration a is positive on $(0, 1)$ and $(2, 4)$ and negative on $(1, 2)$. The particle is speeding up on the interval $(1, 2)$ $[v < 0, a < 0]$ and on $(3, 4)$ $[v > 0, a > 0]$. The particle is slowing down on the interval $(0, 1)$ $[v < 0, a > 0]$ and on $(2, 3)$ $[v < 0, a > 0]$.

7. (a) $h(t) = 2 + 24.5t - 4.9t^2 \quad \Rightarrow \quad v(t) = h'(t) = 24.5 - 9.8t$. The velocity after 2 s is $v(2) = 24.5 - 9.8(2) = 4.9$ m/s and after 4 s is $v(4) = 24.5 - 9.8(4) = -14.7$ m/s.

(b) The projectile reaches its maximum height when the velocity is zero. $v(t) = 0 \Leftrightarrow 24.5 - 9.8t = 0 \Leftrightarrow$

$t = \dfrac{24.5}{9.8} = 2.5$ s.

(c) The maximum height occurs when $t = 2.5$. $h(2.5) = 2 + 24.5(2.5) - 4.9(2.5)^2 = 32.625$ m $\left[\text{or } 32\frac{5}{8}\text{ m}\right]$.

(d) The projectile hits the ground when $h = 0 \Leftrightarrow 2 + 24.5t - 4.9t^2 = 0 \Leftrightarrow$

$t = \dfrac{-24.5 \pm \sqrt{24.5^2 - 4(-4.9)(2)}}{2(-4.9)} \Rightarrow t = t_f \approx 5.08$ s [since $t \geq 0$].

(e) The projectile hits the ground when $t = t_f$. Its velocity is $v(t_f) = 24.5 - 9.8t_f \approx -25.3$ m/s [downward].

8. (a) At maximum height the velocity of the ball is 0 ft/s. $v(t) = s'(t) = 80 - 32t = 0 \Leftrightarrow 32t = 80 \Leftrightarrow t = \frac{5}{2}$.

So the maximum height is $s\left(\frac{5}{2}\right) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = 200 - 100 = 100$ ft.

(b) $s(t) = 80t - 16t^2 = 96 \Leftrightarrow 16t^2 - 80t + 96 = 0 \Leftrightarrow 16(t^2 - 5t + 6) = 0 \Leftrightarrow 16(t-3)(t-2) = 0$.

So the ball has a height of 96 ft on the way up at $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 80 - 32(2) = 16$ ft/s and $v(3) = 80 - 32(3) = -16$ ft/s, respectively.

9. (a) $h(t) = 15t - 1.86t^2 \Rightarrow v(t) = h'(t) = 15 - 3.72t$. The velocity after 2 s is $v(2) = 15 - 3.72(2) = 7.56$ m/s.

(b) $25 = h \Leftrightarrow 1.86t^2 - 15t + 25 = 0 \Leftrightarrow t = \dfrac{15 \pm \sqrt{15^2 - 4(1.86)(25)}}{2(1.86)} \Leftrightarrow t = t_1 \approx 2.35$ or $t = t_2 \approx 5.71$.

The velocities are $v(t_1) = 15 - 3.72t_1 \approx 6.24$ m/s [upward] and $v(t_2) = 15 - 3.72t_2 \approx -6.24$ m/s [downward].

10. (a) $s(t) = t^4 - 4t^3 - 20t^2 + 20t \Rightarrow v(t) = s'(t) = 4t^3 - 12t^2 - 40t + 20$. $v = 20 \Leftrightarrow$

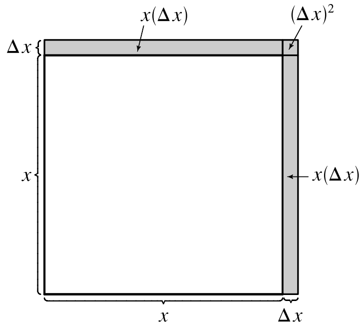
$4t^3 - 12t^2 - 40t + 20 = 20 \Leftrightarrow 4t^3 - 12t^2 - 40t = 0 \Leftrightarrow 4t(t^2 - 3t - 10) = 0 \Leftrightarrow$

$4t(t-5)(t+2) = 0 \Leftrightarrow t = 0$ s or 5 s [for $t \geq 0$].

(b) $a(t) = v'(t) = 12t^2 - 24t - 40$. $a = 0 \Leftrightarrow 12t^2 - 24t - 40 = 0 \Leftrightarrow 4(3t^2 - 6t - 10) = 0 \Leftrightarrow$

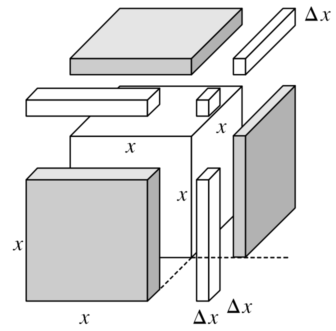
$t = \dfrac{6 \pm \sqrt{6^2 - 4(3)(-10)}}{2(3)} = 1 \pm \dfrac{1}{3}\sqrt{39} \approx 3.08$ s [for $t \geq 0$]. At this time, the acceleration changes from negative to positive and the velocity attains its minimum value.

11. (a) $A(x) = x^2 \Rightarrow A'(x) = 2x$. $A'(15) = 30$ mm^2/mm is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.



(b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx. From the figure, $\Delta A = 2x\,(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x\,(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.

12. (a) $V(x) = x^3 \Rightarrow \dfrac{dV}{dx} = 3x^2$. $\left.\dfrac{dV}{dx}\right|_{x=3} = 3(3)^2 = 27\ \text{mm}^3/\text{mm}$ is the rate at which the volume is increasing as x increases past 3 mm.



(b) The surface area is $S(x) = 6x^2$, so $V'(x) = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx. From the figure, $\Delta V = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$. If Δx is small, then $\Delta V \approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x \approx 3x^2$.

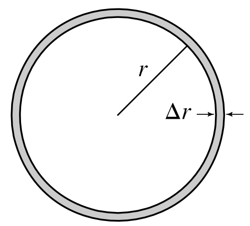
13. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

(i) $\dfrac{A(3) - A(2)}{3 - 2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$ (ii) $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

(iii) $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) $A(r) = \pi r^2 \Rightarrow A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr. Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr, so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically, $\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$. So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.



14. After t seconds the radius is $r = 60t$, so the area is $A(t) = \pi(60t)^2 = 3600\pi t^2 \Rightarrow A'(t) = 7200\pi t \Rightarrow$

(a) $A'(1) = 7200\pi\ \text{cm}^2/\text{s}$ (b) $A'(3) = 21{,}600\pi\ \text{cm}^2/\text{s}$ (c) $A'(5) = 36{,}000\pi\ \text{cm}^2/\text{s}$

As time goes by, the area grows at an increasing rate. In fact, the rate of change is linear with respect to time.

15. $S(r) = 4\pi r^2 \Rightarrow S'(r) = 8\pi r \Rightarrow$

(a) $S'(1) = 8\pi\ \text{ft}^2/\text{ft}$ (b) $S'(2) = 16\pi\ \text{ft}^2/\text{ft}$ (c) $S'(3) = 24\pi\ \text{ft}^2/\text{ft}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

16. (a) Using $V(r) = \frac{4}{3}\pi r^3$, we find that the average rate of change is:

(i) $\dfrac{V(8) - V(5)}{8 - 5} = \dfrac{\frac{4}{3}\pi(512) - \frac{4}{3}\pi(125)}{3} = 172\pi\ \mu\text{m}^3/\mu\text{m}$

(ii) $\dfrac{V(6) - V(5)}{6 - 5} = \dfrac{\frac{4}{3}\pi(216) - \frac{4}{3}\pi(125)}{1} = 121.\overline{3}\pi\ \mu\text{m}^3/\mu\text{m}$

(iii) $\dfrac{V(5.1) - V(5)}{5.1 - 5} = \dfrac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = 102.01\overline{3}\pi\ \mu\text{m}^3/\mu\text{m}$

(b) $V'(r) = 4\pi r^2$, so $V'(5) = 100\pi\ \mu\text{m}^3/\mu\text{m}$.

(c) $V(r) = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V'(r) = 4\pi r^2 = S(r)$. By analogy with Exercise 13(c), we can say that the change in the volume of the spherical shell, ΔV, is approximately equal to its thickness, Δr, times the surface area of the inner sphere. Thus, $\Delta V \approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r \approx 4\pi r^2$.

17. The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6$ kg/m (b) $\rho(2) = 12$ kg/m (c) $\rho(3) = 18$ kg/m

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

18. $V(t) = 5000\left(1 - \frac{1}{40}t\right)^2 \quad \Rightarrow \quad V'(t) = 5000 \cdot 2\left(1 - \frac{1}{40}t\right)\left(-\frac{1}{40}\right) = -250\left(1 - \frac{1}{40}t\right)$

(a) $V'(5) = -250\left(1 - \frac{5}{40}\right) = -218.75$ gal/min (b) $V'(10) = -250\left(1 - \frac{10}{40}\right) = -187.5$ gal/min

(c) $V'(20) = -250\left(1 - \frac{20}{40}\right) = -125$ gal/min (d) $V'(40) = -250\left(1 - \frac{40}{40}\right) = 0$ gal/min

The water is flowing out the fastest at the beginning—when $t = 0$, $V'(t) = -250$ gal/min. The water is flowing out the slowest at the end—when $t = 40$, $V'(t) = 0$. As the tank empties, the water flows out more slowly.

19. The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75$ A (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5$ A

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3}$ s.

20. (a) $F = \dfrac{GmM}{r^2} = (GmM)r^{-2} \quad \Rightarrow \quad \dfrac{dF}{dr} = -2(GmM)r^{-3} = -\dfrac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Given $F'(20{,}000) = -2$, find $F'(10{,}000)$. $-2 = -\dfrac{2GmM}{20{,}000^3} \quad \Rightarrow \quad GmM = 20{,}000^3$.

$$F'(10,000) = -\frac{2(20{,}000^3)}{10{,}000^3} = -2 \cdot 2^3 = -16 \text{ N/km}$$

21. With $m = m_0\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$,

$$F = \frac{d}{dt}(mv) = m\frac{d}{dt}(v) + v\frac{d}{dt}(m) = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \cdot a + v \cdot m_0\left[-\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\right]\left(-\frac{2v}{c^2}\right)\frac{d}{dt}(v)$$

$$= m_0\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \cdot a\left[\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right] = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}$$

Note that we factored out $(1 - v^2/c^2)^{-3/2}$ since $-3/2$ was the lesser exponent. Also note that $\dfrac{d}{dt}(v) = a$.

22. (a) $D(t) = 7 + 5\cos[0.503(t - 6.75)] \quad \Rightarrow \quad D'(t) = -5\sin[0.503(t - 6.75)](0.503) = -2.515\sin[0.503(t - 6.75)]$.
At 3:00 AM, $t = 3$, and $D'(3) = -2.515\sin[0.503(-3.75)] \approx 2.39$ m/h (rising).

(b) At 6:00 AM, $t = 6$, and $D'(6) = -2.515\sin[0.503(-0.75)] \approx 0.93$ m/h (rising).

(c) At 9:00 AM, $t = 9$, and $D'(9) = -2.515\sin[0.503(2.25)] \approx -2.28$ m/h (falling).

(d) At noon, $t = 12$, and $D'(12) = -2.515\sin[0.503(5.25)] \approx -1.21$ m/h (falling).

23. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P.

$$PV = C \quad\Rightarrow\quad V = \frac{C}{P} \quad\Rightarrow\quad \frac{dV}{dP} = -\frac{C}{P^2}.$$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases.

Thus, the volume is decreasing more rapidly at the beginning.

(c) $\beta = -\dfrac{1}{V}\dfrac{dV}{dP} = -\dfrac{1}{V}\left(-\dfrac{C}{P^2}\right) = \dfrac{C}{(PV)P} = \dfrac{C}{CP} = \dfrac{1}{P}$

24. (a) $[\text{C}] = \dfrac{a^2kt}{akt+1} \quad\Rightarrow\quad$ rate of reaction $= \dfrac{d[\text{C}]}{dt} = \dfrac{(akt+1)(a^2k) - (a^2kt)(ak)}{(akt+1)^2} = \dfrac{a^2k(akt+1-akt)}{(akt+1)^2} = \dfrac{a^2k}{(akt+1)^2}$

(b) If $x = [\text{C}]$, then $a - x = a - \dfrac{a^2kt}{akt+1} = \dfrac{a^2kt + a - a^2kt}{akt+1} = \dfrac{a}{akt+1}$.

So $k(a-x)^2 = k\left(\dfrac{a}{akt+1}\right)^2 = \dfrac{a^2k}{(akt+1)^2} = \dfrac{d[\text{C}]}{dt}$ [from part (a)] $= \dfrac{dx}{dt}$.

25. (a) **1920:** $m_1 = \dfrac{1860 - 1750}{1920 - 1910} = \dfrac{110}{10} = 11$, $m_2 = \dfrac{2070 - 1860}{1930 - 1920} = \dfrac{210}{10} = 21$,

$(m_1 + m_2)/2 = (11 + 21)/2 = 16$ million/year

1980: $m_1 = \dfrac{4450 - 3710}{1980 - 1970} = \dfrac{740}{10} = 74$, $m_2 = \dfrac{5280 - 4450}{1990 - 1980} = \dfrac{830}{10} = 83$,

$(m_1 + m_2)/2 = (74 + 83)/2 = 78.5$ million/year

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx -0.000\,284\,900\,3$, $b \approx 0.522\,433\,122\,43$, $c \approx -6.395\,641\,396$, and $d \approx 1720.586\,081$.

(c) $P(t) = at^3 + bt^2 + ct + d \quad\Rightarrow\quad P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d) 1920 corresponds to $t = 20$ and $P'(20) \approx 14.16$ million/year. 1980 corresponds to $t = 80$ and $P'(80) \approx 71.72$ million/year. These estimates are smaller than the estimates in part (a).

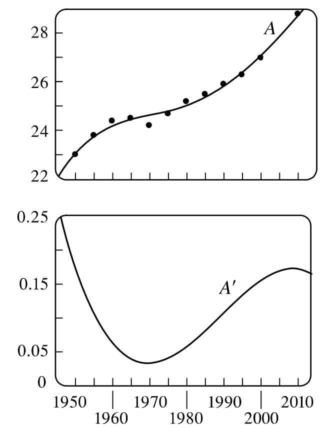
(e) $P'(85) \approx 76.24$ million/year.

26. (a) $A(t) = at^4 + bt^3 + ct^2 + dt + e$, where $a \approx -1.199\,781 \times 10^{-6}$, $b \approx 9.545\,853 \times 10^3$, $c \approx -28.478\,550$, $d \approx 37{,}757.105\,467$, and $e \approx -1.877\,031 \times 10^7$.

(b) $A(t) = at^4 + bt^3 + ct^2 + dt + e \quad\Rightarrow\quad A'(t) = 4at^3 + 3bt^2 + 2ct + d$.

(c) Part (b) gives $A'(1990) \approx 0.106$ years of age per year.

(d)



27. (a) Using $v = \dfrac{P}{4\eta l}(R^2 - r^2)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r:

$v(r) = \dfrac{3000}{4(0.027)3}(0.01^2 - r^2)$. $v(0) = 0.\overline{925}$ cm/s, $v(0.005) = 0.69\overline{4}$ cm/s, $v(0.01) = 0$.

(b) $v(r) = \dfrac{P}{4\eta l}(R^2 - r^2) \quad \Rightarrow \quad v'(r) = \dfrac{P}{4\eta l}(-2r) = -\dfrac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$v'(r) = -\dfrac{3000r}{2(0.027)3}$. $v'(0) = 0$, $v'(0.005) = -92.\overline{592}$ (cm/s)/cm, and $v'(0.01) = -185.\overline{185}$ (cm/s)/cm.

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

28. (a) (i) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-1} \quad \Rightarrow \quad \dfrac{df}{dL} = -\left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-2} = -\dfrac{1}{2L^2}\sqrt{\dfrac{T}{\rho}}$

(ii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{1/2} \quad \Rightarrow \quad \dfrac{df}{dT} = \dfrac{1}{2}\left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{-1/2} = \dfrac{1}{4L\sqrt{T\rho}}$

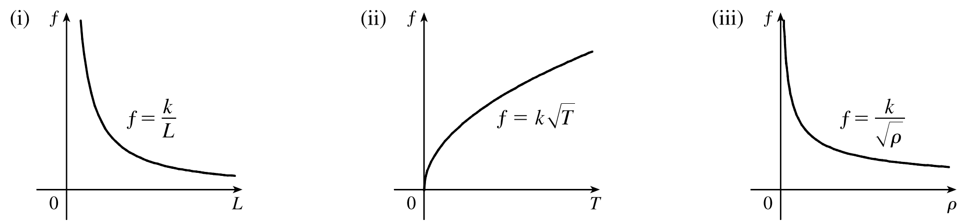
(iii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-1/2} \quad \Rightarrow \quad \dfrac{df}{d\rho} = -\dfrac{1}{2}\left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-3/2} = -\dfrac{\sqrt{T}}{4L\rho^{3/2}}$

(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\dfrac{df}{dL} < 0$ and L is decreasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(ii) $\dfrac{df}{dT} > 0$ and T is increasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(iii) $\dfrac{df}{d\rho} < 0$ and ρ is increasing $\Rightarrow$ f is decreasing $\Rightarrow$ lower note



29. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \quad \Rightarrow \quad C'(x) = 0 + 3(1) + 0.01(2x) + 0.0002(3x^2) = 3 + 0.02x + 0.0006x^2$

(b) $C'(100) = 3 + 0.02(100) + 0.0006(100)^2 = 3 + 2 + 6 = \$11$/pair. $C'(100)$ is the rate at which the cost is increasing as the 100th pair of jeans is produced. It predicts the (approximate) cost of the 101st pair.

(c) The cost of manufacturing the 101st pair of jeans is

$C(101) - C(100) = 2611.0702 - 2600 = 11.0702 \approx \$11.07$. This is close to the marginal cost from part (b).

30. (a) $C(q) = 84 + 0.16q - 0.0006q^2 + 0.000003q^3 \quad \Rightarrow \quad C'(q) = 0.16 - 0.0012q + 0.000009q^2$, and

$C'(100) = 0.16 - 0.0012(100) + 0.000009(100)^2 = 0.13$. This is the rate at which the cost is increasing as the 100th item is produced.

(b) The actual cost of producing the 101st item is $C(101) - C(100) = 97.13030299 - 97 \approx \$0.13$

31. (a) $A(x) = \dfrac{p(x)}{x} \quad \Rightarrow \quad A'(x) = \dfrac{xp'(x) - p(x) \cdot 1}{x^2} = \dfrac{xp'(x) - p(x)}{x^2}$.

$A'(x) > 0 \quad \Rightarrow \quad A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

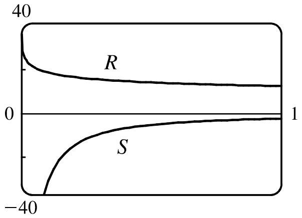
(b) $p'(x)$ is greater than the average productivity $\quad \Rightarrow \quad p'(x) > A(x) \quad \Rightarrow \quad p'(x) > \dfrac{p(x)}{x} \quad \Rightarrow \quad xp'(x) > p(x) \quad \Rightarrow$

$xp'(x) - p(x) > 0 \quad \Rightarrow \quad \dfrac{xp'(x) - p(x)}{x^2} > 0 \quad \Rightarrow \quad A'(x) > 0.$

32. (a) $R = \dfrac{40 + 24x^{0.4}}{1 + 4x^{0.4}} \quad \Rightarrow \quad S = \dfrac{dR}{dx} = \dfrac{(1 + 4x^{0.4})(9.6x^{-0.6}) - (40 + 24x^{0.4})(1.6x^{-0.6})}{(1 + 4x^{0.4})^2}$

$$= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} = -\frac{54.4x^{-0.6}}{(1 + 4x^{0.4})^2}$$

(b)



At low levels of brightness, R is quite large $[R(0) = 40]$ and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

33. $PV = nRT \quad \Rightarrow \quad T = \dfrac{PV}{nR} = \dfrac{PV}{(10)(0.0821)} = \dfrac{1}{0.821}(PV)$. Using the Product Rule, we have

$\dfrac{dT}{dt} = \dfrac{1}{0.821}\left[P(t)V'(t) + V(t)P'(t)\right] = \dfrac{1}{0.821}\left[(8)(-0.15) + (10)(0.10)\right] \approx -0.2436 \text{ K/min.}$

34. $f(r) = 2\sqrt{Dr} \quad \Rightarrow \quad f'(r) = 2 \cdot \frac{1}{2}(Dr)^{-1/2} \cdot D = \dfrac{D}{\sqrt{Dr}} = \sqrt{\dfrac{D}{r}}$. $f'(r)$ is the rate of change of the wave speed with respect to the reproductive rate.

35. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is, $\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$.

(b) “The caribou go extinct” means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\dfrac{dC}{dt} = aC - bCW$ and $\dfrac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ **(1)** and $-0.05W + 0.0001CW = 0$ **(2)**. Adding 10 times **(2)** to **(1)** eliminates the CW-terms and gives us $0.05C - 0.5W = 0 \quad \Rightarrow \quad C = 10W$. Substituting $C = 10W$ into **(1)** results in $0.05(10W) - 0.001(10W)W = 0 \quad \Leftrightarrow \quad 0.5W - 0.01W^2 = 0 \quad \Leftrightarrow \quad 50W - W^2 = 0 \quad \Leftrightarrow$ $W(50 - W) = 0 \quad \Leftrightarrow \quad W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs (C, W) that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

36. (a) If $dP/dt = 0$, the population is stable (it is constant).

(b) $\dfrac{dP}{dt} = 0 \quad\Rightarrow\quad \beta P = r_0\left(1 - \dfrac{P}{P_c}\right)P \quad\Rightarrow\quad \dfrac{\beta}{r_0} = 1 - \dfrac{P}{P_c} \quad\Rightarrow\quad \dfrac{P}{P_c} = 1 - \dfrac{\beta}{r_0} \quad\Rightarrow\quad P = P_c\left(1 - \dfrac{\beta}{r_0}\right).$

If $P_c = 10{,}000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10{,}000\left(1 - \frac{4}{5}\right) = 2000$.

(c) If $\beta = 0.05$, then $P = 10{,}000\left(1 - \frac{5}{5}\right) = 0$. There is no stable population.

2.8 Related Rates

1. $V = x^3 \quad\Rightarrow\quad \dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt} = 3x^2\,\dfrac{dx}{dt}$

2. (a) $A = \pi r^2 \quad\Rightarrow\quad \dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt} = 2\pi r\,\dfrac{dr}{dt}$ (b) $\dfrac{dA}{dt} = 2\pi r\,\dfrac{dr}{dt} = 2\pi(30\text{ m})(1\text{ m/s}) = 60\pi\text{ m}^2/\text{s}$

3. Let s denote the side of a square. The square's area A is given by $A = s^2$. Differentiating with respect to t gives us $\dfrac{dA}{dt} = 2s\,\dfrac{ds}{dt}$. When $A = 16$, $s = 4$. Substitution 4 for s and 6 for $\dfrac{ds}{dt}$ gives us $\dfrac{dA}{dt} = 2(4)(6) = 48\text{ cm}^2/\text{s}$.

4. $A = \ell w \quad\Rightarrow\quad \dfrac{dA}{dt} = \ell\cdot\dfrac{dw}{dt} + w\cdot\dfrac{d\ell}{dt} = 20(3) + 10(8) = 140\text{ cm}^2/\text{s}$.

5. $V = \pi r^2 h = \pi(5)^2 h = 25\pi h \quad\Rightarrow\quad \dfrac{dV}{dt} = 25\pi\,\dfrac{dh}{dt} \quad\Rightarrow\quad 3 = 25\pi\,\dfrac{dh}{dt} \quad\Rightarrow\quad \dfrac{dh}{dt} = \dfrac{3}{25\pi}\text{ m/min.}$

6. $V = \frac{4}{3}\pi r^3 \quad\Rightarrow\quad \dfrac{dV}{dt} = \frac{4}{3}\pi\cdot 3r^2\,\dfrac{dr}{dt} \quad\Rightarrow\quad \dfrac{dV}{dt} = 4\pi\left(\frac{1}{2}\cdot 80\right)^2(4) = 25{,}600\pi\text{ mm}^3/\text{s}$.

7. $S = 4\pi r^2 \quad\Rightarrow\quad \dfrac{dS}{dt} = 4\pi\cdot 2r\,\dfrac{dr}{dt} \quad\Rightarrow\quad \dfrac{dS}{dt} = 4\pi\cdot 2\cdot 8\cdot 2 = 128\pi\text{ cm}^2/\text{min.}$

8. (a) $A = \frac{1}{2}ab\sin\theta \quad\Rightarrow\quad \dfrac{dA}{dt} = \frac{1}{2}ab\cos\theta\,\dfrac{d\theta}{dt} = \frac{1}{2}(2)(3)\left(\cos\frac{\pi}{3}\right)(0.2) = 3\left(\frac{1}{2}\right)(0.2) = 0.3\text{ cm}^2/\text{min.}$

(b) $A = \frac{1}{2}ab\sin\theta \quad\Rightarrow$

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2}a\left(b\cos\theta\,\frac{d\theta}{dt} + \sin\theta\,\frac{db}{dt}\right) = \tfrac{1}{2}(2)\left[3\left(\cos\tfrac{\pi}{3}\right)(0.2) + \left(\sin\tfrac{\pi}{3}\right)(1.5)\right]\\ &= 3\left(\tfrac{1}{2}\right)(0.2) + \tfrac{1}{2}\sqrt{3}\left(\tfrac{3}{2}\right) = 0.3 + \tfrac{3}{4}\sqrt{3}\text{ cm}^2/\text{min}\quad[\approx 1.6]\end{aligned}$$

(c) $A = \frac{1}{2}ab\sin\theta \quad\Rightarrow$

$$\begin{aligned}\frac{dA}{dt} &= \frac{1}{2}\left(\frac{da}{dt}\,b\sin\theta + a\,\frac{db}{dt}\sin\theta + ab\cos\theta\,\frac{d\theta}{dt}\right) \qquad \text{[by Exercise 2.3.87(a)]}\\ &= \tfrac{1}{2}\left[(2.5)(3)\left(\tfrac{1}{2}\sqrt{3}\right) + (2)(1.5)\left(\tfrac{1}{2}\sqrt{3}\right) + (2)(3)\left(\tfrac{1}{2}\right)(0.2)\right]\\ &= \left(\tfrac{15}{8}\sqrt{3} + \tfrac{3}{4}\sqrt{3} + 0.3\right) = \left(\tfrac{21}{8}\sqrt{3} + 0.3\right)\text{ cm}^2/\text{min}\quad[\approx 4.85]\end{aligned}$$

Note how this answer relates to the answer in part (a) [θ changing] and part (b) [b and θ changing].

9. (a) $y = \sqrt{2x+1}$ and $\dfrac{dx}{dt} = 3 \quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{1}{2}(2x+1)^{-1/2}\cdot 2\cdot 3 = \dfrac{3}{\sqrt{2x+1}}$. When $x = 4$, $\dfrac{dy}{dt} = \dfrac{3}{\sqrt{9}} = 1$.

(b) $y = \sqrt{2x+1} \quad\Rightarrow\quad y^2 = 2x+1 \quad\Rightarrow\quad 2x = y^2 - 1 \quad\Rightarrow\quad x = \frac{1}{2}y^2 - \frac{1}{2}$ and $\dfrac{dy}{dt} = 5 \quad\Rightarrow$

$\dfrac{dx}{dt} = \dfrac{dx}{dy}\dfrac{dy}{dt} = y\cdot 5 = 5y$. When $x = 12$, $y = \sqrt{25} = 5$, so $\dfrac{dx}{dt} = 5(5) = 25$.

10. (a) $\dfrac{d}{dt}\left(4x^2 + 9y^2\right) = \dfrac{d}{dt}(36) \quad\Rightarrow\quad 8x\,\dfrac{dx}{dt} + 18y\,\dfrac{dy}{dt} = 0 \quad\Rightarrow\quad 4x\,\dfrac{dx}{dt} + 9y\,\dfrac{dy}{dt} = 0 \quad\Rightarrow$

$4(2)\,\dfrac{dx}{dt} + 9\left(\dfrac{2}{3}\sqrt{5}\right)\left(\dfrac{1}{3}\right) = 0 \quad\Rightarrow\quad 8\,\dfrac{dx}{dt} = -2\sqrt{5} \quad\Rightarrow\quad \dfrac{dx}{dt} = -\dfrac{1}{4}\sqrt{5}$

(b) $4x\,\dfrac{dx}{dt} + 9y\,\dfrac{dy}{dt} = 0 \quad\Rightarrow\quad 4(-2)(3) + 9\left(\dfrac{2}{3}\sqrt{5}\right)\dfrac{dy}{dt} = 0 \quad\Rightarrow\quad 6\sqrt{5}\,\dfrac{dy}{dt} = 24 \quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{4}{\sqrt{5}}$

11. $\dfrac{d}{dt}\left(x^2 + y^2 + z^2\right) = \dfrac{d}{dt}(9) \quad\Rightarrow\quad 2x\,\dfrac{dx}{dt} + 2y\,\dfrac{dy}{dt} + 2z\,\dfrac{dz}{dt} = 0 \quad\Rightarrow\quad x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt} + z\,\dfrac{dz}{dt} = 0.$

If $\dfrac{dx}{dt} = 5$, $\dfrac{dy}{dt} = 4$ and $(x, y, z) = (2, 2, 1)$, then $2(5) + 2(4) + 1\,\dfrac{dz}{dt} = 0 \quad\Rightarrow\quad \dfrac{dz}{dt} = -18.$

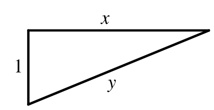
12. $\dfrac{d}{dt}(xy) = \dfrac{d}{dt}(8) \quad\Rightarrow\quad x\,\dfrac{dy}{dt} + y\,\dfrac{dx}{dt} = 0.$ If $\dfrac{dy}{dt} = -3$ cm/s and $(x, y) = (4, 2)$, then $4(-3) + 2\,\dfrac{dx}{dt} = 0 \quad\Rightarrow$

$\dfrac{dx}{dt} = 6$. Thus, the x-coordinate is increasing at a rate of 6 cm/s.

13. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt = 500$ mi/h.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 2$ mi.

(c)



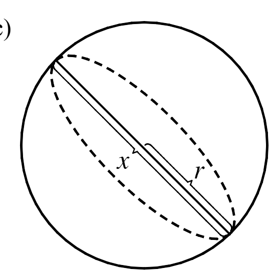
(d) By the Pythagorean Theorem, $y^2 = x^2 + 1 \quad\Rightarrow\quad 2y\,(dy/dt) = 2x\,(dx/dt)$.

(e) $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{x}{y}(500)$. Since $y^2 = x^2 + 1$, when $y = 2$, $x = \sqrt{3}$, so $\dfrac{dy}{dt} = \dfrac{\sqrt{3}}{2}(500) = 250\sqrt{3} \approx 433$ mi/h.

14. (a) Given: the rate of decrease of the surface area is 1 cm^2/min. If we let t be time (in minutes) and S be the surface area (in cm^2), then we are given that $dS/dt = -1$ cm^2/s.

(b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x = 10$ cm.

(c)



(d) If the radius is r and the diameter $x = 2r$, then $r = \frac{1}{2}x$ and $S = 4\pi r^2 = 4\pi\left(\frac{1}{2}x\right)^2 = \pi x^2 \quad \Rightarrow$

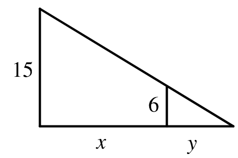
$$\frac{dS}{dt} = \frac{dS}{dx}\frac{dx}{dt} = 2\pi x\,\frac{dx}{dt}.$$

(e) $-1 = \dfrac{dS}{dt} = 2\pi x\,\dfrac{dx}{dt} \quad \Rightarrow \quad \dfrac{dx}{dt} = -\dfrac{1}{2\pi x}$. When $x = 10$, $\dfrac{dx}{dt} = -\dfrac{1}{20\pi}$. So the rate of decrease is $\dfrac{1}{20\pi}$ cm/min.

15. (a) Given: a man 6 ft tall walks away from a street light mounted on a 15-ft-tall pole at a rate of 5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt = 5$ ft/s.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of his shadow (in ft), then we want to find $\dfrac{d}{dt}(x+y)$ when $x = 40$ ft.

(c)



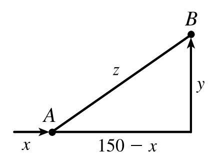
(d) By similar triangles, $\dfrac{15}{6} = \dfrac{x+y}{y} \quad \Rightarrow \quad 15y = 6x + 6y \quad \Rightarrow \quad 9y = 6x \quad \Rightarrow \quad y = \frac{2}{3}x.$

(e) The tip of the shadow moves at a rate of $\dfrac{d}{dt}(x+y) = \dfrac{d}{dt}\left(x + \dfrac{2}{3}x\right) = \dfrac{5}{3}\dfrac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$ ft/s.

16. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

(b) Unknown: the rate at which the distance between the ships is changing at 4:00 PM. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

(c)

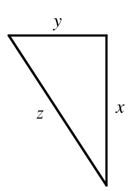


(d) $z^2 = (150 - x)^2 + y^2 \quad \Rightarrow \quad 2z\,\dfrac{dz}{dt} = 2(150 - x)\left(-\dfrac{dx}{dt}\right) + 2y\,\dfrac{dy}{dt}$

(e) At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \quad \Rightarrow \quad z = \sqrt{(150-140)^2 + 100^2} = \sqrt{10{,}100}$.

So $\dfrac{dz}{dt} = \dfrac{1}{z}\left[(x - 150)\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right] = \dfrac{-10(35) + 100(25)}{\sqrt{10{,}100}} = \dfrac{215}{\sqrt{101}} \approx 21.4$ km/h.

17.



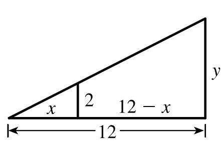
We are given that $\dfrac{dx}{dt} = 60$ mi/h and $\dfrac{dy}{dt} = 25$ mi/h. $z^2 = x^2 + y^2 \quad \Rightarrow$

$$2z\,\frac{dz}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \quad \Rightarrow \quad z\,\frac{dz}{dt} = x\,\frac{dx}{dt} + y\,\frac{dy}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{1}{z}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right).$$

After 2 hours, $x = 2\,(60) = 120$ and $y = 2\,(25) = 50 \quad \Rightarrow \quad z = \sqrt{120^2 + 50^2} = 130$,

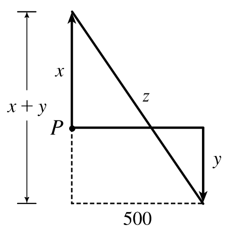
so $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right) = \dfrac{120(60) + 50(25)}{130} = 65$ mi/h.

18.



We are given that $\dfrac{dx}{dt} = 1.6$ m/s. By similar triangles, $\dfrac{y}{12} = \dfrac{2}{x} \quad \Rightarrow \quad y = \dfrac{24}{x} \quad \Rightarrow$

$\dfrac{dy}{dt} = -\dfrac{24}{x^2}\dfrac{dx}{dt} = -\dfrac{24}{x^2}(1.6)$. When $x = 8$, $\dfrac{dy}{dt} = -\dfrac{24(1.6)}{64} = -0.6$ m/s, so the shadow is decreasing at a rate of 0.6 m/s.

19. 

We are given that $\dfrac{dx}{dt} = 4$ ft/s and $\dfrac{dy}{dt} = 5$ ft/s. $z^2 = (x+y)^2 + 500^2 \quad \Rightarrow$

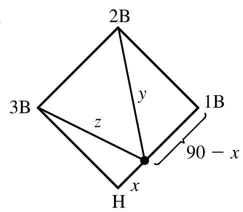
$2z\,\dfrac{dz}{dt} = 2(x+y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. 15 minutes after the woman starts, we have

$x = (4 \text{ ft/s})(20 \text{ min})(60 \text{ s/min}) = 4800$ ft and $y = 5 \cdot 15 \cdot 60 = 4500 \quad \Rightarrow$

$z = \sqrt{(4800 + 4500)^2 + 500^2} = \sqrt{86{,}740{,}000}$, so

$$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{4800 + 4500}{\sqrt{86{,}740{,}000}}(4+5) = \frac{837}{\sqrt{8674}} \approx 8.99 \text{ ft/s}.$$

20. We are given that $\dfrac{dx}{dt} = 24$ ft/s.

(a) 

$y^2 = (90 - x)^2 + 90^2 \quad \Rightarrow \quad 2y\,\dfrac{dy}{dt} = 2(90 - x)\left(-\dfrac{dx}{dt}\right)$. When $x = 45$,

$y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, so $\dfrac{dy}{dt} = \dfrac{90 - x}{y}\left(-\dfrac{dx}{dt}\right) = \dfrac{45}{45\sqrt{5}}(-24) = -\dfrac{24}{\sqrt{5}}$,

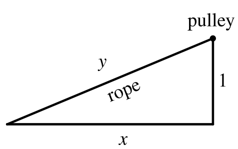
so the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.7$ ft/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer—and we do.

$z^2 = x^2 + 90^2 \quad \Rightarrow \quad 2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt}$. When $x = 45$, $z = 45\sqrt{5}$, so $\dfrac{dz}{dt} = \dfrac{45}{45\sqrt{5}}(24) = \dfrac{24}{\sqrt{5}} \approx 10.7$ ft/s.

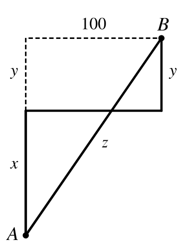
21. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\dfrac{dh}{dt} = 1$ cm/min and $\dfrac{dA}{dt} = 2$ cm^2/min. Using the Product Rule, we have $\dfrac{dA}{dt} = \dfrac{1}{2}\left(b\,\dfrac{dh}{dt} + h\,\dfrac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have $100 = \frac{1}{2}b(10) \quad \Rightarrow \quad \frac{1}{2}b = 10 \quad \Rightarrow$

$b = 20$, so $2 = \dfrac{1}{2}\left(20 \cdot 1 + 10\,\dfrac{db}{dt}\right) \quad \Rightarrow \quad 4 = 20 + 10\,\dfrac{db}{dt} \quad \Rightarrow \quad \dfrac{db}{dt} = \dfrac{4 - 20}{10} = -1.6$ cm/min.

22. 

Given $\dfrac{dy}{dt} = -1$ m/s, find $\dfrac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \quad \Rightarrow \quad 2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} \quad \Rightarrow$

$\dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt} = -\dfrac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so $\dfrac{dx}{dt} = -\dfrac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\dfrac{\sqrt{65}}{8} \approx 1.01$ m/s.

23. 

We are given that $\dfrac{dx}{dt} = 35$ km/h and $\dfrac{dy}{dt} = 25$ km/h. $z^2 = (x+y)^2 + 100^2 \quad \Rightarrow$

$2z\,\dfrac{dz}{dt} = 2(x+y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \quad \Rightarrow$

$z = \sqrt{(140 + 100)^2 + 100^2} = \sqrt{67{,}600} = 260$, so

$$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{140 + 100}{260}(35 + 25) = \frac{720}{13} \approx 55.4 \text{ km/h}.$$

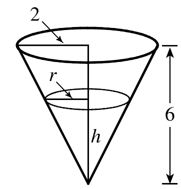
24. The distance z of the particle to the origin is given by $z = \sqrt{x^2 + y^2}$, so $z^2 = x^2 + [2\sin(\pi x/2)]^2 \quad \Rightarrow$

$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 4 \cdot 2\sin\left(\dfrac{\pi}{2}x\right)\cos\left(\dfrac{\pi}{2}x\right) \cdot \dfrac{\pi}{2}\dfrac{dx}{dt} \quad \Rightarrow \quad z\,\dfrac{dz}{dt} = x\,\dfrac{dx}{dt} + 2\pi\sin\left(\dfrac{\pi}{2}x\right)\cos\left(\dfrac{\pi}{2}x\right)\dfrac{dx}{dt}$. When

$(x,y)=\left(\frac{1}{3},1\right)$, $z=\sqrt{\left(\frac{1}{3}\right)^2+1^2}=\sqrt{\frac{10}{9}}=\frac{1}{3}\sqrt{10}$, so $\frac{1}{3}\sqrt{10}\,\frac{dz}{dt}=\frac{1}{3}\sqrt{10}+2\pi\,\sin\frac{\pi}{6}\,\cos\frac{\pi}{6}\cdot\sqrt{10}\quad\Rightarrow$

$\frac{1}{3}\frac{dz}{dt}=\frac{1}{3}+2\pi\left(\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{3}\right)\quad\Rightarrow\quad\frac{dz}{dt}=1+\frac{3\sqrt{3}\,\pi}{2}$ cm/s.

25. If $C=$ the rate at which water is pumped in, then $\frac{dV}{dt}=C-10{,}000$, where

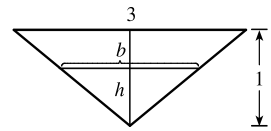


$V=\frac{1}{3}\pi r^2h$ is the volume at time t. By similar triangles, $\frac{r}{2}=\frac{h}{6}\quad\Rightarrow\quad r=\frac{1}{3}h\quad\Rightarrow$

$V=\frac{1}{3}\pi\left(\frac{1}{3}h\right)^2h=\frac{\pi}{27}h^3\quad\Rightarrow\quad\frac{dV}{dt}=\frac{\pi}{9}h^2\,\frac{dh}{dt}$. When $h=200$ cm,

$\frac{dh}{dt}=20\text{ cm/min}$, so $C-10{,}000=\frac{\pi}{9}(200)^2(20)\quad\Rightarrow\quad C=10{,}000+\frac{800{,}000}{9}\pi\approx 289{,}253\text{ cm}^3/\text{min}$.

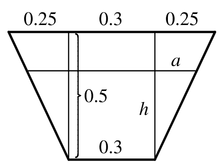
26. By similar triangles, $\frac{3}{1}=\frac{b}{h}$, so $b=3h$. The trough has volume



$V=\frac{1}{2}bh(10)=5(3h)h=15h^2\quad\Rightarrow\quad 12=\frac{dV}{dt}=30h\,\frac{dh}{dt}\quad\Rightarrow\quad\frac{dh}{dt}=\frac{2}{5h}$.

When $h=\frac{1}{2}$, $\frac{dh}{dt}=\frac{2}{5\cdot\frac{1}{2}}=\frac{4}{5}$ ft/min.

27.



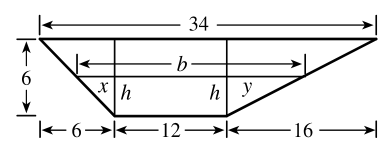
The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1+\text{base}_2)(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is $V=(10)\frac{1}{2}[0.3+(0.3+2a)]h$.

By similar triangles, $\frac{a}{h}=\frac{0.25}{0.5}=\frac{1}{2}$, so $2a=h\quad\Rightarrow\quad V=5(0.6+h)h=3h+5h^2$.

Now $\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt}\quad\Rightarrow\quad 0.2=(3+10h)\frac{dh}{dt}\quad\Rightarrow\quad\frac{dh}{dt}=\frac{0.2}{3+10h}$. When $h=0.3$,

$\frac{dh}{dt}=\frac{0.2}{3+10(0.3)}=\frac{0.2}{6}\text{ m/min}=\frac{1}{30}\text{ m/min}$ or $\frac{10}{3}$ cm/min.

28. The figure is drawn without the top 3 feet.



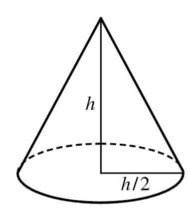
$V=\frac{1}{2}(b+12)h(20)=10(b+12)h$ and, from similar triangles,

$\frac{x}{h}=\frac{6}{6}$ and $\frac{y}{h}=\frac{16}{6}=\frac{8}{3}$, so $b=x+12+y=h+12+\frac{8h}{3}=12+\frac{11h}{3}$.

Thus, $V=10\left(24+\frac{11h}{3}\right)h=240h+\frac{110h^2}{3}$ and so $0.8=\frac{dV}{dt}=\left(240+\frac{220}{3}h\right)\frac{dh}{dt}$.

When $h=5$, $\frac{dh}{dt}=\frac{0.8}{240+5(220/3)}=\frac{3}{2275}\approx 0.00132$ ft/min.

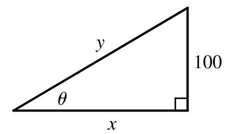
29. We are given that $\frac{dV}{dt}=30\text{ ft}^3/\text{min}$. $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2h=\frac{\pi h^3}{12}\quad\Rightarrow$



$\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt}\quad\Rightarrow\quad 30=\frac{\pi h^2}{4}\frac{dh}{dt}\quad\Rightarrow\quad\frac{dh}{dt}=\frac{120}{\pi h^2}$.

When $h=10$ ft, $\frac{dh}{dt}=\frac{120}{10^2\pi}=\frac{6}{5\pi}\approx 0.38$ ft/min.

30. We are given $dx/dt = 8$ ft/s. $\cot\theta = \dfrac{x}{100} \Rightarrow x = 100\cot\theta \Rightarrow$



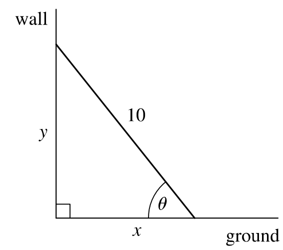
$\dfrac{dx}{dt} = -100\csc^2\theta\,\dfrac{d\theta}{dt} \Rightarrow \dfrac{d\theta}{dt} = -\dfrac{\sin^2\theta}{100}\cdot 8$. When $y = 200$, $\sin\theta = \dfrac{100}{200} = \dfrac{1}{2} \Rightarrow$

$\dfrac{d\theta}{dt} = -\dfrac{(1/2)^2}{100}\cdot 8 = -\dfrac{1}{50}$ rad/s. The angle is decreasing at a rate of $\frac{1}{50}$ rad/s.

31. The area A of an equilateral triangle with side s is given by $A = \frac{1}{4}\sqrt{3}\,s^2$.

$\dfrac{dA}{dt} = \frac{1}{4}\sqrt{3}\cdot 2s\,\dfrac{ds}{dt} = \frac{1}{4}\sqrt{3}\cdot 2(30)(10) = 150\sqrt{3}\ \text{cm}^2/\text{min}.$

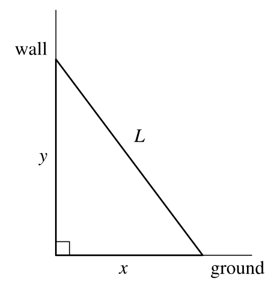
32. $\cos\theta = \dfrac{x}{10} \Rightarrow -\sin\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{10}\,\dfrac{dx}{dt}$. From Example 2, $\dfrac{dx}{dt} = 1$ and



when $x = 6$, $y = 8$, so $\sin\theta = \dfrac{8}{10}$.

Thus, $-\dfrac{8}{10}\,\dfrac{d\theta}{dt} = \dfrac{1}{10}(1) \Rightarrow \dfrac{d\theta}{dt} = -\dfrac{1}{8}$ rad/s.

33. From the figure and given information, we have $x^2 + y^2 = L^2$, $\dfrac{dy}{dt} = -0.15\,\text{m/s}$, and



$\dfrac{dx}{dt} = 0.2\,\text{m/s}$ when $x = 3\,\text{m}$. Differentiating implicitly with respect to t, we get

$x^2 + y^2 = L^2 \Rightarrow 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \Rightarrow y\dfrac{dy}{dt} = -x\dfrac{dx}{dt}$. Substituting the given information gives us $y(-0.15) = -3(0.2) \Rightarrow y = 4\,\text{m}$. Thus, $3^2 + 4^2 = L^2 \Rightarrow$ $L^2 = 25 \Rightarrow L = 5\,\text{m}$.

34. According to the model in Example 2, $\dfrac{dy}{dt} = -\dfrac{x}{y}\,\dfrac{dx}{dt} \to -\infty$ as $y \to 0$, which doesn't make physical sense. For example, the model predicts that for sufficiently small y, the tip of the ladder moves at a speed greater than the speed of light. Therefore the model is not appropriate for small values of y. What actually happens is that the tip of the ladder leaves the wall at some point in its descent. For a discussion of the true situation see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal,* 27, (1), January 1996, pages 49–54. Also see "On Mathematical and Physical Ladders" by M. Freeman and P. Palffy-Muhoray in the *American Journal of Physics,* 53 (3), March 1985, pages 276–277.

35. The area A of a sector of a circle with radius r and angle θ is given by $A = \frac{1}{2}r^2\theta$. Here r is constant and θ varies, so $\dfrac{dA}{dt} = \dfrac{1}{2}r^2\,\dfrac{d\theta}{dt}$. The minute hand rotates through $360^\circ = 2\pi$ radians each hour, so $\dfrac{dA}{dt} = \dfrac{1}{2}r^2(2\pi) = \pi r^2\ \text{cm}^2/\text{h}$. This answer makes sense because the minute hand sweeps through the full area of a circle, πr^2, each hour.

36. The volume of a hemisphere is $\frac{2}{3}\pi r^3$, so the volume of a hemispherical basin of radius 30 cm is $\frac{2}{3}\pi(30)^3 = 18{,}000\pi\ \text{cm}^3$. If the basin is half full, then $V = \pi\left(rh^2 - \frac{1}{3}h^3\right) \Rightarrow 9000\pi = \pi\left(30h^2 - \frac{1}{3}h^3\right) \Rightarrow \frac{1}{3}h^3 - 30h^2 + 9000 = 0 \Rightarrow$

$h = H \approx 19.58$ [from a graph or numerical rootfinder; the other two solutions are less than 0 and greater than 30].

$V = \pi\left(30h^2 - \frac{1}{3}h^3\right) \Rightarrow \frac{dV}{dt} = \pi\left(60h\,\frac{dh}{dt} - h^2\,\frac{dh}{dt}\right) \Rightarrow \left(2\,\frac{\text{L}}{\text{min}}\right)\left(1000\,\frac{\text{cm}^3}{\text{L}}\right) = \pi(60h - h^2)\,\frac{dh}{dt} \Rightarrow$

$\frac{dh}{dt} = \frac{2000}{\pi(60H - H^2)} \approx 0.804$ cm/min.

37. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\,\frac{dV}{dt} + V\,\frac{dP}{dt} = 0 \Rightarrow$ $\frac{dV}{dt} = -\frac{V}{P}\,\frac{dP}{dt}$. When $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$, so we have $\frac{dV}{dt} = -\frac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of 80 cm^3/min.

38. $PV^{1.4} = C \Rightarrow P \cdot 1.4V^{0.4}\,\frac{dV}{dt} + V^{1.4}\,\frac{dP}{dt} = 0 \Rightarrow \frac{dV}{dt} = -\frac{V^{1.4}}{P \cdot 1.4V^{0.4}}\,\frac{dP}{dt} = -\frac{V}{1.4P}\,\frac{dP}{dt}$.

When $V = 400$, $P = 80$ and $\frac{dP}{dt} = -10$, so we have $\frac{dV}{dt} = -\frac{400}{1.4(80)}(-10) = \frac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7} \approx 36$ cm^3/min.

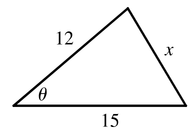
39. With $R_1 = 80$ and $R_2 = 100$, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{80} + \frac{1}{100} = \frac{180}{8000} = \frac{9}{400}$, so $R = \frac{400}{9}$. Differentiating $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ with respect to t, we have $-\frac{1}{R^2}\,\frac{dR}{dt} = -\frac{1}{R_1^2}\,\frac{dR_1}{dt} - \frac{1}{R_2^2}\,\frac{dR_2}{dt} \Rightarrow \frac{dR}{dt} = R^2\left(\frac{1}{R_1^2}\,\frac{dR_1}{dt} + \frac{1}{R_2^2}\,\frac{dR_2}{dt}\right)$. When $R_1 = 80$ and $R_2 = 100$, $\frac{dR}{dt} = \frac{400^2}{9^2}\left[\frac{1}{80^2}(0.3) + \frac{1}{100^2}(0.2)\right] = \frac{107}{810} \approx 0.132\ \Omega/\text{s}$.

40. We want to find $\frac{dB}{dt}$ when $L = 18$ using $B = 0.007W^{2/3}$ and $W = 0.12L^{2.53}$.

$$\frac{dB}{dt} = \frac{dB}{dW}\,\frac{dW}{dL}\,\frac{dL}{dt} = \left(0.007 \cdot \tfrac{2}{3}W^{-1/3}\right)(0.12 \cdot 2.53 \cdot L^{1.53})\left(\frac{20 - 15}{10{,}000{,}000}\right)$$

$$= \left[0.007 \cdot \tfrac{2}{3}(0.12 \cdot 18^{2.53})^{-1/3}\right](0.12 \cdot 2.53 \cdot 18^{1.53})\left(\frac{5}{10^7}\right) \approx 1.045 \times 10^{-8}\ \text{g/yr}$$

41. We are given $d\theta/dt = 2°/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,

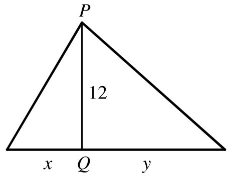


$x^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta = 369 - 360\cos\theta \Rightarrow$

$2x\,\frac{dx}{dt} = 360\sin\theta\,\frac{d\theta}{dt} \Rightarrow \frac{dx}{dt} = \frac{180\sin\theta}{x}\,\frac{d\theta}{dt}$. When $\theta = 60°$,

$x = \sqrt{369 - 360\cos 60°} = \sqrt{189} = 3\sqrt{21}$, so $\frac{dx}{dt} = \frac{180\sin 60°}{3\sqrt{21}}\,\frac{\pi}{90} = \frac{\pi\sqrt{3}}{3\sqrt{21}} = \frac{\sqrt{7}\,\pi}{21} \approx 0.396$ m/min.

42. Using Q for the origin, we are given $\frac{dx}{dt} = -2$ ft/s and need to find $\frac{dy}{dt}$ when $x = -5$.



Using the Pythagorean Theorem twice, we have $\sqrt{x^2 + 12^2} + \sqrt{y^2 + 12^2} = 39$, the total length of the rope. Differentiating with respect to t, we get

$\frac{x}{\sqrt{x^2 + 12^2}}\,\frac{dx}{dt} + \frac{y}{\sqrt{y^2 + 12^2}}\,\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -\frac{x\sqrt{y^2 + 12^2}}{y\sqrt{x^2 + 12^2}}\,\frac{dx}{dt}$.

Now when $x = -5$, $39 = \sqrt{(-5)^2 + 12^2} + \sqrt{y^2 + 12^2} = 13 + \sqrt{y^2 + 12^2} \quad \Leftrightarrow \quad \sqrt{y^2 + 12^2} = 26$, and

$y = \sqrt{26^2 - 12^2} = \sqrt{532}$. So when $x = -5$, $\dfrac{dy}{dt} = -\dfrac{(-5)(26)}{\sqrt{532}\,(13)}(-2) = -\dfrac{10}{\sqrt{133}} \approx -0.87$ ft/s.

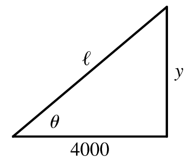
So cart B is moving towards Q at about 0.87 ft/s.

43. (a) By the Pythagorean Theorem, $4000^2 + y^2 = \ell^2$. Differentiating with respect to t,

we obtain $2y\,\dfrac{dy}{dt} = 2\ell\,\dfrac{d\ell}{dt}$. We know that $\dfrac{dy}{dt} = 600$ ft/s, so when $y = 3000$ ft,

$\ell = \sqrt{4000^2 + 3000^2} = \sqrt{25{,}000{,}000} = 5000$ ft

and $\dfrac{d\ell}{dt} = \dfrac{y}{\ell}\,\dfrac{dy}{dt} = \dfrac{3000}{5000}(600) = \dfrac{1800}{5} = 360$ ft/s.



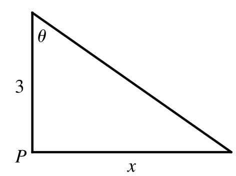
(b) Here $\tan\theta = \dfrac{y}{4000} \quad \Rightarrow \quad \dfrac{d}{dt}(\tan\theta) = \dfrac{d}{dt}\left(\dfrac{y}{4000}\right) \quad \Rightarrow \quad \sec^2\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{4000}\,\dfrac{dy}{dt} \quad \Rightarrow \quad \dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{4000}\,\dfrac{dy}{dt}$. When

$y = 3000$ ft, $\dfrac{dy}{dt} = 600$ ft/s, $\ell = 5000$ and $\cos\theta = \dfrac{4000}{\ell} = \dfrac{4000}{5000} = \dfrac{4}{5}$, so $\dfrac{d\theta}{dt} = \dfrac{(4/5)^2}{4000}(600) = 0.096$ rad/s.

44. We are given that $\dfrac{d\theta}{dt} = 4(2\pi) = 8\pi$ rad/min. $x = 3\tan\theta \quad \Rightarrow$

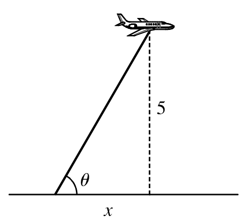
$\dfrac{dx}{dt} = 3\sec^2\theta\,\dfrac{d\theta}{dt}$. When $x = 1$, $\tan\theta = \frac{1}{3}$, so $\sec^2\theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$

and $\dfrac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80}{3}\pi \approx 83.8$ km/min.



45. $\cot\theta = \dfrac{x}{5} \quad \Rightarrow \quad -\csc^2\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{5}\,\dfrac{dx}{dt} \quad \Rightarrow \quad -\left(\csc\dfrac{\pi}{3}\right)^2\left(-\dfrac{\pi}{6}\right) = \dfrac{1}{5}\,\dfrac{dx}{dt} \quad \Rightarrow$

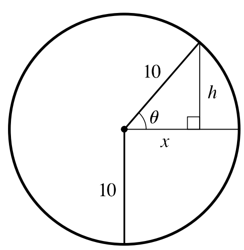
$\dfrac{dx}{dt} = \dfrac{5\pi}{6}\left(\dfrac{2}{\sqrt{3}}\right)^2 = \dfrac{10}{9}\pi$ km/min $[\approx 130$ mi/h$]$



46. We are given that $\dfrac{d\theta}{dt} = \dfrac{2\pi \text{ rad}}{2 \text{ min}} = \pi$ rad/min. By the Pythagorean Theorem, when

$h = 6$, $x = 8$, so $\sin\theta = \frac{6}{10}$ and $\cos\theta = \frac{8}{10}$. From the figure, $\sin\theta = \dfrac{h}{10} \quad \Rightarrow$

$h = 10\sin\theta$, so $\dfrac{dh}{dt} = 10\cos\theta\,\dfrac{d\theta}{dt} = 10\left(\dfrac{8}{10}\right)\pi = 8\pi$ m/min.

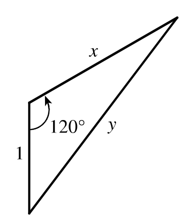


47. We are given that $\dfrac{dx}{dt} = 300$ km/h. By the Law of Cosines,

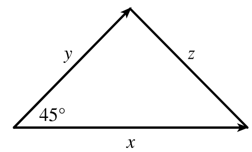
$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120^\circ = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1$, so

$2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} + \dfrac{dx}{dt} \quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{2x+1}{2y}\,\dfrac{dx}{dt}$. After 1 minute, $x = \frac{300}{60} = 5$ km $\quad \Rightarrow$

$y = \sqrt{5^2 + 5 + 1} = \sqrt{31}$ km $\quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{2(5)+1}{2\sqrt{31}}(300) = \dfrac{1650}{\sqrt{31}} \approx 296$ km/h.



48. We are given that $\dfrac{dx}{dt} = 3$ mi/h and $\dfrac{dy}{dt} = 2$ mi/h. By the Law of Cosines,



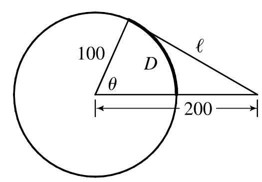
$z^2 = x^2 + y^2 - 2xy\cos 45° = x^2 + y^2 - \sqrt{2}\,xy \quad \Rightarrow$

$2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} - \sqrt{2}\,x\dfrac{dy}{dt} - \sqrt{2}\,y\dfrac{dx}{dt}$. After 15 minutes $\left[= \frac{1}{4}\text{ h}\right]$,

we have $x = \frac{3}{4}$ and $y = \frac{2}{4} = \frac{1}{2} \quad \Rightarrow \quad z^2 = \left(\frac{3}{4}\right)^2 + \left(\frac{2}{4}\right)^2 - \sqrt{2}\left(\frac{3}{4}\right)\left(\frac{2}{4}\right) \quad \Rightarrow \quad z = \dfrac{\sqrt{13 - 6\sqrt{2}}}{4}$ and

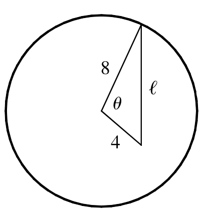
$$\frac{dz}{dt} = \frac{2}{\sqrt{13-6\sqrt{2}}}\left[2\left(\tfrac{3}{4}\right)3 + 2\left(\tfrac{1}{2}\right)2 - \sqrt{2}\left(\tfrac{3}{4}\right)2 - \sqrt{2}\left(\tfrac{1}{2}\right)3\right] = \frac{2}{\sqrt{13-6\sqrt{2}}}\,\frac{13-6\sqrt{2}}{2} = \sqrt{13-6\sqrt{2}} \approx 2.125 \text{ mi/h}.$$

49. Let the distance between the runner and the friend be ℓ. Then by the Law of Cosines,



$\ell^2 = 200^2 + 100^2 - 2\cdot 200\cdot 100\cdot\cos\theta = 50{,}000 - 40{,}000\cos\theta$ $(\star)$. Differentiating implicitly with respect to t, we obtain $2\ell\dfrac{d\ell}{dt} = -40{,}000(-\sin\theta)\dfrac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so $\theta = \dfrac{1}{100}D \quad \Rightarrow \quad \dfrac{d\theta}{dt} = \dfrac{1}{100}\dfrac{dD}{dt} = \dfrac{7}{100}$. To substitute into the expression for $\dfrac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell = 200$, which we find from $(\star)$: $200^2 = 50{,}000 - 40{,}000\cos\theta \quad \Leftrightarrow$ $\cos\theta = \frac{1}{4} \quad \Rightarrow \quad \sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\dfrac{d\ell}{dt} = 40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right) \quad \Rightarrow$ $d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

50. The hour hand of a clock goes around once every 12 hours or, in radians per hour,



$\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rad/h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2 = 4^2 + 8^2 - 2\cdot 4\cdot 8\cdot\cos\theta = 80 - 64\cos\theta$ $(\star)$.

Differentiating implicitly with respect to t, we get $2\ell\dfrac{d\ell}{dt} = -64(-\sin\theta)\dfrac{d\theta}{dt}$. At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use $(\star)$ to find ℓ at 1:00: $\ell = \sqrt{80 - 64\cos\frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$.

Substituting, we get $2\ell\dfrac{d\ell}{dt} = 64\sin\dfrac{\pi}{6}\left(-\dfrac{11\pi}{6}\right) \quad \Rightarrow \quad \dfrac{d\ell}{dt} = \dfrac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80-32\sqrt{3}}} = -\dfrac{88\pi}{3\sqrt{80-32\sqrt{3}}} \approx -18.6.$

So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h ≈ 0.005 mm/s.

2.9 Linear Approximations and Differentials

1. $f(x) = x^3 - x^2 + 3 \quad \Rightarrow \quad f'(x) = 3x^2 - 2x$, so $f(-2) = -9$ and $f'(-2) = 16$. Thus,

$L(x) = f(-2) + f'(-2)(x - (-2)) = -9 + 16(x + 2) = 16x + 23.$

2. $f(x) = \sin x \quad \Rightarrow \quad f'(x) = \cos x$, so $f\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $f'\left(\frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}$. Thus,

$L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}\,x + \frac{1}{2} - \frac{1}{12}\sqrt{3}\,\pi.$

3. $f(x) = \sqrt{x} \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^{-1/2} = 1/(2\sqrt{x})$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. Thus,

$L(x) = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4) = 2 + \frac{1}{4}x - 1 = \frac{1}{4}x + 1.$

4. $f(x) = 2/\sqrt{x^2 - 5} = 2(x^2 - 5)^{-1/2} \quad \Rightarrow \quad f'(x) = 2\left(-\frac{1}{2}\right)(x^2 - 5)^{-3/2}(2x) = -\dfrac{2x}{(x^2 - 5)^{3/2}}$, so $f(3) = 1$ and

$f'(3) = -\frac{3}{4}$. Thus, $L(x) = f(3) + f'(3)(x - 3) = 1 - \frac{3}{4}(x - 3) = -\frac{3}{4}x + \frac{13}{4}.$

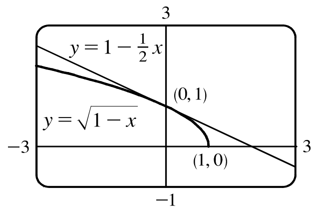
5. $f(x) = \sqrt{1 - x} \quad \Rightarrow \quad f'(x) = \dfrac{-1}{2\sqrt{1 - x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$.

Therefore,

$\sqrt{1 - x} = f(x) \approx f(0) + f'(0)(x - 0) = 1 + \left(-\frac{1}{2}\right)(x - 0) = 1 - \frac{1}{2}x.$

So $\sqrt{0.9} = \sqrt{1 - 0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$

and $\sqrt{0.99} = \sqrt{1 - 0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995.$

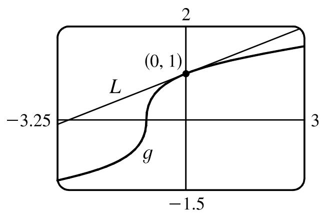


6. $g(x) = \sqrt[3]{1 + x} = (1 + x)^{1/3} \quad \Rightarrow \quad g'(x) = \frac{1}{3}(1 + x)^{-2/3}$, so $g(0) = 1$ and

$g'(0) = \frac{1}{3}$. Therefore, $\sqrt[3]{1 + x} = g(x) \approx g(0) + g'(0)(x - 0) = 1 + \frac{1}{3}x.$

So $\sqrt[3]{0.95} = \sqrt[3]{1 + (-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\overline{3}$,

and $\sqrt[3]{1.1} = \sqrt[3]{1 + 0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\overline{3}.$

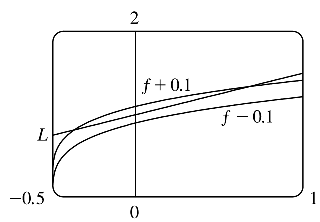


7. $f(x) = \sqrt[4]{1 + 2x} \quad \Rightarrow \quad f'(x) = \frac{1}{4}(1 + 2x)^{-3/4}(2) = \frac{1}{2}(1 + 2x)^{-3/4}$, so

$f(0) = 1$ and $f'(0) = \frac{1}{2}$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 + \frac{1}{2}x.$

We need $\sqrt[4]{1 + 2x} - 0.1 < 1 + \frac{1}{2}x < \sqrt[4]{1 + 2x} + 0.1$, which is true when

$-0.368 < x < 0.677.$

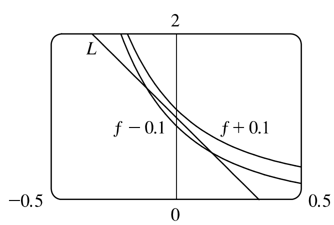


8. $f(x) = (1 + x)^{-3} \quad \Rightarrow \quad f'(x) = -3(1 + x)^{-4}$, so $f(0) = 1$ and

$f'(0) = -3$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - 3x$. We need

$(1 + x)^{-3} - 0.1 < 1 - 3x < (1 + x)^{-3} + 0.1$, which is true when

$-0.116 < x < 0.144.$

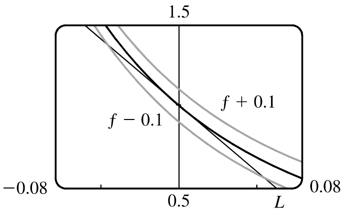


9. $f(x) = \dfrac{1}{(1+2x)^4} = (1+2x)^{-4} \quad \Rightarrow$

$f'(x) = -4(1+2x)^{-5}(2) = \dfrac{-8}{(1+2x)^5}$, so $f(0) = 1$ and $f'(0) = -8$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + (-8)(x-0) = 1 - 8x$.

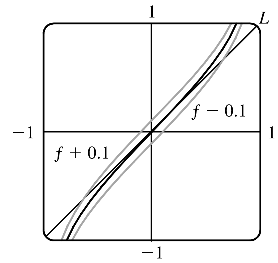
We need $\dfrac{1}{(1+2x)^4} - 0.1 < 1 - 8x < \dfrac{1}{(1+2x)^4} + 0.1$, which is true when $-0.045 < x < 0.055$.



10. $f(x) = \tan x \quad \Rightarrow \quad f'(x) = \sec^2 x$, so $f(0) = 0$ and $f'(0) = 1$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

We need $\tan x - 0.1 < x < \tan x + 0.1$, which is true when $-0.63 < x < 0.63$.



11. (a) The differential dy is defined in terms of dx by the equation $dy = f'(x)\,dx$. For $y = f(x) = (x^2-3)^{-2}$,

$f'(x) = -2(x^2-3)^{-3}(2x) = -\dfrac{4x}{(x^2-3)^3}$, so $dy = -\dfrac{4x}{(x^2-3)^3}\,dx$.

(b) For $y = f(t) = \sqrt{1-t^4}$, $f'(t) = \frac{1}{2}(1-t^4)^{-1/2}(-4t^3) = -\dfrac{2t^3}{\sqrt{1-t^4}}$, so $dy = -\dfrac{2t^3}{\sqrt{1-t^4}}\,dt$.

12. (a) For $y = f(u) = \dfrac{1+2u}{1+3u}$, $f'(u) = \dfrac{(1+3u)(2)-(1+2u)(3)}{(1+3u)^2} = \dfrac{-1}{(1+3u)^2}$, so $dy = \dfrac{-1}{(1+3u)^2}\,du$.

(b) For $y = f(\theta) = \theta^2 \sin 2\theta$, $f'(\theta) = \theta^2(\cos 2\theta)(2) + (\sin 2\theta)(2\theta)$, so $dy = 2\theta(\theta\cos 2\theta + \sin 2\theta)\,d\theta$.

13. (a) For $y = f(t) = \tan\sqrt{t}$, $f'(t) = \sec^2\sqrt{t} \cdot \dfrac{1}{2}t^{-1/2} = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}$, so $dy = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}\,dt$.

(b) For $y = f(v) = \dfrac{1-v^2}{1+v^2}$,

$$f'(v) = \frac{(1+v^2)(-2v)-(1-v^2)(2v)}{(1+v^2)^2} = \frac{-2v[(1+v^2)+(1-v^2)]}{(1+v^2)^2} = \frac{-2v(2)}{(1+v^2)^2} = \frac{-4v}{(1+v^2)^2},$$

so $dy = \dfrac{-4v}{(1+v^2)^2}\,dv$.

14. (a) For $y = f(t) = \sqrt{t-\cos t}$, $f'(t) = \frac{1}{2}(t-\cos t)^{-1/2}(1+\sin t) = \dfrac{1+\sin t}{2\sqrt{t-\cos t}}$, so $dy = \dfrac{1+\sin t}{2\sqrt{t-\cos t}}\,dt$.

(b) For $y = f(x) = \dfrac{1}{x}\sin x$, $f'(x) = \dfrac{1}{x}\cos x - \dfrac{1}{x^2}\sin x = \dfrac{x\cos x - \sin x}{x^2}$, so $dy = \dfrac{x\cos x - \sin x}{x^2}\,dx$.

15. (a) $y = \tan x \quad \Rightarrow \quad dy = \sec^2 x\,dx$

(b) When $x = \pi/4$ and $dx = -0.1$, $dy = [\sec(\pi/4)]^2(-0.1) = \left(\sqrt{2}\right)^2(-0.1) = -0.2$.

16. (a) $y = \cos \pi x \quad \Rightarrow \quad dy = -\sin \pi x \cdot \pi\, dx = -\pi \sin \pi x\, dx$

(b) $x = \frac{1}{3}$ and $dx = -0.02 \quad \Rightarrow \quad dy = -\pi \sin \frac{\pi}{3}(-0.02) = \pi\left(\sqrt{3}/2\right)(0.02) = 0.01\pi\sqrt{3} \approx 0.054.$

17. (a) $y = \sqrt{3 + x^2} \quad \Rightarrow \quad dy = \dfrac{1}{2}(3 + x^2)^{-1/2}(2x)\, dx = \dfrac{x}{\sqrt{3 + x^2}}\, dx$

(b) $x = 1$ and $dx = -0.1 \quad \Rightarrow \quad dy = \dfrac{1}{\sqrt{3 + 1^2}}(-0.1) = \dfrac{1}{2}(-0.1) = -0.05.$

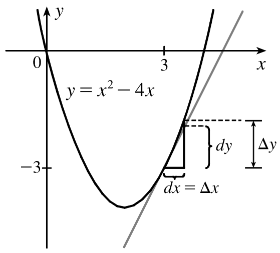
18. (a) $y = \dfrac{x + 1}{x - 1} \quad \Rightarrow \quad dy = \dfrac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}\, dx = \dfrac{-2}{(x - 1)^2}\, dx$

(b) $x = 2$ and $dx = 0.05 \quad \Rightarrow \quad dy = \dfrac{-2}{(2 - 1)^2}(0.05) = -2(0.05) = -0.1.$

19. $y = f(x) = x^2 - 4x,\ x = 3,\ \Delta x = 0.5 \quad \Rightarrow$

$\Delta y = f(3.5) - f(3) = -1.75 - (-3) = 1.25$

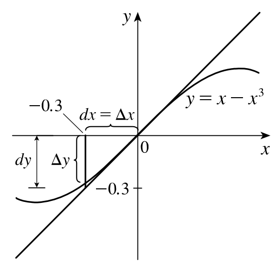
$dy = f'(x)\, dx = (2x - 4)\, dx = (6 - 4)(0.5) = 1$



20. $y = f(x) = x - x^3,\ x = 0,\ \Delta x = -0.3 \quad \Rightarrow$

$\Delta y = f(-0.3) - f(0) = -0.273 - 0 = -0.273$

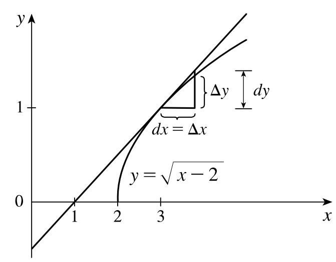
$dy = f'(x)\, dx = (1 - 3x^2)\, dx = (1 - 0)(-0.3) = -0.3$



21. $y = f(x) = \sqrt{x - 2},\ x = 3,\ \Delta x = 0.8 \quad \Rightarrow$

$\Delta y = f(3.8) - f(3) = \sqrt{1.8} - 1 \approx 0.34$

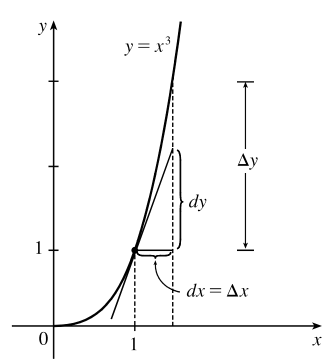
$dy = f'(x)\, dx = \dfrac{1}{2\sqrt{x - 2}}\, dx = \dfrac{1}{2(1)}(0.8) = 0.4$



22. $y = x^3,\ x = 1,\ \Delta x = 0.5 \quad \Rightarrow$

$\Delta y = (1.5)^3 - 1^3 = 3.375 - 1 = 2.375.$

$dy = 3x^2\, dx = 3(1)^2(0.5) = 1.5$



23. To estimate $(1.999)^4$, we'll find the linearization of $f(x) = x^4$ at $a = 2$. Since $f'(x) = 4x^3$, $f(2) = 16$, and $f'(2) = 32$, we have $L(x) = 16 + 32(x - 2)$. Thus, $x^4 \approx 16 + 32(x - 2)$ when x is near 2, so $(1.999)^4 \approx 16 + 32(1.999 - 2) = 16 - 0.032 = 15.968$.

24. $y = f(x) = 1/x \quad \Rightarrow \quad dy = -1/x^2\,dx$. When $x = 4$ and $dx = 0.002$, $dy = -\frac{1}{16}(0.002) = -\frac{1}{8000}$, so $\frac{1}{4.002} \approx f(4) + dy = \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000} = 0.249875$.

25. $y = f(x) = \sqrt[3]{x} \quad \Rightarrow \quad dy = \frac{1}{3}x^{-2/3}\,dx$. When $x = 1000$ and $dx = 1$, $dy = \frac{1}{3}(1000)^{-2/3}(1) = \frac{1}{300}$, so $\sqrt[3]{1001} = f(1001) \approx f(1000) + dy = 10 + \frac{1}{300} = 10.00\overline{3} \approx 10.003$.

26. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad dy = \frac{1}{2}x^{-1/2}\,dx$. When $x = 100$ and $dx = 0.5$, $dy = \frac{1}{2}(100)^{-1/2}\left(\frac{1}{2}\right) = \frac{1}{40}$, so $\sqrt{100.5} = f(100.5) \approx f(100) + dy = 10 + \frac{1}{40} = 10.025$.

27. $y = f(x) = \tan x \quad \Rightarrow \quad dy = \sec^2 x\,dx$. When $x = 0^\circ$ [i.e., 0 radians] and $dx = 2^\circ$ [i.e., $\frac{\pi}{90}$ radians], $dy = (\sec^2 0)\left(\frac{\pi}{90}\right) = 1^2\left(\frac{\pi}{90}\right) = \frac{\pi}{90}$, so $\tan 2^\circ = f(2^\circ) \approx f(0^\circ) + dy = 0 + \frac{\pi}{90} = \frac{\pi}{90} \approx 0.0349$.

28. $y = f(x) = \cos x \quad \Rightarrow \quad dy = -\sin x\,dx$. When $x = 30^\circ$ $[\pi/6]$ and $dx = -1^\circ$ $[-\pi/180]$, $dy = \left(-\sin\frac{\pi}{6}\right)\left(-\frac{\pi}{180}\right) = -\frac{1}{2}\left(-\frac{\pi}{180}\right) = \frac{\pi}{360}$, so $\cos 29^\circ = f(29^\circ) \approx f(30^\circ) + dy = \frac{1}{2}\sqrt{3} + \frac{\pi}{360} \approx 0.875$.

29. $y = f(x) = \sec x \quad \Rightarrow \quad f'(x) = \sec x\,\tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is $f(0) + f'(0)(x - 0) = 1 + 0(x) = 1$. Since 0.08 is close to 0, approximating $\sec 0.08$ with 1 is reasonable.

30. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad f'(x) = 1/(2\sqrt{x}\,)$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. The linear approximation of f at 4 is $f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$. Now $f(4.02) = \sqrt{4.02} \approx 2 + \frac{1}{4}(0.02) = 2 + 0.005 = 2.005$, so the approximation is reasonable.

31. (a) If x is the edge length, then $V = x^3 \quad \Rightarrow \quad dV = 3x^2\,dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3. The relative error is calculated by dividing the change in V, ΔV, by V. We approximate ΔV with dV.

Relative error $= \dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3x^2\,dx}{x^3} = 3\,\dfrac{dx}{x} = 3\left(\dfrac{0.1}{30}\right) = 0.01$.

Percentage error = relative error $\times$ 100% $= 0.01 \times 100\% = 1\%$.

(b) $S = 6x^2 \quad \Rightarrow \quad dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2.

Relative error $= \dfrac{\Delta S}{S} \approx \dfrac{dS}{S} = \dfrac{12x\,dx}{6x^2} = 2\,\dfrac{dx}{x} = 2\left(\dfrac{0.1}{30}\right) = 0.00\overline{6}$.

Percentage error = relative error $\times$ 100% $= 0.00\overline{6} \times 100\% = 0.\overline{6}\%$.

32. (a) $A = \pi r^2 \;\Rightarrow\; dA = 2\pi r\,dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30\text{ cm}^2$.

(b) Relative error $= \dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{2\pi r\,dr}{\pi r^2} = \dfrac{2\,dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{0.2}{12} = \dfrac{1}{60} = 0.01\overline{6}$.

Percentage error $=$ relative error $\times 100\% = 0.01\overline{6} \times 100\% = 1.\overline{6}\%$.

33. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so

$r = \dfrac{C}{2\pi} \;\Rightarrow\; S = 4\pi\left(\dfrac{C}{2\pi}\right)^2 = \dfrac{C^2}{\pi} \;\Rightarrow\; dS = \dfrac{2}{\pi}C\,dC$. When $C = 84$ and $dC = 0.5$, $dS = \dfrac{2}{\pi}(84)(0.5) = \dfrac{84}{\pi}$,

so the maximum error is about $\dfrac{84}{\pi} \approx 27\text{ cm}^2$. Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012 = 1.2\%$

(b) $V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi\left(\dfrac{C}{2\pi}\right)^3 = \dfrac{C^3}{6\pi^2} \;\Rightarrow\; dV = \dfrac{1}{2\pi^2}C^2\,dC$. When $C = 84$ and $dC = 0.5$,

$dV = \dfrac{1}{2\pi^2}(84)^2(0.5) = \dfrac{1764}{\pi^2}$, so the maximum error is about $\dfrac{1764}{\pi^2} \approx 179\text{ cm}^3$.

The relative error is approximately $\dfrac{dV}{V} = \dfrac{1764/\pi^2}{(84)^3/(6\pi^2)} = \dfrac{1}{56} \approx 0.018 = 1.8\%$.

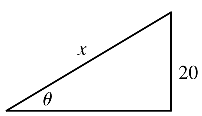
34. For a hemispherical dome, $V = \frac{2}{3}\pi r^3 \;\Rightarrow\; dV = 2\pi r^2\,dr$. When $r = \frac{1}{2}(50) = 25$ m and $dr = 0.05\text{ cm} = 0.0005$ m, $dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8} \approx 2\text{ m}^3$.

35. (a) $V = \pi r^2 h \;\Rightarrow\; \Delta V \approx dV = 2\pi r h\,dr = 2\pi r h\,\Delta r$

(b) The error is

$\Delta V - dV = [\pi(r+\Delta r)^2 h - \pi r^2 h] - 2\pi r h\,\Delta r = \pi r^2 h + 2\pi r h\,\Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi r h\,\Delta r = \pi(\Delta r)^2 h$.

36. (a) $\sin\theta = \dfrac{20}{x} \;\Rightarrow\; x = 20\csc\theta \;\Rightarrow$



$dx = 20(-\csc\theta\,\cot\theta)\,d\theta = -20\csc 30^\circ\,\cot 30^\circ\,(\pm 1^\circ)$

$= -20(2)(\sqrt{3})\left(\pm\dfrac{\pi}{180}\right) = \pm\dfrac{2\sqrt{3}}{9}\pi$

So the maximum error is about $\pm\frac{2}{9}\sqrt{3}\,\pi \approx \pm 1.21$ cm.

(b) The relative error is $\dfrac{\Delta x}{x} \approx \dfrac{dx}{x} = \dfrac{\pm\frac{2}{9}\sqrt{3}\,\pi}{20(2)} = \pm\dfrac{\sqrt{3}}{180}\pi \approx \pm 0.03$, so the percentage error is approximately $\pm 3\%$.

37. $V = RI \;\Rightarrow\; I = \dfrac{V}{R} \;\Rightarrow\; dI = -\dfrac{V}{R^2}\,dR$. The relative error in calculating I is $\dfrac{\Delta I}{I} \approx \dfrac{dI}{I} = \dfrac{-(V/R^2)\,dR}{V/R} = -\dfrac{dR}{R}$.

Hence, the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

38. $F = kR^4 \;\Rightarrow\; dF = 4kR^3\,dR \;\Rightarrow\; \dfrac{dF}{F} = \dfrac{4kR^3\,dR}{kR^4} = 4\left(\dfrac{dR}{R}\right)$. Thus, the relative change in F is about 4 times the relative change in R. So a 5% increase in the radius corresponds to a 20% increase in blood flow.

39. (a) $dc = \dfrac{dc}{dx}\,dx = 0\,dx = 0$ (b) $d(cu) = \dfrac{d}{dx}(cu)\,dx = c\,\dfrac{du}{dx}\,dx = c\,du$

(c) $d(u+v) = \dfrac{d}{dx}(u+v)\,dx = \left(\dfrac{du}{dx} + \dfrac{dv}{dx}\right)dx = \dfrac{du}{dx}\,dx + \dfrac{dv}{dx}\,dx = du + dv$

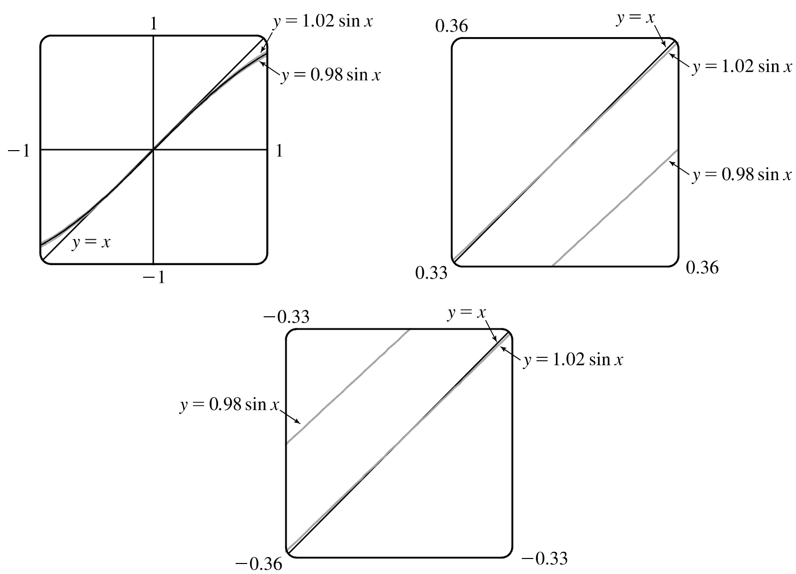
(d) $d(uv) = \dfrac{d}{dx}(uv)\,dx = \left(u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}\right)dx = u\,\dfrac{dv}{dx}\,dx + v\,\dfrac{du}{dx}\,dx = u\,dv + v\,du$

(e) $d\left(\dfrac{u}{v}\right) = \dfrac{d}{dx}\left(\dfrac{u}{v}\right)\,dx = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}\,dx = \dfrac{v\,\dfrac{du}{dx}\,dx - u\,\dfrac{dv}{dx}\,dx}{v^2} = \dfrac{v\,du - u\,dv}{v^2}$

(f) $d\,(x^n) = \dfrac{d}{dx}\,(x^n)\,dx = nx^{n-1}\,dx$

40. (a) $f(x) = \sin x \;\;\Rightarrow\;\; f'(x) = \cos x$, so $f(0) = 0$ and $f'(0) = 1$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

(b)



We want to know the values of x for which $y = x$ approximates $y = \sin x$ with less than a 2% difference; that is, the values of x for which

$$\left|\frac{x - \sin x}{\sin x}\right| < 0.02 \quad\Leftrightarrow\quad -0.02 < \frac{x - \sin x}{\sin x} < 0.02 \quad\Leftrightarrow$$

$$\begin{cases} -0.02\sin x < x - \sin x < 0.02\sin x & \text{if } \sin x > 0 \\ -0.02\sin x > x - \sin x > 0.02\sin x & \text{if } \sin x < 0 \end{cases} \quad\Leftrightarrow\quad \begin{cases} 0.98\sin x < x < 1.02\sin x & \text{if } \sin x > 0 \\ 1.02\sin x < x < 0.98\sin x & \text{if } \sin x < 0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $x = 0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y = x$ intersects $y = 1.02\sin x$ at $x \approx 0.344$. By symmetry, they also intersect at $x \approx -0.344$ (see the third figure). Converting 0.344 radians to degrees, we get $0.344\left(\frac{180°}{\pi}\right) \approx 19.7° \approx 20°$, which verifies the statement.

41. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x-1) = 5 + 2(x-1) = 2x + 3$.

$f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

42. (a) $g'(x) = \sqrt{x^2+5} \;\Rightarrow\; g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95-2) = -4 + 3(-0.05) = -4.15$.
$g(2.05) \approx g(2) + g'(2)(2.05-2) = -4 + 3(0.05) = -3.85$.

(b) The formula $g'(x) = \sqrt{x^2+5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g. Hence, the estimates in part (a) are too small.

LABORATORY PROJECT Taylor Polynomials

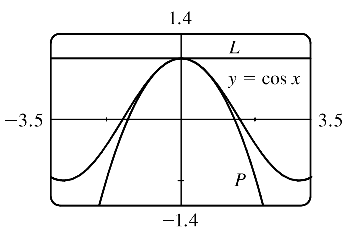
1. We first write the functions described in conditions (i), (ii), and (iii):

$$\begin{aligned} P(x) &= A + Bx + Cx^2 & f(x) &= \cos x \\ P'(x) &= B + 2Cx & f'(x) &= -\sin x \\ P''(x) &= 2C & f''(x) &= -\cos x \end{aligned}$$

So, taking $a = 0$, our three conditions become

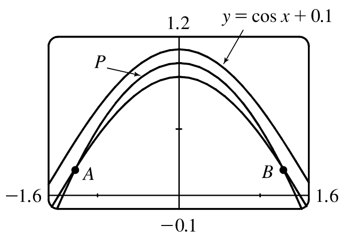
$$\begin{aligned} P(0) &= f(0): & A &= \cos 0 = 1 \\ P'(0) &= f'(0): & B &= -\sin 0 = 0 \\ P''(0) &= f''(0): & 2C &= -\cos 0 = -1 \;\Rightarrow\; C = -\tfrac{1}{2} \end{aligned}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $\left|\cos x - \left(1 - \frac{1}{2}x^2\right)\right| < 0.1 \;\Leftrightarrow\; -0.1 < \cos x - \left(1 - \frac{1}{2}x^2\right) < 0.1 \;\Leftrightarrow$
$0.1 > \left(1 - \frac{1}{2}x^2\right) - \cos x > -0.1 \;\Leftrightarrow\; \cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \;\Leftrightarrow\; \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1$.



From the figure we see that this is true between A and B. Zooming in or using an intersect feature, we find that the x-coordinates of B and A are about ± 1.26. Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

3. If $P(x) = A + B(x-a) + C(x-a)^2$, then $P'(x) = B + 2C(x-a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$P(a) = f(a): \quad A = f(a)$$
$$P'(a) = f'(a): \quad B = f'(a)$$
$$P''(a) = f''(a): \quad 2C = f''(a) \quad \Rightarrow \quad C = \tfrac{1}{2}f''(a)$$

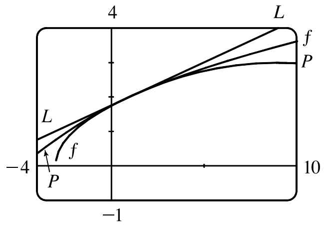
Thus, $P(x) = A + B(x-a) + C(x-a)^2$ can be written in the form $P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

4. From Example 2.9.1, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and $f'(x) = \frac{1}{2}(x+3)^{-1/2}$.

So $f''(x) = -\frac{1}{4}(x+3)^{-3/2} \quad \Rightarrow \quad f''(1) = -\frac{1}{32}$.

From Problem 3, the quadratic approximation $P(x)$ is

$\sqrt{x+3} \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2$.



The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373...	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118...	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015...	2.04937500

5. $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$. If we put $x = a$ in this equation, then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain $T_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1}$. Substituting $x = a$ gives $T_n'(a) = c_1$. Differentiating again, we have $T_n''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a^2) + \cdots + (n-1)nc_n(x-a)^{n-2}$ and so $T_n''(a) = 2c_2$. Continuing in this manner, we get $T_n'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3}$ and $T_n'''(a) = 2\cdot 3c_3$. By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain $T_n^{(4)}(a) = 2\cdot 3\cdot 4c_4$ and in general, for any integer k between 1 and n, $T_n^{(k)}(a) = 2\cdot 3\cdot 4\cdot 5\cdot \cdots \cdot kc_k = k!\,c_k \quad \Rightarrow$ $c_k = \dfrac{T_n^{(k)}(a)}{k!}$. Because we want T_n and f to have the same derivatives at a, we require that $c_k = \dfrac{f^{(k)}(a)}{k!}$ for $k = 1, 2, \ldots, n$.

6. $T_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

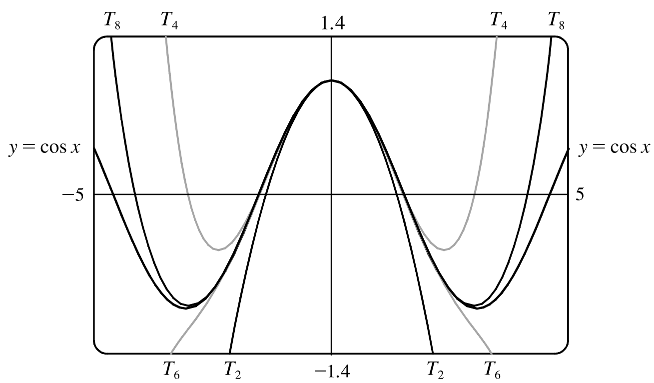
$$f(x) = \cos x \qquad f(0) = \cos 0 = 1$$
$$f'(x) = -\sin x \qquad f'(0) = -\sin 0 = 0$$
$$f''(x) = -\cos x \qquad f''(0) = -1$$
$$f'''(x) = \sin x \qquad f'''(0) = 0$$
$$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$$

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$.

From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

$$\begin{aligned} T_8(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8 \\ &= 1 + 0\cdot x + \frac{-1}{2!}x^2 + 0\cdot x^3 + \frac{1}{4!}x^4 + 0\cdot x^5 + \frac{-1}{6!}x^6 + 0\cdot x^7 + \frac{1}{8!}x^8 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \end{aligned}$$

and the desired approximation is $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$. The Taylor polynomials T_2, T_4, and T_6 consist of the initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \frac{x^2}{2!}$, $T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, and $T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$. We graph T_2, T_4, T_6, T_8, and f:



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

2 Review

TRUE-FALSE QUIZ

1. False. See the note after Theorem 2.2.4.

2. True. This is the Sum Rule.

3. False. See the warning before the Product Rule.

4. True. This is the Chain Rule.

5. True. $\frac{d}{dx}\sqrt{f(x)} = \frac{d}{dx}[f(x)]^{1/2} = \frac{1}{2}[f(x)]^{-1/2}\, f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

6. False. $\frac{d}{dx}\, f(\sqrt{x}\,) = f'(\sqrt{x}\,) \cdot \frac{1}{2}x^{-1/2} = \frac{f'(\sqrt{x}\,)}{2\sqrt{x}}$, which is not $\frac{f'(x)}{2\sqrt{x}}$.

7. False. $f(x) = |x^2 + x| = x^2 + x$ for $x \geq 0$ or $x \leq -1$ and $|x^2 + x| = -(x^2 + x)$ for $-1 < x < 0$.
So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \geq -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.

8. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim\limits_{x \to r} f(x) = f(r)$.

9. True. $g(x) = x^5 \Rightarrow g'(x) = 5x^4 \Rightarrow g'(2) = 5(2)^4 = 80$, and by the definition of the derivative, $\lim\limits_{x \to 2} \dfrac{g(x) - g(2)}{x - 2} = g'(2) = 5(2)^4 = 80$.

10. False. $\dfrac{d^2y}{dx^2}$ is the second derivative while $\left(\dfrac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\dfrac{d^2y}{dx^2} = 0$, but $\left(\dfrac{dy}{dx}\right)^2 = 1$.

11. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x + 2)$. [The given equation, $y - 4 = 2x(x + 2)$, is not even linear!]

12. True. $\dfrac{d}{dx}(\tan^2 x) = 2\tan x \sec^2 x$, and $\dfrac{d}{dx}(\sec^2 x) = 2\sec x(\sec x \tan x) = 2\tan x \sec^2 x$.

Or: $\dfrac{d}{dx}(\sec^2 x) = \dfrac{d}{dx}(1 + \tan^2 x) = \dfrac{d}{dx}(\tan^2 x)$.

13. True. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$, which is a polynomial.

14. True. If $r(x) = \dfrac{p(x)}{q(x)}$, then $r'(x) = \dfrac{q(x)p'(x) - p(x)q'(x)}{[q(x)]^2}$, which is a quotient of polynomials, that is, a rational function.

15. True. $f(x) = (x^6 - x^4)^5$ is a polynomial of degree 30, so its 31st derivative, $f^{(31)}(x)$, is 0.

EXERCISES

1. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1 + h]$ is

$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{1 + 2(1+h) + (1+h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10 + h}{4}$$

So for the following intervals the average velocities are:

(i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = (10 + 2)/4 = 3$ m/s

(ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = (10 + 1)/4 = 2.75$ m/s

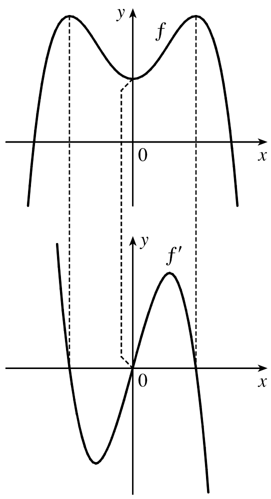
(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = (10 + 0.5)/4 = 2.625$ m/s

(iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = (10 + 0.1)/4 = 2.525$ m/s

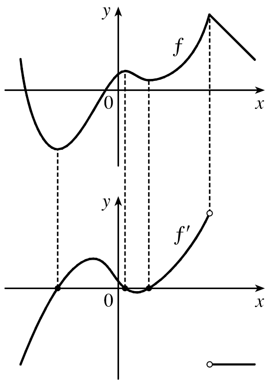
(b) When $t = 1$, the instantaneous velocity is $\lim\limits_{h \to 0} \dfrac{s(1+h) - s(1)}{h} = \lim\limits_{h \to 0} \dfrac{10 + h}{4} = \dfrac{10}{4} = 2.5$ m/s.

2. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

3.



4.



5. The graph of a has tangent lines with positive slope for $x < 0$ and negative slope for $x > 0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a. The graph of c has horizontal tangent lines to the left and right of the x-axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c. Therefore, a is the graph of f, c is the graph of f', and b is the graph of f''.

6. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.

7. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

8. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$, $F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

(c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

9. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h\to 0}\dfrac{P(t+h)-P(t)}{h} \approx \dfrac{P(t+h)-P(t)}{h}$ for small values of h.

For 1950: $P'(1950) \approx \dfrac{P(1960)-P(1950)}{1960-1950} = \dfrac{35.7-31.1}{10} = 0.46$

For 1960: We estimate $P'(1960)$ by using $h=-10$ and $h=10$, and then average the two results to obtain a final estimate.

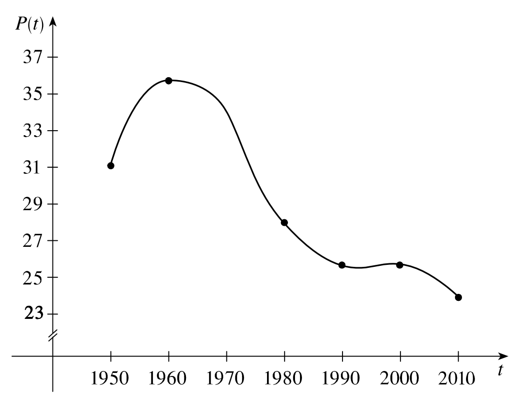
$h=-10 \quad\Rightarrow\quad P'(1960) \approx \dfrac{P(1950)-P(1960)}{1950-1960} = \dfrac{31.1-35.7}{-10} = 0.46$

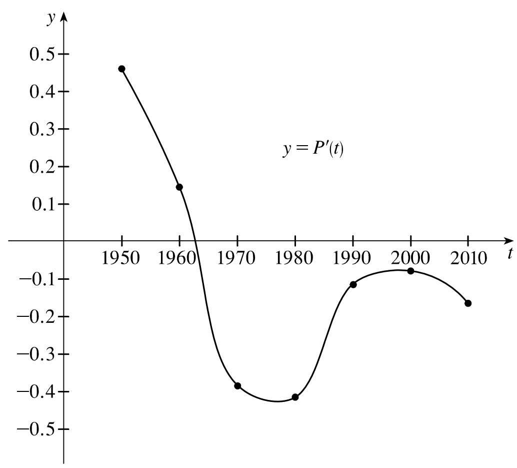
$h=10 \quad\Rightarrow\quad P'(1960) \approx \dfrac{P(1970)-P(1960)}{1970-1960} = \dfrac{34.0-35.7}{10} = -0.17$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46+(-0.17)] = 0.145$.

t	1950	1960	1970	1980	1990	2000	2010
$P'(t)$	0.460	0.145	−0.385	−0.415	−0.115	−0.085	−0.170

(c)





(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, 1995, and 2005.

10. $f(x) = \dfrac{4-x}{3+x} \quad\Rightarrow$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{4-(x+h)}{3+(x+h)} - \dfrac{4-x}{3+x}}{h} = \lim_{h\to 0}\frac{(4-x-h)(3+x)-(4-x)(3+x+h)}{h(3+x+h)(3+x)}$$

$$= \lim_{h\to 0}\frac{-7h}{h(3+x+h)(3+x)} = \lim_{h\to 0}\frac{-7}{(3+x+h)(3+x)} = -\frac{7}{(3+x)^2}$$

11. $f(x) = x^3+5x+4 \quad\Rightarrow$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3+5(x+h)+4-(x^3+5x+4)}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3+5h}{h} = \lim_{h\to 0}(3x^2+3xh+h^2+5) = 3x^2+5$$

12. (a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{3-5(x+h)} - \sqrt{3-5x}}{h} \dfrac{\sqrt{3-5(x+h)} + \sqrt{3-5x}}{\sqrt{3-5(x+h)} + \sqrt{3-5x}}$

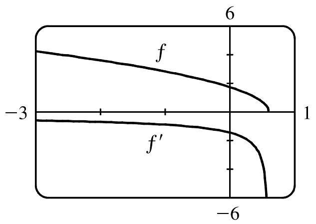
$$= \lim_{h\to 0} \frac{[3-5(x+h)]-(3-5x)}{h\left(\sqrt{3-5(x+h)}+\sqrt{3-5x}\right)} = \lim_{h\to 0} \frac{-5}{\sqrt{3-5(x+h)}+\sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

(b) Domain of f: (the radicand must be nonnegative) $3-5x \geq 0 \quad \Rightarrow$

$5x \leq 3 \quad \Rightarrow \quad x \in \left(-\infty, \frac{3}{5}\right]$

Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero;

$x \in \left(-\infty, \frac{3}{5}\right)$



(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.

13. $y = (x^2+x^3)^4 \quad \Rightarrow \quad y' = 4(x^2+x^3)^3(2x+3x^2) = 4(x^2)^3(1+x)^3x(2+3x) = 4x^7(x+1)^3(3x+2)$

14. $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}} = x^{-1/2} - x^{-3/5} \quad \Rightarrow \quad y' = -\dfrac{1}{2}x^{-3/2} + \dfrac{3}{5}x^{-8/5}$ or $\dfrac{3}{5x\sqrt[5]{x^3}} - \dfrac{1}{2x\sqrt{x}}$ or $\dfrac{1}{10}x^{-8/5}(-5x^{1/10}+6)$

15. $y = \dfrac{x^2-x+2}{\sqrt{x}} = x^{3/2} - x^{1/2} + 2x^{-1/2} \quad \Rightarrow \quad y' = \dfrac{3}{2}x^{1/2} - \dfrac{1}{2}x^{-1/2} - x^{-3/2} = \dfrac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x}} - \dfrac{1}{\sqrt{x^3}}$

16. $y = \dfrac{\tan x}{1+\cos x} \quad \Rightarrow \quad y' = \dfrac{(1+\cos x)\sec^2 x - \tan x(-\sin x)}{(1+\cos x)^2} = \dfrac{(1+\cos x)\sec^2 x + \tan x\,\sin x}{(1+\cos x)^2}$

17. $y = x^2\sin \pi x \quad \Rightarrow \quad y' = x^2(\cos \pi x)\pi + (\sin \pi x)(2x) = x(\pi x \cos \pi x + 2\sin \pi x)$

18. $y = \left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}} \quad \Rightarrow \quad y' = \sqrt{7}\left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}-1}\left(1 - \dfrac{2}{x^3}\right)$

19. $y = \dfrac{t^4-1}{t^4+1} \quad \Rightarrow \quad y' = \dfrac{(t^4+1)4t^3 - (t^4-1)4t^3}{(t^4+1)^2} = \dfrac{4t^3[(t^4+1)-(t^4-1)]}{(t^4+1)^2} = \dfrac{8t^3}{(t^4+1)^2}$

20. $y = \sin(\cos x) \quad \Rightarrow \quad y' = \cos(\cos x)(-\sin x) = -\sin x\,\cos(\cos x)$

21. $y = \tan\sqrt{1-x} \quad \Rightarrow \quad y' = \left(\sec^2\sqrt{1-x}\right)\left(\dfrac{1}{2\sqrt{1-x}}\right)(-1) = -\dfrac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$

22. Using the Reciprocal Rule, $g(x) = \dfrac{1}{f(x)} \quad \Rightarrow \quad g'(x) = -\dfrac{f'(x)}{[f(x)]^2}$, we have $y = \dfrac{1}{\sin(x-\sin x)} \quad \Rightarrow$

$y' = -\dfrac{\cos(x-\sin x)(1-\cos x)}{\sin^2(x-\sin x)}$.

23. $\dfrac{d}{dx}(xy^4+x^2y) = \dfrac{d}{dx}(x+3y) \quad \Rightarrow \quad x\cdot 4y^3y' + y^4\cdot 1 + x^2\cdot y' + y\cdot 2x = 1 + 3y' \quad \Rightarrow$

$y'(4xy^3+x^2-3) = 1-y^4-2xy \quad \Rightarrow \quad y' = \dfrac{1-y^4-2xy}{4xy^3+x^2-3}$

24. $y = \sec(1+x^2) \quad \Rightarrow \quad y' = 2x\sec(1+x^2)\,\tan(1+x^2)$

25. $y = \dfrac{\sec 2\theta}{1 + \tan 2\theta} \quad \Rightarrow$

$$y' = \frac{(1 + \tan 2\theta)(\sec 2\theta \ \tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1 + \tan 2\theta)^2} = \frac{2 \sec 2\theta \, [(1 + \tan 2\theta) \tan 2\theta - \sec^2 2\theta]}{(1 + \tan 2\theta)^2}$$
$$= \frac{2 \sec 2\theta \, (\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1 + \tan 2\theta)^2} = \frac{2 \sec 2\theta \, (\tan 2\theta - 1)}{(1 + \tan 2\theta)^2} \qquad [1 + \tan^2 x = \sec^2 x]$$

26. $\dfrac{d}{dx} (x^2 \cos y + \sin 2y) = \dfrac{d}{dx} (xy) \quad \Rightarrow \quad x^2(-\sin y \cdot y') + (\cos y)(2x) + \cos 2y \cdot 2y' = x \cdot y' + y \cdot 1 \quad \Rightarrow$

$y'(-x^2 \sin y + 2 \cos 2y - x) = y - 2x \cos y \quad \Rightarrow \quad y' = \dfrac{y - 2x \cos y}{2 \cos 2y - x^2 \sin y - x}$

27. $y = (1 - x^{-1})^{-1} \quad \Rightarrow$

$y' = -1(1 - x^{-1})^{-2}[-(-1x^{-2})] = -(1 - 1/x)^{-2} x^{-2} = -((x - 1)/x)^{-2} x^{-2} = -(x - 1)^{-2}$

28. $y = \dfrac{1}{\sqrt[3]{x + \sqrt{x}}} = \left(x + \sqrt{x}\right)^{-1/3} \quad \Rightarrow \quad y' = -\frac{1}{3}\left(x + \sqrt{x}\right)^{-4/3}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

29. $\sin(xy) = x^2 - y \quad \Rightarrow \quad \cos(xy)(xy' + y \cdot 1) = 2x - y' \quad \Rightarrow \quad x \cos(xy) y' + y' = 2x - y \cos(xy) \quad \Rightarrow$

$y'[x \cos(xy) + 1] = 2x - y \cos(xy) \quad \Rightarrow \quad y' = \dfrac{2x - y \cos(xy)}{x \cos(xy) + 1}$

30. $y = \sqrt{\sin\sqrt{x}} \quad \Rightarrow \quad y' = \frac{1}{2}\left(\sin\sqrt{x}\right)^{-1/2}\left(\cos\sqrt{x}\right)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\cos\sqrt{x}}{4\sqrt{x \sin\sqrt{x}}}$

31. $y = \cot(3x^2 + 5) \quad \Rightarrow \quad y' = -\csc^2(3x^2 + 5)(6x) = -6x \csc^2(3x^2 + 5)$

32. $y = \dfrac{(x + \lambda)^4}{x^4 + \lambda^4} \quad \Rightarrow \quad y' = \dfrac{(x^4 + \lambda^4)(4)(x + \lambda)^3 - (x + \lambda)^4(4x^3)}{(x^4 + \lambda^4)^2} = \dfrac{4(x + \lambda)^3(\lambda^4 - \lambda x^3)}{(x^4 + \lambda^4)^2}$

33. $y = \sqrt{x} \cos \sqrt{x} \quad \Rightarrow$

$$y' = \sqrt{x} \left(\cos \sqrt{x}\right)' + \cos \sqrt{x} \left(\sqrt{x}\right)' = \sqrt{x} \left[-\sin \sqrt{x} \left(\tfrac{1}{2} x^{-1/2}\right)\right] + \cos \sqrt{x} \left(\tfrac{1}{2} x^{-1/2}\right)$$
$$= \tfrac{1}{2} x^{-1/2} \left(-\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x}\right) = \frac{\cos \sqrt{x} - \sqrt{x} \sin \sqrt{x}}{2\sqrt{x}}$$

34. $y = (\sin mx)/x \quad \Rightarrow \quad y' = (mx \cos mx - \sin mx)/x^2$

35. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \quad \Rightarrow \quad y' = 2[\tan(\sin\theta)] \cdot \sec^2(\sin\theta) \cdot \cos\theta$

36. $x \tan y = y - 1 \quad \Rightarrow \quad \tan y + (x \sec^2 y)\, y' = y' \quad \Rightarrow \quad y' = \dfrac{\tan y}{1 - x \sec^2 y}$

37. $y = (x \tan x)^{1/5} \quad \Rightarrow \quad y' = \frac{1}{5}(x \tan x)^{-4/5}(\tan x + x \sec^2 x)$

38. $y = \dfrac{(x - 1)(x - 4)}{(x - 2)(x - 3)} = \dfrac{x^2 - 5x + 4}{x^2 - 5x + 6} \quad \Rightarrow \quad y' = \dfrac{(x^2 - 5x + 6)(2x - 5) - (x^2 - 5x + 4)(2x - 5)}{(x^2 - 5x + 6)^2} = \dfrac{2(2x - 5)}{(x - 2)^2(x - 3)^2}$

39. $y = \sin\left(\tan\sqrt{1 + x^3}\right) \quad \Rightarrow \quad y' = \cos\left(\tan\sqrt{1 + x^3}\right)\left(\sec^2\sqrt{1 + x^3}\right)\left[3x^2/\left(2\sqrt{1 + x^3}\right)\right]$

40. $y = \sin^2\left(\cos\sqrt{\sin \pi x}\right) = \left[\sin\left(\cos\sqrt{\sin \pi x}\right)\right]^2 \quad \Rightarrow$

$$\begin{aligned} y' &= 2\left[\sin\left(\cos\sqrt{\sin \pi x}\right)\right]\left[\sin\left(\cos\sqrt{\sin \pi x}\right)\right]' = 2\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\left(\cos\sqrt{\sin \pi x}\right)' \\ &= 2\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\left(-\sin\sqrt{\sin \pi x}\right)\left(\sqrt{\sin \pi x}\right)' \\ &= -2\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\sin\sqrt{\sin \pi x}\cdot \tfrac{1}{2}(\sin \pi x)^{-1/2}(\sin \pi x)' \\ &= \frac{-\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\sin\sqrt{\sin \pi x}}{\sqrt{\sin \pi x}}\cdot \cos \pi x \cdot \pi \\ &= \frac{-\pi \sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\sin\sqrt{\sin \pi x}\,\cos \pi x}{\sqrt{\sin \pi x}} \end{aligned}$$

41. $f(t) = \sqrt{4t+1} \quad \Rightarrow \quad f'(t) = \frac{1}{2}(4t+1)^{-1/2}\cdot 4 = 2(4t+1)^{-1/2} \quad \Rightarrow$

$f''(t) = 2(-\frac{1}{2})(4t+1)^{-3/2}\cdot 4 = -4/(4t+1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\frac{4}{27}$.

42. $g(\theta) = \theta\sin\theta \quad \Rightarrow \quad g'(\theta) = \theta\cos\theta + \sin\theta\cdot 1 \quad \Rightarrow \quad g''(\theta) = \theta(-\sin\theta) + \cos\theta\cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$,

so $g''(\pi/6) = 2\cos(\pi/6) - (\pi/6)\sin(\pi/6) = 2\left(\sqrt{3}/2\right) - (\pi/6)(1/2) = \sqrt{3} - \pi/12$.

43. $x^6 + y^6 = 1 \quad \Rightarrow \quad 6x^5 + 6y^5y' = 0 \quad \Rightarrow \quad y' = -x^5/y^5 \quad \Rightarrow$

$$y'' = -\frac{y^5(5x^4) - x^5(5y^4y')}{(y^5)^2} = -\frac{5x^4y^4\left[y - x(-x^5/y^5)\right]}{y^{10}} = -\frac{5x^4\left[(y^6+x^6)/y^5\right]}{y^6} = -\frac{5x^4}{y^{11}}$$

44. $f(x) = (2-x)^{-1} \quad \Rightarrow \quad f'(x) = (2-x)^{-2} \quad \Rightarrow \quad f''(x) = 2(2-x)^{-3} \quad \Rightarrow \quad f'''(x) = 2\cdot 3(2-x)^{-4} \quad \Rightarrow$

$f^{(4)}(x) = 2\cdot 3\cdot 4(2-x)^{-5}$. In general, $f^{(n)}(x) = 2\cdot 3\cdot 4\cdot\cdots\cdot n(2-x)^{-(n+1)} = \dfrac{n!}{(2-x)^{(n+1)}}$.

45. $\displaystyle\lim_{x\to 0}\frac{\sec x}{1-\sin x} = \frac{\sec 0}{1-\sin 0} = \frac{1}{1-0} = 1$

46. $\displaystyle\lim_{t\to 0}\frac{t^3}{\tan^3 2t} = \lim_{t\to 0}\frac{t^3\cos^3 2t}{\sin^3 2t} = \lim_{t\to 0}\cos^3 2t\cdot\frac{1}{8\dfrac{\sin^3 2t}{(2t)^3}} = \lim_{t\to 0}\frac{\cos^3 2t}{8\left(\lim\limits_{t\to 0}\dfrac{\sin 2t}{2t}\right)^3} = \frac{1}{8\cdot 1^3} = \frac{1}{8}$

47. $y = 4\sin^2 x \quad \Rightarrow \quad y' = 4\cdot 2\sin x\cos x$. At $\left(\frac{\pi}{6}, 1\right)$, $y' = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line is $y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$, or $y = 2\sqrt{3}\,x + 1 - \pi\sqrt{3}/3$.

48. $y = \dfrac{x^2-1}{x^2+1} \quad \Rightarrow \quad y' = \dfrac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$.

At $(0,-1)$, $y' = 0$, so an equation of the tangent line is $y + 1 = 0(x-0)$, or $y = -1$.

49. $y = \sqrt{1+4\sin x} \quad \Rightarrow \quad y' = \frac{1}{2}(1+4\sin x)^{-1/2}\cdot 4\cos x = \dfrac{2\cos x}{\sqrt{1+4\sin x}}$.

At $(0,1)$, $y' = \dfrac{2}{\sqrt{1}} = 2$, so an equation of the tangent line is $y - 1 = 2(x-0)$, or $y = 2x+1$.

The slope of the normal line is $-\frac{1}{2}$, so an equation of the normal line is $y - 1 = -\frac{1}{2}(x-0)$, or $y = -\frac{1}{2}x + 1$.

50. $x^2 + 4xy + y^2 = 13 \quad \Rightarrow \quad 2x + 4(xy' + y \cdot 1) + 2yy' = 0 \quad \Rightarrow \quad x + 2xy' + 2y + yy' = 0 \quad \Rightarrow$

$2xy' + yy' = -x - 2y \quad \Rightarrow \quad y'(2x + y) = -x - 2y \quad \Rightarrow \quad y' = \dfrac{-x - 2y}{2x + y}$.

At $(2, 1)$, $y' = \dfrac{-2 - 2}{4 + 1} = -\dfrac{4}{5}$, so an equation of the tangent line is $y - 1 = -\frac{4}{5}(x - 2)$, or $y = -\frac{4}{5}x + \frac{13}{5}$.

The slope of the normal line is $\frac{5}{4}$, so an equation of the normal line is $y - 1 = \frac{5}{4}(x - 2)$, or $y = \frac{5}{4}x - \frac{3}{2}$.

51. (a) $f(x) = x\sqrt{5 - x} \quad \Rightarrow$

$$f'(x) = x\left[\frac{1}{2}(5 - x)^{-1/2}(-1)\right] + \sqrt{5 - x} = \frac{-x}{2\sqrt{5 - x}} + \sqrt{5 - x} \cdot \frac{2\sqrt{5 - x}}{2\sqrt{5 - x}} = \frac{-x}{2\sqrt{5 - x}} + \frac{2(5 - x)}{2\sqrt{5 - x}}$$

$$= \frac{-x + 10 - 2x}{2\sqrt{5 - x}} = \frac{10 - 3x}{2\sqrt{5 - x}}$$

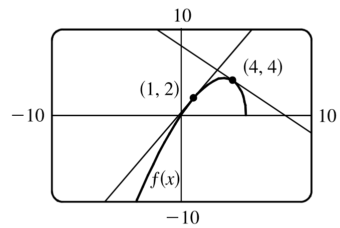
(b) At $(1, 2)$: $f'(1) = \frac{7}{4}$.

So an equation of the tangent line is $y - 2 = \frac{7}{4}(x - 1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

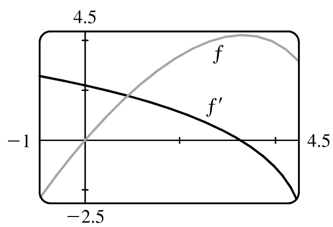
At $(4, 4)$: $f'(4) = -\frac{2}{2} = -1$.

So an equation of the tangent line is $y - 4 = -1(x - 4)$ or $y = -x + 8$.

(c)



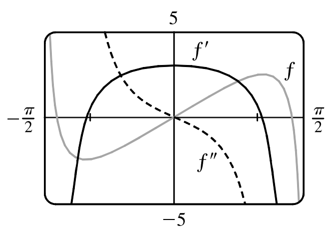
(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

52. (a) $f(x) = 4x - \tan x \quad \Rightarrow \quad f'(x) = 4 - \sec^2 x \quad \Rightarrow \quad f''(x) = -2\sec x\,(\sec x\,\tan x) = -2\sec^2 x\,\tan x$.

(b)



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f''.

53. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x = 0 \quad \Leftrightarrow \quad \cos x = \sin x$ and $0 \le x \le 2\pi \quad \Leftrightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

54. $x^2 + 2y^2 = 1 \quad \Rightarrow \quad 2x + 4yy' = 0 \quad \Rightarrow \quad y' = -x/(2y) = 1 \quad \Leftrightarrow \quad x = -2y$. Since the points lie on the ellipse, we have $(-2y)^2 + 2y^2 = 1 \quad \Rightarrow \quad 6y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{6}}$. The points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

55. $y = f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$. We know that $f'(-1) = 6$ and $f'(5) = -2$, so $-2a + b = 6$ and $10a + b = -2$. Subtracting the first equation from the second gives $12a = -8 \quad \Rightarrow \quad a = -\frac{2}{3}$. Substituting $-\frac{2}{3}$ for a in the first equation gives $b = \frac{14}{3}$. Now $f(1) = 4 \quad \Rightarrow \quad 4 = a + b + c$, so $c = 4 + \frac{2}{3} - \frac{14}{3} = 0$ and hence, $f(x) = -\frac{2}{3}x^2 + \frac{14}{3}x$.

56. If $y = f(x) = \dfrac{x}{x+1}$, then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is

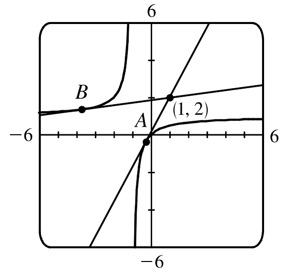
$y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x-a)$. This line passes through $(1, 2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1-a) \quad \Leftrightarrow$

$2(a+1)^2 - a(a+1) = 1 - a \quad \Leftrightarrow \quad 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \quad \Leftrightarrow \quad a^2 + 4a + 1 = 0.$

The quadratic formula gives the roots of this equation as $a = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$,

so there are two such tangent lines. Since

$$f(-2 \pm \sqrt{3}) = \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}}$$
$$= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2},$$



the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1 - \sqrt{3}}{2}\right) \approx (-0.27, -0.37)$

and $B\left(-2 - \sqrt{3}, \frac{1 + \sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.

57. $f(x) = (x-a)(x-b)(x-c) \quad \Rightarrow \quad f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$.

So $\dfrac{f'(x)}{f(x)} = \dfrac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c}$.

58. (a) $\cos 2x = \cos^2 x - \sin^2 x \quad \Rightarrow \quad -2\sin 2x = -2\cos x \sin x - 2\sin x \cos x \quad \Leftrightarrow \quad \sin 2x = 2\sin x \cos x$

(b) $\sin(x+a) = \sin x \, \cos a + \cos x \, \sin a \quad \Rightarrow \quad \cos(x+a) = \cos x \, \cos a - \sin x \, \sin a$.

59. (a) $S(x) = f(x) + g(x) \quad \Rightarrow \quad S'(x) = f'(x) + g'(x) \quad \Rightarrow \quad S'(1) = f'(1) + g'(1) = 3 + 1 = 4$

(b) $P(x) = f(x)\,g(x) \quad \Rightarrow \quad P'(x) = f(x)\,g'(x) + g(x)\,f'(x) \quad \Rightarrow$

$P'(2) = f(2)\,g'(2) + g(2)f'(2) = 1(4) + 1(2) = 4 + 2 = 6$

(c) $Q(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad Q'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2} \quad \Rightarrow$

$Q'(1) = \dfrac{g(1)\,f'(1) - f(1)\,g'(1)}{[g(1)]^2} = \dfrac{3(3) - 2(1)}{3^2} = \dfrac{9-2}{9} = \dfrac{7}{9}$

(d) $C(x) = f(g(x)) \quad \Rightarrow \quad C'(x) = f'(g(x))\,g'(x) \quad \Rightarrow \quad C'(2) = f'(g(2))\,g'(2) = f'(1) \cdot 4 = 3 \cdot 4 = 12$

60. (a) $P(x) = f(x)\,g(x) \quad \Rightarrow \quad P'(x) = f(x)\,g'(x) + g(x)\,f'(x) \quad \Rightarrow$

$P'(2) = f(2)\,g'(2) + g(2)\,f'(2) = (1)\left(\frac{6-0}{3-0}\right) + (4)\left(\frac{0-3}{3-0}\right) = (1)(2) + (4)(-1) = 2 - 4 = -2$

(b) $Q(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad Q'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2} \quad \Rightarrow$

$Q'(2) = \dfrac{g(2)\,f'(2) - f(2)\,g'(2)}{[g(2)]^2} = \dfrac{(4)(-1) - (1)(2)}{4^2} = \dfrac{-6}{16} = -\dfrac{3}{8}$

(c) $C(x) = f(g(x)) \quad \Rightarrow \quad C'(x) = f'(g(x))g'(x) \quad \Rightarrow$

$C'(2) = f'(g(2))g'(2) = f'(4)g'(2) = \left(\frac{6-0}{5-3}\right)(2) = (3)(2) = 6$

61. $f(x) = x^2 g(x) \quad\Rightarrow\quad f'(x) = x^2 g'(x) + g(x)(2x) = x[xg'(x) + 2g(x)]$

62. $f(x) = g(x^2) \quad\Rightarrow\quad f'(x) = g'(x^2)(2x) = 2xg'(x^2)$

63. $f(x) = [g(x)]^2 \quad\Rightarrow\quad f'(x) = 2[g(x)] \cdot g'(x) = 2g(x)\,g'(x)$

64. $f(x) = x^a g(x^b) \quad\Rightarrow\quad f'(x) = ax^{a-1}g(x^b) + x^a g'(x^b)(bx^{b-1}) = ax^{a-1}g(x^b) + bx^{a+b-1}g'(x^b)$

65. $f(x) = g(g(x)) \quad\Rightarrow\quad f'(x) = g'(g(x))\,g'(x)$

66. $f(x) = \sin(g(x)) \quad\Rightarrow\quad f'(x) = \cos(g(x)) \cdot g'(x)$

67. $f(x) = g(\sin x) \quad\Rightarrow\quad f'(x) = g'(\sin x) \cdot \cos x$

68. $f(x) = g\left(\tan\sqrt{x}\right) \quad\Rightarrow$

$$f'(x) = g'\left(\tan\sqrt{x}\right) \cdot \frac{d}{dx}\left(\tan\sqrt{x}\right) = g'\left(\tan\sqrt{x}\right) \cdot \sec^2\sqrt{x} \cdot \frac{d}{dx}\left(\sqrt{x}\right) = \frac{g'\left(\tan\sqrt{x}\right)\sec^2\sqrt{x}}{2\sqrt{x}}$$

69. $h(x) = \dfrac{f(x)\,g(x)}{f(x) + g(x)} \quad\Rightarrow$

$$\begin{aligned} h'(x) &= \frac{[f(x) + g(x)]\,[f(x)\,g'(x) + g(x)\,f'(x)] - f(x)\,g(x)\,[f'(x) + g'(x)]}{[f(x) + g(x)]^2} \\ &= \frac{[f(x)]^2\,g'(x) + f(x)\,g(x)\,f'(x) + f(x)\,g(x)\,g'(x) + [g(x)]^2\,f'(x) - f(x)\,g(x)\,f'(x) - f(x)\,g(x)\,g'(x)}{[f(x) + g(x)]^2} \\ &= \frac{f'(x)\,[g(x)]^2 + g'(x)\,[f(x)]^2}{[f(x) + g(x)]^2} \end{aligned}$$

70. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}} \quad\Rightarrow\quad h'(x) = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2\sqrt{f(x)/g(x)}\,[g(x)]^2} = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2[g(x)]^{3/2}\sqrt{f(x)}}$

71. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \quad\Rightarrow$

$$h'(x) = f'(g(\sin 4x)) \cdot \frac{d}{dx}\,(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \frac{d}{dx}\,(\sin 4x) = f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4).$$

72. (a) $x = \sqrt{b^2 + c^2 t^2} \quad\Rightarrow\quad v(t) = x' = \left[1/\left(2\sqrt{b^2 + c^2 t^2}\right)\right] 2c^2 t = c^2 t/\sqrt{b^2 + c^2 t^2} \quad\Rightarrow$

$$a(t) = v'(t) = \frac{c^2\sqrt{b^2 + c^2 t^2} - c^2 t\left(c^2 t/\sqrt{b^2 + c^2 t^2}\right)}{b^2 + c^2 t^2} = \frac{b^2 c^2}{(b^2 + c^2 t^2)^{3/2}}$$

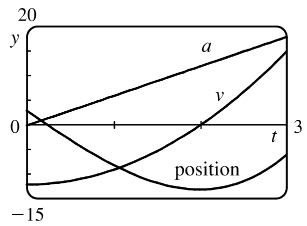
(b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

73. (a) $y = t^3 - 12t + 3 \quad\Rightarrow\quad v(t) = y' = 3t^2 - 12 \quad\Rightarrow\quad a(t) = v'(t) = 6t$

(b) $v(t) = 3(t^2 - 4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3) - y(2) = -6 - (-13) = 7$,

Distance downward $= y(0) - y(2) = 3 - (-13) = 16$. Total distance $= 7 + 16 = 23$.

(d) 

(e) The particle is speeding up when v and a have the same sign, that is, when $t > 2$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 2$.

74. (a) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dh = \frac{1}{3}\pi r^2 \quad [r \text{ constant}]$

(b) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dr = \frac{2}{3}\pi r h \quad [h \text{ constant}]$

75. The linear density ρ is the rate of change of mass m with respect to length x.

$m = x\left(1 + \sqrt{x}\right) = x + x^{3/2} \quad \Rightarrow \quad \rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2}\sqrt{4} = 4$ kg/m.

76. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \quad \Rightarrow \quad C'(x) = 2 - 0.04x + 0.00021x^2$

(b) $C'(100) = 2 - 4 + 2.1 = \$0.10/\text{unit}$. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

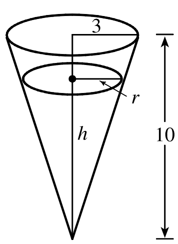
(c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \$0.10107$, slightly larger than $C'(100)$.

77. If $x =$ edge length, then $V = x^3 \quad \Rightarrow \quad dV/dt = 3x^2\,dx/dt = 10 \quad \Rightarrow \quad dx/dt = 10/(3x^2)$ and $S = 6x^2 \quad \Rightarrow$ $dS/dt = (12x)\,dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}\ \text{cm}^2/\text{min}$.

78. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2 h$ and, from similar triangles, $\dfrac{r}{h} = \dfrac{3}{10} \quad \Rightarrow \quad V = \dfrac{\pi}{3}\left(\dfrac{3h}{10}\right)^2 h = \dfrac{3\pi}{100}h^3$, so

$$2 = \frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi\,(5)^2} = \frac{8}{9\pi}\ \text{cm/s}$$

when $h = 5$.

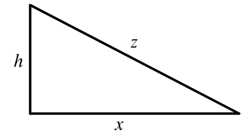


79. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \quad \Rightarrow$

$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 2h\,\dfrac{dh}{dt} \quad \Rightarrow \quad \dfrac{dz}{dt} = \dfrac{1}{z}(15x + 5h)$. When $t = 3$,

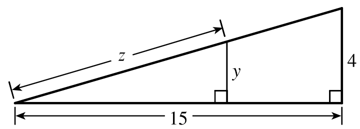
$h = 45 + 3(5) = 60$ and $x = 15(3) = 45 \quad \Rightarrow \quad z = \sqrt{45^2 + 60^2} = 75$,

so $\dfrac{dz}{dt} = \dfrac{1}{75}[15(45) + 5(60)] = 13$ ft/s.



80. We are given $dz/dt = 30$ ft/s. By similar triangles, $\dfrac{y}{z} = \dfrac{4}{\sqrt{241}} \quad \Rightarrow$

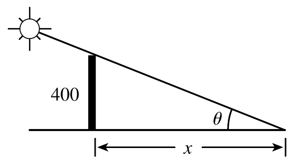
$y = \dfrac{4}{\sqrt{241}}z$, so $\dfrac{dy}{dt} = \dfrac{4}{\sqrt{241}}\dfrac{dz}{dt} = \dfrac{120}{\sqrt{241}} \approx 7.7$ ft/s.



81. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \quad\Rightarrow$

$x = 400\cot\theta \quad\Rightarrow\quad \dfrac{dx}{dt} = -400\csc^2\theta\,\dfrac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$,

$\dfrac{dx}{dt} = -400(2)^2(-0.25) = 400$ ft/h.

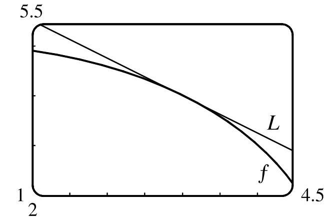


82. (a) $f(x) = \sqrt{25 - x^2} \quad\Rightarrow\quad f'(x) = \dfrac{-2x}{2\sqrt{25 - x^2}} = -x(25 - x^2)^{-1/2}$.

So the linear approximation to $f(x)$ near 3

is $f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3)$.

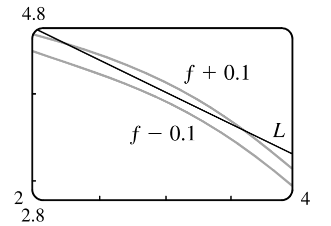
(b)



(c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$ and

$4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that these both

hold for $2.24 < x < 3.66$.



83. (a) $f(x) = \sqrt[3]{1 + 3x} = (1 + 3x)^{1/3} \quad\Rightarrow\quad f'(x) = (1 + 3x)^{-2/3}$, so the linearization of f at $a = 0$ is

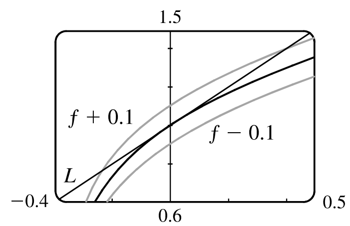
$L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1 + 3x} \approx 1 + x \quad\Rightarrow$

$\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)} \approx 1 + (0.01) = 1.01$.

(b) The linear approximation is $\sqrt[3]{1 + 3x} \approx 1 + x$, so for the required accuracy

we want $\sqrt[3]{1 + 3x} - 0.1 < 1 + x < \sqrt[3]{1 + 3x} + 0.1$. From the graph,

it appears that this is true when $-0.235 < x < 0.401$.

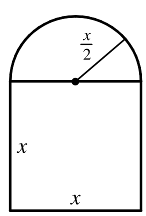


84. $y = x^3 - 2x^2 + 1 \quad\Rightarrow\quad dy = (3x^2 - 4x)\,dx$. When $x = 2$ and $dx = 0.2$, $dy = \left[3(2)^2 - 4(2)\right](0.2) = 0.8$.

85. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2 \quad\Rightarrow\quad dA = \left(2 + \frac{\pi}{4}\right)x\,dx$. When $x = 60$

and $dx = 0.1$, $dA = \left(2 + \frac{\pi}{4}\right)60(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is

approximately $12 + \frac{3\pi}{2} \approx 16.7\ \text{cm}^2$.



86. $\displaystyle\lim_{x\to 1}\frac{x^{17} - 1}{x - 1} = \left[\frac{d}{dx}x^{17}\right]_{x=1} = 17(1)^{16} = 17$

87. $\displaystyle\lim_{h\to 0}\frac{\sqrt[4]{16 + h} - 2}{h} = \left[\frac{d}{dx}\sqrt[4]{x}\right]_{x=16} = \frac{1}{4}x^{-3/4}\Big|_{x=16} = \frac{1}{4\left(\sqrt[4]{16}\right)^3} = \frac{1}{32}$

88. $\displaystyle\lim_{\theta\to\pi/3}\frac{\cos\theta-0.5}{\theta-\pi/3}=\left[\frac{d}{d\theta}\cos\theta\right]_{\theta=\pi/3}=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$

89.
$$\begin{aligned}\lim_{x\to0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}&=\lim_{x\to0}\frac{\left(\sqrt{1+\tan x}-\sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\\&=\lim_{x\to0}\frac{(1+\tan x)-(1+\sin x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}=\lim_{x\to0}\frac{\sin x\,(1/\cos x-1)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\cdot\frac{\cos x}{\cos x}\\&=\lim_{x\to0}\frac{\sin x\,(1-\cos x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x}\cdot\frac{1+\cos x}{1+\cos x}\\&=\lim_{x\to0}\frac{\sin x\cdot\sin^2 x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}\\&=\left(\lim_{x\to0}\frac{\sin x}{x}\right)^3\lim_{x\to0}\frac{1}{\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}\\&=1^3\cdot\frac{1}{\left(\sqrt{1}+\sqrt{1}\right)\cdot1\cdot(1+1)}=\frac{1}{4}\end{aligned}$$

90. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain $f(g(x))=x \;\Rightarrow\; f'(g(x))\,g'(x)=1 \;\Rightarrow\; g'(x)=\dfrac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression gives $g'(x)=\dfrac{1}{1+[f(g(x))]^2}$. But the first given equation states that $f(g(x))=x$, so $g'(x)=\dfrac{1}{1+x^2}$.

91. $\dfrac{d}{dx}[f(2x)]=x^2 \;\Rightarrow\; f'(2x)\cdot2=x^2 \;\Rightarrow\; f'(2x)=\frac{1}{2}x^2$. Let $t=2x$. Then $f'(t)=\frac{1}{2}\left(\frac{1}{2}t\right)^2=\frac{1}{8}t^2$, so $f'(x)=\frac{1}{8}x^2$.

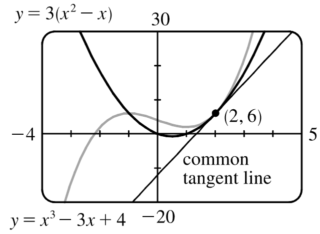
92. Let (b,c) be on the curve, that is, $b^{2/3}+c^{2/3}=a^{2/3}$. Now $x^{2/3}+y^{2/3}=a^{2/3} \;\Rightarrow\; \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\dfrac{dy}{dx}=0$, so $\dfrac{dy}{dx}=-\dfrac{y^{1/3}}{x^{1/3}}=-\left(\dfrac{y}{x}\right)^{1/3}$, so at (b,c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is $y-c=-(c/b)^{1/3}(x-b)$ or $y=-(c/b)^{1/3}x+(c+b^{2/3}c^{1/3})$. Setting $y=0$, we find that the x-intercept is $b^{1/3}c^{2/3}+b=b^{1/3}(c^{2/3}+b^{2/3})=b^{1/3}a^{2/3}$ and setting $x=0$ we find that the y-intercept is $c+b^{2/3}c^{1/3}=c^{1/3}(c^{2/3}+b^{2/3})=c^{1/3}a^{2/3}$. So the length of the tangent line between these two points is

$$\begin{aligned}\sqrt{(b^{1/3}a^{2/3})^2+(c^{1/3}a^{2/3})^2}&=\sqrt{b^{2/3}a^{4/3}+c^{2/3}a^{4/3}}=\sqrt{(b^{2/3}+c^{2/3})a^{4/3}}\\&=\sqrt{a^{2/3}a^{4/3}}=\sqrt{a^2}=a=\text{constant}\end{aligned}$$

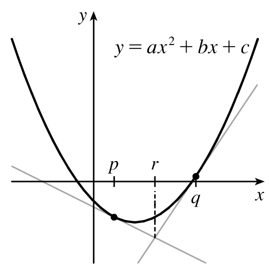
☐ PROBLEMS PLUS

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3} \quad \Rightarrow \quad a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

2. $y = x^3 - 3x + 4 \quad \Rightarrow \quad y' = 3x^2 - 3$, and $y = 3(x^2 - x) \quad \Rightarrow \quad y' = 6x - 3$. The slopes of the tangents of the two curves are equal when $3x^2 - 3 = 6x - 3$; that is, when $x = 0$ or 2. At $x = 0$, both tangents have slope -3, but the curves do not intersect. At $x = 2$, both tangents have slope 9 and the curves intersect at $(2, 6)$. So there is a common tangent line at $(2, 6)$, $y = 9x - 12$.



3.



We must show that r (in the figure) is halfway between p and q, that is, $r = (p + q)/2$. For the parabola $y = ax^2 + bx + c$, the slope of the tangent line is given by $y' = 2ax + b$. An equation of the tangent line at $x = p$ is $y - (ap^2 + bp + c) = (2ap + b)(x - p)$. Solving for y gives us

$$y = (2ap + b)x - 2ap^2 - bp + (ap^2 + bp + c)$$

or

$$y = (2ap + b)x + c - ap^2 \qquad \textbf{(1)}$$

Similarly, an equation of the tangent line at $x = q$ is

$$y = (2aq + b)x + c - aq^2 \qquad \textbf{(2)}$$

We can eliminate y and solve for x by subtracting equation **(1)** from equation **(2)**.

$$\begin{aligned} [(2aq + b) - (2ap + b)]x - aq^2 + ap^2 &= 0 \\ (2aq - 2ap)x &= aq^2 - ap^2 \\ 2a(q - p)x &= a(q^2 - p^2) \\ x &= \frac{a(q + p)(q - p)}{2a(q - p)} = \frac{p + q}{2} \end{aligned}$$

Thus, the x-coordinate of the point of intersection of the two tangent lines, namely r, is $(p + q)/2$.

4. We could differentiate and then simplify or we can simplify and then differentiate. The latter seems to be the simpler method.

$$\begin{aligned} \frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} &= \frac{\sin^2 x}{1 + \dfrac{\cos x}{\sin x}} \cdot \frac{\sin x}{\sin x} + \frac{\cos^2 x}{1 + \dfrac{\sin x}{\cos x}} \cdot \frac{\cos x}{\cos x} = \frac{\sin^3 x}{\sin x + \cos x} + \frac{\cos^3 x}{\cos x + \sin x} \\ &= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \quad [\text{factor sum of cubes}] \quad = \frac{(\sin x + \cos x)(\sin^2 x - \sin x\,\cos x + \cos^2 x)}{\sin x + \cos x} \\ &= \sin^2 x - \sin x\,\cos x + \cos^2 x = 1 - \sin x\,\cos x = 1 - \tfrac{1}{2}(2\sin x\,\cos x) = 1 - \tfrac{1}{2}\sin 2x \end{aligned}$$

Thus, $\dfrac{d}{dx}\left(\dfrac{\sin^2 x}{1 + \cot x} + \dfrac{\cos^2 x}{1 + \tan x}\right) = \dfrac{d}{dx}\left(1 - \frac{1}{2}\sin 2x\right) = -\frac{1}{2}\cos 2x \cdot 2 = -\cos 2x$.

5. Using $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, we recognize the given expression, $f(x) = \lim\limits_{t \to x} \dfrac{\sec t - \sec x}{t - x}$, as $g'(x)$ with $g(x) = \sec x$. Now $f'(\frac{\pi}{4}) = g''(\frac{\pi}{4})$, so we will find $g''(x)$. $g'(x) = \sec x \tan x \quad \Rightarrow$

$g''(x) = \sec x \sec^2 x + \tan x \sec x \tan x = \sec x(\sec^2 x + \tan^2 x)$, so $g''(\frac{\pi}{4}) = \sqrt{2}(\sqrt{2}^2 + 1^2) = \sqrt{2}(2+1) = 3\sqrt{2}$.

6. Using $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$, we see that for the given equation, $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{ax+b} - 2}{x} = \dfrac{5}{12}$, we have $f(x) = \sqrt[3]{ax+b}$, $f(0) = 2$, and $f'(0) = \frac{5}{12}$. Now $f(0) = 2 \quad \Leftrightarrow \quad \sqrt[3]{b} = 2 \quad \Leftrightarrow \quad b = 8$. Also $f'(x) = \frac{1}{3}(ax+b)^{-2/3} \cdot a$, so

$f'(0) = \frac{5}{12} \quad \Leftrightarrow \quad \frac{1}{3}(8)^{-2/3} \cdot a = \frac{5}{12} \quad \Leftrightarrow \quad \frac{1}{3}(\frac{1}{4})a = \frac{5}{12} \quad \Leftrightarrow \quad a = 5$.

7. We use mathematical induction. Let S_n be the statement that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned} \frac{d}{dx}(\sin^4 x + \cos^4 x) &= 4\sin^3 x\,\cos x - 4\cos^3 x\,\sin x = 4\sin x\,\cos x\left(\sin^2 x - \cos^2 x\right)x \\ &= -4\sin x\,\cos x\,\cos 2x = -2\sin 2x\,\cos 2 = -\sin 4x = \sin(-4x) \\ &= \cos\left(\tfrac{\pi}{2} - (-4x)\right) = \cos\left(\tfrac{\pi}{2} + 4x\right) = 4^{n-1}\cos\left(4x + n\tfrac{\pi}{2}\right) \text{ when } n = 1 \end{aligned}$$

Now assume S_k is true, that is, $\dfrac{d^k}{dx^k}\left(\sin^4 x + \cos^4 x\right) = 4^{k-1}\cos\left(4x + k\frac{\pi}{2}\right)$. Then

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(\sin^4 x + \cos^4 x) &= \frac{d}{dx}\left[\frac{d^k}{dx^k}(\sin^4 x + \cos^4 x)\right] = \frac{d}{dx}\left[4^{k-1}\cos\left(4x + k\tfrac{\pi}{2}\right)\right] \\ &= -4^{k-1}\sin\left(4x + k\tfrac{\pi}{2}\right) \cdot \frac{d}{dx}\left(4x + k\tfrac{\pi}{2}\right) = -4^k\sin\left(4x + k\tfrac{\pi}{2}\right) \\ &= 4^k\sin\left(-4x - k\tfrac{\pi}{2}\right) = 4^k\cos\left(\tfrac{\pi}{2} - \left(-4x - k\tfrac{\pi}{2}\right)\right) = 4^k\cos\left(4x + (k+1)\tfrac{\pi}{2}\right) \end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another proof: First write

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\,\cos^2 x = 1 - \tfrac{1}{2}\sin^2 2x = 1 - \tfrac{1}{4}(1 - \cos 4x) = \tfrac{3}{4} + \tfrac{1}{4}\cos 4x$$

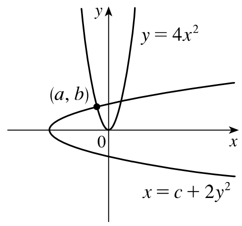
Then we have $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = \dfrac{d^n}{dx^n}\left(\dfrac{3}{4} + \dfrac{1}{4}\cos 4x\right) = \dfrac{1}{4} \cdot 4^n\cos\left(4x + n\dfrac{\pi}{2}\right) = 4^{n-1}\cos\left(4x + n\dfrac{\pi}{2}\right)$.

8.
$$\begin{aligned} \lim_{x \to a}\frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} &= \lim_{x \to a}\left[\frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}\right] = \lim_{x \to a}\left[\frac{f(x) - f(a)}{x - a} \cdot \left(\sqrt{x} + \sqrt{a}\right)\right] \\ &= \lim_{x \to a}\frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}\left(\sqrt{x} + \sqrt{a}\right) = f'(a) \cdot \left(\sqrt{a} + \sqrt{a}\right) = 2\sqrt{a}\,f'(a) \end{aligned}$$

9. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the points $\left(\pm x_0, x_0^2\right)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\dfrac{1}{\pm 2x_0}$ and pass through $\left(\pm x_0, x_0^2\right)$ respectively, so the normals have the equations $y - x_0^2 = -\frac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \frac{1}{2x_0}(x + x_0)$. The common y-intercept is $x_0^2 + \frac{1}{2}$. We want to find the value of x_0 for which the distance from $\left(0, x_0^2 + \frac{1}{2}\right)$ to $\left(x_0, x_0^2\right)$ equals 1. The square of the distance is $(x_0 - 0)^2 + \left[x_0^2 - \left(x_0^2 + \frac{1}{2}\right)\right]^2 = x_0^2 + \frac{1}{4} = 1 \quad \Leftrightarrow \quad x_0 = \pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $\left(0, \frac{5}{4}\right)$.

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$. Solving with the equation of the parabola, $y = x^2$, we get $x^2 + (x^2 - a)^2 = 1 \quad \Leftrightarrow \quad x^2 + x^4 - 2ax^2 + a^2 = 1 \quad \Leftrightarrow$ $x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4(a^2 - 1) = 0 \quad \Leftrightarrow$ $1 - 4a + 4a^2 - 4a^2 + 4 = 0 \quad \Leftrightarrow \quad 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $\left(0, \frac{5}{4}\right)$.

10. See the figure. The parabolas $y = 4x^2$ and $x = c + 2y^2$ intersect each other at right angles at the point (a, b) if and only if (a, b) satisfies both equations and the tangent lines at (a, b) are perpendicular. $y = 4x^2 \quad \Rightarrow \quad y' = 8x$ and $x = c + 2y^2 \quad \Rightarrow \quad 1 = 4y\,y' \quad \Rightarrow \quad y' = \frac{1}{4y}$, so at (a, b) we must

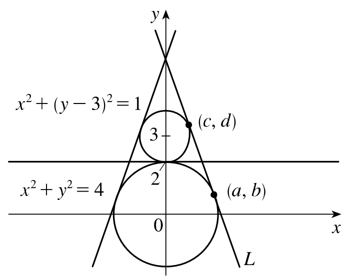


have $8a = -\frac{1}{1/(4b)} \quad \Rightarrow \quad 8a = -4b \quad \Rightarrow \quad b = -2a$. Since (a, b) is on both parabolas, we have **(1)** $b = 4a^2$ and **(2)** $a = c + 2b^2$. Substituting $-2a$ for b in **(1)** gives us $-2a = 4a^2 \quad \Rightarrow \quad 4a^2 + 2a = 0 \quad \Rightarrow \quad 2a(2a + 1) = 0 \quad \Rightarrow \quad a = 0$ or $a = -\frac{1}{2}$.

If $a = 0$, then $b = 0$ and $c = 0$, and the tangent lines at $(0, 0)$ are $y = 0$ and $x = 0$.

If $a = -\frac{1}{2}$, then $b = -2\left(-\frac{1}{2}\right) = 1$ and $-\frac{1}{2} = c + 2(1)^2 \quad \Rightarrow \quad c = -\frac{5}{2}$, and the tangent lines at $\left(-\frac{1}{2}, 1\right)$ are $y - 1 = -4\left(x + \frac{1}{2}\right)$ [or $y = -4x - 1$] and $y - 1 = \frac{1}{4}\left(x + \frac{1}{2}\right)$ $\left[\text{or } y = \frac{1}{4}x + \frac{9}{8}\right]$.

11. See the figure. Clearly, the line $y = 2$ is tangent to both circles at the point $(0, 2)$. We'll look for a tangent line L through the points (a, b) and (c, d), and if such a line exists, then its reflection through the y-axis is another such line. The slope of L is the same at (a, b) and (c, d). Find those slopes: $x^2 + y^2 = 4 \quad \Rightarrow$ $2x + 2y\,y' = 0 \quad \Rightarrow \quad y' = -\frac{x}{y} \quad \left[= -\frac{a}{b}\right]$ and $x^2 + (y - 3)^2 = 1 \quad \Rightarrow$ $2x + 2(y - 3)y' = 0 \quad \Rightarrow \quad y' = -\frac{x}{y - 3} \quad \left[= -\frac{c}{d - 3}\right]$.



Now an equation for L can be written using either point-slope pair, so we get $y - b = -\frac{a}{b}(x - a)$ $\left[\text{or } y = -\frac{a}{b}x + \frac{a^2}{b} + b\right]$ and $y - d = -\frac{c}{d - 3}(x - c)$ $\left[\text{or } y = -\frac{c}{d - 3}x + \frac{c^2}{d - 3} + d\right]$. The slopes are equal, so $-\frac{a}{b} = -\frac{c}{d - 3} \quad \Leftrightarrow$

$d - 3 = \dfrac{bc}{a}$. Since (c, d) is a solution of $x^2 + (y-3)^2 = 1$, we have $c^2 + (d-3)^2 = 1$, so $c^2 + \left(\dfrac{bc}{a}\right)^2 = 1 \;\Rightarrow$

$a^2c^2 + b^2c^2 = a^2 \;\Rightarrow\; c^2(a^2 + b^2) = a^2 \;\Rightarrow\; 4c^2 = a^2$ [since (a, b) is a solution of $x^2 + y^2 = 4$] $\Rightarrow\; a = 2c$.

Now $d - 3 = \dfrac{bc}{a} \;\Rightarrow\; d = 3 + \dfrac{bc}{2c}$, so $d = 3 + \dfrac{b}{2}$. The y-intercepts are equal, so $\dfrac{a^2}{b} + b = \dfrac{c^2}{d-3} + d \;\Leftrightarrow$

$\dfrac{a^2}{b} + b = \dfrac{(a/2)^2}{b/2} + \left(3 + \dfrac{b}{2}\right) \;\Leftrightarrow\; \left[\dfrac{a^2}{b} + b = \dfrac{a^2}{2b} + 3 + \dfrac{b}{2}\right](2b) \;\Leftrightarrow\; 2a^2 + 2b^2 = a^2 + 6b + b^2 \;\Leftrightarrow$

$a^2 + b^2 = 6b \;\Leftrightarrow\; 4 = 6b \;\Leftrightarrow\; b = \frac{2}{3}$. It follows that $d = 3 + \dfrac{b}{2} = \dfrac{10}{3}$, $a^2 = 4 - b^2 = 4 - \frac{4}{9} = \frac{32}{9} \;\Rightarrow\; a = \frac{4}{3}\sqrt{2}$,

and $c^2 = 1 - (d-3)^2 = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9} \;\Rightarrow\; c = \frac{2}{3}\sqrt{2}$. Thus, L has equation $y - \dfrac{2}{3} = -\dfrac{(4/3)\sqrt{2}}{2/3}\left(x - \dfrac{4}{3}\sqrt{2}\right) \;\Leftrightarrow$

$y - \frac{2}{3} = -2\sqrt{2}\left(x - \frac{4}{3}\sqrt{2}\right) \;\Leftrightarrow\; y = -2\sqrt{2}\,x + 6$. Its reflection has equation $y = 2\sqrt{2}\,x + 6$.

In summary, there are three lines tangent to both circles: $y = 2$ touches at $(0, 2)$, L touches at $\left(\frac{4}{3}\sqrt{2}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}\sqrt{2}, \frac{10}{3}\right)$, and its reflection through the y-axis touches at $\left(-\frac{4}{3}\sqrt{2}, \frac{2}{3}\right)$ and $\left(-\frac{2}{3}\sqrt{2}, \frac{10}{3}\right)$.

12. $f(x) = \dfrac{x^{46} + x^{45} + 2}{1 + x} = \dfrac{x^{45}(x+1) + 2}{x+1} = \dfrac{x^{45}(x+1)}{x+1} + \dfrac{2}{x+1} = x^{45} + 2(x+1)^{-1}$, so

$f^{(46)}(x) = (x^{45})^{(46)} + 2\left[(x+1)^{-1}\right]^{(46)}$. The forty-sixth derivative of any forty-fifth degree polynomial is 0, so

$(x^{45})^{46} = 0$. Thus, $f^{(46)}(x) = 2\left[(-1)(-2)(-3)\cdots(-46)(x+1)^{-47}\right] = 2(46!)(x+1)^{-47}$ and $f^{(46)}(3) = 2(46!)(4)^{-47}$ or $(46!)2^{-93}$.

13. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $= 360 \cdot (2\pi \text{ rad})/(60 \text{ s}) = 12\pi$ rad/s.] Then the position of A as a function of time is

$A = (40\cos\theta, 40\sin\theta) = (40\cos 12\pi t, 40\sin 12\pi t)$, so $\sin\alpha = \dfrac{y}{1.2\text{ m}} = \dfrac{40\sin\theta}{120} = \dfrac{\sin\theta}{3} = \dfrac{1}{3}\sin 12\pi t$.

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha \cdot \dfrac{d\alpha}{dt} = \dfrac{1}{3} \cdot 12\pi \cdot \cos 12\pi t = 4\pi\cos\theta$. When $\theta = \dfrac{\pi}{3}$, we have

$\sin\alpha = \dfrac{1}{3}\sin\theta = \dfrac{\sqrt{3}}{6}$, so $\cos\alpha = \sqrt{1 - \left(\dfrac{\sqrt{3}}{6}\right)^2} = \sqrt{\dfrac{11}{12}}$ and $\dfrac{d\alpha}{dt} = \dfrac{4\pi\cos\frac{\pi}{3}}{\cos\alpha} = \dfrac{2\pi}{\sqrt{11/12}} = \dfrac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56$ rad/s.

(b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2\,|OA|\,|OP|\cos\theta \;\Rightarrow$

$120^2 = 40^2 + |OP|^2 - 2 \cdot 40\,|OP|\cos\theta \;\Rightarrow\; |OP|^2 - (80\cos\theta)\,|OP| - 12{,}800 = 0 \;\Rightarrow$

$|OP| = \frac{1}{2}\left(80\cos\theta \pm \sqrt{6400\cos^2\theta + 51{,}200}\right) = 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8} = 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$ cm

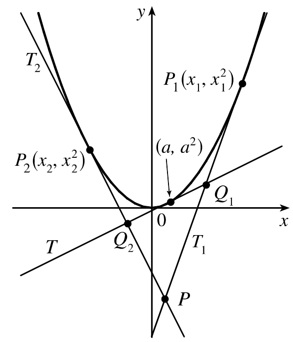
[since $|OP| > 0$]. As a check, note that $|OP| = 160$ cm when $\theta = 0$ and $|OP| = 80\sqrt{2}$ cm when $\theta = \frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x = 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\right)$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = 40\left(-\sin\theta - \frac{2\cos\theta\sin\theta}{2\sqrt{8 + \cos^2\theta}}\right) \cdot 12\pi = -480\pi\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8 + \cos^2\theta}}\right) \text{ cm/s.}$$

In particular, $dx/dt = 0$ cm/s when $\theta = 0$ and $dx/dt = -480\pi$ cm/s when $\theta = \frac{\pi}{2}$.

14. The equation of T_1 is $y - x_1^2 = 2x_1(x - x_1) = 2x_1x - 2x_1^2$ or $y = 2x_1x - x_1^2$.



The equation of T_2 is $y = 2x_2x - x_2^2$. Solving for the point of intersection, we get $2x(x_1 - x_2) = x_1^2 - x_2^2 \;\Rightarrow\; x = \frac{1}{2}(x_1 + x_2)$. Therefore, the coordinates of P are $\left(\frac{1}{2}(x_1 + x_2), x_1x_2\right)$. So if the point of contact of T is $\left(a, a^2\right)$, then Q_1 is $\left(\frac{1}{2}(a + x_1), ax_1\right)$ and Q_2 is $\left(\frac{1}{2}(a + x_2), ax_2\right)$. Therefore,

$|PQ_1|^2 = \frac{1}{4}(a - x_2)^2 + x_1^2(a - x_2)^2 = (a - x_2)^2\left(\frac{1}{4} + x_1^2\right)$ and

$|PP_1|^2 = \frac{1}{4}(x_1 - x_2)^2 + x_1^2(x_1 - x_2)^2 = (x_1 - x_2)^2\left(\frac{1}{4} + x_1^2\right)$.

So $\dfrac{|PQ_1|^2}{|PP_1|^2} = \dfrac{(a - x_2)^2}{(x_1 - x_2)^2}$, and similarly $\dfrac{|PQ_2|^2}{|PP_2|^2} = \dfrac{(x_1 - a)^2}{(x_1 - x_2)^2}$. Finally, $\dfrac{|PQ_1|}{|PP_1|} + \dfrac{|PQ_2|}{|PP_2|} = \dfrac{a - x_2}{x_1 - x_2} + \dfrac{x_1 - a}{x_1 - x_2} = 1$.

15. It seems from the figure that as P approaches the point $(0, 2)$ from the right, $x_T \to \infty$ and $y_T \to 2^+$. As P approaches the point $(3, 0)$ from the left, it appears that $x_T \to 3^+$ and $y_T \to \infty$. So we guess that $x_T \in (3, \infty)$ and $y_T \in (2, \infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N \in (0, 3)$, and $y_N \in (-\infty, 0)$ or $(-2, 0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \;\Rightarrow\; \dfrac{2x}{9} + \dfrac{2y}{4}y' = 0$, so $y' = -\dfrac{4}{9}\dfrac{x}{y}$. So at the point (x_0, y_0) on the ellipse, an equation of the tangent line is $y - y_0 = -\dfrac{4}{9}\dfrac{x_0}{y_0}(x - x_0)$ or $4x_0x + 9y_0y = 4x_0^2 + 9y_0^2$. This can be written as $\dfrac{x_0x}{9} + \dfrac{y_0y}{4} = \dfrac{x_0^2}{9} + \dfrac{y_0^2}{4} = 1$, because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is $\dfrac{x_0x}{9} + \dfrac{y_0y}{4} = 1$.

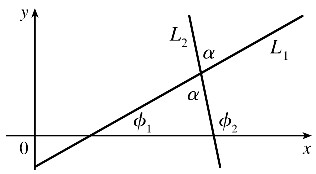
Therefore, the x-intercept x_T for the tangent line is given by $\dfrac{x_0x_T}{9} = 1 \;\Leftrightarrow\; x_T = \dfrac{9}{x_0}$, and the y-intercept y_T is given by $\dfrac{y_0y_T}{4} = 1 \;\Leftrightarrow\; y_T = \dfrac{4}{y_0}$.

So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$. At the point (x_0, y_0) on the ellipse, the slope of the normal line is $-\dfrac{1}{y'(x_0, y_0)} = \dfrac{9}{4}\dfrac{y_0}{x_0}$, and its equation is $y - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x - x_0)$. So the x-intercept x_N for the normal line is given by $0 - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x_N - x_0) \;\Rightarrow\; x_N = -\dfrac{4x_0}{9} + x_0 = \dfrac{5x_0}{9}$, and the y-intercept y_N is given by $y_N - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(0 - x_0) \;\Rightarrow\; y_N = -\dfrac{9y_0}{4} + y_0 = -\dfrac{5y_0}{4}$.

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $\left(0, \frac{5}{3}\right)$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $\left(-\frac{5}{2}, 0\right)$.

16. $\displaystyle\lim_{x \to 0} \frac{\sin(3 + x)^2 - \sin 9}{x} = f'(3)$ where $f(x) = \sin x^2$. Now $f'(x) = (\cos x^2)(2x)$, so $f'(3) = 6\cos 9$.

17. (a) If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2, then $m_1 = \tan\,\phi_1$ and $m_2 = \tan\,\phi_2$. The triangle in the figure shows that $\phi_1 + \alpha + (180^\circ - \phi_2) = 180^\circ$ and so $\alpha = \phi_2 - \phi_1$. Therefore, using the identity for $\tan(x-y)$, we have $\tan\alpha = \tan(\phi_2 - \phi_1) = \dfrac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2\tan\phi_1}$ and so $\tan\,\alpha = \dfrac{m_2 - m_1}{1 + m_1 m_2}$.



(b) (i) The parabolas intersect when $x^2 = (x-2)^2 \;\Rightarrow\; x = 1$. If $y = x^2$, then $y' = 2x$, so the slope of the tangent to $y = x^2$ at $(1,1)$ is $m_1 = 2(1) = 2$. If $y = (x-2)^2$, then $y' = 2(x-2)$, so the slope of the tangent to $y = (x-2)^2$ at $(1,1)$ is $m_2 = 2(1-2) = -2$. Therefore, $\tan\alpha = \dfrac{m_2 - m_1}{1 + m_1 m_2} = \dfrac{-2-2}{1+2(-2)} = \dfrac{4}{3}$ and so $\alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53^\circ$ [or 127°].

(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$ intersect when $x^2 - 4x + (x^2 - 3) + 3 = 0 \;\Leftrightarrow\; 2x(x-2) = 0 \;\Rightarrow\; x = 0$ or 2, but 0 is extraneous. If $x = 2$, then $y = \pm 1$. If $x^2 - y^2 = 3$ then $2x - 2yy' = 0 \;\Rightarrow\; y' = x/y$ and $x^2 - 4x + y^2 + 3 = 0 \;\Rightarrow\; 2x - 4 + 2yy' = 0 \;\Rightarrow\; y' = \dfrac{2-x}{y}$. At $(2,1)$ the slopes are $m_1 = 2$ and $m_2 = 0$, so $\tan\alpha = \frac{0-2}{1+2\cdot 0} = -2 \;\Rightarrow\; \alpha \approx 117^\circ$. At $(2,-1)$ the slopes are $m_1 = -2$ and $m_2 = 0$, so $\tan\alpha = \dfrac{0-(-2)}{1+(-2)(0)} = 2 \;\Rightarrow\; \alpha \approx 63^\circ$ [or 117°].

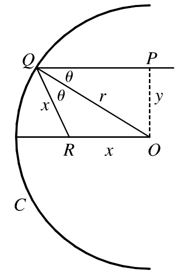
18. $y^2 = 4px \;\Rightarrow\; 2yy' = 4p \;\Rightarrow\; y' = 2p/y \;\Rightarrow\;$ slope of tangent at $P(x_1, y_1)$ is $m_1 = 2p/y_1$. The slope of FP is $m_2 = \dfrac{y_1}{x_1 - p}$, so by the formula from Problem 17(a),

$$\tan\alpha = \frac{\dfrac{y_1}{x_1-p} - \dfrac{2p}{y_1}}{1 + \left(\dfrac{2p}{y_1}\right)\left(\dfrac{y_1}{x_1-p}\right)} \cdot \frac{y_1\,(x_1-p)}{y_1\,(x_1-p)} = \frac{y_1^2 - 2p(x_1-p)}{y_1(x_1-p) + 2py_1} = \frac{4px_1 - 2px_1 + 2p^2}{x_1y_1 - py_1 + 2py_1}$$

$$= \frac{2p(p+x_1)}{y_1(p+x_1)} = \frac{2p}{y_1} = \text{slope of tangent at } P = \tan\beta$$

Since $0 \le \alpha, \beta \le \frac{\pi}{2}$, this proves that $\alpha = \beta$.

19. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so $|QR| = |RO| = x$. By the Law of Cosines, $x^2 = x^2 + r^2 - 2rx\cos\theta$. Hence, $2rx\cos\theta = r^2$, so $x = \dfrac{r^2}{2r\cos\theta} = \dfrac{r}{2\cos\theta}$. Note that as $y \to 0^+$, $\theta \to 0^+$ (since $\sin\theta = y/r$), and hence $x \to \dfrac{r}{2\cos 0} = \dfrac{r}{2}$. Thus, as P is taken closer and closer to the x-axis, the point R approaches the midpoint of the radius AO.



20. $$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f(x)-0}{g(x)-0} = \lim_{x\to 0}\frac{f(x)-f(0)}{g(x)-g(0)} = \lim_{x\to 0}\frac{\dfrac{f(x)-f(0)}{x-0}}{\dfrac{g(x)-g(0)}{x-0}} = \frac{\displaystyle\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}}{\displaystyle\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}} = \frac{f'(0)}{g'(0)}$$

21. $\lim\limits_{x\to 0} \dfrac{\sin(a+2x) - 2\sin(a+x) + \sin a}{x^2}$

$$= \lim_{x\to 0} \frac{\sin a\, \cos 2x + \cos a\, \sin 2x - 2\sin a\, \cos x - 2\cos a\, \sin x + \sin a}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin a\,(\cos 2x - 2\cos x + 1) + \cos a\,(\sin 2x - 2\sin x)}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin a\,(2\cos^2 x - 1 - 2\cos x + 1) + \cos a\,(2\sin x\, \cos x - 2\sin x)}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin a\,(2\cos x)(\cos x - 1) + \cos a\,(2\sin x)(\cos x - 1)}{x^2}$$

$$= \lim_{x\to 0} \frac{2(\cos x - 1)[\sin a\, \cos x + \cos a\, \sin x](\cos x + 1)}{x^2(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{-2\sin^2 x\,[\sin(a+x)]}{x^2(\cos x + 1)} = -2 \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \frac{\sin(a+x)}{\cos x + 1} = -2(1)^2\, \frac{\sin(a+0)}{\cos 0 + 1} = -\sin a$$

22. Suppose that $y = mx + c$ is a tangent line to the ellipse. Then it intersects the ellipse at only one point, so the discriminant of the equation $\dfrac{x^2}{a^2} + \dfrac{(mx+c)^2}{b^2} = 1 \quad\Leftrightarrow\quad (b^2 + a^2m^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0$ must be 0; that is,

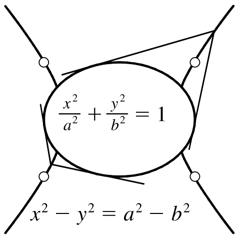
$$0 = (2mca^2)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 4a^4c^2m^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4c^2m^2 + 4a^4b^2m^2$$
$$= 4a^2b^2(a^2m^2 + b^2 - c^2)$$

Therefore, $a^2m^2 + b^2 - c^2 = 0$.

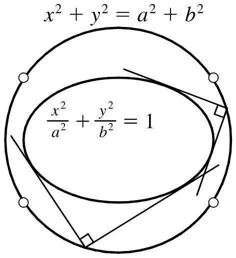
Now if a point (α, β) lies on the line $y = mx + c$, then $c = \beta - m\alpha$, so from above,

$$0 = a^2m^2 + b^2 - (\beta - m\alpha)^2 = (a^2 - \alpha^2)m^2 + 2\alpha\beta m + b^2 - \beta^2 \quad\Leftrightarrow\quad m^2 + \frac{2\alpha\beta}{a^2 - \alpha^2}m + \frac{b^2 - \beta^2}{a^2 - \alpha^2} = 0.$$

(a) Suppose that the two tangent lines from the point (α, β) to the ellipse have slopes m and $\dfrac{1}{m}$. Then m and $\dfrac{1}{m}$ are roots of the equation $z^2 + \dfrac{2\alpha\beta}{a^2 - \alpha^2}z + \dfrac{b^2 - \beta^2}{a^2 - \alpha^2} = 0$. This implies that $(z - m)\left(z - \dfrac{1}{m}\right) = 0 \quad\Leftrightarrow\quad z^2 - \left(m + \dfrac{1}{m}\right)z + m\left(\dfrac{1}{m}\right) = 0$, so equating the constant terms in the two quadratic equations, we get $\dfrac{b^2 - \beta^2}{a^2 - \alpha^2} = m\left(\dfrac{1}{m}\right) = 1$, and hence $b^2 - \beta^2 = a^2 - \alpha^2$. So (α, β) lies on the hyperbola $x^2 - y^2 = a^2 - b^2$.



(b) If the two tangent lines from the point (α, β) to the ellipse have slopes m and $-\dfrac{1}{m}$, then m and $-\dfrac{1}{m}$ are roots of the quadratic equation, and so $(z - m)\left(z + \dfrac{1}{m}\right) = 0$, and equating the constant terms as in part (a), we get $\dfrac{b^2 - \beta^2}{a^2 - \alpha^2} = -1$, and hence $b^2 - \beta^2 = \alpha^2 - a^2$. So the point (α, β) lies on the circle $x^2 + y^2 = a^2 + b^2$.



23. $y = x^4 - 2x^2 - x \quad \Rightarrow \quad y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + (-3a^4 + 2a^2)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \quad \Rightarrow \quad (a - b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \quad \Rightarrow \quad 3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \quad \Rightarrow \quad a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = (a^2 + ab + b^2) - (a^2 + b^2) = 1 - \dfrac{2}{3} = \dfrac{1}{3}$, so $b = \dfrac{1}{3a}$. Hence, $a^2 + \dfrac{1}{9a^2} = \dfrac{2}{3}$, so $9a^4 + 1 = 6a^2 \quad \Rightarrow$ $0 = 9a^4 - 6a^2 + 1 = (3a^2 - 1)^2$. So $3a^2 - 1 = 0 \quad \Rightarrow \quad a^2 = \dfrac{1}{3} \quad \Rightarrow \quad b^2 = \dfrac{1}{9a^2} = \dfrac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

24. Suppose that the normal lines at the three points $\left(a_1, a_1^2\right)$, $\left(a_2, a_2^2\right)$, and $\left(a_3, a_3^2\right)$ intersect at a common point. Now if one of the a_i is 0 (suppose $a_1 = 0$) then by symmetry $a_2 = -a_3$, so $a_1 + a_2 + a_3 = 0$. So we can assume that none of the a_i is 0.

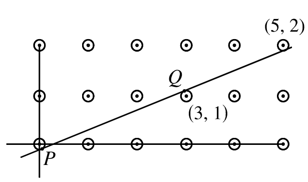
The slope of the tangent line at $\left(a_i, a_i^2\right)$ is $2a_i$, so the slope of the normal line is $-\dfrac{1}{2a_i}$ and its equation is $y - a_i^2 = -\dfrac{1}{2a_i}(x - a_i)$. We solve for the x-coordinate of the intersection of the normal lines from $\left(a_1, a_1^2\right)$ and $\left(a_2, a_2^2\right)$:

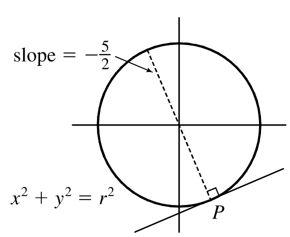
$$y = a_1^2 - \frac{1}{2a_1}(x - a_1) = a_2^2 - \frac{1}{2a_2}(x - a_2) \quad \Rightarrow \quad x\left(\frac{1}{2a_2} - \frac{1}{2a_1}\right) = a_2^2 - a_1^2 \quad \Rightarrow$$

$x\left(\dfrac{a_1 - a_2}{2a_1a_2}\right) = (-a_1 - a_2)(a_1 + a_2) \quad \Leftrightarrow \quad x = -2a_1a_2(a_1 + a_2)$ **(1)**. Similarly, solving for the x-coordinate of the intersections of the normal lines from $\left(a_1, a_1^2\right)$ and $\left(a_3, a_3^2\right)$ gives $x = -2a_1a_3(a_1 + a_3)$ **(2)**.

Equating **(1)** and **(2)** gives $a_2(a_1 + a_2) = a_3(a_1 + a_3) \quad \Leftrightarrow \quad a_1(a_2 - a_3) = a_3^2 - a_2^2 = -(a_2 + a_3)(a_2 - a_3) \quad \Leftrightarrow$ $a_1 = -(a_2 + a_3) \quad \Leftrightarrow \quad a_1 + a_2 + a_3 = 0$.

25. Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3, 1)$ and the circles centered at $(0, 0)$ and $(5, 2)$.



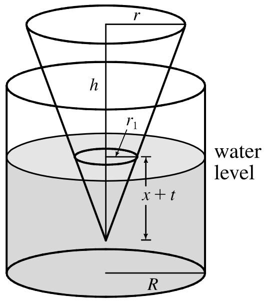


To find P, the point at which the line is tangent to the circle at $(0, 0)$, we simultaneously solve $x^2 + y^2 = r^2$ and

$y = -\frac{5}{2}x \quad \Rightarrow \quad x^2 + \frac{25}{4}x^2 = r^2 \quad \Rightarrow \quad x^2 = \frac{4}{29}r^2 \quad \Rightarrow \quad x = \frac{2}{\sqrt{29}}r, y = -\frac{5}{\sqrt{29}}r$. To find Q, we either use symmetry or solve $(x-3)^2 + (y-1)^2 = r^2$ and $y - 1 = -\frac{5}{2}(x-3)$. As above, we get $x = 3 - \frac{2}{\sqrt{29}}r, y = 1 + \frac{5}{\sqrt{29}}r$. Now the slope of the line PQ is $\frac{2}{5}$, so $m_{PQ} = \dfrac{1 + \frac{5}{\sqrt{29}}r - \left(-\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \frac{2}{\sqrt{29}}r} = \dfrac{1 + \frac{10}{\sqrt{29}}r}{3 - \frac{4}{\sqrt{29}}r} = \dfrac{\sqrt{29} + 10r}{3\sqrt{29} - 4r} = \dfrac{2}{5} \quad \Rightarrow$

$5\sqrt{29} + 50r = 6\sqrt{29} - 8r \quad \Leftrightarrow \quad 58r = \sqrt{29} \quad \Leftrightarrow \quad r = \frac{\sqrt{29}}{58}$. So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r = \frac{\sqrt{29}}{58} \approx 0.093$.

26.



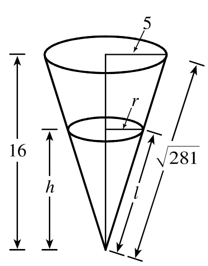
Assume the axes of the cone and the cylinder are parallel. Let H denote the initial height of the water. When the cone has been dropping for t seconds, the water level has risen x centimeters, so the tip of the cone is $x + 1t$ centimeters below the water line. We want to find dx/dt when $x + t = h$ (when the cone is completely submerged). Using similar triangles, $\dfrac{r_1}{x+t} = \dfrac{r}{h} \quad \Rightarrow \quad r_1 = \dfrac{r}{h}(x+t)$.

volume of water and cone at time t	$=$	original volume of water	$+$	volume of submerged part of cone
$\pi R^2(H+x)$	$=$	$\pi R^2 H$	$+$	$\frac{1}{3}\pi r_1^2(x+t)$
$\pi R^2 H + \pi R^2 x$	$=$	$\pi R^2 H$	$+$	$\frac{1}{3}\pi \dfrac{r^2}{h^2}(x+t)^3$
$3h^2R^2x$	$=$	$r^2(x+t)^3$		

Differentiating implicitly with respect to t gives us $3h^2R^2\dfrac{dx}{dt} = r^2\left[3(x+t)^2\dfrac{dx}{dt} + 3(x+t)^2\dfrac{dt}{dt}\right] \quad \Rightarrow$

$\dfrac{dx}{dt} = \dfrac{r^2(x+t)^2}{h^2R^2 - r^2(x+t)^2} \quad \Rightarrow \quad \left.\dfrac{dx}{dt}\right|_{x+t=h} = \dfrac{r^2h^2}{h^2R^2 - r^2h^2} = \dfrac{r^2}{R^2 - r^2}$. Thus, the water level is rising at a rate of $\dfrac{r^2}{R^2 - r^2}$ cm/s at the instant the cone is completely submerged.

27.



By similar triangles, $\dfrac{r}{5} = \dfrac{h}{16} \quad \Rightarrow \quad r = \dfrac{5h}{16}$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\dfrac{5h}{16}\right)^2 h = \dfrac{25\pi}{768}h^3$, so $\dfrac{dV}{dt} = \dfrac{25\pi}{256}h^2\dfrac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added ($2\text{ cm}^3/\text{min}$) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k is a proportionality constant). Thus, $\dfrac{dV}{dt} = 2 - k\pi rl$.

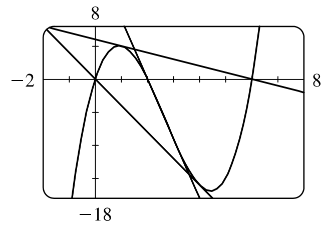
Equating the two expressions for $\dfrac{dV}{dt}$ and substituting $h = 10$, $\dfrac{dh}{dt} = -0.3$, $r = \dfrac{5(10)}{16} = \dfrac{25}{8}$, and $\dfrac{l}{\sqrt{281}} = \dfrac{10}{16} \quad \Leftrightarrow$

$l = \dfrac{5}{8}\sqrt{281}$, we get $\dfrac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi\dfrac{25}{8}\cdot\dfrac{5}{8}\sqrt{281} \quad \Leftrightarrow \quad \dfrac{125k\pi\sqrt{281}}{64} = 2 + \dfrac{750\pi}{256}$. Solving for k gives us

$k = \dfrac{256 + 375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in; that is, $\dfrac{dV}{dt} = 0$. Thus, the rate at which we should pour the liquid into the container is

$$k\pi rl = \frac{256 + 375\pi}{250\pi\sqrt{281}} \cdot \pi \cdot \frac{25}{8} \cdot \frac{5\sqrt{281}}{8} = \frac{256 + 375\pi}{128} \approx 11.204 \text{ cm}^3\text{/min}$$

28. (a) $f(x) = x(x-2)(x-6) = x^3 - 8x^2 + 12x \quad \Rightarrow$



$f'(x) = 3x^2 - 16x + 12$. The average of the first pair of zeros is $(0+2)/2 = 1$. At $x = 1$, the slope of the tangent line is $f'(1) = -1$, so an equation of the tangent line has the form $y = -1x + b$. Since $f(1) = 5$, we have $5 = -1 + b \quad \Rightarrow \quad b = 6$ and the tangent has equation $y = -x + 6$. Similarly, at $x = \dfrac{0+6}{2} = 3$, $y = -9x + 18$; at $x = \dfrac{2+6}{2} = 4$, $y = -4x$. From the graph, we see that each tangent line drawn at the average of two zeros intersects the graph of f at the third zero.

(b) A CAS gives $f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$ or $f'(x) = 3x^2 - 2(a+b+c)x + ab + ac + bc$. Using the `Simplify` command, we get $f'\left(\dfrac{a+b}{2}\right) = -\dfrac{(a-b)^2}{4}$ and $f\left(\dfrac{a+b}{2}\right) = -\dfrac{(a-b)^2}{8}(a+b-2c)$, so an equation of the tangent line at $x = \dfrac{a+b}{2}$ is $y = -\dfrac{(a-b)^2}{4}\left(x - \dfrac{a+b}{2}\right) - \dfrac{(a-b)^2}{8}(a+b-2c)$. To find the x-intercept, let $y = 0$ and use the `Solve` command. The result is $x = c$.

Using Derive, we can begin by authoring the expression $(x-a)(x-b)(x-c)$. Now load the utility file `DifferentiationApplications`. Next we author tangent $(\#1, x, (a+b)/2)$—this is the command to find an equation of the tangent line of the function in #1 whose independent variable is x at the x-value $(a+b)/2$. We then simplify that expression and obtain the equation $y = \#4$. The form in expression #4 makes it easy to see that the x-intercept is the third zero, namely c. In a similar fashion we see that b is the x-intercept for the tangent line at $(a+c)/2$ and a is the x-intercept for the tangent line at $(b+c)/2$.

```
#1:  (x - a)·(x - b)·(x - c)
#2:  LOAD(C:\Program Files\TI Education\Derive 6\Math\DifferentiationApplications.mth

                                           a + b
#3:  TANGENT ((x - a)·(x - b)·(x - c), x,  ─────)
                                             2

                                               2           2
                                           (a   - 2·a·b + b )·(c-x)
#4:                                        ───────────────────────
                                                      4
```

2 Derivatives

2.1 Derivatives and Rates of Change

SUGGESTED TIME AND EMPHASIS

1–2 classes   Essential material

POINTS TO STRESS

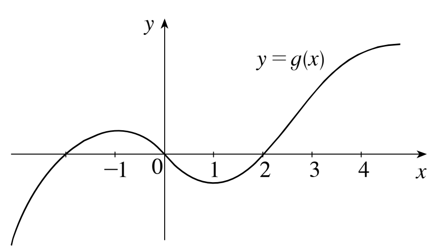
1. The slope of the tangent line as the limit of the slopes of secant lines (visually, numerically, algebraically).
2. Physical examples of instantaneous rates of change (velocity, reaction rate, marginal cost, and so on) and their units.
3. The derivative notations $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ and $f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$.
4. Using f' to write an equation of the tangent line to a curve at a given point.
5. Using f' as an approximate rate of change when working with discrete data.

QUIZ QUESTIONS

- **TEXT QUESTION** Why is it necessary to take a limit when computing the slope of the tangent line?
  ANSWER There are several possible answers here. Examples include the following:
  - By definition, the slope of the tangent line is the limit of the slopes of secant lines.
  - You don't know where to draw the tangent line unless you pick two points very close together.

  The idea is to get them thinking about this question.
- **DRILL QUESTION** For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

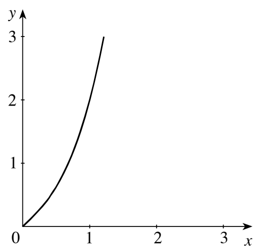
  $0 \qquad g'(-2) \qquad g'(0) \qquad g'(2) \qquad g'(4)$

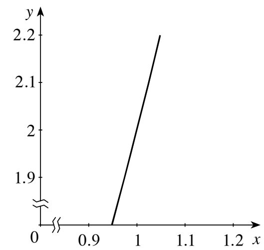


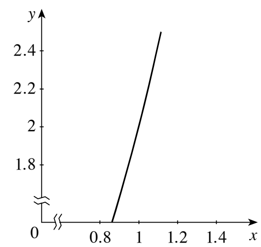
ANSWER $g'(0) < 0 < g'(4) < g'(-2) < g'(2)$

MATERIALS FOR LECTURE

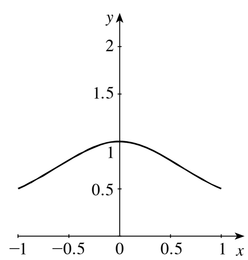
- Review the geometry of the tangent line, and the concept of "locally linear". Estimate the slope of the line tangent to $y = x^3 + x$ at $(1, 2)$ by looking at the slopes of the lines between $x = 0.9$ and $x = 1.1$, $x = 0.99$ and $x = 1.01$, and so forth. Illustrate these secant lines on a graph of the function, redrawing the figure when necessary to illustrate the "zooming in" process.

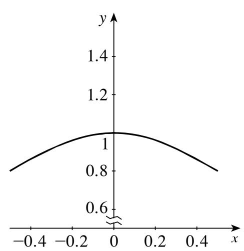


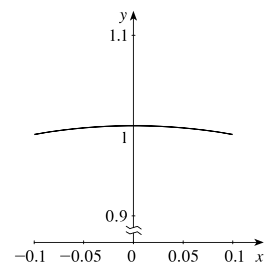




Similarly examine $y = \frac{1}{x^2+1}$ at $(0, 1)$.



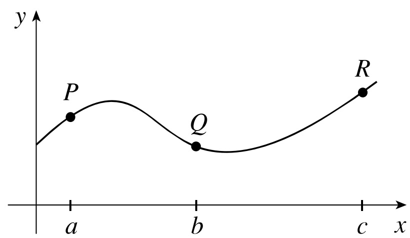




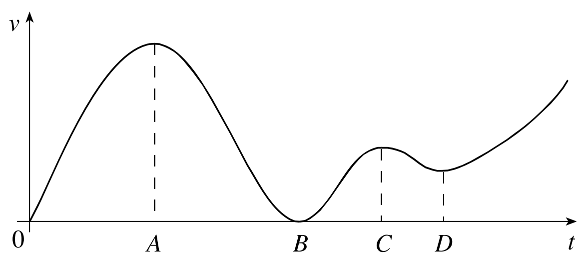
- If "A Jittery Function" was covered in Section 1.7, look at $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$ Poll the class: Is there a tangent line at $x = 0$? Then examine what happens if you look at the limits of the secant lines.
- Have students estimate the slope of the tangent line to $y = \sin x$ at various points. Foreshadow the concept of concavity by asking them some open-ended questions such as the following: What happens to the function when the slope of the tangent is increasing? Decreasing? Zero? Slowly changing?
- Discuss how physical situations can be translated into statements about derivatives. For example, the budget deficit can be viewed as the derivative of the national debt. Describe the units of derivatives in real world situations. The budget deficit, for example, is measured in billions of dollars per year. Another example: if $s(d)$ represents the sales figures for a magazine given d dollars of advertising, where s is the number of magazines sold, then $s'(d)$ is in magazines per dollar spent. Describe enough examples to make the pattern evident.
- Note that the text shows that if $f(x) = x^2 - 8x + 9$, then $f'(a) = 2a - 8$. Thus, $f'(55) = 102$ and $f'(100) = 192$. Demonstrate that these quantities cannot be easily estimated from a graph of the function. Foreshadow the treatment of a as a variable in Section 2.2.
- If a function models discrete data and the quantities involved are orders of magnitude larger than 1, we can use the approximation $f'(x) \approx f(x+1) - f(x)$. (That is, we can use $h = 1$ in the limit definition of the derivative.) For example, let $f(t)$ be the total population of the world, where t is measured in years since 1800. Then $f(211)$ is the world population in 2011, $f(212)$ is the total population in 2012, and $f'(211)$ is approximately the change in population from 2011 to 2012. In business, if $f(n)$ is the total cost of producing n objects, $f'(n)$ approximates the cost of producing the $(n+1)$th object.

WORKSHOP/DISCUSSION

- "Thumbnail" derivative estimates: graph a function on the board and have the class call out rough values of the derivative. Is it larger than 1? About 1? Between 0 and 1? About 0? Between -1 and 0? About -1? Smaller than -1? This is good preparation for Group Work 2 ("Oiling Up Your Calculators").
- Draw a function like the following, and first estimate slopes of secant lines between $x = a$ and $x = b$, and between $x = b$ and $x = c$. Then order the five quantities $f'(a)$, $f'(b)$, $f'(c)$, m_{PQ}, and m_{QR} in decreasing order. [Answer: $f'(b) < m_{PQ} < m_{QR} < f'(c) < f'(a)$.]



- Start the following problem with the students: A car is travelling down a highway away from its starting location with distance function $d(t) = 8\left(t^3 - 6t^2 + 12t\right)$, where t is in hours, and d is in miles.
  1. How far has the car travelled after 1, 2, and 3 hours?
  2. What is the average velocity over the intervals [0, 1], [1, 2], and [2, 3]?
- Consider a car's velocity function described by the graph below.



  1. Ask the students to determine when the car was stopped.
  2. Ask the students when the car was accelerating (that is, when the velocity was increasing). When was the car decelerating?
  3. Ask the students to describe what is happening at times A, C, and D in terms of both velocity and acceleration. What is happening at time B?
- Estimate the slope of the tangent line to $y = \sin x$ at $x = 1$ by looking at the following table of values.

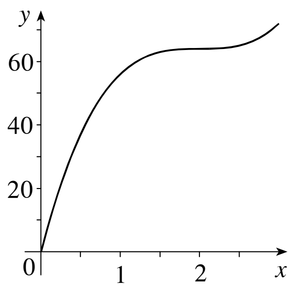
x	$\sin x$	$\dfrac{\sin x - \sin 1}{x - 1}$
0	0	0.841471
0.5	0.4794	0.724091
0.9	0.7833	0.581441
0.99	0.8360	0.544501
0.999	0.8409	0.540723
1.0001	0.8415	0.540260
1.001	0.8420	0.539881

- Demonstrate some sample computations similar to Example 4, such as finding the derivative of $f(t) = \sqrt{1+t}$ at $t = 3$, or of $g(x) = x - x^2$ at $x = 1$.

GROUP WORK 1: FOLLOW THAT CAR

Start this problem by giving the students the function $d(t) = 8\left(t^3 - 6t^2 + 12t\right)$ and having them sketch its graph. Ask them how far the car has traveled after 1, 2, and 3 hours, and then show them how to compute the average velocity for [0, 1], [1, 2], and [2, 3].

ANSWERS

1. 

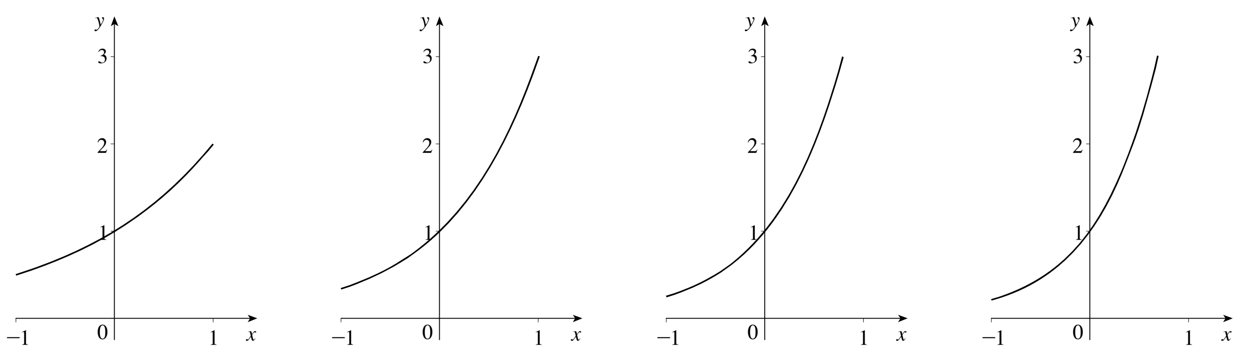
2. It appears to stop at $t = 2$.
3. 8 mi/h, 2 mi/h, 0.08 mi/h
4. 0 mi/h. This is where the car stops.

GROUP WORK 2: OILING UP YOUR CALCULATORS

As long as the students have the ability to graph a function on their calculators and to estimate the slope of a curve at a point, they don't need to have been exposed to the exponential function to do this activity. The exponential function and the number e will be covered in Chapter 6, and this exercise is a good initial introduction to the concept.

ANSWERS

1. If the students do this numerically, they should be able to get some pretty good estimates of $\ln 3 \approx 1.098612$. If they use graphs, they should be able to get 1.1 as an estimate.
2. 0.7 is a good estimate from a graph.
3. As a increases, the slope of the curve at $x = 0$ is increasing, as can be seen below.



4. The slope is less than 1 at $a = 2$ and greater than 1 at $a = 3$. Now apply the Intermediate Value Theorem.
5. The students are estimating e and should get 2.72 at a minimum level of accuracy.

GROUP WORK 3: CONNECT THE DOTS

Closure is particularly important on this activity. At this point in the course, many students will have the impression that all reasonable estimates are equally valid, so it is crucial that students discuss Problem 4. If there is student interest, this table can generate a rich discussion. Can A' ever be negative? What would that mean in real terms? What would $(A')'$ mean in real terms in this instance?

ANSWERS

1. $A'(3500) \approx 0.06$ %/\$ It is likely to be an overestimate, because the function lies below its tangent line near $p = 3500$.
2. After spending \$3500, consumer approval is increasing at the rate of about 0.06 % for every additional dollar spent.
3. Percent per dollar
4. $A'(\$3550) \approx 0.06$ %/\$. This is a better estimate because the same figures now give a two-sided approximation of the limit of the difference quotient.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 5, 9, 11, 14, 22, 23, 33, 40, 48

SAMPLE ASSIGNMENT 3, 5, 9, 11, 14, 17, 22, 23, 33, 40, 48, 53, 59

EXERCISE	D	A	N	G
3		×		×
5		×		
9				×
11				×
14	×	×		
17		×		
22		×		
23		×		
33		×	×	
40				×
48				×
53	×	×		
59		×		

GROUP WORK 1, SECTION 2.1

Follow that Car

Here, we continue with the analysis of the distance $d(t) = 8\left(t^3 - 6t^2 + 12t\right)$ of a car, where d is in miles and t is in hours.

1. Draw a graph of $d(t)$ from $t = 0$ to $t = 3$.

2. Does the car ever stop?

3. What is the average velocity over [1, 3]? over [1.5, 2.5]? over [1.9, 2.1]?

4. Estimate the instantaneous velocity at $t = 2$. Give a physical interpretation of your answer.

GROUP WORK 2, SECTION 2.1

Oiling Up Your Calculators

1. Use your calculator to graph $y = 3^x$. Estimate the slope of the line tangent to this curve at $x = 0$ using a method of your choosing.

2. Use your calculator to graph $y = 2^x$. Estimate the slope of the line tangent to this curve at $x = 0$ using a method of your choosing.

3. It is a fact that, as a increases, the slope of the line tangent to $y = a^x$ at $x = 0$ also increases in a continuous way. Geometrically, why should this be the case?

4. Prove that there is a special value of a for which the slope of the line tangent to $y = a^x$ at $x = 0$ is 1.

5. By trial and error, find an estimate of this special value of a, accurate to two decimal places.

GROUP WORK 3, SECTION 2.1

Connect the Dots

A company does a study on the effect of production value p of an advertisement on its consumer approval rating A. After interviewing eight focus groups, they come up with the following data:

Production Value	**Consumer Approval**
\$1000	32%
\$2000	33%
\$3000	46%
\$3500	55%
\$3600	61%
\$3800	65%
\$4000	69%
\$5000	70%

Assume that $A(p)$ gives the consumer approval percentage as a function of p.

1. Estimate $A'(\$3500)$. Is this likely to be an overestimate or an underestimate?

2. Interpret your answer to Problem 1 in real terms. What does your estimate of $A'(\$3500)$ tell you?

3. What are the units of $A'(p)$?

4. Estimate $A'(\$3550)$. Is your estimate better or worse than your estimate of $A'(\$3500)$? Why?

WRITING PROJECT **Early Methods for Finding Tangents**

The history of calculus is a fascinating and too-often neglected subject. Most people who study history never see calculus, and vice versa. We recommend assigning this section as extra credit to any motivated class, and possibly as a required group project, especially for a class consisting of students who are not science or math majors.

The students will need clear instructions detailing what their final result should look like. For example, recommend a page or two about Fermat's or Barrow's life and career, followed by two or three technical pages describing the alternate method of finding tangent lines as in the project's directions, and completed by a final half page of meaningful conclusion.

2.2 The Derivative as a Function

SUGGESTED TIME AND EMPHASIS

2 classes Essential material

POINTS TO STRESS

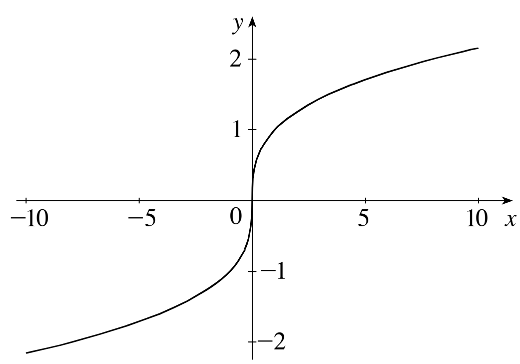
1. The concept of a differentiable function interpreted visually, algebraically, and descriptively.
2. Obtaining the derivative function f' by first considering the derivative at a point x, and then treating x as a variable.
3. How a function can fail to be differentiable.
4. Sketching the derivative function given a graph of the original function.
5. Second and higher derivatives

QUIZ QUESTIONS

TEXT QUESTION The previous section discussed the derivative $f'(a)$ for some function f. This section discusses the derivative $f'(x)$ for some function f. What is the difference, and why is it significant enough to merit separate sections?

ANSWER a is considered a constant, x is considered a variable. So $f'(a)$ is a number (the slope of the tangent line) and $f'(x)$ is a function.

DRILL QUESTION Consider the graph of $f(x) = \sqrt[3]{x}$. Is this function defined at $x = 0$? Continuous at $x = 0$? Differentiable at $x = 0$? Why?

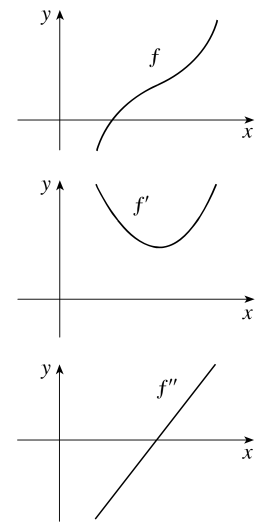


ANSWER It is defined and continuous, but not differentiable because it has a vertical tangent.

MATERIALS FOR LECTURE

- Ask the class this question: "If you were in a car, blindfolded, ears plugged, all five senses neutralized, what quantities would you still be able to perceive?" (Answers: They could feel the second derivative of motion, acceleration. They could also feel the third derivative of motion, "jerk".) Many students incorrectly add velocity to this list. Stress that acceleration is perceived as a force (hence $F = ma$) and that "jerk" causes the uncomfortable sensation when the car stops suddenly.
- Review definitions of differentiability, continuity, and the existence of a limit.
- Sketch f' from a graphical representation of $f(x) = \left|x^2 - 4\right|$, noting where f' does not exist. Then sketch $(f')'$ from the graph of f'. Point out that differentiability implies continuity, and not vice versa.

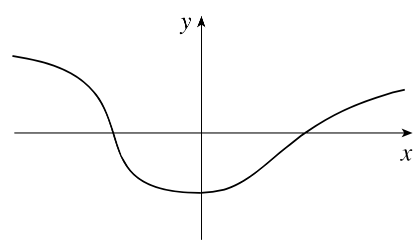
- Examine graphs of f and f' aligned vertically as shown. If you wish to foreshadow f'', add its graph below.
  Discuss what it means for f' to be positive, negative or zero.
  Then discuss what it means for f' to be increasing, decreasing or constant.



- If the group work "A Jittery Function" was covered in Section 1.7, then examine the differentiability of $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational} \end{cases}$ at $x = 0$ and elsewhere, if you have not already done so.
- Show that if $f(x) = x^4 - x^2 + x + 1$, then $f^{(5)}(x) \equiv 0$. Conclude that if $f(x)$ is a polynomial of degree m, then $f^{(m+1)}(x) \equiv 0$.

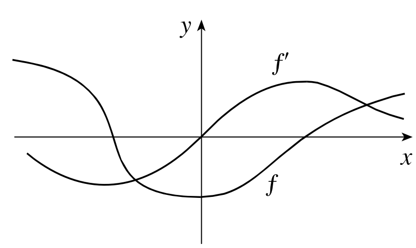
WORKSHOP/DISCUSSION

- Estimate derivatives from the graph of $f(x) = \sin x$. Do this at various points, and plot the results on the blackboard. See if the class can recognize the graph as a graph of the cosine curve.
- Given the graph of f below, have students determine where f has a horizontal tangent, where f' is positive, where f' is negative, where f' is increasing (this may require some additional discussion), and where f' is decreasing. Then have them sketch the graph of f'.



  TEC has more exercises of this type using a wide variety of functions.

  ANSWER There is a horizontal tangent near $x = 0$. f' is positive to the right of 0, negative to the left. f' is increasing between the x-intercepts, and decreasing outside of them.



- Compute $f'(x)$ and $g'(x)$ if $f(x) = x^2 + x + 2$ and $g(x) = x^2 + x + 4$. Point out that $f'(x) = g'(x)$ and discuss why the constant term is not important. Next, compute $h'(x)$ if $h(x) = x^2 + 2x + 2$. Point out that the graph of $h'(x)$ is just the graph of $f'(x)$ shifted up one unit, so the linear term just shifts derivatives. TEC contains more explorations on how the coefficients in polynomials and other functions affect first and second derivatives.

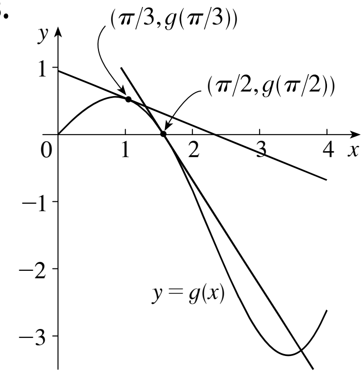
- Consider the function $f(x) = \sqrt{|x|}$. Show that it is not differentiable at 0 in two ways: by inspection (it has a cusp); and by computing the left- and right-hand limits of $f'(x)$ at $x = 0$ ($\lim_{x\to 0^+} f'(x) = \infty$, $\lim_{x\to 0^-} f'(x) = -\infty$).
- TEC can be used to develop students' ability to look at the graph of a function and visualize the graph of that function's derivative. The key feature of this module is that it allows the students to mark various features of the derivative *directly on the graph of the function* (for example, where the derivative is positive or negative). Then, after using this information and sketching a graph of the derivative, they can view the actual graph of the derivative and check their work.

GROUP WORK 1: TANGENT LINES AND THE DERIVATIVE FUNCTION

This simple activity reinforces that although we are moving to thinking of the derivative as a function of x, it is still the slope of the line tangent to the graph of f.

ANSWERS

1, 3.



2. $y = -\frac{\pi}{2}\left(x - \frac{\pi}{2}\right)$

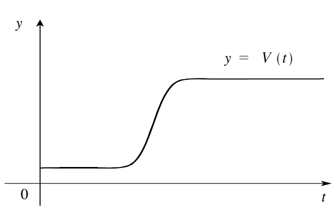
4. $y = \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{6}\right)\left(x - \frac{\pi}{3}\right) + \frac{\pi}{6}$

GROUP WORK 2: THE REVENGE OF ORVILLE REDENBACHER

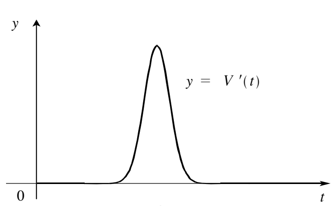
In an advanced class, or a class in which one group has finished far ahead of the others, ask the students to repeat the activity substituting "$D(t)$, the density function" for $V(t)$.

ANSWERS

1.

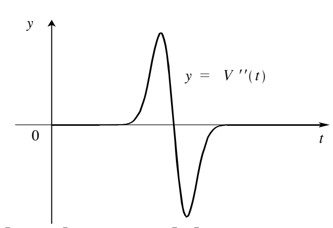


2.



Units are cm^3/s.

3.



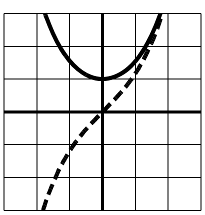
When the second derivative crosses the x-axis, the first derivative has a maximum, meaning the popcorn is expanding the fastest.

GROUP WORK 3: THE DERIVATIVE FUNCTION

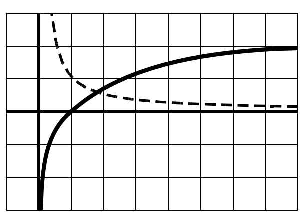
Give each group of between three and five students the picture of all eight graphs. They are to sketch the derivative functions by first estimating the slopes at points, and plotting the values of $f'(x)$. Each group should also be given a large copy of one of the graphs, perhaps on acetate. When they are ready, with this information they can draw the derivative graph on the same axes. For closure, project their solutions on the wall and point out salient features. Perhaps the students will notice that the derivatives turn out to be positive when their corresponding functions are increasing. Concavity can even be introduced at this time. Large copies of the answers are provided, in case the instructor wishes to overlay them on top of students' answers for reinforcement. Note that the derivative of graph 6 ($y = e^x$) is itself. Also note that the derivative of graph 1 ($y = \cosh x$) is *not* a straight line. Leave at least 15 minutes for closure. The whole activity should take about 45–60 minutes, but it is really, truly worth the time.

If a group finishes early, have them discuss where f' is increasing and where it is decreasing. Also show that where f is increasing, f' is positive, and where f is decreasing, f' is negative.

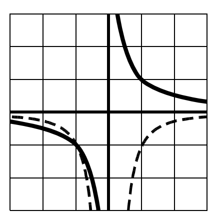
ANSWER (larger answer graphs are included after the group work)



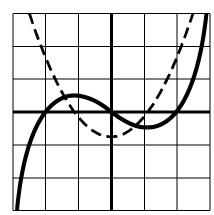
Graph 1



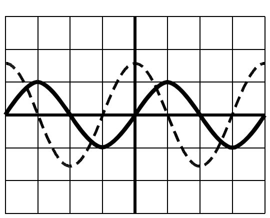
Graph 2



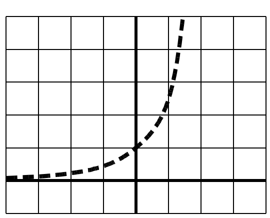
Graph 3



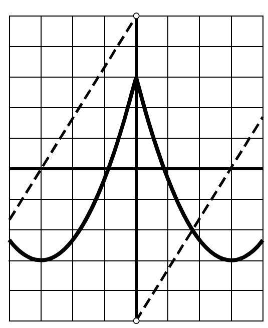
Graph 4



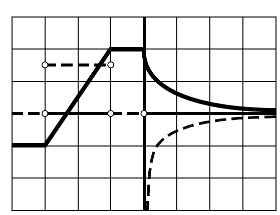
Graph 5



Graph 6



Graph 7



Graph 8

HOMEWORK PROBLEMS

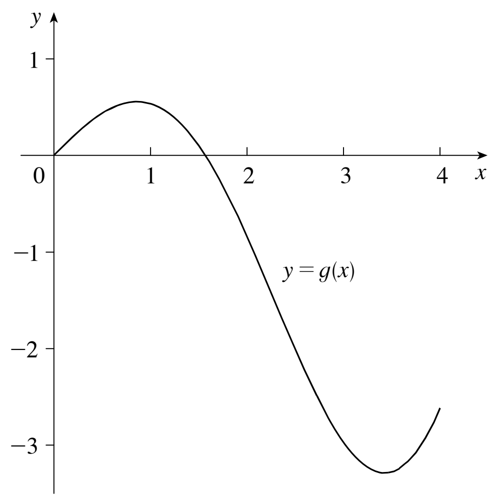
CORE EXERCISES 1, 3, 5, 8, 11, 19, 33, 50

SAMPLE ASSIGNMENT 1, 3, 5, 7, 8, 11, 16, 17, 19, 33, 42, 50, 53

EXERCISE	D	A	N	G
1				×
3				×
5				×
7	×		×	
8				×
11				×
16		×		
17		×		
19		×		
33		×		
42		×		
50		×		
53		×		×

Tangent Lines and the Derivative Function

The following is a graph of $g(x) = x \cos x$.



It is a fact that the derivative of this function is $g'(x) = \cos x - x \sin x$.

1. Sketch the line tangent to $g(x)$ at $x = \frac{\pi}{2} \approx 1.57$ on the graph above.

2. Find an equation of the tangent line at $x = \frac{\pi}{2}$.

3. Now sketch the line tangent to $g(x)$ at $x = \frac{\pi}{3} \approx 1.05$.

4. Find an equation of the tangent line at $x = \frac{\pi}{3}$.

GROUP WORK 2, SECTION 2.2

The Revenge of Orville Redenbacher

1. Consider a single kernel of popcorn in a microwave oven. Let $V(t)$ be the volume in cm^3 of the kernel at time t seconds. Draw a graph of $V(t)$, including as much detail as you can, up to the time that the kernel is taken from the oven.

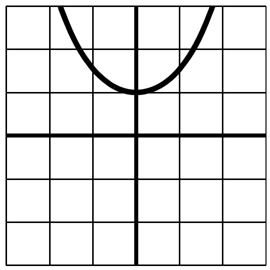
2. Now sketch a graph of the derivative function $V'(t)$. What are the units of $V'(t)$?

3. Finally, sketch a graph of $V''(t)$. What does it mean when this graph crosses the x-axis?

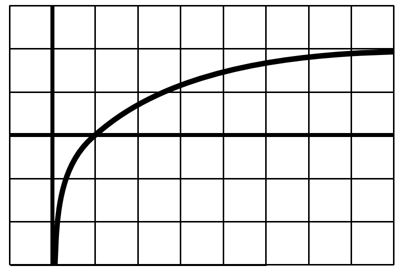
GROUP WORK 3, SECTION 2.2

The Derivative Function

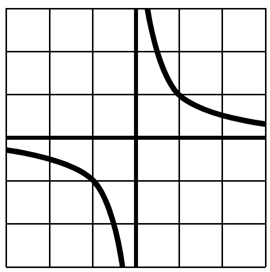
The graphs of several functions f are shown below. For each function, estimate the slope of the graph of f at various points. From your estimates, sketch graphs of f'.



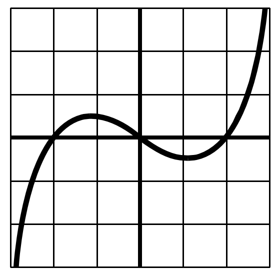
Graph 1



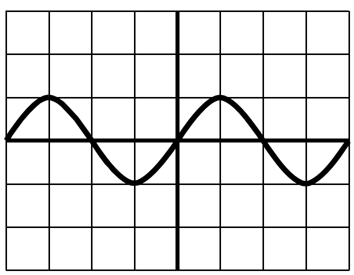
Graph 2



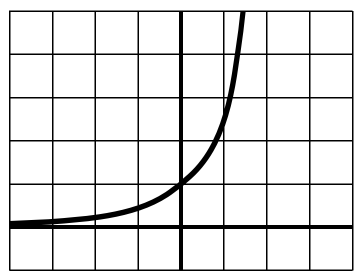
Graph 3



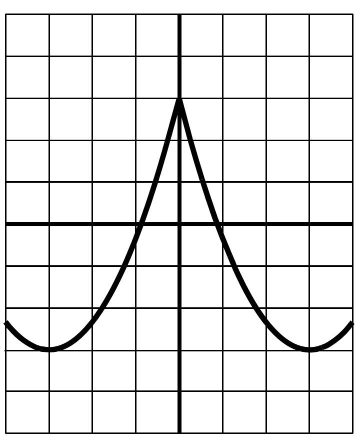
Graph 4



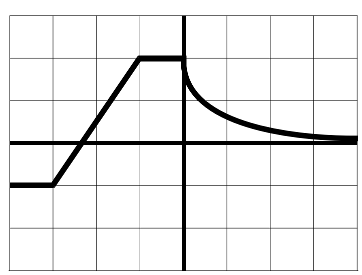
Graph 5



Graph 6

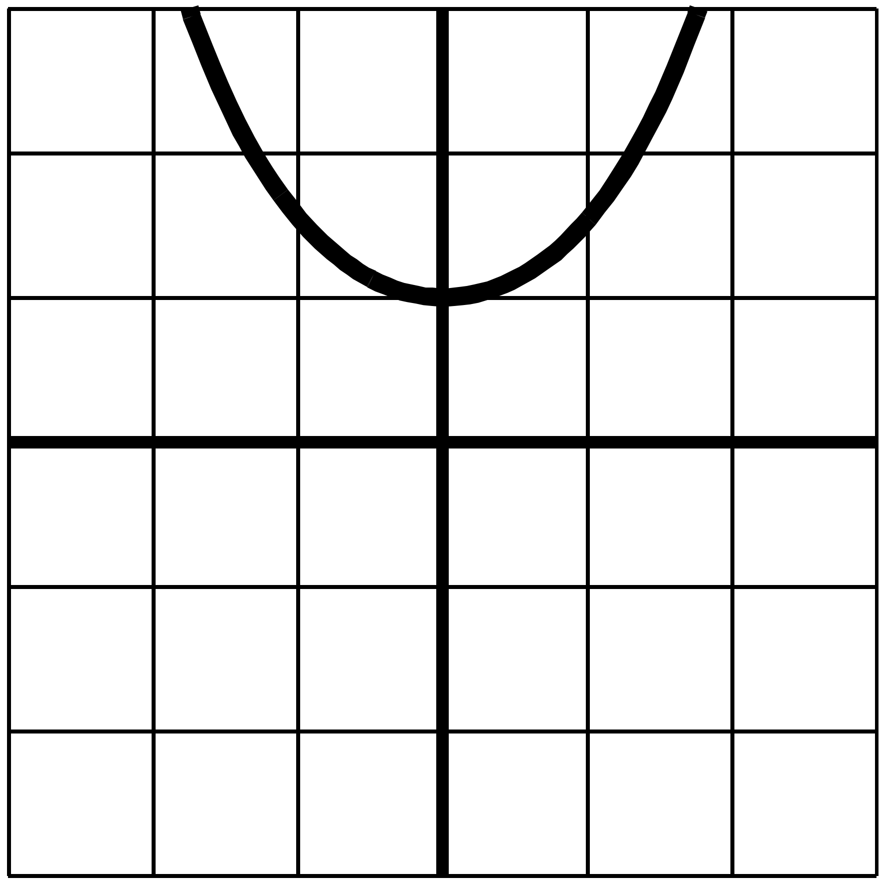


Graph 7

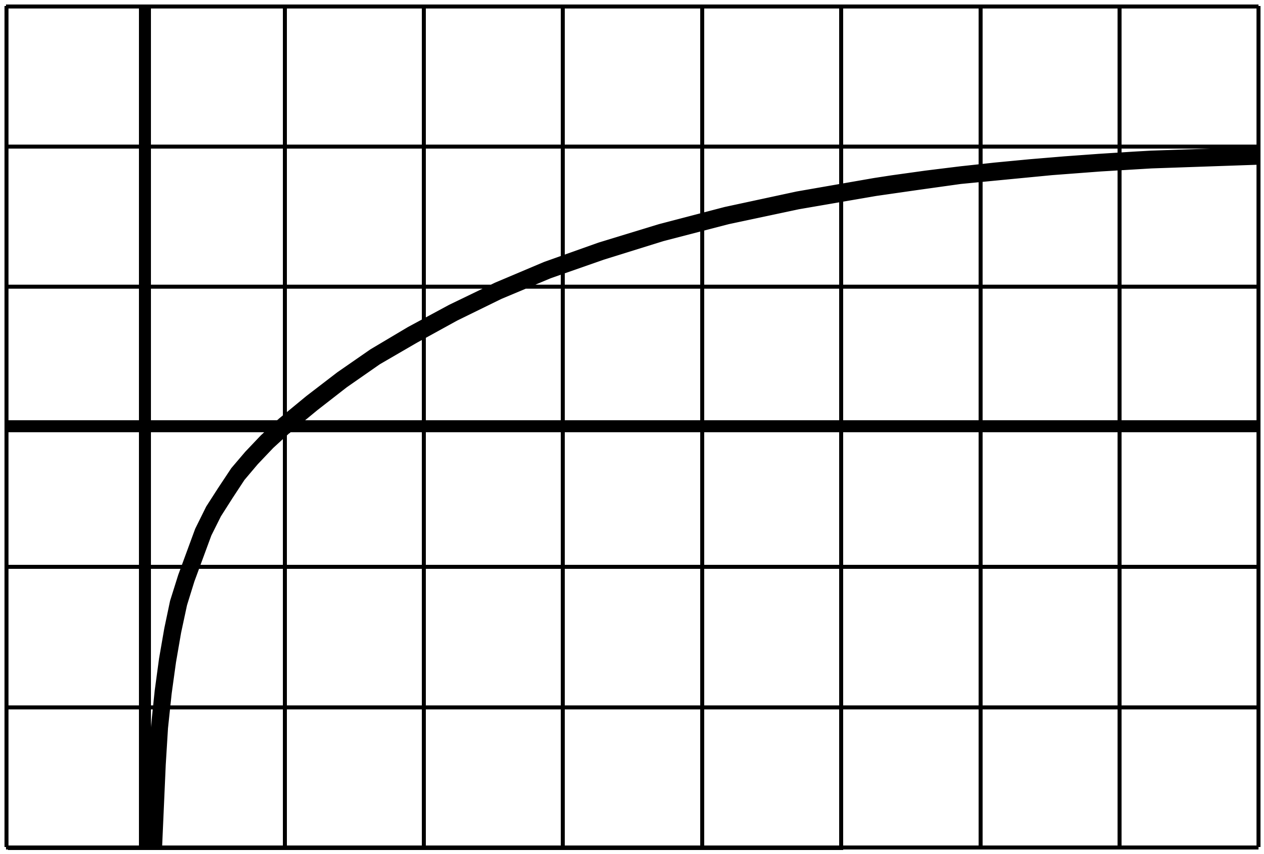


Graph 8

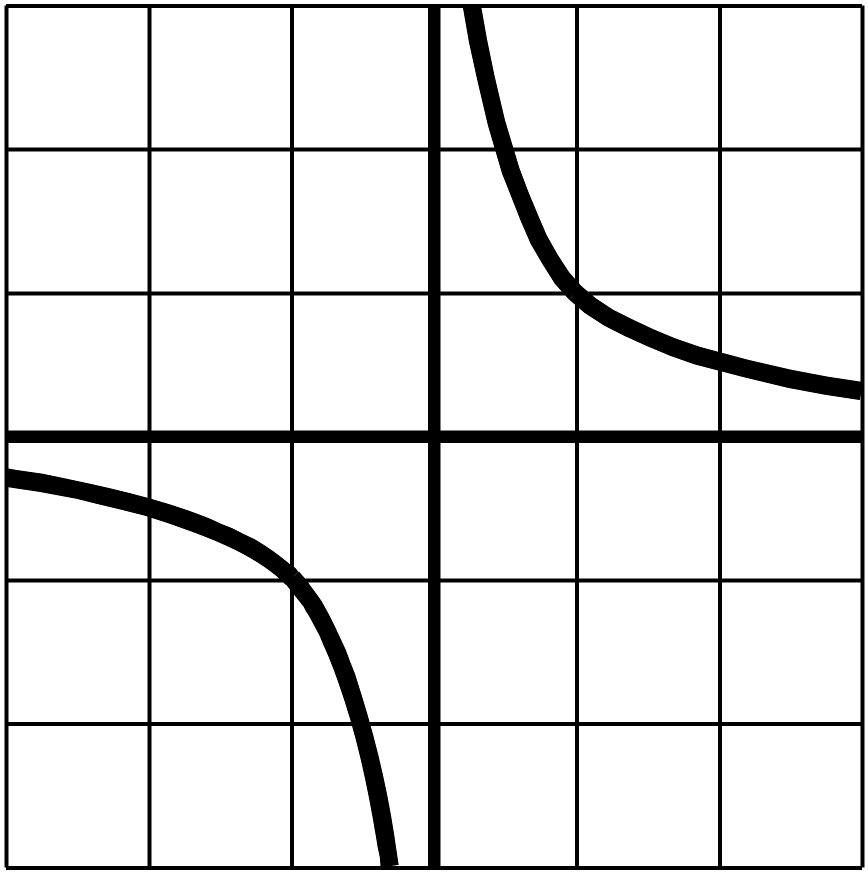
Graph 1



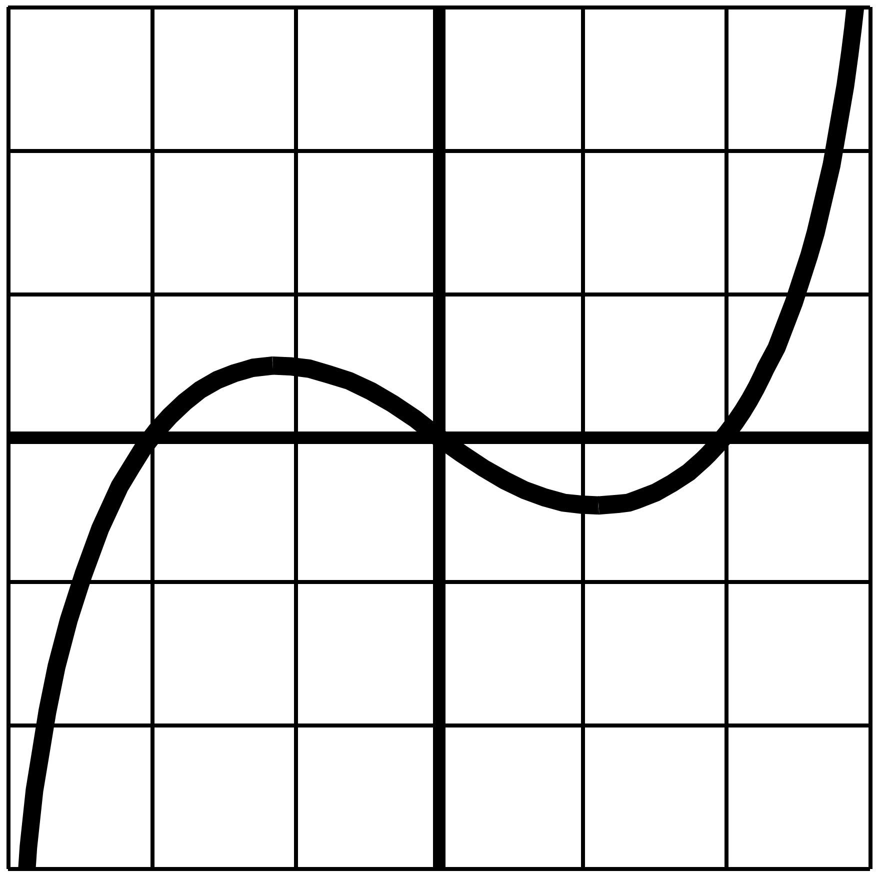
Graph 2



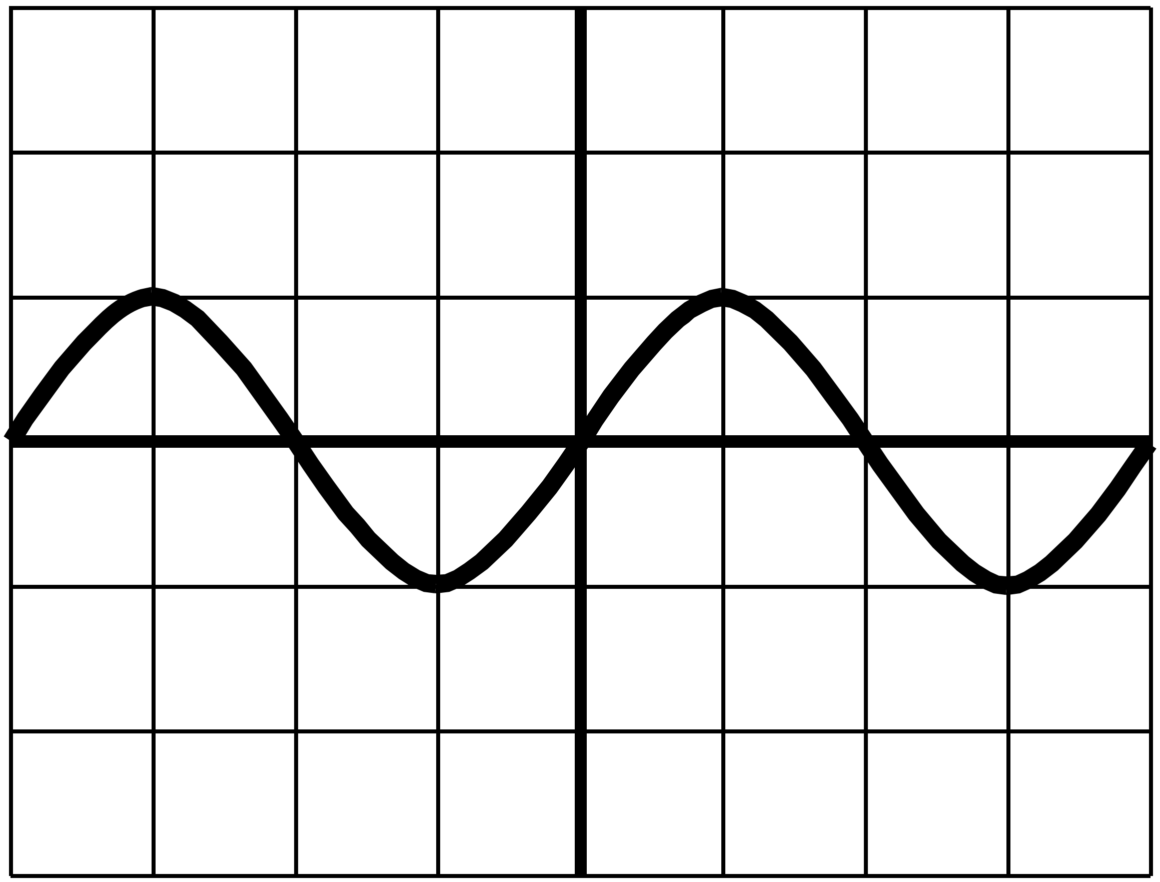
Graph 3



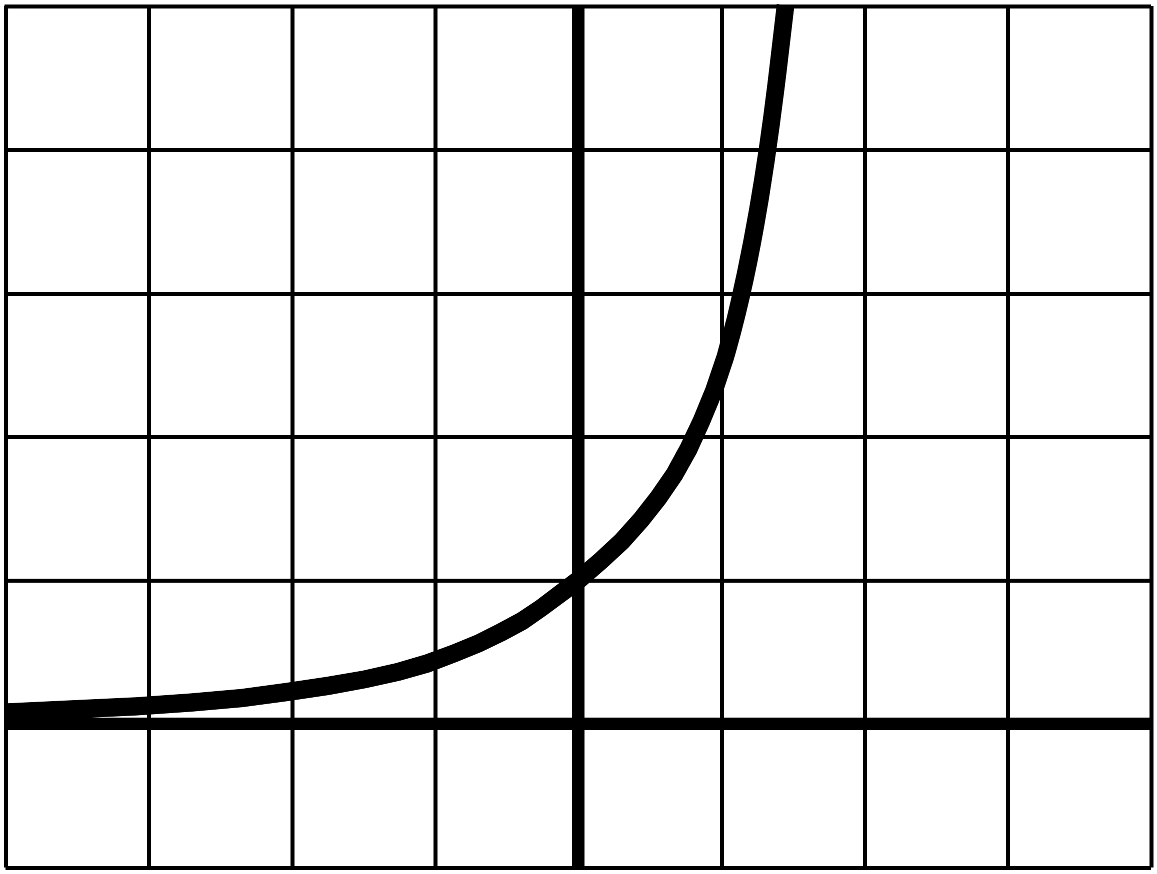
Graph 4



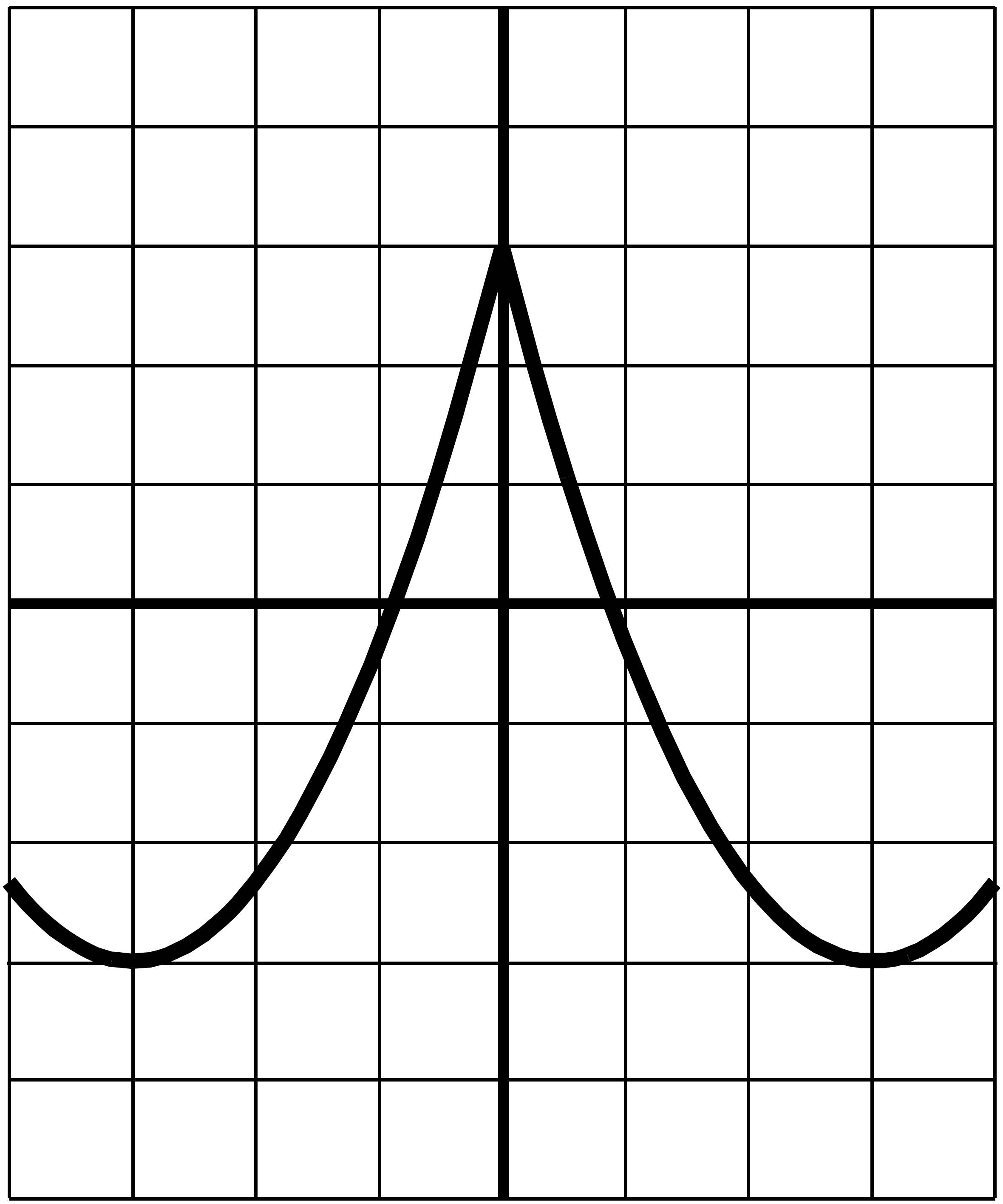
Graph 5



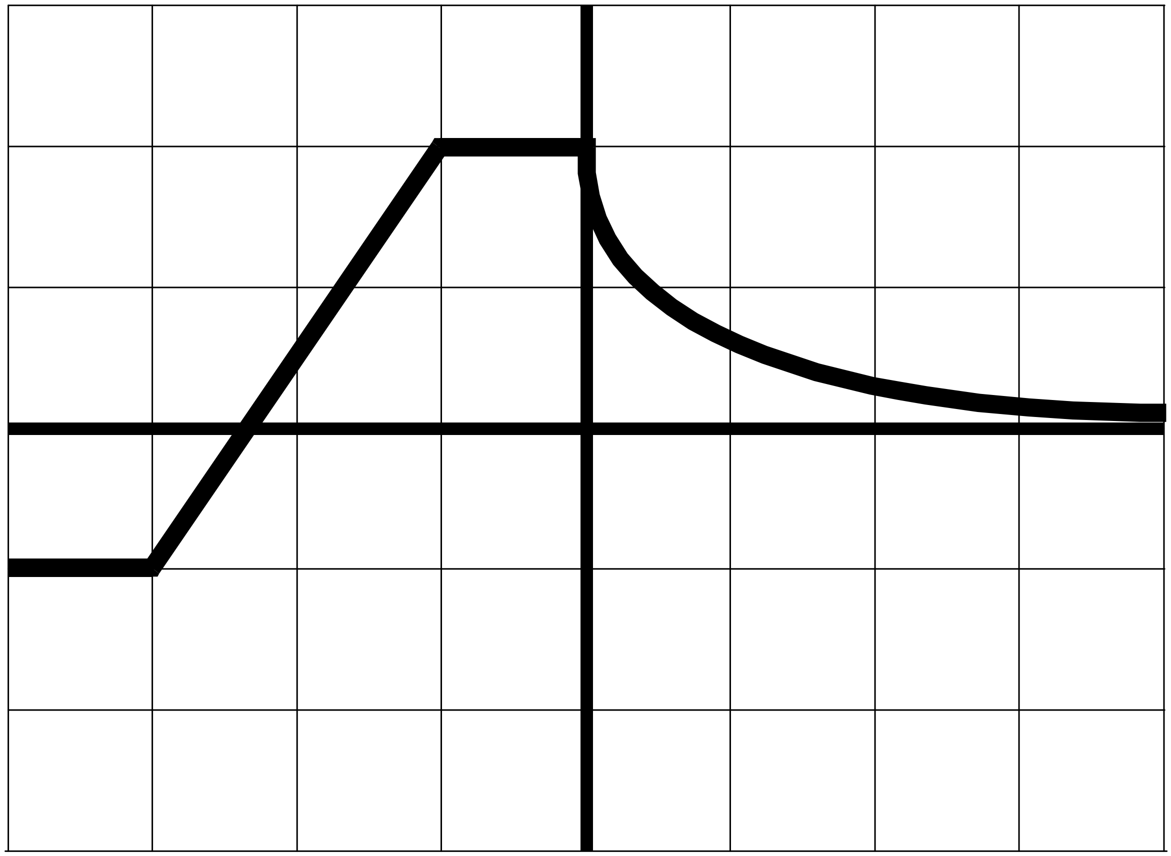
Graph 6



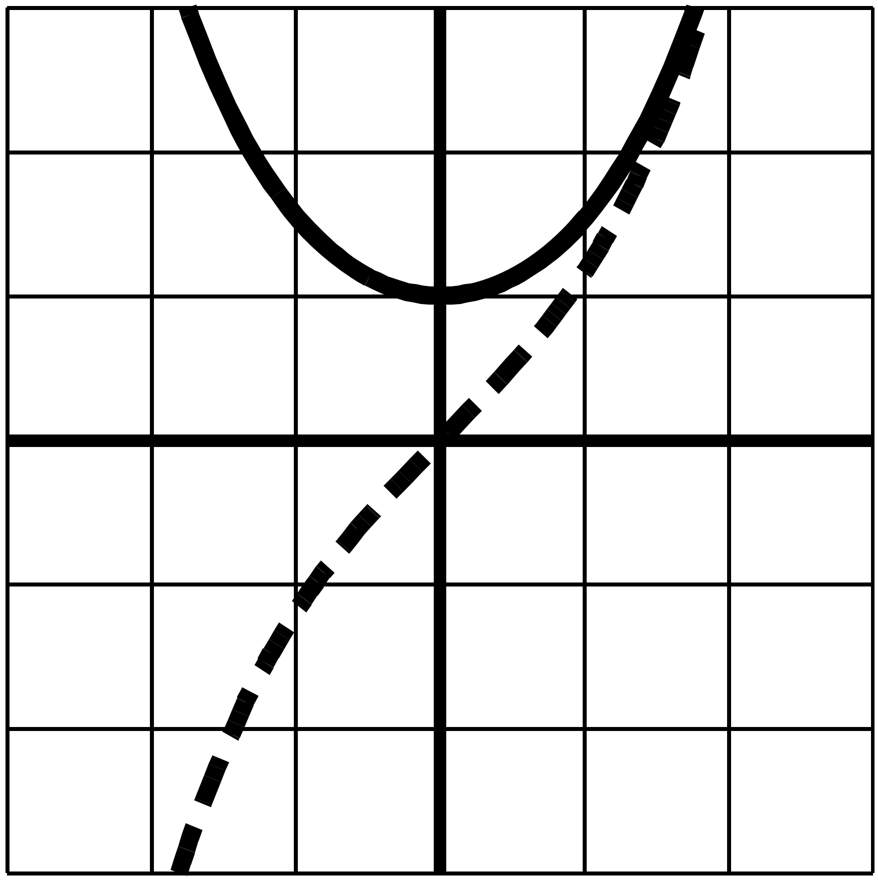
Graph 7



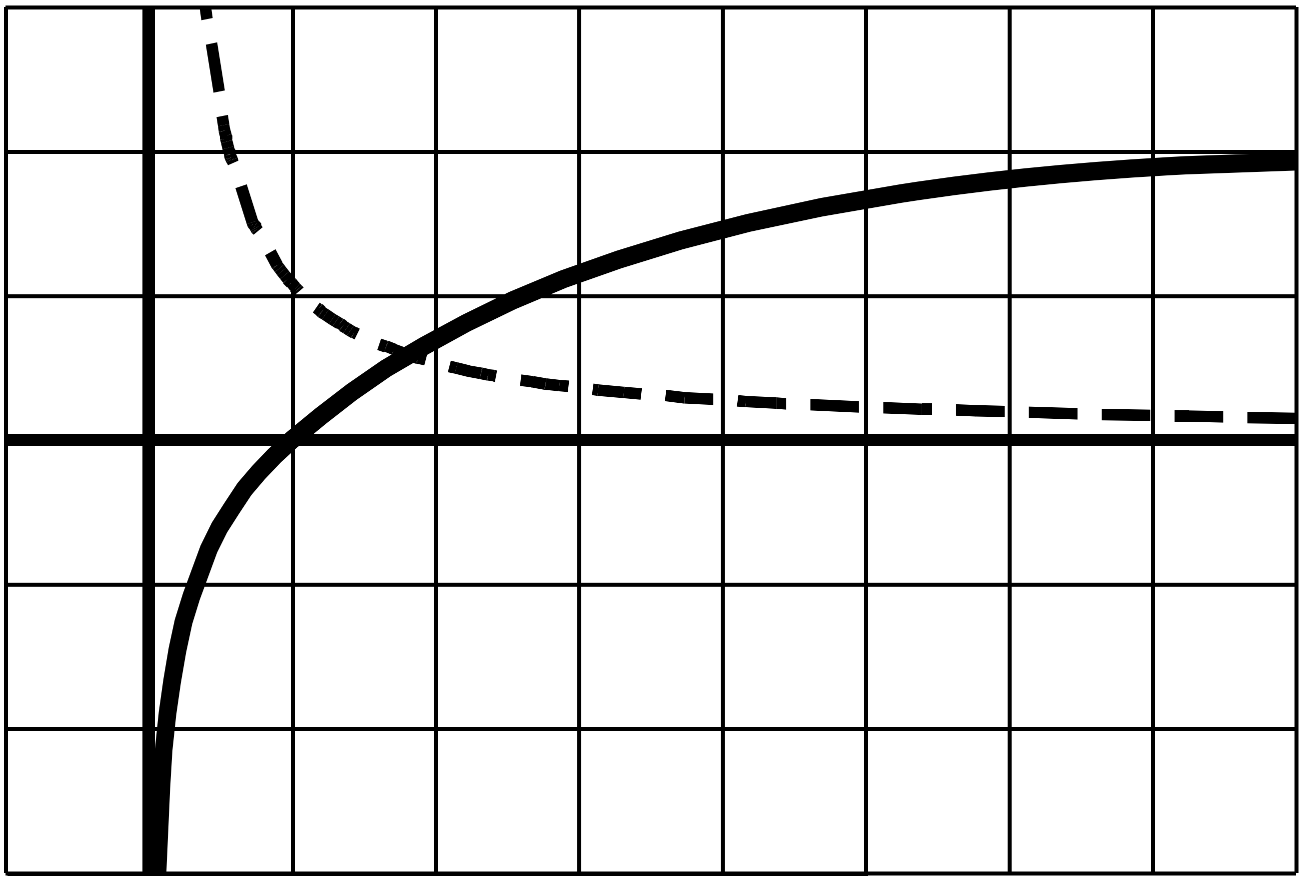
Graph 8



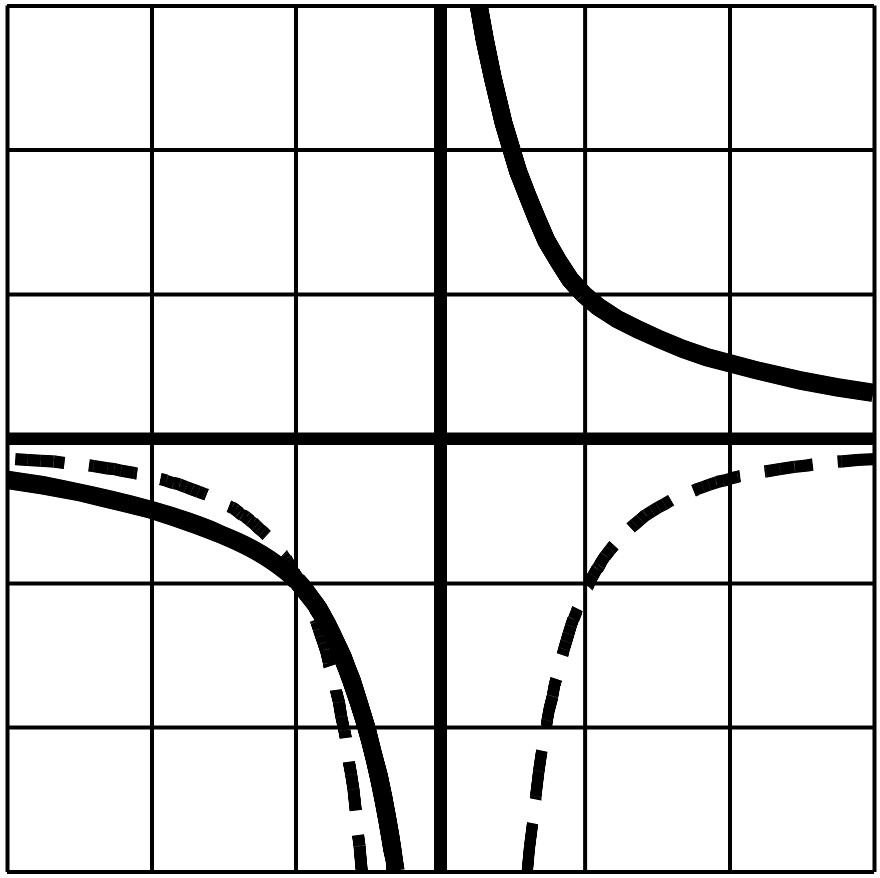
Answer 1



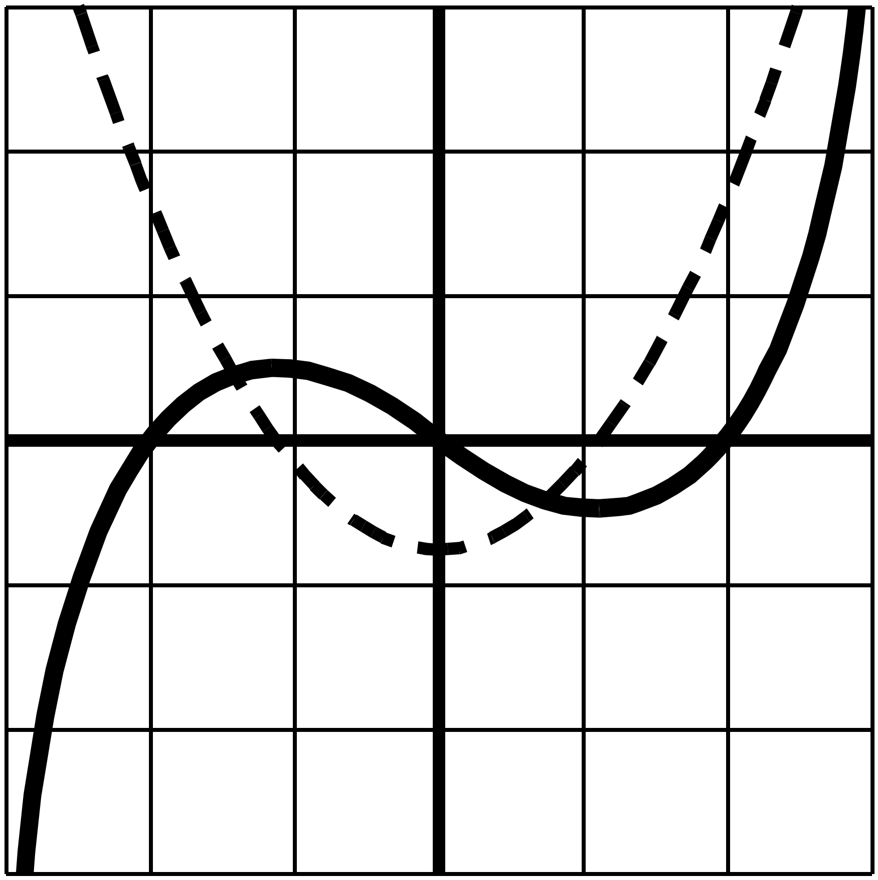
Answer 2



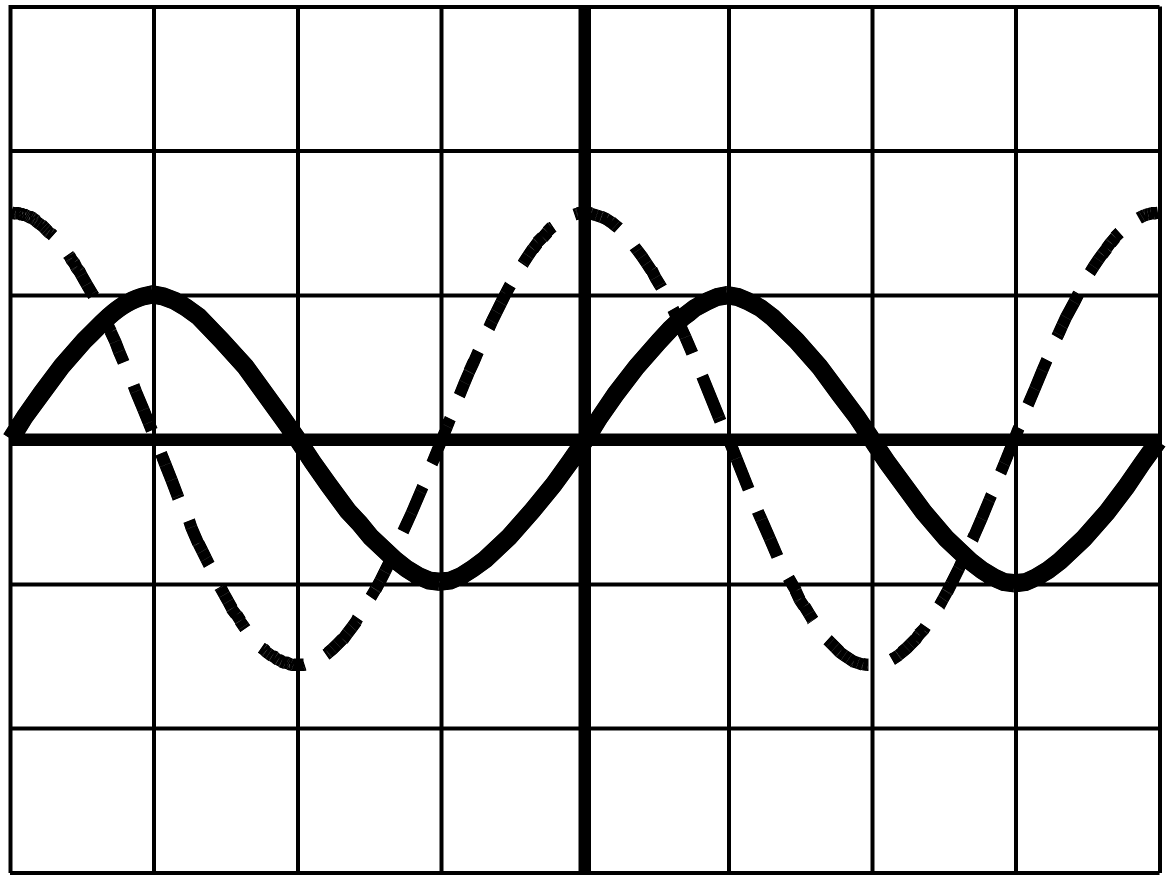
Answer 3



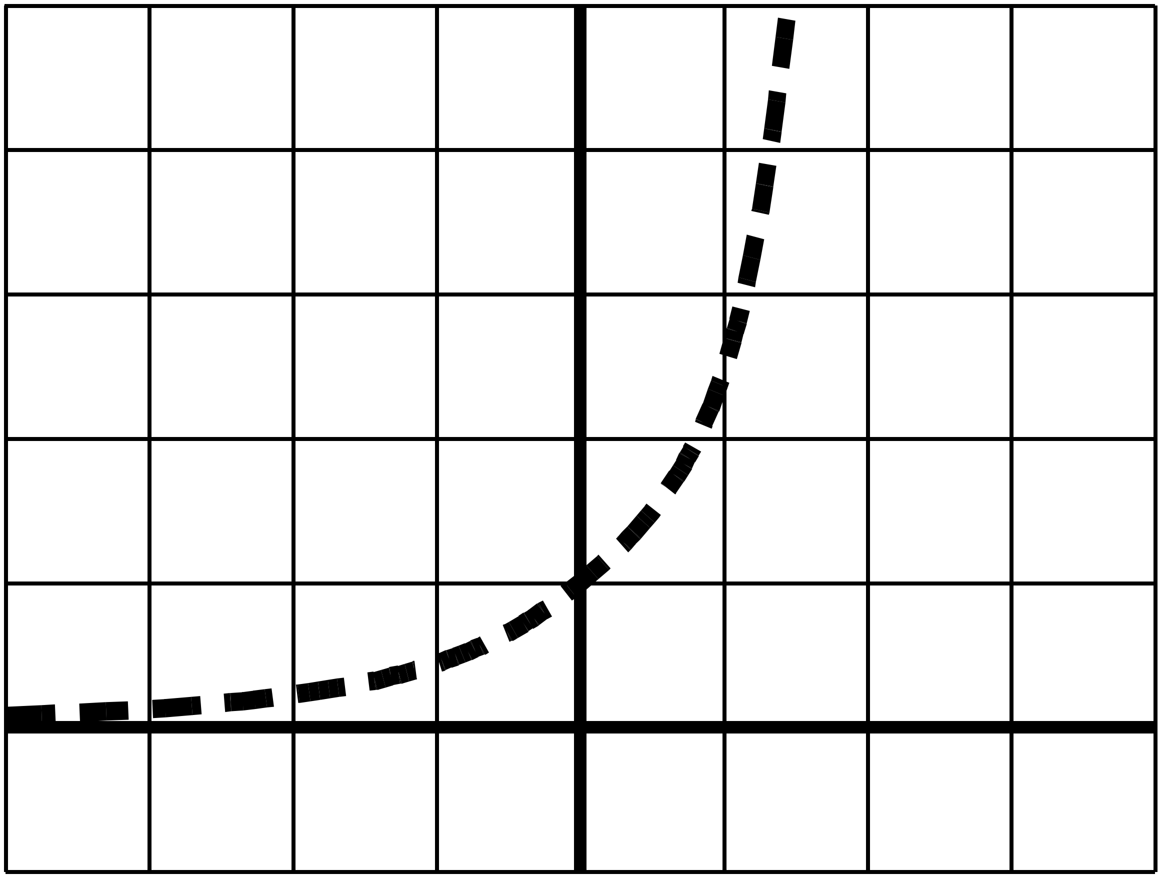
Answer 4



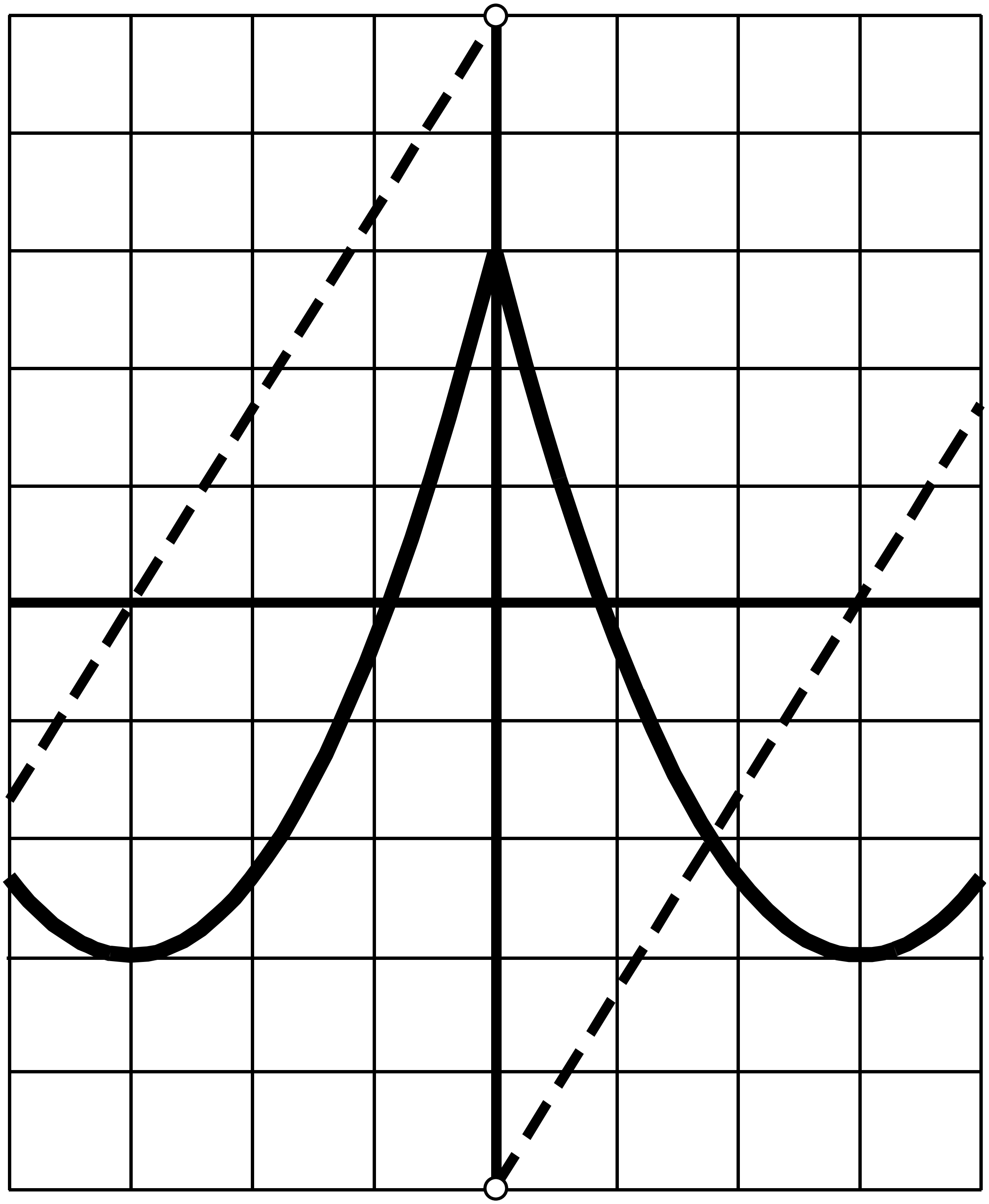
Answer 5



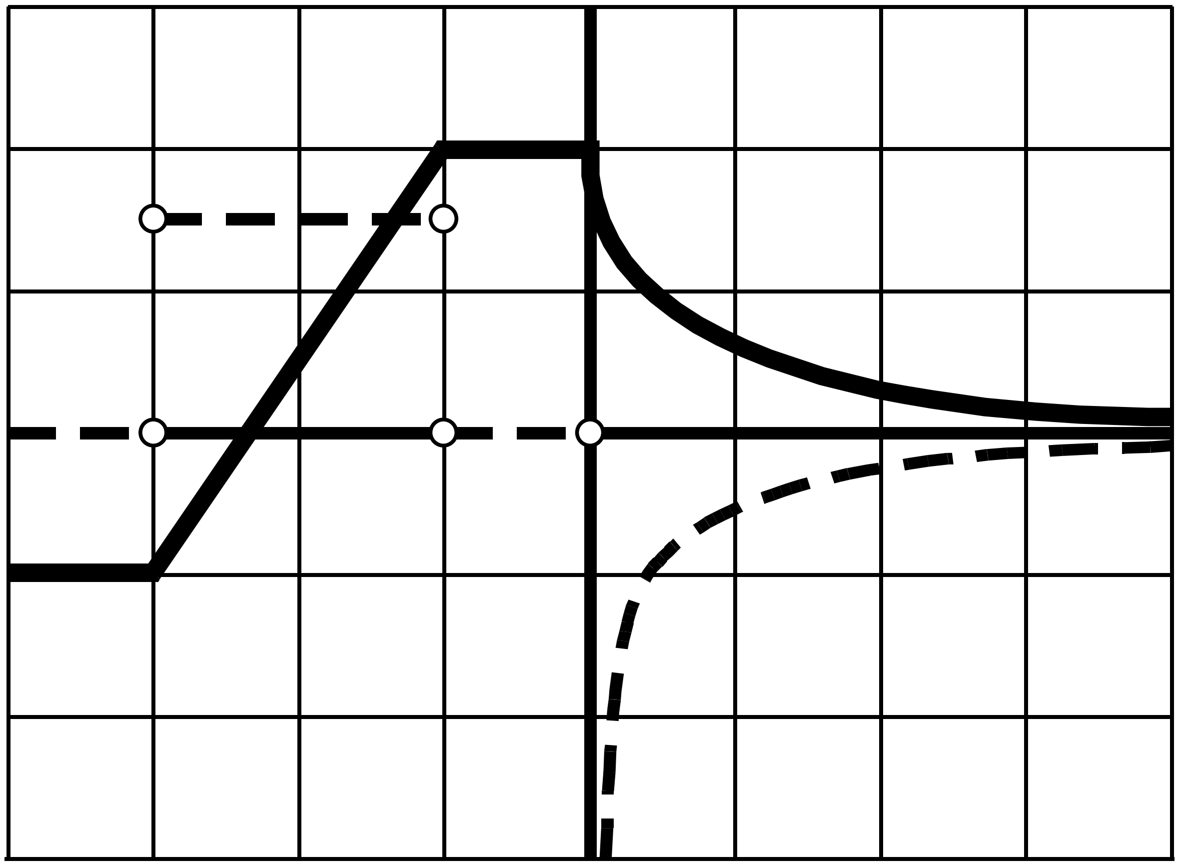
Answer 6



Answer 7



Answer 8



SUGGESTED TIME AND EMPHASIS

2–3 classes Essential material

POINTS TO STRESS

1. The Power, Constant Multiple, Sum and Difference Rules, and how they are developed from the limit definition of the derivative.
2. Justification of the Product and Quotient Rules.
3. The computation of derivatives using the above rules.

QUIZ QUESTIONS

- **TEXT QUESTION** Why don't we use the Quotient Rule every time we encounter a quotient?
  ANSWER Sometimes algebraic simplification can make the problem much easier.
- **DRILL QUESTION** Compute the derivative of $\left(x^6 - \frac{1}{8}x^4\right)\left(\sqrt{x} + \pi\right)$.
  ANSWER $\dfrac{x^6 - \frac{1}{8}x^4}{2\sqrt{x}} + \left(6x^5 - \frac{1}{2}x^3\right)\left(\sqrt{x} + \pi\right)$

MATERIALS FOR LECTURE

- As an introductory exercise, draw the function $f(x) = \dfrac{x^3}{3}$. Ask the students to estimate slopes at several points, perhaps using secant lines. Create a table of x versus $f'(x)$ and try to get them to see the pattern. Then review the idea of the derivative function. Similarly, examine the derivatives of $f(x) = 5x + 2$ and $f(x) = 3$.
- Let $f(x) = x^3 + 2x^2 + 3x + 4$. Find a point a, both visually and algebraically, where $f'(a) = 2$. Then ask them to find where the tangent line to the function $f(x) = x^3 - x + 1$ is parallel to the line $y = x$.
- Derive the Product Rule, and show its relationship to the Constant Multiple Rule (For example, one can find $\left[3e^x\right]'$ using *either* rule, but $\left[xe^x\right]'$ requires the Product Rule.)
- State and demonstrate a proof of the Quotient Rule via the Reciprocal Rule:

  Let $fg = 1$. Then by the Product Rule, $f'g + g'f = 0 \quad \Rightarrow \quad f'g = -g'f \quad \Rightarrow \quad f' = -\dfrac{g'f}{g} = -\dfrac{g'}{g^2}$ since $f = \dfrac{1}{g}$. This is the Reciprocal Rule: If $f = \dfrac{1}{g}$, then $f' = -\dfrac{g'}{g^2}$.

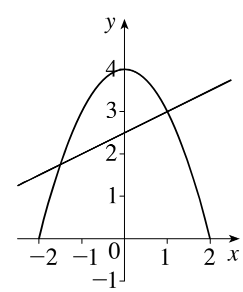
  This result allows us to prove the Quotient Rule:

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \left(f \cdot \frac{1}{g}\right)' = f'\left(\frac{1}{g}\right) + f\left(\frac{1}{g}\right)' \quad \text{(by the Product Rule)}\\ &= \frac{f'}{g} + f\left(-\frac{g'}{g^2}\right) \quad \text{(by the Reciprocal Rule)}\\ &= \frac{f'g - fg'}{g^2}\end{aligned}$$

- Show that, if $f(x) = x^4 - x^2 + x + 1$, then $f^{(5)}(x) \equiv 0$. Conclude that if $f(x)$ is a polynomial of degree m, then $f^{(m+1)}(x) \equiv 0$.

WORKSHOP/DISCUSSION

- Do a complex-looking differentiation that requires algebraic simplification, such as $f(x) = \dfrac{x^2\sqrt[3]{x} + x\sqrt{x^3} - \left(\sqrt{2}x\right)^2}{\left(x^{2/3}\right)^2}$
- After the students have mastered the basics of the Power Rule, have them differentiate some notationally tricky functions such as x^π, $\sqrt[\pi]{x}$, and $\pi^{\sqrt{2}}$.
- Give some examples in which the automatic use of the Quotient Rule is not the best strategy to follow, for example, $f(x) = \dfrac{x^2 + \sqrt{x} - \sqrt[3]{x}}{x}$, $g(x) = \dfrac{x^3 - 2x}{17}$, or $h(x) = \dfrac{3}{x}$. The idea is to get the students to think and simplify first (if they can) before using any of the rules.
- Do an example like Exercise 53. If you actually use the Witch of Agnesi, the students may be interested to hear the history of the curve: Italian mathematician Maria Agnesi (1718–1799) was a scholar whose first paper was published when she was nine years old. She called a particular curve *versiera*, or "turning curve". John Colson from Cambridge confused the word with *avversiera*, or "wife of the devil," and translated it "witch".
- Graph $f(x) = 4 - x^2$ and compute the equations of the tangent line and the normal line at $x = 1$. Draw those lines and point out that, as predicted, they are perpendicular.



GROUP WORK 1: DOING A LOT WITH A LITTLE

This exercise starts out by showing what can be done with the Power Rule, and ends by foreshadowing the Chain Rule. The first page should be handed out separately, and then the second sheet handed out to groups who finish early. Emphasize that the solution to Problem 5 should resemble that of Problem 4 in form. If a group finishes both sheets far ahead of the others, ask them to figure out a formula for the derivative of $f(x) = (g(x))^n$, and to come up with a few examples to check their formula. (Notice that when we state the Power Rule, we allow n to be any real number.)

ANSWERS (Notation may vary)

1. $f'(x) = 10x^9 + 7x^8 + 4x^7 - 35x^6 - 1.98x^5 + 5\pi x^4 - 4\sqrt{2}x^3$

2. $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$, $g'(x) = -\dfrac{3}{x^4} + \dfrac{3}{4\sqrt[4]{x^7}}$, $h'(x) = \frac{9}{2}x^{7/2} - x^{-3/2}$

3. $f'(x) = 64x^3$, $g'(x) = 15x^{14}$

4. This follows immediately when the given functions are expanded.

5. $f'(x) = n(kx)^{n-1}k$, $g'(x) = n\left(x^k\right)^{n-1} \cdot kx^{k-1}$

GROUP WORK 2: FIND THE ERROR

This is the first of several exercises where students will try to find mistakes in somebody else's reasoning. When first faced with a task like this, some students will pick a line towards the end, show it is false, and then consider the task completed. It is important to stress you want them to find the reasoning error; what the person who did the work did incorrectly to get that false line.

If a student still doesn't understand the idea, put it this way: "The person who wrote this listens to what you just said, and says, 'What did I do wrong?' Can you give an answer that will help that person avoid making similar mistakes in the future?"

ANSWER The function "$\underbrace{x + x + \cdots + x}_{x \text{ times}}$" is defined only for integer values of x and is thus not a differentiable function.

GROUP WORK 3: BACK AND FORTH

This exercise foreshadows antiderivatives and gives students an opportunity to practice using the derivative rules they've learned so far.

The students pair up, and decide who is A and who is B. Seat the A's on one side of the room and the B's on the other side. All the A's get one sheet, and all the B's get the other sheet. The students compute five derivatives, without simplifying, and write their answers in the space provided. Emphasize that they should write only their unsimplified answers, not the work leading up to them, in the blanks. Then they trade papers with their partner and try to undo what their partner has done, that is, find the antiderivative.

If a pair finishes early, have them repeat the exercise, making up their own functions, and simplifying at will.

When closing this exercise, have the class notice that there was no way to recover the constant terms in Problems 1 and 5. Ask what this implies about the general problem of finding a function whose derivative is equal to a given function.

ANSWERS

FORM A $f'(x) = 20x^3 + 3x$ (the 4 is unrecoverable), $g'(x) = x^{-1/2} - x^{-3/4}$, $h'(x) = \left(x^2 + 2x + 4\right)\left(3x^2 - 1\right) + (2x + 2)\left(x^3 - x - 3\right)$, $j'(x) = \dfrac{\left(\sqrt{x} + 1\right)\left(4x^3 - 4\right) - \left[\left(x^4 - 4x + 3\right) / \left(2\sqrt{x}\right)\right]}{\left(\sqrt{x} + 1\right)^2}$, $k'(x) = -x^{-4/3}$ (the 42 is unrecoverable)

FORM B $f'(x) = -6x^2 + 8\sqrt{x}$ (the 8 is unrecoverable), $g'(x) = \frac{1}{5}\left[\left(3x^2\right)\left(x^3 + x\right) + \left(x^3 + 1\right)\left(3x^2 + 1\right) + 12x\right]$,
$h'(x) = \left(x^3 + x^2 + 2x\right)\left(10x - 8x^3 + 8\right) + \left(5x^2 - 2x^4 + 8x\right)\left(3x^2 + 2x + 2\right)$,
$j'(x) = 1 + 2x$ $(x \neq 0)$, $k'(x) = -\frac{22}{3}x^{-2/3}$

GROUP WORK 4: SPARSE DATA

This exercise allows the students to practice the rules they have learned, with a minimum of algebraic manipulation. The students should work on these problems in groups of three or four, perhaps choosing groups of students with similar algebraic proficiency. Problem 5 uses the General Power Rule, which was illustrated in Group Work 1.

ANSWERS **1.** 0 **2.** -48 **3.** $\frac{43}{25}$ **4.** -18 **5.** $\frac{1}{3}$

HOMEWORK PROBLEMS

CORE EXERCISES 2, 5, 12, 18, 24, 26, 32, 50, 51, 60, 101

SAMPLE ASSIGNMENT 2, 5, 12, 18, 24, 26, 32, 32, 35, 47, 50, 51, 60, 61, 68, 73, 90, 94, 101, 105

EXERCISE	D	A	N	G
2		×		
5		×		
12		×		
18		×		
24		×		
26		×		
32		×		
32		×		
35		×		
47				×
50		×		×
51		×		
60		×		
61	×			
68			×	×
73				×
90		×		
94		×		
101		×		
105		×		

GROUP WORK 1, SECTION 2.3

Doing a Lot with a Little

Section 2.3 introduces the Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$, where n is any real number. The good news is that this rule, combined with the Constant Multiple and Sum Rules, allows us to take the derivative of even the most formidable polynomial with ease! To demonstrate this power, try Problem 1:

1. *A formidable polynomial:*

$$f(x) = x^{10} + \tfrac{7}{9}x^9 + \tfrac{1}{2}x^8 - 5x^7 - 0.33x^6 + \pi x^5 - \sqrt{2}x^4 - 42$$

Its derivative:

$$f'(x) =$$

The ability to differentiate polynomials is only one of the things we've gained by establishing the Power Rule. Using some basic definitions, and a touch of algebra, there are all kinds of functions that can be differentiated using the Power Rule.

2. *All kinds of functions:*

$$f(x) = \sqrt[3]{x} + \sqrt[5]{2} \qquad g(x) = \frac{1}{x^3} - \frac{1}{\sqrt[4]{x^3}} \qquad h(x) = \frac{x^5 - 3\sqrt{x} + 2}{\sqrt{x}}$$

Their derivatives:

$$f'(x) = \qquad g'(x) = \qquad h'(x) =$$

Unfortunately, there are some deceptive functions that look like they should be straightforward applications of the Power and Constant Multiple Rules, but actually require a little thought.

3. *Some deceptive functions:*

$$f(x) = (2x)^4 \qquad g(x) = \left(x^3\right)^5$$

Their derivatives:

$$f'(x) = \qquad g'(x) =$$

The process you used to take the derivative of the functions in Problem 3 can be generalized. In the first case, $f(x) = (2x)^4$, we had a function that was of the form $(kx)^n$, where k and n were constants ($k = 2$ and $n = 4$). In the second case, $g(x) = \left(x^3\right)^5$, we had a function of the form $\left(x^k\right)^n$. Now we are going to find a pattern, similar to the Power Rule, that will allow us to find the derivatives of these functions as well.

4. Show that your answers to Problem 3 can also be written in this form:

$$f'(x) = 4(2x)^3 \cdot 2 \qquad g'(x) = 5\left(x^3\right)^4 \cdot 3x^2$$

And now it is time to generalize the Power Rule. Consider the two general functions, and try to find expressions for the derivatives similar in form to those given in Problem 4. You may assume that n is an integer.

5. *Two general functions*:

$$f(x) = (kx)^n \qquad g(x) = \left(x^k\right)^n$$

Their derivatives:

$$f'(x) =$$
$$g'(x) =$$

GROUP WORK 2, SECTION 2.3

Find the Error

It is a bright Spring morning. You have just finished your Chemistry lab, and have a Physics class starting in a half hour, so you have a little bit of time to sit on a park bench and relax by leafing through your *Calculus* book. Suddenly, you notice a wild-eyed, hungry-looking stranger looking over your shoulder.

"Lies! Lies!" he yells. "That book there is filled with nothing but lies!"

"Why, you are mistaken," you explain. "My *Calculus* book is chock-a-block with knowledge and useful wisdom."

"Oh yeah? Well what would your calculus book say about THIS?" he demands, and hands you a piece of paper with the following written on it:

For $x > 0$:

$$\begin{aligned}
x &= \underbrace{1+1+\cdots+1}_{x \text{ times}} \\
x^2 &= \underbrace{x+x+\cdots+x}_{x \text{ times}} \\
D\left(x^2\right) &= D\left(\underbrace{x+x+\cdots+x}_{x \text{ times}}\right) \\
D\left(x^2\right) &= \underbrace{D(x)+D(x)+\cdots+D(x)}_{x \text{ times}} \\
2x &= \underbrace{1+1+\cdots+1}_{x \text{ times}} \\
2x &= x \\
2 &= 1
\end{aligned}$$

"Put THAT in your pipe and smoke it!" At that, the gentleman runs off, screaming, "I'll be back!" into the wind.

Is all of mathematics wrong? Is two really equal to one? Are "two for one" specials really no bargain at all? Is "six of one" really not "half a dozen of the other"? Or is there a mistake in your new friend's reasoning? If so, what is it?

GROUP WORK 3, SECTION 2.3

Back and Forth (Form A)

Compute the following derivatives. Write your answers at the bottom of this sheet, where indicated. When finished, fold the top of the page backward along the dotted line and hand to your partner.

Do not simplify.

1. $f(x) = 5x^4 + \frac{3}{2}x^2 - 4$

2. $g(x) = 2\sqrt{x} - 4\sqrt[4]{x}$

3. $h(x) = \left(x^2 + 2x + 4\right)\left(x^3 - x - 3\right)$

4. $j(x) = \dfrac{x^4 - 4x + 3}{\sqrt{x} + 1}$

5. $k(x) = \dfrac{3}{\sqrt[3]{x}} + 42$

- -

ANSWERS

$f'(x) =$

$g'(x) =$

$h'(x) =$

$j'(x) =$

$k'(x) =$

GROUP WORK 3, SECTION 2.3

Back and Forth (Form B)

Compute the following derivatives. Write your answers at the bottom of this sheet, where indicated. When finished, fold the top of the page backward along the dotted line and hand to your partner.

Do not simplify.

1. $f(x) = -2x^3 + \frac{\sqrt{8}}{2}x^2 - 8$

2. $g(x) = \dfrac{(x^3+1)(x^3+x)+6x^2}{5}$

3. $h(x) = (x^3+x^2+2x)(5x^2-2x^4+8x)$

4. $j(x) = \dfrac{x^2+x^3}{x}$

5. $k(x) = \sqrt{11} - 22\sqrt[3]{x}$

- -

ANSWERS

$f'(x) =$

$g'(x) =$

$h'(x) =$

$j'(x) =$

$k'(x) =$

GROUP WORK 4, SECTION 2.3

Sparse Data

Assume that $f(x)$ and $g(x)$ are differentiable functions about which we know very little. In fact, assume that all we know about these functions is the following table of data:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	3	1	-5	8
-1	-9	7	4	1
0	5	9	9	-3
1	3	-3	2	6
2	-5	3	8	?

This isn't a lot of information. For example, we can't compute $f'(3)$ with any degree of accuracy. But we are still able to figure some things out, using the rules of differentiation.

1. Let $h(x) = \left(\sqrt[3]{x}\right)^4 f(x)$. What is $h'(0)$?

2. Let $j(x) = -4f(x)g(x)$. What is $j'(1)$?

3. Let $k(x) = \dfrac{xf(x)}{g(x)}$. What is $k'(-2)$?

4. Let $l(x) = x^3 g(x)$. If $l'(2) = -48$, what is $g'(2)$?

5. Let $m(x) = \dfrac{1}{f(x)}$. What is $m'(1)$?

APPLIED PROJECT **Building a Better Roller Coaster**

This project models a typical hill in a roller coaster ride using two lines as the sides and a parabola for the peak area. It also discusses how to smooth this model to have a continuous second derivative by using cubic connecting functions between the parabola and the two lines. A computer algebra system is needed to solve the resulting equations. In their report, students should address the question, "*Why* do we want the second derivative to be continuous?"

2.4 Derivatives of Trigonometric Functions

SUGGESTED TIME AND EMPHASIS

1 class Essential material

POINTS TO STRESS

Formulas for the derivatives of the standard trigonometric functions.

QUIZ QUESTIONS

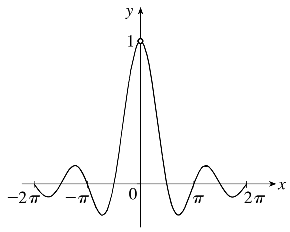
- **TEXT QUESTION** Why does the text bother going through all the fuss of computing $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$ and $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta}$?
  ANSWER When deriving the formulas for the derivatives for $\sin \theta$ and $\cos \theta$, these limits arise when taking the limits of the difference quotients. These computations are necessary to finish the derivations.
- **DRILL QUESTION** What is $\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - \tan\left(\frac{\pi}{4}\right)}{h}$?
  (A) 2 (B) $-\frac{\sqrt{2}}{2}$ (C) 0 (D) 1 (E) Does not exist
  ANSWER (A)

MATERIALS FOR LECTURE

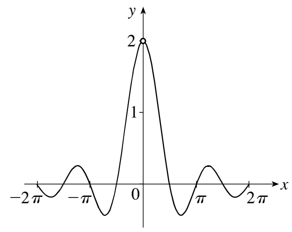
- Many students may need a review of notation: $\sin^2 x = (\sin x)^2$, $\sin x^2 = \sin\left(x^2\right)$, $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$, $\sin x^{-1} = \sin \frac{1}{x}$, but $\sin^{-1} x$ represents the inverse sine of x, $\arcsin x$, and not any of the previous functions.
- Demonstrate simple harmonic motion in different ways such as observing the end of a vertical spring, marking the edge of a spinning disk, or swinging an object on a chain.
- Have the students set their calculators to degrees and approximate the derivative of $\cos x$ at $x = \frac{\pi}{2}$ by zooming in on the graph of $\cos x$. Repeat the exercise with their calculators set to radians. Discuss the reason why the answers are different, and why only one is considered correct. Show how the slope of the tangent to the graph of $\sin x$ at $x = 0$ is *not* 1 if the x-axis is calibrated in degrees instead of radians.
  ANSWER The derivation of $(\sin \theta)' = \cos \theta$ involved using the fundamental trigonometric limit, which assumed θ was in radians.

WORKSHOP/DISCUSSION

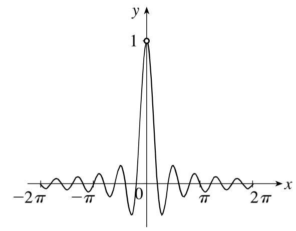
- Demonstrate that $\lim_{x \to 0} \frac{\sin ax}{ax} = 1$ for any positive a. Then ask students to find $\lim_{x \to 0} \frac{\sin ax}{x}$. Show how this argument can be extended to derive the formulas $\frac{d}{dx} \sin ax = a \cos ax$ and $\frac{d}{dx} \cos ax = -a \sin ax$. Finally, demonstrate that your results make sense by drawing graphs of $\frac{\sin ax}{ax}$ and $\frac{\sin ax}{x}$ for various values of a.



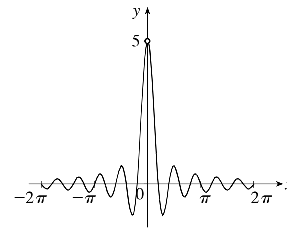
$y = \dfrac{\sin 2x}{2x}$



$y = \dfrac{\sin 2x}{x}$



$y = \dfrac{\sin 5x}{5x}$



$y = \dfrac{\sin 5x}{x}$

- Consider $f(x) = \frac{1}{2}x + \cos x$, $0 \le x \le 2\pi$. Discuss local maxima and minima of $f(x)$. Repeat for $g(x) = \frac{9}{10}x + \cos x$ and $h(x) = x + \cos x$. Discuss why h is qualitatively different from f and g.

GROUP WORK 1: THE MAGNIFICENT SIX

After showing the students that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, it is possible to use the Quotient Rule to derive the trigonometric derivatives on their own, and the process of deriving these formulas is good practice at using the rules learned so far.

ANSWERS

1. $\cos x$ **2.** $-\sin x$ **3.** $\sec^2 x$

4. $\tan x \sec x$ **5.** $-\csc^2 x$ **6.** $-\cot x \csc x$

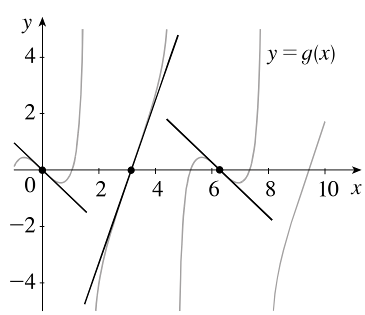
GROUP WORK 2: USING OUR NEW KNOWLEDGE

ANSWERS

1. $-1, 3, -1$

2. $y = -x$, $y = 3x - 3\pi$, $y = -x + 2\pi$

3.



4. There is no tangent line at $y = \frac{\pi}{2}$ because the function has a vertical asymptote there.

GROUP WORK 3: WHEN THE LIGHTS GO DOWN IN THE CITY

This activity will help the students understand the relationship between a trigonometric function in the abstract, and a trigonometric function as a model for real situations.

Creative use of technology can be encouraged here. It is important to stress to the students that Problem 2 assumes that they are looking at only a one-month window. Problem 6 foreshadows the technique of linear approximation covered in Section 2.9.

ANSWERS

1. Maximum: 1, minimum: 20 **2.** It is part of a cosine curve. **3.** December **4.** May
5. 3.095 minutes per day (0.5158 hours per day) **6.** $3.095 \cdot 31 = 95.94$, accurate to within about 1%.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 7, 21, 28, 42, 41, 44

SAMPLE ASSIGNMENT 3, 7, 21, 28, 33, 37, 41, 42, 44

EXERCISE	D	A	N	G
3		×		
7		×		
21		×		
28		×		×
33		×		
37	×	×		
41		×		
42	×	×		
44	×			

GROUP WORK 1, SECTION 2.4

The Magnificent Six

The derivative of $f(x) = \sin x$ was derived for you in class. From this one piece of information, it is possible to figure out formulas for the derivatives of the other five trigonometric functions. Using the trigonometric identities you know, compute the following derivatives. Simplify your answers as much as possible.

1. $(\sin x)' =$

2. $(\cos x)' =$

3. $(\tan x)' =$

4. $(\sec x)' =$

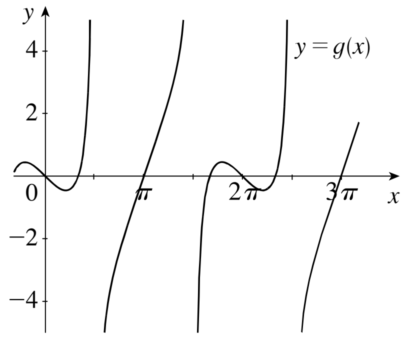
5. $(\cot x)' =$

6. $(\csc x)' =$

GROUP WORK 2, SECTION 2.4

Using Our New Knowledge

The following is a graph of $g(x) = \tan x - 2\sin x$.



There are some things we can say about the graph just by looking at the picture, although our intuition may sometimes mislead us.

1. Compute $g'(0)$, $g'(\pi)$, and $g'(2\pi)$.

2. Find equations of the lines tangent to this curve at $x = 0$, $x = \pi$, and $x = 2\pi$.

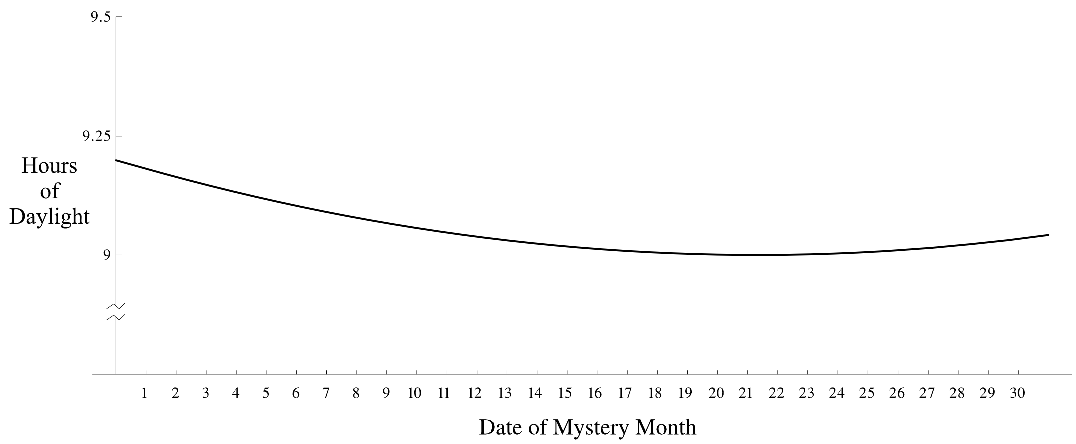
3. Graph the equations you found in Problem 2, and make sure they look as they should.

4. What happens when you try to find the equation of the line tangent to this curve at $x = \frac{\pi}{2}$? Why?

GROUP WORK 3, SECTION 2.4

When the Lights Go Down in the City

The number of hours of daylight in Summitville, Canada varies between 9 hours and 15 hours per day. A model for the number of daylight hours on day t is $D(t) = 12 - 3\cos(0.0172(t + 11))$, $0 < t \leq 365$. ($t = 1$ corresponds to January 1.) The graph for a particular month looks like this:



1. On approximately what day of the month does this graph achieve its minimum? Its maximum?

2. Why does this graph have the shape that it does?

3. What month is this graph likely to represent?

4. For which month would you expect to see a graph shaped like this one, only upside-down?

5. How rapidly are we gaining daylight 90 days after the minimum occurs?

6. A newspaper in Summitville states that during the period of 31 days starting from day 68 after the minimum, we gain 1 hour and 35 minutes of sunlight. Use the rate of change computed in Problem 5 to estimate the change in hours of sunlight over this period. How close is your estimate to the figure reported in the newspaper?

2.5 The Chain Rule

SUGGESTED TIME AND EMPHASIS

$1\frac{1}{2}$–2 classes   Essential material

POINTS TO STRESS

1. A justification of the Chain Rule by interpreting derivatives as rates of change.
2. The use of the Chain Rule to compute derivatives.

QUIZ QUESTIONS

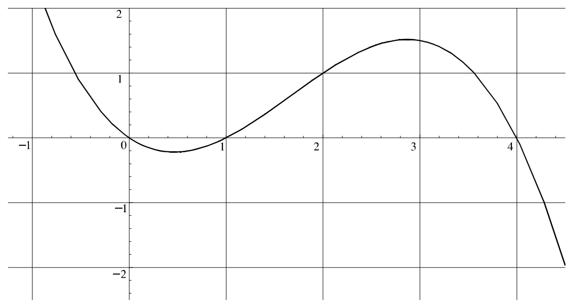
- **TEXT QUESTION** The text presents the two forms of the Chain Rule: $(f(g(x)))' = f'(g(x))\,g'(x)$ and $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$. Do these two equations say the same thing? Explain your answer.
  ANSWER They do. Let $y = f(u)$ and $u = g(x)$. Then the statement $f(g(x))' = f'(g(x))\,g'(x)$ becomes $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.
- **DRILL QUESTION** Compute $\dfrac{d}{dx}\sin x^2$ and $\dfrac{d}{dx}\sin^2 x$.
  ANSWER $2x\cos x^2$, $2\sin x\cos x$

MATERIALS FOR LECTURE

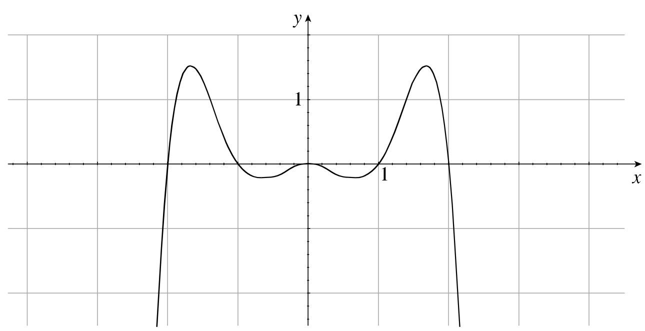
- The following is one way to introduce the Chain Rule:
  Before formally discussing the rule, do two examples of differentiating multi-nested functions. Explain to the students that you aren't going to justify anything yet, but that you just want them to see the pattern before getting into the material. After every step, say something like, "The derivative of $\sin x$ is $\cos x$, so the derivative of the sine of this stuff is the cosine of this stuff, times the derivative of what's left." After the students have seen the pattern with functions like $\left(\sin\left(\cos\left(x^2+4x+5\right)\right)\right)^{33}$, you should justify the Chain Rule and discuss the details.
- Show how to compute derivatives, using the Chain Rule, in one line. Take the derivative of $\sin\left(x^4+1\right)$, first by using the Chain Rule explicitly [$f(u) = \sin u$, $u(x) = x^4+1$], and then by inspection [the derivative of $\sin\left(x^4+1\right)$, which is $\cos\left(x^4+1\right)$ times the derivative of x^4+1, which is $4x^3$.]
- Address the question: "Where do you stop when using the Chain Rule?" For example, why is it false that $\dfrac{d}{dx}\sin\left(x^5+4x^2\right) \stackrel{?}{=} \left[\cos\left(x^5+4x^2\right)\right]\left(5x^4+8x\right)\left(20x^3+8\right)\left(60x^2\right)(120x)(120)$?
  One way to help students decide "when to stop" is to draw their attention to the text's Reference Page 5 (Differentiation Rules). One stops when the derivative is one of the primitive rules such as the ones on that page.
- Justify the Chain Rule using rate of change arguments, such as the following: One factory converts sugar to chocolate ($c = 8s$) and another converts chocolate to candy bars ($b = 16c$). Finding the rate at which sugar is converted to candy bars can be used to help justify the Chain Rule, particularly if the units of the relevant quantities are emphasized.

WORKSHOP/DISCUSSION

- Compute some derivatives, such as those of $\sec\left(x^2+x\right)$, $\left(\dfrac{\cos x}{x^2+1}\right)^3$, $\sqrt[3]{x+\cos x^2}$, and $\cos\left(\sin\left(x^2\right)\right)$.
- Compute the equation of the line tangent to $y = \cos\left(x + \dfrac{\pi}{\sqrt{x+1}}\right)$ at $(0, -1)$.
- Draw the following graph of $f(x)$ (or copy it onto a transparency).



Tell the students that $f'(0) = -1$, $f'(1) = \frac{3}{4}$, $f'(2) = 1$, and $f'(4) = -3$. Define $g(x) = f\left(x^2\right)$. First compute $g(-1)$, $g(0)$, $g(1)$, $g\left(\sqrt{2}\right)$, and $g(2)$. Then compute $g'(0)$, $g'(-2)$, and $g'(2)$. Finally, sketch $g(x)$ as below. Use the graph to verify the values for $g(x)$ and $g'(x)$ computed above.



GROUP WORK 1: UNBROKEN CHAIN

This is meant to be a gentle introduction to the mechanics of taking derivatives using the Chain Rule. You may be surprised at the difficulty some groups have with Problem 4 of the activity, but by the end they all should be ready to go home and practice.

Start by "warming the class up" as a large group by having them take the derivatives of functions like $x^{3.24}$, $\sin x$, $\sqrt{x}$, $\tan x$, and so on. This quick review is important, because the activity works best if their mental focus is on the Chain Rule, as opposed to formulas they should already know.

While helping the individual groups, don't volunteer that the answers to most of the questions are supersets of the previous questions. They are supposed to discover this pattern for themselves.

If a group finishes early, give them a function like $\cos\left(x^2+\sqrt{x}\right)\sin(1/x)$ to try.

When they are finished, write the solutions to Problems 4, 6, and 7 on the board. Ask the students if they need you to write the solutions to the earlier ones. After they say "no", try to get them to explain why it isn't necessary. (If they say "yes", refuse and ask them why you are refusing.)

ANSWERS

1. $\cos 3x \cdot 3$ **2.** $3(\sin 3x)^2 \cos 3x \cdot 3$ **3.** $3(\sin 3x)^2 \cos 3x \cdot 3 + 5$

4. $5\left[(\sin 3x)^3 + 5x\right]^4 \left[3(\sin 3x)^2 (\cos 3x)\, 3 + 5\right]$ (check their parentheses carefully)

5. $1 - x^{-2}$ **6.** $\frac{1}{2}\left[x + (1/x)\right]^{-1/2}(1 - x^{-2})$

7. $\left[(\sin 3x)^3 + 5x\right]^5 \frac{1}{2}\left(x + \frac{1}{x}\right)^{-1/2}(1 - x^{-2}) + 5\left[(\sin 3x)^3 + 5x\right]^4 \left[3(\sin 3x)^2 (\cos 3x)\, 3 + 5\right]\left(\sqrt{x + \frac{1}{x}}\right)$

(If the students don't write out the answer to Part 7, instead referring to the answers to previous parts, don't penalize them; they have gotten the point.)

GROUP WORK 2: CHAIN RULE WITHOUT FORMULAS

This exercise works best with pairs or groups of three. Before handing it out, write both forms of the Chain Rule on the board. If a group finishes early, ask them where $h' = 0$ and over which intervals h' is constant. (This turns out to be a tricky problem.)

ANSWERS **1.** $f'(3)\, g'(1) \approx 3$ **2.** $f'(0)\, g'(0) \approx -\frac{3}{2}$ **3.** $g'(2)$ does not exist, so $h'(2)$ does not exist.

GROUP WORK 3: EXAMINING A STRANGE GRAPH

Have the students first answer the questions just by looking at the graph, and then go back and verify their intuition using calculus. If the students find this curve interesting, you can point out another interesting property. Consider the line segment going from $(0, -1)$ to $(0, 1)$. The curve gets arbitrarily close to *every* point on this segment, although it never actually touches the segment. If we consider the combined segment and curve we get a mathematical object that is "connected" but not "path connected".

If a group finishes early, perhaps ask them to figure out what the graph of $\tan(1/x)$ will look like, and to verify their guess using their calculators.

ANSWERS

1. $y' = -\dfrac{\cos(1/x)}{x^2}$. As $x \to \infty$, $y' \to 0$. Therefore the function has a horizontal asymptote. Or, one can argue that as $x \to \infty$, $1/x \to 0$, so $\sin(1/x) \to 0$.

2. The function does not approach a specific y-value as $x \to 0$. (One can look at either the function or its derivative as $x \to 0$.)

3. The slope of the curve approaches 0.

4. The slope oscillates, but its peaks and valleys get larger and larger without bound as $x \to 0$.

HOMEWORK PROBLEMS

CORE EXERCISES 4, 7, 10, 12, 24, 44, 53, 67

SAMPLE ASSIGNMENT 4, 7, 10, 12, 19, 24, 44, 53, 63, 65, 67, 68, 73, 80, 84

EXERCISE	D	A	N	G
4		×		
7		×		
10		×		
12		×		
19		×		
24		×		
44		×		
53		×		
63			×	×
65	×		×	
67	×			
68		×		
73	×			
80	×			
84	×	×		

GROUP WORK 1, SECTION 2.5

Unbroken Chain

For each of the following functions of x, write the equation for the derivative function. This will go a lot more smoothly if you remember the Sum, Product, Quotient, and Chain Rules... especially the Chain Rule! Please do us both a favor and don't simplify the answers.

1. $f(x) = \sin 3x$ $\qquad f'(x) =$

2. $g(x) = (\sin 3x)^3$ $\qquad g'(x) =$

3. $h(x) = (\sin 3x)^3 + 5x$ $\qquad h'(x) =$

4. $j(x) = \left[(\sin 3x)^3 + 5x\right]^5$ $\qquad j'(x) =$

5. $k(x) = x + \dfrac{1}{x}$ $\qquad k'(x) =$

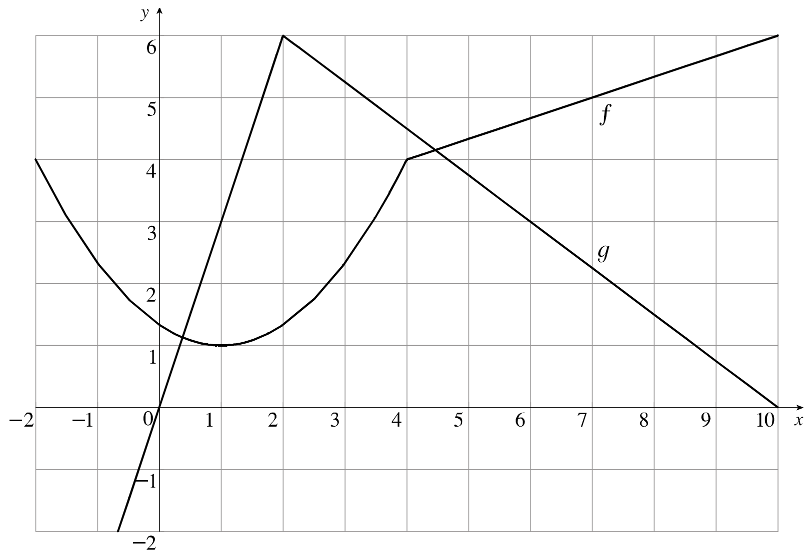
6. $l(x) = \sqrt{x + \dfrac{1}{x}}$ $\qquad l'(x) =$

7. $m(x) = \left(\sqrt{x + \dfrac{1}{x}}\right)\left[(\sin 3x)^3 + 5x\right]^5$ $\qquad m'(x) =$

GROUP WORK 2, SECTION 2.5

Chain Rule Without Formulas

Consider the functions f and g given by the following graph:



Define $h = f \circ g$.

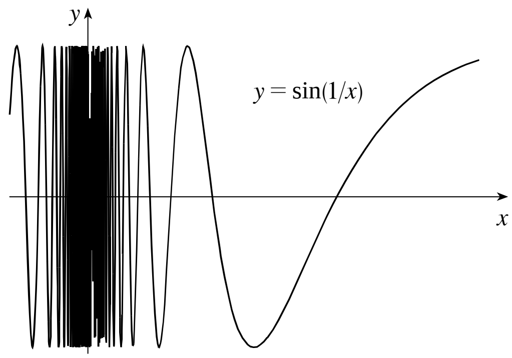
1. Compute $h'(1)$.

2. Compute $h'(0)$.

3. Does $h'(2)$ exist?

GROUP WORK 3, SECTION 2.5
Examining a Strange Graph

Several times in this course, we have looked at the graph of $y = \sin(1/x)$.



There are some things we can say about the graph just by looking at the picture, although our intuition may deceive us.

1. As we move farther and farther to the right, does the graph oscillate forever, or does it approach some y-value?

2. As we move closer and closer to zero, does the graph oscillate forever, or approach some y-value?

3. What happens to the slope of the curve as we go farther and farther to the right?

4. What happens to the slope of the curve as we approach zero?

Since intuition could fail us, please consider the function $y = \sin(1/x)$ directly, and prove that your answers to the above questions are correct. If it turns out that you were wrong above, then correct your answer and note why your intuition led you astray.

SPECIAL SECTION **Derivative Hangman**

I recommend doing this activity just after covering the Chain Rule, for a class of students who need more practice computing derivatives. It is designed to keep all the students involved and practicing both computing derivatives, and checking their work. Divide the class into teams of 4–6 students each. Put blanks representing the letters of a mystery word or phrase on the board. The game then proceeds as follows:

One representative from each team goes to the blackboard. The teacher then puts up a function either on the blackboard, or using the overhead projector. Everyone in the room tries to compute the derivative. The people at the board cannot speak, but their teammates can work together, speaking quietly.

The first person at the board to compute the derivative slaps the board, blows a whistle, or claps their hands. The teacher calls on him or her to state the solution. Then each other team gets a chance to accept the answer, or challenge.

The team that wins (first to have their representative get it right, or first to challenge successfully) gets to guess a letter of the puzzle. If they guess A, for example, all instances of A in the mystery phrase are filled in:

__ __ A __ __ A __ __ __   __ __ __ __ __ __ A

Whether or not their letter was in the phrase, they then get a chance to guess at the puzzle ("QUADRATIC FORMULA", in this case). If they get it right, the round is over and they win. If not, each team sends up a new representative and the game continues.

If this game is officiated with care and enthusiasm, all the students will be involved and working every time a new problem is put on the board.

APPLIED PROJECT **Where Should a Pilot Start Descent?**

This project can be used as an out-of-class assignment, or as an extended in-class exercise. At this point in the course, some students may be asking about opportunities for extra credit, and an oral report based on this project would be a worthwhile extra-credit activity.

The project includes a computation of the minimum distance from the airport at which an airplane should begin its descent. A nice addition to this project would be the actual figure (or range of figures) used by a local airport, obtained by a few well-placed telephone calls.

2.6 Implicit Differentiation

SUGGESTED TIME AND EMPHASIS

1 class   Essential material

POINTS TO STRESS

1. The concepts of implicit functions and implicit curves.
2. The technique of implicit differentiation.

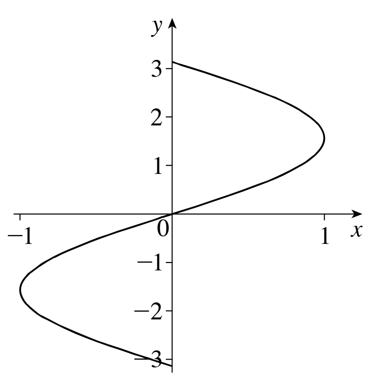
QUIZ QUESTIONS

- **TEXT QUESTION** Describe what is being illustrated by Figure 3. Make sure your answer is as complete as possible.
  ANSWER The implicit curve $x^3 + y^3 = 6xy$ does not define a function. Figure 3 illustrates several functions, each of which is implicitly defined by $x^3 + y^3 = 6xy$.

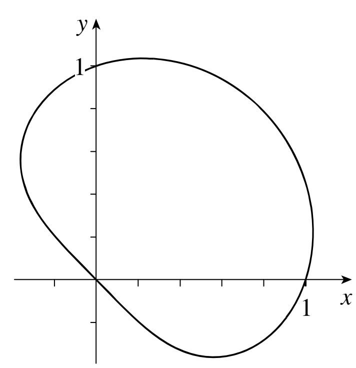
- **DRILL QUESTION** If $x^2 + xy = 10$, find $\frac{dy}{dx}$ when $x = 2$.
  ANSWER $-\frac{7}{2}$

MATERIALS FOR LECTURE

- Go over the definition of implicit curves, and the method of implicit differentiation. A good starting example is the curve defined by $x = \sin y$ (which can be easily graphed and visualized). Another example is the curve $x + y = \left(x^2 + y^2\right)^2$, which can be graphed using polar coordinates.
  ANSWER

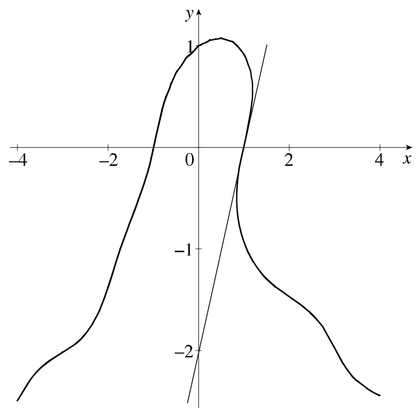


$x = \sin y$

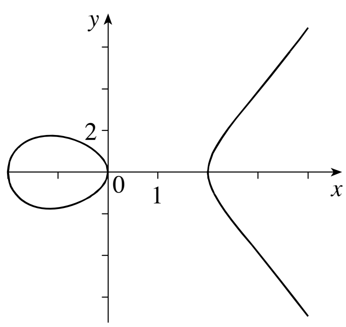


$x + y = \left(x^2 + y^2\right)^2,$
$r = \sqrt[3]{\cos\theta + \sin\theta}$

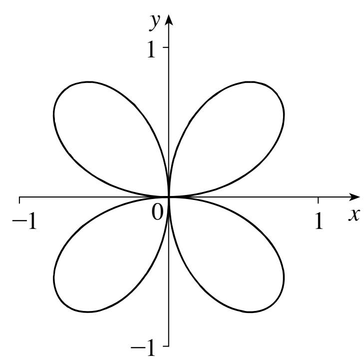
- Derive the equation of the line tangent to the curve $x^2 - \sin(xy) + y^3 = 1$ at the point $(1, 0)$. Sketch the curve as below and draw the tangent line.
  ANSWER The tangent line is $y = 2x - 2$.



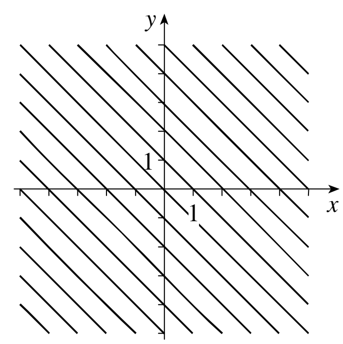
- Display some interesting looking implicit curves such as the following:



$y^2 = x^3 - 4x$
an elliptic curve



$x^6 + y^6 = 4x^2y^2 - 3y^2x^4 - 3x^2y^4$
a four-leaved rose



$\sin(\pi(x + y)) = 0$

Have the students figure out a test to see if a given point is on the implicit curve. For example, is $(2, 0)$ on the first graph? Is $(0.6, 0.2)$ on the second? Is $(1.2, 2.8)$ on the third? Have the students determine the slopes of the lines in the third graph, and show that they are parallel.
ANSWER Substituting the coordinates into the equations shows that $(2, 0)$ is on the first graph, $(0.6, 0.2)$ is not on the second, and $(1.2, 2.8)$ is on the third. The lines on the third graph all have slope -1, and are therefore parallel.

WORKSHOP/DISCUSSION

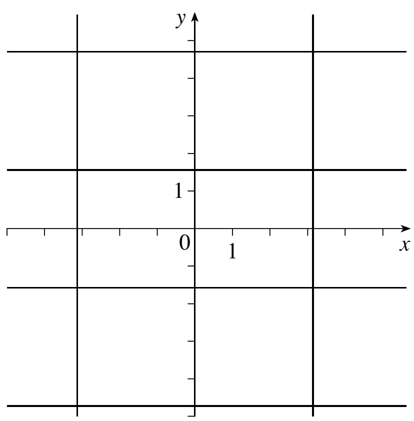
- If the students have access to appropriate graphing technology, have them try to come up with interesting-looking implicit curves. Perhaps have an award for the most aesthetically pleasing one.
- Consider $r^2 + 2s\sqrt{t} = rt$. Show the students how to compute dr/dt when s is held constant, dr/ds and ds/dr when t is held constant, and dt/ds when r is held constant.
- Have the students differentiate $y^2 = x^7 - 6x$ implicitly, and then differentiate $y = \sqrt{x^7 - 6x}$ using the Chain Rule.
- If $f(x)^4 = (x + f(x))^3$ and $f(1) = 2$, find $f'(1)$.

GROUP WORK 1: IMPLICIT CURVES

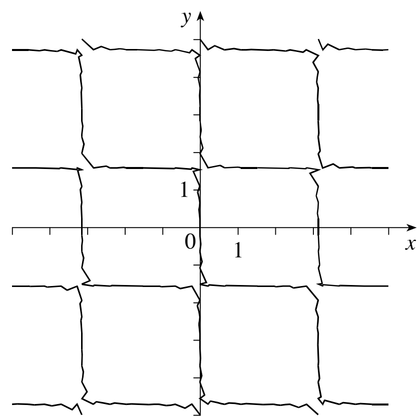
Computer algebra systems are notoriously bad at graphing implicit functions. Even simple functions such as the ones described above in Materials for Lecture point 5 are often poorly graphed by implicit function plotters. This activity describes an implicit curve which many calculators graph inaccurately, but which can be analyzed using a little bit of algebra.

ANSWERS

1. All lines of the form $x = \pi k$, $y = \frac{\pi}{2} + \pi k$, k an integer.



2. Maple gives the graph below.



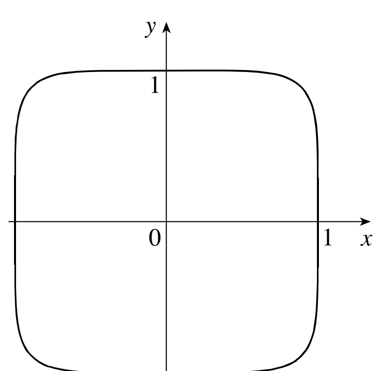
3. $dy/dx = 0$ or is undefined when $x = \pi k$. The derivative must be taken carefully to obtain this result.

GROUP WORK 2: CIRCLES AND ASTROIDS

The basic idea of this activity is for students to visualize flat circles and astroids, and to compute slopes by implicit differentiation. The question about where the slope is 1 or -1 can be addressed first visually and then analytically. As a follow-up question, students can be asked to show that the answers are always the points of intersection with the lines $y = x$ and $y = -x$.

ANSWERS

1. $\dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^5$. The slope of the tangent is 1 at $\left(\pm 2^{-1/6}, \mp 2^{-1/6}\right)$ and -1 at $\left(\pm 2^{-1/6}, \pm 2^{-1/6}\right)$.



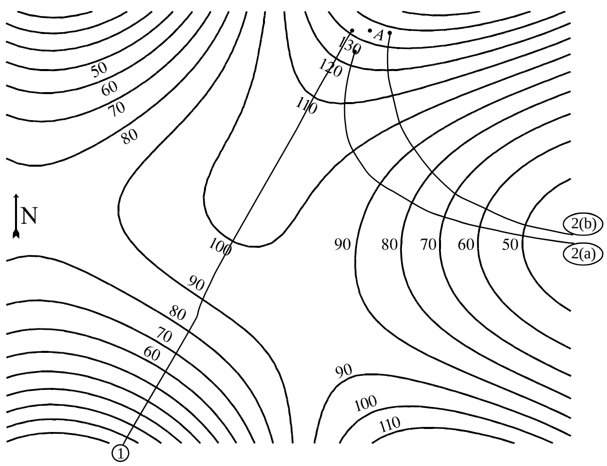
2. If $p/q = \frac{4}{3}$, the slope is 1 at $\left(\pm 2^{-3/4}, \mp 2^{-3/4}\right)$ and -1 at $\left(\pm 2^{-3/4}, \pm 2^{-3/4}\right)$. If $p/q = \frac{2}{5}$, the slope is 1 at $\left(\pm 2^{-5/2}, \mp 2^{-5/2}\right)$ and -1 at $\left(\pm 2^{-5/2}, \pm 2^{-5/2}\right)$.

GROUP WORK 3: A WALK IN THE PARK

Before beginning this activity, discuss the concepts of orthogonal trajectories (discussed in Exercises 49–52) and path of steepest descent. Perhaps do a quick example on the blackboard, and then hand out the activity. Problem 4 requires some deep reasoning.

ANSWERS

1, 2.



3. The steepest descent lines are always perpendicular to the contour lines.

4. Yes, there are. There are precarious balance points between the paths that go to one valley or the other. These are *points of unstable equilibrium*.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 10, 18, 22, 25, 32, 48, 49, 56

SAMPLE ASSIGNMENT 3, 10, 18, 22, 25, 32, 44, 48, 49, 51, 56, 59

EXERCISE	D	A	N	G
3		×		
10		×		
18		×		
22		×		
25		×		
32		×		
44	×			
48	×	×		
49		×		×
51		×		
56	×			
59		×		

GROUP WORK 1, SECTION 2.6

Implicit Curves

Consider the implicit function $\sin x \cos y = 0$.

1. Without using technology, graph this function. You have to think carefully, but you can get it.

2. If you have access to technology that can graph implicit functions, have it graph this function. Do you get a good graph?

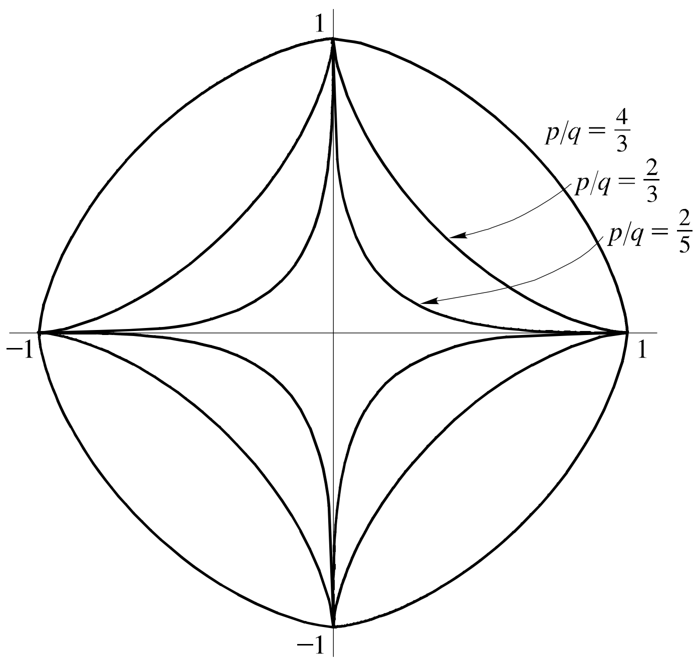
3. Use implicit differentiation to compute $\frac{dy}{dx}$. Does your graph confirm or contradict your answer?

GROUP WORK 2, SECTION 2.6

Circles and Astroids

1. Consider the "flat" circle $x^6 + y^6 = 1$. At what point(s) is the slope of the tangent line equal to 1? Where is it equal to -1?

2. Below are some curves $x^{p/q} + y^{p/q} = 1$, where p is even and q is odd. These curves are sometimes called *astroids* when $p/q < 1$.

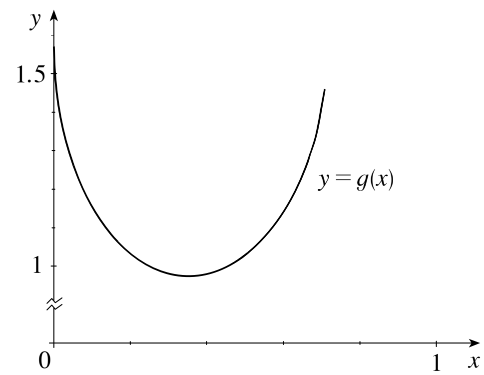


At what point(s) is the slope of the tangent line equal to 1 or -1 if $p/q = \frac{4}{3}$? How about if $p/q = \frac{2}{5}$?

GROUP WORK 3, SECTION 2.6

Looking for the Minimum

The graph of $g(x) = \arcsin\left(x^2 + e^{-x}\right)$ is shown below. Clearly there is a minimum value somewhere between $x = 0.2$ and $x = 0.4$.



1. Find a formula for $g'(x)$.

2. Find an equation of the line tangent to this curve at $x = 0.34$. (Round all numbers to three significant figures.)

3. Does the minimum value of $g(x)$ occur to the left or to the right of $x = 0.34$? How do you know?

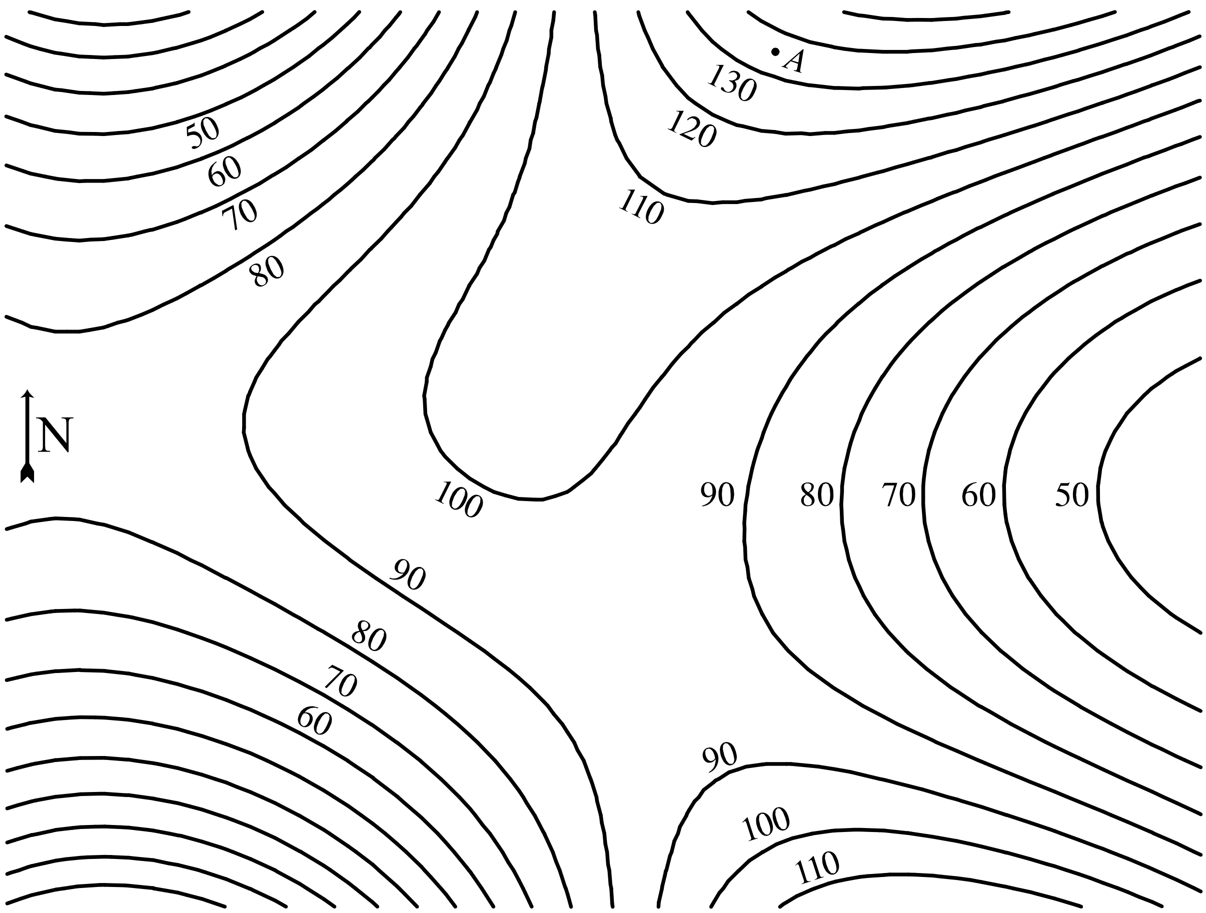
4. Find an equation of the line tangent to the curve at $x = 0.36$. Does the minimum value of $g(x)$ lie to the left or to the right of $x = 0.36$?

5. Estimate the location of the minimum value of $g(x)$. Then use technology to see how close your estimate is to the actual location.

GROUP WORK 4, SECTION 2.6

A Walk in the Park

The following is a contour map of a region in Orange Rock National Park.



1. Suppose you start a little to the west of point A. Draw the path of steepest descent from this point to the edge of the map.
2. (a) Now start a little bit southwest of point A, and trace the path of steepest descent.

   (b) Repeat this starting at a point a little east of point A.
3. What assumptions are you making in drawing your paths?

4. Are there any paths starting near point A that do *not* fall into one of the three valleys that are in the park? Explain your reasoning.

LABORATORY PROJECT **Families of Implicit Curves**

This exciting project puts the abilities of a CAS to use quite nicely. Students should be encouraged to take the last part of Problem 1(b) seriously by exploring many values of c, not just the ones explicitly mentioned. With a CAS, this takes only a few keystrokes. In Problem 2, students should be encouraged to play with the equation by putting constants in front of other terms and noting what effect this has on the graph.

2.7 Rates of Change in the Natural and Social Sciences

SUGGESTED TIME AND EMPHASIS

1 class Essential material

POINTS TO STRESS

1. The concepts of average and instantaneous rate of change.
2. Some uses of derivatives in physics and in other disciplines.

QUIZ QUESTIONS

- **TEXT QUESTION** This section discusses many different kinds of examples. What is the main idea underlying them all?

  ANSWER All of them involve expressing quantities as an average rate of change, and then using the idea of the derivative to compute an instantaneous rate of change.
- **DRILL QUESTION** The magnitude F of the force exerted by the Earth on an object is inversely proportional to the square of the distance r from that body to the center of the Earth.

  (a) Write an equation expressing F as a function of r.

  (b) Write an equation expressing dF/dr as a function of r.

  (c) What is the physical meaning of dF/dr?

  ANSWER

  (a) $F = \frac{k}{r^2}$ (b) $\frac{dF}{dr} = -\frac{2k}{r^3}$

  (c) dF/dr tells how fast the force changes as a result of a slight change in the object's distance from the center of the Earth.

MATERIALS FOR LECTURE

- Bring in a taut string, rubber band, violin, or guitar. Illustrate that when the string is plucked, the pitch depends on the length. Discuss Exercise 28, solving it as a class.
- Go over Examples 6 and 7 in detail (or different examples, based on the makeup of the student population).
- Foreshadow Exercise 35 by defining "stable population" and discussing some of the underlying concepts.

WORKSHOP/DISCUSSION

- Discuss some of the issues involved in using a continuous function to model discrete data. For example, ask if taking the derivative of a step function like "cost" is a valid thing to do.
- Do a velocity/distance linear motion problem, such as the one below:

  Let $s(t) = t^4 - 8t^3 + 18t^2$ be the distance function for a particle.

  1. Find the position at $t = 1$, $t = 2$, $t = 3$, and $t = 6$.
  2. Find the velocity at $t = 2$ and $t = 4$.
  3. Determine when the particle is at rest. When is the acceleration zero?
  4. Find the total distance traveled on the intervals [0, 1], [0, 2], [0, 3], and [0, 6].
  5. When is the particle speeding up? Slowing down? This motion can be visualized and analyzed graphically.

GROUP WORK 1: FOLLOW THAT PARTICLE!

Students are asked to analyze the motion of a typical particle.

ANSWERS

1. 0, 3, 22, ≈ 1.1
2. $v(t) = -4t^3 + 15t^2 - 1$, -1, 10, 27, ≈ -118.6
3. At rest: at $t \approx 3.7$. Moving forward: $0 \le t \lesssim 3.7$
4. $\int_1^2 |f(x)|\,dx$ is larger
5. $a(t) = -12t^2 + 30t$
6. Speeding up: $0 < t < 2.5$. Slowing down: $2.5 < t < 5$.

GROUP WORK 2

To help with the homework assignment, put the students into groups, ideally grouping similar majors together, and have each group work on a different problem from the upcoming assignment. After finishing their work, each group should present their solution to the class. Each student will then have a start on several of the problems from the assignment.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 5, 14, 20, 28

SAMPLE ASSIGNMENT 3, 5, 14, 20, 28, 29, 42, 49

EXERCISE	D	A	N	G
3		×		×
5	×		×	
14	×	×		
20	×	×		
28	×	×		
29	×	×		
42	×	×		
49	×	×		

GROUP WORK 1, SECTION 2.7

Follow That Particle!

For 4.95 seconds, a particle moves in a straight line according to the position function

$$f(t) = \left(t^3 + 1\right)(5 - t) - 5$$

where t is measured in seconds and f in feet.

Answer the following questions. You can visualize this motion and verify many of your answers using a graph. First attempt all the problems by hand, and then graph the position function to verify your answers.

1. What is the position of the particle at $t = 0$, $t = 1$, $t = 2$, $t = 4.95$?

2. Find the velocity of the particle at time t. What is the velocity of the particle at $t = 0$, $t = 1$, $t = 2$, $t = 4.95$?

3. When is the particle at rest? When is the particle moving forward?

4. Find the total distance traveled by the particle on the intervals [0, 1] and [1, 2]. Which is larger and why?

5. Find the acceleration of the particle at time t.

6. When was the particle speeding up? Slowing down?

2.8 Related Rates

SUGGESTED TIME AND EMPHASIS

1 class   Recommended material

POINTS TO STRESS

1. The concept of related rates (first two paragraphs of the text).
2. The classic procedure for handling related rates, including the warning to the side of the procedure in the text.
3. The value of careful diagrams and good notation.

QUIZ QUESTIONS

- **TEXT QUESTION** In Example 2 in the text, what is the physical meaning of the negative sign in the expression $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$?

  ANSWER The value of y is getting smaller, because the ladder is moving downward.
- **DRILL QUESTION** If one side of a rectangle, a, is increasing at a rate of 3 inches per minute while the other side, b, is decreasing at a rate of 3 inches per minute, which of the following must be true about the area A of the rectangle?

  (A) A is always increasing

  (B) A is always decreasing

  (C) A is decreasing only when $a < b$

  (D) A is decreasing only when $a > b$

  (E) A is constant.

  ANSWER (D)

MATERIALS FOR LECTURE

- Begin with a quick review of implicit differentiation, particularly when an implicit function in x and y is differentiated with respect to time or some other third variable. Have the students read the first two paragraphs of the section, and try to see why implicit differentiation is going to be useful in solving related rates problems. Then present a sample problem such as Exercise 14, using the strategy outlined in the text. Deliberately start to make the error referred to, to see if the students catch it.
- Bring balloons into class, and show the students (or have them discover for themselves) how the radius naturally grows more slowly as time goes on, assuming air comes in at a constant rate (for example, one breath every 30 seconds).
- Revisit Example 2 in the text. Compute the velocity of the ladder when it is $\frac{1}{1000}$ inch off the ground ($y = 0.001$). Show how that at some point, the tip of the ladder will exceed the speed of light. Have the students discuss what they think the problem is. (This can be done even with a large class; give them a few minutes.) Since the conclusion that "the tip really does exceed the speed of light" is impossible, the only possible conclusion to draw is that the model is faulty. Take a yardstick and actually do the experiment. (The tip of the yardstick does not stay in contact with the wall.) If the room is such that the students cannot all see the result of the experiment, have a few volunteers come up to watch and describe what happens, and encourage the students to try the experiment at home with a ruler or other similar object.

WORKSHOP/DISCUSSION

- Work this problem with the class: You are blowing a bubble with bubble gum and can blow air into the bubble at a rate of 3 in^3/s.
  (a) At what rate is the volume V increasing with respect to the radius when the radius r is 1 inch? When the radius is 3 inches?
  (b) How fast is the radius increasing with respect to time when $r = 1$ inch? When $r = 3$ inches?
  (c) Suppose you increase your effort when $r = 3$ inches and begin to blow in air at a rate of 4 in^3/s. How fast is the radius increasing now?
- Do some challenging related rates problems, such as the ones in the later exercises.
- Many children notice that when they eat a spherical lollipop (as opposed to the disk-shaped kind) it seems like at first they can lick and lick and lick without it seeming to get smaller, and then toward the end it disappears quickly. If they tell an adult, it is usually attributed to imagination or the subjectivity of passing time. Have the students try to come up with a mathematical explanation.
  ANSWER If a student is licking at a constant rate, dV/dt is constant. However, the perceived change in size of the lollipop is based on the *diameter* of the sphere, which decreases more quickly near the end.

GROUP WORK 1: FIND THE ERROR

This activity illustrates a common error that many students make. You may want to project the problem on an overhead, and give the class a few minutes to discuss it. The activity can stand alone, or be handed out as a warm-up.

GROUP WORK 2: NOBODY ESCAPES THE CUBE

This is a good introduction to related rates problems, requiring the students to express the volume of a cube in terms of its surface area.

ANSWERS **1.** 2 in^2/s **2.** $\frac{1}{2}$ in^3/s

GROUP WORK 3: THE SWIMMING POOL

The students shouldn't work on this activity until they've had a chance to see or try some basic related rates problems. Be prepared to give plenty of guidance to the students.

ANSWERS

1. $V = \begin{cases} 500h + \frac{125}{8}h^2 & \text{if } 0 < h < 16 \\ 1500h - 12{,}000 & \text{if } 16 \le h < 20 \end{cases} \quad \Rightarrow \quad \dfrac{dV}{dt} = \begin{cases} 500 + \frac{125}{4}h & \text{if } 0 < h < 16 \\ 1500\frac{dh}{dt} & \text{if } 16 \le h < 20 \end{cases}$

2. You would need dV/dt, the rate at which the pool is being filled. Note that you would not need h; if you knew dV/dt and the pool was empty at $t = 0$, you could calculate V and then compute h.

HOMEWORK PROBLEMS

CORE EXERCISES 1, 4, 7, 13, 17, 19, 31, 39, 42, 49

SAMPLE ASSIGNMENT 1, 4, 7, 13, 17, 19, 23, 31, 39, 41, 42, 49

EXERCISE	D	A	N	G
1		×		
4		×		
7		×		
13		×		
17		×		
19		×		
23		×		
31	×	×		
39	×			
41				×
42				×
49				×

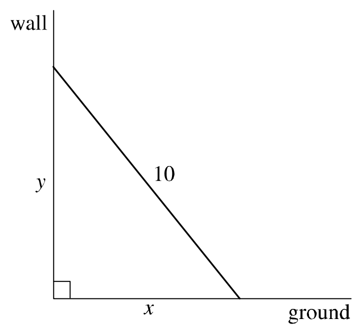
GROUP WORK 1, SECTION 2.8

Find the Error

It is a beautiful Spring evening. You and your wild-eyed, hungry-looking friends are sitting around, reading your Calculus books. You arrive at the following:

> **EXAMPLE 2** A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Your enthused roommates don't read the rest of the example, preferring to do the problem on their own. This is how they proceed:



"We want to find dy/dt. So we set up

$$x^2 + y^2 = 100$$

Now, we want dy/dt when $dx/dt = 1$ and $x = 6$. Substituting $x = 6$ gives us

$$36 + y^2 = 100 \text{ or } y^2 = 64$$

Now we take derivatives:

$$2y\frac{dy}{dt} = 0$$

giving $dy/dt = 0$."

The problem is, of course, that this answer doesn't make any sense.

1. Why does their answer not make any sense?

2. What error did they make? How could they correct it?

GROUP WORK 2, SECTION 2.8

Nobody Escapes the Cube

We are designing a computer graphic in which we zoom in on a cube. The volume V, surface area S, and side length x of the cube are all varying with respect to time. With this information, compute the following quantities, using the steps described in the text:

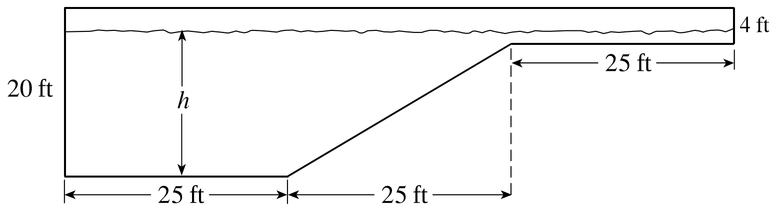
1. dS/dt when $x = 2$ inches and $dV/dt = 1\ \text{in}^3/\text{s}$.

2. dV/dt when $x = 2$ inches and $dS/dt = 1\ \text{in}^2/\text{s}$.

GROUP WORK 3, SECTION 2.8

The Swimming Pool

We wish to find the change in volume of a 20-foot-wide pool as it fills up with water. A cross-section of the pool is shown below.



1. Express dV/dt in terms of h, V, and dh/dt.

2. What additional information would you need to find dh/dt at $t = 10$ minutes?

2.9 Linear Approximations and Differentials

SUGGESTED TIME AND EMPHASIS

1 class Essential material (linear approximation) and optional material (differentials)

POINTS TO STRESS

1. The general equation of a line tangent to the graph of a function, and its use in approximating that function near a point.
2. The differential as the difference between the linearization of a function and the function itself.

QUIZ QUESTIONS

- **TEXT QUESTION** What is the difference between the function $L(x)$ defined in the text and the equation of the tangent line $y = f(a) + f'(a)(x - a)$?
  ANSWER None
- **DRILL QUESTION** Write the equation of the straight line that best approximates the graph of $y = x + \cos x$ at the point $(0, 1)$.
  ANSWER $y = x + 1$

MATERIALS FOR LECTURE

- Discuss the motivation for studying linear approximations. Ask, "Why use an approximation to a function when a computer can find the answer precisely?"
  ANSWERS
  1. A common modeling technique is to assume a function is locally linear, and then use the linear equation in calculations, since it is easier to manipulate.
  2. It is often easier physically to measure the derivative of a function than the function itself. Then the derivative measurements can be used to obtain an approximation of the function.
  3. When measuring a real phenomenon, there is often no easy-to-understand function that can be written in a line or two, and the best that can be obtained is a set of sample data points. The "underlying" function must be approximated.
  4. In the real world, the input to functions can be noisy or wiggly. It is easier to handle small input fluctuations if we assume that the output varies linearly.
  5. When a function is called thousands of times by a computer program, as occurs in computer graphics applications, the small time savings from using a linear function can result in savings of hours or even days.
- Discuss the meaning of the phrase "approximating along the tangent line" and its connections to linear approximation. Then present examples of linear approximation, such as $\sin x \approx x$ for x near 1 and $x + \cos x \approx x + 1$ for x near 0.
- Raise the question, "What if we want a more accurate model of a function?" Foreshadow the quadratic approximation (Taylor polynomial of order two) as an extension of the linear approximation. (The linear approximation matches the function in the first derivative, so how can you make a function match the second derivative as well?)

- Graph $y = \sin x$ with its approximations at $x = 0$ and $x = \frac{\pi}{4}$. Discuss which is "better".
- To illustrate how controversial differentials once were, cite the quotation from Bishop Berkeley (1734) on differentials: "And what are these evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?"
- Bring in a carpenter's level. Show how, when the level is held perfectly straight, it can be used to measure acceleration. (The bubble moves when the level is accelerated, and returns to center at constant velocity). This can be done on an overhead projector, if the floor is flat. This is the principle used to make a simple accelerometer. Then discuss how, given acceleration measurements, it is possible to approximate velocity using the technique of linear approximation.

WORKSHOP/DISCUSSION

- Let $f(x) = x^{1.857}$. Find the linear approximation of $f(x)$ at $a = 1$ and use it to approximate f at $x = 1.1$, $x = 1.01$, and $x = 1.001$. Compare the approximations to the actual values the calculator gives for f at these points.
- In Example 1, discuss why we base our linear approximation at $x = 1$ rather than at $x = 0.99$ or 1.01.
- Practice using linear approximations with $y = \dfrac{1}{\sqrt{x}}$ at $x = 4$, and use differentials to approximate Δy for $\Delta x = -1$ and $\Delta x = 1$.
- Have the students try to find a linear approximation for $|x|$ near $x = 0$, and explain why it is impossible.

GROUP WORK 1: FOUR VARIATIONS ON A THEME

This activity explores four different functions that have identical linear approximations near $x = 0$.

ANSWERS

1. $y = x$ in all cases.

2.

Function	Function Value at $x = 0.1$	Approximation at $x = 0.1$
f	0.09545	0.1
g	0.101	0.1
h	0.11007	0.1
j	0.09983	0.1

3. If the students need to, they can check the approximations for $x = 0.2$ or $x = 0.3$. The best approximation is the one to $j(x)$, and the worst is the one to $h(x)$. This is immediate from looking at the graphs. Notice that j and g have inflection points at $x = 0$.

GROUP WORK 2: LINEAR APPROXIMATION

Some students may try to find approximations of the derivative functions. They should be reminded that we are approximating f, using the graph of f' as an aid.

ANSWERS

1. $f(x) \approx 1.75(x-2)+4$, so $f(1.98) \approx 3.965$ and $f(2.02) \approx 4.035$.
2. The graph of f lies below its tangent line, so the approximations are overestimates.
3. The estimates are both 7, because the function is horizontal when $x=3$.

HOMEWORK PROBLEMS

CORE EXERCISES 3, 13, 17, 25, 33, 39

SAMPLE ASSIGNMENT 3, 5, 10, 13, 17, 25, 33, 38, 39, 41

EXERCISE	D	A	N	G
3		×		
5		×		×
10		×		
13		×		
17		×		
25		×		
33	×	×		
38	×	×		
39		×		
41		×		×

GROUP WORK 1, SECTION 2.9

Four Variations on a Theme

Consider the following four functions:

$$f(x) = -1 + \sqrt{2x + 1} \qquad g(x) = x^3 + x \qquad h(x) = \tan^2 x + x \qquad j(x) = \sin x$$

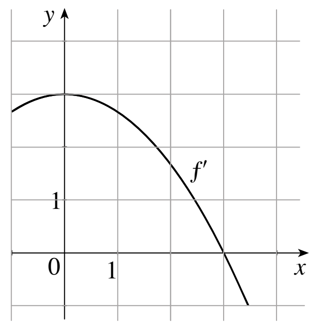
1. Find the linearizations of f, g, h, and j at $a = 0$.

2. Compute the values of each of these functions at $x = 0.1$ and the values of their linearizations.

3. For which function is the approximation best? For which is it worst? Why?

GROUP WORK 2, SECTION 2.9
Linear Approximation

Consider this graph of $f'(x)$, the *derivative* of $f(x)$.



1. Suppose that $f(2) = 4$. Approximate $f(1.98)$ and $f(2.02)$ as best you can. Don't just guess. Show your work.

2. Determine whether your approximations were overestimates or underestimates.

3. Suppose you also know that $f(3) = 7$. Can you approximate $f(2.98)$ and $f(3.02)$? Explain your answer.

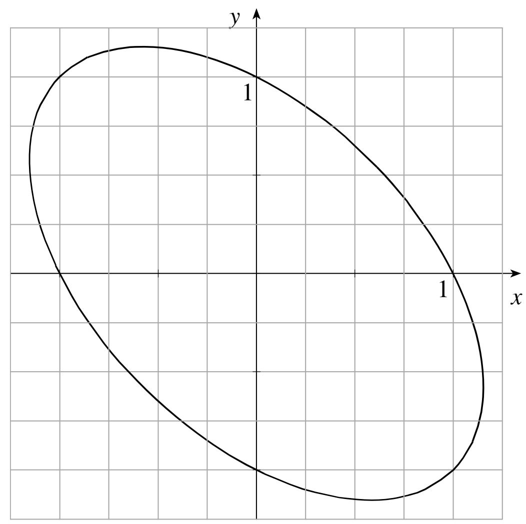
LABORATORY PROJECT **Taylor Polynomials**

This project provides a solid early introduction to Taylor polynomials as extensions of the tangent line approximation concept. A few examples involving $\cos x$ and $\sqrt{x+3}$ are explored in more detail. Students may be asked to explore their own function, and see what happens. Have them go beyond just working through the six questions, and try to demonstrate that they understand the pretty concept introduced in this project.

2 SAMPLE EXAM

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. Consider the graph of $x^2 + xy + y^2 = 1$.



(a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

(b) Find all points where the tangent line is horizontal.

(c) Find all points where the tangent line is parallel to the line $y = -x$.

2. Let $f(x) = 7\sin(x + \pi) + \cos 2x$.

(a) Compute $f'(x)$, $f''(x)$, $f^3(x)$, and $f^4(x)$.

(b) Compute $f^{13}(0)$.

3. Assume that $f(x)$ and $g(x)$ are differential functions that we know very little about. In fact, assume that all we know of these function is the following table of data:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-2	3	1	-5	8
-1	-9	7	4	1
0	5	9	9	-3
1	3	-3	2	6
2	-5	3	8	0

(a) Let $h(x) = g(x)\sin x$. What is $h'(0)$?

(b) Let $j(x) = \left[f(x) + x^2\right]^3$. What is $j'(1)$?

(c) Let $l(x) = \frac{\tan \pi x}{g(x)}$. What is $l'(-1)$?

4. Let $u(x)$ be an always positive function such that $u'(x) < 0$ for all real numbers.

(a) Let $f(x) = [u(x)]^2$. For what values of x will $f(x)$ be increasing?

(b) Let $g(x) = u(u(x))$. For what values of x will $g(x)$ be increasing?

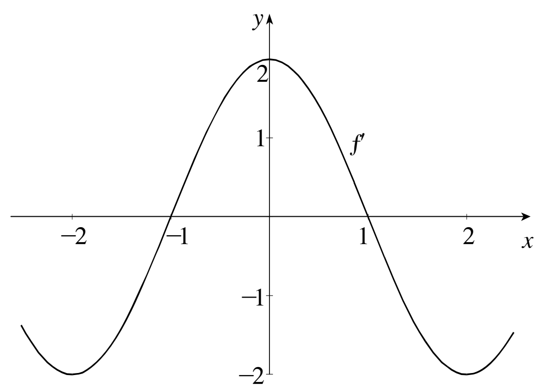
5. Let $f(x) = -x^3 - 2x^2 + x + 1$ and $g(x) = \sin x + 1$.

(a) Find the equation of the line tangent to $f(x)$ at $x = 0$.

(b) Show that $g(x)$ has the same tangent line as $f(x)$ at $x = 0$.

(c) Does this tangent line give a better approximation of $f(x)$ or $g(x)$ at $x = 1$? Give reasons for your answer.

6. The following is a graph of f', the derivative of some function f.



(a) Where is f increasing?

(b) Where does f have a local minimum? Where does f have a local maximum?

(c) Where is f concave up?

(d) Assuming that $f(0) = -1$, sketch a possible graph of f.

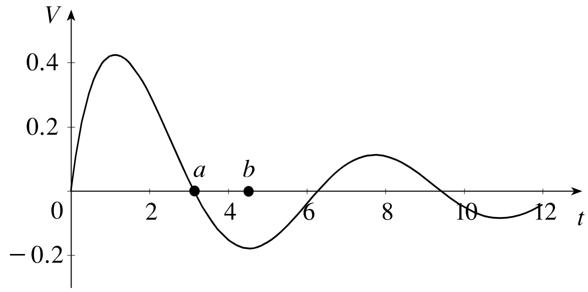
7. As a spherical raindrop evaporates, its volume changes at a rate proportional to its surface area A.

(a) If the constant of proportionality is K, find the rate of change of the radius r when $r = 4$.

(b) Show that the rate of change of the radius is always constant.

(c) Does part (b) mean that the rate of change of the volume is always constant? Why or why not?

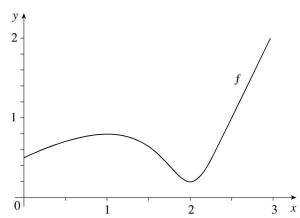
8. The voltage across a resistor R is given by $V(t) = \dfrac{1}{1+t}\sin t$. A graph of $V(t)$ is shown below.



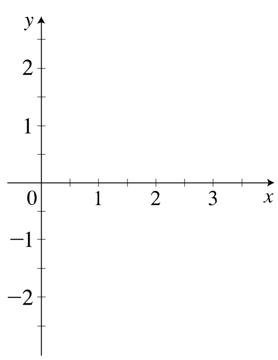
(a) How fast is the voltage changing after 2 seconds?

(b) Would you be better off using the linear approximation at $x = a$ to estimate $V(b)$, or using the linear approximation at $x = b$ to estimate $V(a)$? Justify your answer.

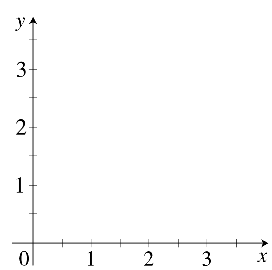
9. Let f be the function whose graph is given below.



(a) Sketch a plausible graph of f'.



(b) Sketch a plausible graph of a function F such that $F' = f$ and $F(0) = 1$.



10. Suppose that the line tangent to the graph of $y = f(x)$ at $x = 3$ passes through the points $(-2, 3)$ and $(4, -1)$.

(a) Find $f'(3)$.

(b) Find $f(3)$.

(c) What is the equation of the line tangent to f at 3?

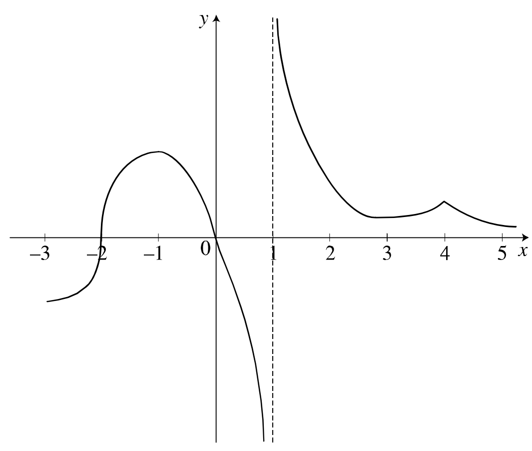
11. Each of the following limits represent the derivative of a function f at some point a. State a formula for f and the value of the point a.

(a) $\lim_{h\to 0} \dfrac{(3+h)^2 - 9}{h}$

(b) $\lim_{x\to 3} \dfrac{(x+1)^{3/2} - 8}{x-3}$

(c) $\lim_{h\to 0} \dfrac{\sin(\pi(2+h)) - 0}{h}$

12. The graph of $f(x)$ is given below. For which value(s) of x is $f(x)$ not differentiable? Justify your answer(s).



2 SAMPLE EXAM SOLUTIONS

1. (a) $2x + x\dfrac{dy}{dx} + y + 2y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$

(b) Set $y + 2x = 0$ and $y = -2x$. Then $x^2 - 2x^2 + 4x^2 = 1 \Leftrightarrow 3x^2 = 1 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}} \Leftrightarrow y = \mp\frac{2}{\sqrt{3}}$, so the points are $\left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

(c) Set $-\dfrac{2x+y}{x+2y} = -1$ to get $y = x$. Then $x^2 + x^2 + x^2 = 3x^2 = 1 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}} \Leftrightarrow y = \pm\frac{1}{\sqrt{3}}$, so the points are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

2. (a) $f'(x) = 7\cos(x+\pi) - 2\sin 2x$; $f''(x) = -7\sin(x+\pi) - 4\cos 2x$; $f^{(3)}(x) = -7\cos(x+\pi) + 8\sin 2x$; $f^{(4)}(x) = 7\sin(x+\pi) + 16\cos 2x$

(b) $f^{(13)}(x) = 7\cos(x+\pi) - 2^{13}\sin 2x$; $f^{(13)}(0) = 7\cos\pi = -7$

3. (a) $h'(x) = g'(x)\sin x + g(x)\cos x$; $h'(0) = g(0) = 9$

(b) $j'(x) = 3\left(f(x) + x^2\right)^2\left(f'(x) + 2x\right)$; $j'(1) = 3\left(f(1)+1\right)^2\left(f'(1)+2\right) = 3\cdot 4^2\cdot 4 = 192$

(c) $l'(x) = \dfrac{g(x)\cdot\pi\sec^2\pi x - g'(x)\tan\pi x}{g(x)^2}$; $l'(-1) = \dfrac{7\pi - 0}{7^2} = \dfrac{\pi}{7}$

4. (a) $f'(x) = 2u(x)u'(x) < 0$ for all x, since $u(x) > 0$ and $u'(x) < 0$. Never increasing.

(b) $g'(x) = u'(u(x))\cdot u'(x) > 0$, since $u'(u(x))$ and $u'(x) < 0$. Always increasing.

5. (a) $f'(x) = -3x^2 - 4x + 1$; $f'(0) = 1$, $f(0) = 1$. Tangent line is $y = 1 + 1\cdot x = 1 + x$

(b) $g'(x) = \cos x$; $g'(0) = 1$, $g(0) = 1$. Tangent line is $y = 1 + x$

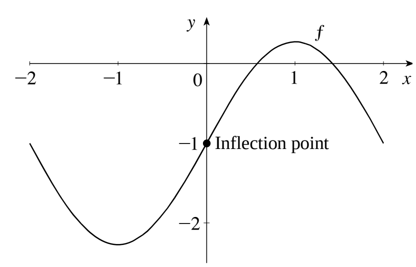
(c) At $x = 1$, $f(1) = -1$, $g(1) = \sin 1 + 1 \approx 1.841$. The tangent line approximation is $y = 1 + 1 = 2$. This is better for $g(x)$ at $x = 1$.

6. (a) f is increasing on $(-1, 1)$.

(b) Local minimum at $x = -1$; local maximum at $x = 1$

(c) f is concave up where $f'(x)$ is increasing, that is, on $(-2, 0)$.

(d)



7. (a) $\dfrac{dV}{dt} = KA$. $V = \frac{4}{3}\pi r^3$, so $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. Since $A = 4\pi r^2$, we have $K4\pi r^2 = 4\pi r^2\dfrac{dr}{dt}$. Thus, $\dfrac{dr}{dt} = K$.

(b) By part (a), $\dfrac{dr}{dt} = K$ is constant. $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt} = 4K\pi r^2$. So $\dfrac{dV}{dt}$ depends on r^2 and is not constant.

8. (a) $V'(t) = \dfrac{1}{1+t}\cos t - \dfrac{1}{(1+t^2)}\sin t$; $V'(2) = \frac{1}{3}\cos t - \frac{1}{9}\sin 2 \approx -0.240$

(b) The tangent line at $x = b$ is horizontal. So the estimate for $V(a)$ using this linear approximation is $V(b)$, which is not very good. Thus, it is better to use the linear approximation at $x = a$ to estimate $V(b)$.

9. (a) Answers will vary. Look for:

(i) zeros at 1 and 2

(ii) f' positive for $x \in (0, 1)$ and $(2, 4)$

(iii) f' negative for $x \in (1, 2)$

(iv) f' flattens out for $x > 2.5$

(b) Answers will vary. Look for

(i) $F(0) = 1$

(ii) F always increasing

(ii) F is never perfectly flat

(iv) F is closest to being flat at $x = 2$

(v) F is concave up for $x \in (0, 1)$ and $x \in (2, 4)$

(vi) F is concave down for $x \in (1, 2)$

10. (a) $\frac{3-(-1)}{-2-4} = -\frac{2}{3}$

(b) The equation of the tangent line is $y - 3 = -\frac{2}{3}(x+2)$, so $f(3) = -\frac{2}{3}(3+2) + 3 = -\frac{1}{3}$.

(c) The equation of the tangent line is $y - 3 = -\frac{2}{3}(x+2)$.

11. (a) $f(x) = x^2$, $a = 3$ (b) $f(x) = (x+1)^{3/2}$, $a = 3$ (c) $f(x) = \sin(\pi x)$, $a = 2$

12. f isn't differentiable at $x = 1$, because it is not continuous there; at $x = -2$, because it has a vertical tangent there; and at $x = 4$, because it has a cusp there.