Complete Solutions Manual

for

SINGLE VARIABLE CALCULUS

SEVENTH EDITION

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PREFACE

This *Complete Solutions Manual* contains solutions to all exercises in the text *Single Variable Calculus,* Seventh Edition, by James Stewart. A student version of this manual is also available; it contains solutions to the odd-numbered exercises in each section, the review sections, the True-False Quizzes, and the Problem Solving sections, as well as solutions to all the exercises in the Concept Checks. No solutions to the projects appear in the student version. It is our hope that by browsing through the solutions, professors will save time in determining appropriate assignments for their particular class.

We use some nonstandard notation in order to save space. If you see a symbol that you don't recognize, refer to the Table of Abbreviations and Symbols on page v.

We appreciate feedback concerning errors, solution correctness or style, and manual style. Any comments may be sent directly to jeff.cole@anokaramsey.edu, or in care of the publisher: Brooks/Cole, Cengage Learning, 20 Davis Drive, Belmont CA 94002-3098.

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ABBREVIATIONS AND SYMBOLS

CD	concave downward
CU	concave upward
D	the domain of f
FDT	First Derivative Test
HA	horizontal asymptote(s)
I	interval of convergence
I/D	Increasing/Decreasing Test
IP	inflection point(s)
R	radius of convergence
VA	vertical asymptote(s)
$\overset{\text{CAS}}{=}$	indicates the use of a computer algebra system.
$\overset{\text{H}}{=}$	indicates the use of l'Hospital's Rule.
$\overset{j}{=}$	indicates the use of Formula j in the Table of Integrals in the back endpapers.
$\overset{\text{s}}{=}$	indicates the use of the substitution $\{u = \sin x, du = \cos x\, dx\}$.
$\overset{\text{c}}{=}$	indicates the use of the substitution $\{u = \cos x, du = -\sin x\, dx\}$.

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DIAGNOSTIC TESTS

Test A Algebra

1. (a) $(-3)^4 = (-3)(-3)(-3)(-3) = 81$

(b) $-3^4 = -(3)(3)(3)(3) = -81$

(c) $3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$

(d) $\dfrac{5^{23}}{5^{21}} = 5^{23-21} = 5^2 = 25$

(e) $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

(f) $16^{-3/4} = \dfrac{1}{16^{3/4}} = \dfrac{1}{\left(\sqrt[4]{16}\right)^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

2. (a) Note that $\sqrt{200} = \sqrt{100 \cdot 2} = 10\sqrt{2}$ and $\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$. Thus $\sqrt{200} - \sqrt{32} = 10\sqrt{2} - 4\sqrt{2} = 6\sqrt{2}$.

(b) $(3a^3b^3)(4ab^2)^2 = 3a^3b^3 16a^2b^4 = 48a^5b^7$

(c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2} = \left(\dfrac{x^2y^{-1/2}}{3x^{3/2}y^3}\right)^2 = \dfrac{(x^2y^{-1/2})^2}{(3x^{3/2}y^3)^2} = \dfrac{x^4y^{-1}}{9x^3y^6} = \dfrac{x^4}{9x^3y^6y} = \dfrac{x}{9y^7}$

3. (a) $3(x+6) + 4(2x-5) = 3x + 18 + 8x - 20 = 11x - 2$

(b) $(x+3)(4x-5) = 4x^2 - 5x + 12x - 15 = 4x^2 + 7x - 15$

(c) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \sqrt{a}\,\sqrt{b} + \sqrt{a}\,\sqrt{b} - \left(\sqrt{b}\right)^2 = a - b$

Or: Use the formula for the difference of two squares to see that $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a - b$.

(d) $(2x+3)^2 = (2x+3)(2x+3) = 4x^2 + 6x + 6x + 9 = 4x^2 + 12x + 9$.

Note: A quicker way to expand this binomial is to use the formula $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 2x$ and $b = 3$:
$(2x+3)^2 = (2x)^2 + 2(2x)(3) + 3^2 = 4x^2 + 12x + 9$

(e) See Reference Page 1 for the binomial formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Using it, we get
$(x+2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3 = x^3 + 6x^2 + 12x + 8$.

4. (a) Using the difference of two squares formula, $a^2 - b^2 = (a+b)(a-b)$, we have
$4x^2 - 25 = (2x)^2 - 5^2 = (2x+5)(2x-5)$.

(b) Factoring by trial and error, we get $2x^2 + 5x - 12 = (2x-3)(x+4)$.

(c) Using factoring by grouping and the difference of two squares formula, we have
$x^3 - 3x^2 - 4x + 12 = x^2(x-3) - 4(x-3) = (x^2-4)(x-3) = (x-2)(x+2)(x-3)$.

(d) $x^4 + 27x = x(x^3 + 27) = x(x+3)(x^2 - 3x + 9)$

This last expression was obtained using the sum of two cubes formula, $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ with $a = x$ and $b = 3$. [See Reference Page 1 in the textbook.]

(e) The smallest exponent on x is $-\frac{1}{2}$, so we will factor out $x^{-1/2}$.
$3x^{3/2} - 9x^{1/2} + 6x^{-1/2} = 3x^{-1/2}(x^2 - 3x + 2) = 3x^{-1/2}(x-1)(x-2)$

(f) $x^3y - 4xy = xy(x^2 - 4) = xy(x-2)(x+2)$

5. (a) $\dfrac{x^2+3x+2}{x^2-x-2}=\dfrac{(x+1)(x+2)}{(x+1)(x-2)}=\dfrac{x+2}{x-2}$

(b) $\dfrac{2x^2-x-1}{x^2-9}\cdot\dfrac{x+3}{2x+1}=\dfrac{(2x+1)(x-1)}{(x-3)(x+3)}\cdot\dfrac{x+3}{2x+1}=\dfrac{x-1}{x-3}$

(c) $\dfrac{x^2}{x^2-4}-\dfrac{x+1}{x+2}=\dfrac{x^2}{(x-2)(x+2)}-\dfrac{x+1}{x+2}=\dfrac{x^2}{(x-2)(x+2)}-\dfrac{x+1}{x+2}\cdot\dfrac{x-2}{x-2}=\dfrac{x^2-(x+1)(x-2)}{(x-2)(x+2)}$

$$=\frac{x^2-(x^2-x-2)}{(x+2)(x-2)}=\frac{x+2}{(x+2)(x-2)}=\frac{1}{x-2}$$

(d) $\dfrac{\frac{y}{x}-\frac{x}{y}}{\frac{1}{y}-\frac{1}{x}}=\dfrac{\frac{y}{x}-\frac{x}{y}}{\frac{1}{y}-\frac{1}{x}}\cdot\dfrac{xy}{xy}=\dfrac{y^2-x^2}{x-y}=\dfrac{(y-x)(y+x)}{-(y-x)}=\dfrac{y+x}{-1}=-(x+y)$

6. (a) $\dfrac{\sqrt{10}}{\sqrt{5}-2}=\dfrac{\sqrt{10}}{\sqrt{5}-2}\cdot\dfrac{\sqrt{5}+2}{\sqrt{5}+2}=\dfrac{\sqrt{50}+2\sqrt{10}}{\left(\sqrt{5}\right)^2-2^2}=\dfrac{5\sqrt{2}+2\sqrt{10}}{5-4}=5\sqrt{2}+2\sqrt{10}$

(b) $\dfrac{\sqrt{4+h}-2}{h}=\dfrac{\sqrt{4+h}-2}{h}\cdot\dfrac{\sqrt{4+h}+2}{\sqrt{4+h}+2}=\dfrac{4+h-4}{h\left(\sqrt{4+h}+2\right)}=\dfrac{h}{h\left(\sqrt{4+h}+2\right)}=\dfrac{1}{\sqrt{4+h}+2}$

7. (a) $x^2+x+1=\left(x^2+x+\frac{1}{4}\right)+1-\frac{1}{4}=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

(b) $2x^2-12x+11=2(x^2-6x)+11=2(x^2-6x+9-9)+11=2(x^2-6x+9)-18+11=2(x-3)^2-7$

8. (a) $x+5=14-\frac{1}{2}x \iff x+\frac{1}{2}x=14-5 \iff \frac{3}{2}x=9 \iff x=\frac{2}{3}\cdot 9 \iff x=6$

(b) $\dfrac{2x}{x+1}=\dfrac{2x-1}{x} \Rightarrow 2x^2=(2x-1)(x+1) \iff 2x^2=2x^2+x-1 \iff x=1$

(c) $x^2-x-12=0 \iff (x+3)(x-4)=0 \iff x+3=0$ or $x-4=0 \iff x=-3$ or $x=4$

(d) By the quadratic formula, $2x^2+4x+1=0 \iff$

$$x=\frac{-4\pm\sqrt{4^2-4(2)(1)}}{2(2)}=\frac{-4\pm\sqrt{8}}{4}=\frac{-4\pm 2\sqrt{2}}{4}=\frac{2\left(-2\pm\sqrt{2}\right)}{4}=\frac{-2\pm\sqrt{2}}{2}=-1\pm\tfrac{1}{2}\sqrt{2}.$$

(e) $x^4-3x^2+2=0 \iff (x^2-1)(x^2-2)=0 \iff x^2-1=0$ or $x^2-2=0 \iff x^2=1$ or $x^2=2 \iff$ $x=\pm 1$ or $x=\pm\sqrt{2}$

(f) $3\,|x-4|=10 \iff |x-4|=\frac{10}{3} \iff x-4=-\frac{10}{3}$ or $x-4=\frac{10}{3} \iff x=\frac{2}{3}$ or $x=\frac{22}{3}$

(g) Multiplying through $2x(4-x)^{-1/2}-3\sqrt{4-x}=0$ by $(4-x)^{1/2}$ gives $2x-3(4-x)=0 \iff$ $2x-12+3x=0 \iff 5x-12=0 \iff 5x=12 \iff x=\frac{12}{5}$.

9. (a) $-4<5-3x\le 17 \iff -9<-3x\le 12 \iff 3>x\ge -4$ or $-4\le x<3$.
In interval notation, the answer is $[-4,3)$.

(b) $x^2<2x+8 \iff x^2-2x-8<0 \iff (x+2)(x-4)<0$. Now, $(x+2)(x-4)$ will change sign at the critical values $x=-2$ and $x=4$. Thus the possible intervals of solution are $(-\infty,-2)$, $(-2,4)$, and $(4,\infty)$. By choosing a single test value from each interval, we see that $(-2,4)$ is the only interval that satisfies the inequality.

(c) The inequality $x(x-1)(x+2)>0$ has critical values of $-2, 0$, and 1. The corresponding possible intervals of solution are $(-\infty,-2)$, $(-2,0)$, $(0,1)$ and $(1,\infty)$. By choosing a single test value from each interval, we see that both intervals $(-2,0)$ and $(1,\infty)$ satisfy the inequality. Thus, the solution is the union of these two intervals: $(-2,0)\cup(1,\infty)$.

(d) $|x-4|<3 \quad\Leftrightarrow\quad -3<x-4<3 \quad\Leftrightarrow\quad 1<x<7$. In interval notation, the answer is $(1,7)$.

(e) $\dfrac{2x-3}{x+1}\le 1 \quad\Leftrightarrow\quad \dfrac{2x-3}{x+1}-1\le 0 \quad\Leftrightarrow\quad \dfrac{2x-3}{x+1}-\dfrac{x+1}{x+1}\le 0 \quad\Leftrightarrow\quad \dfrac{2x-3-x-1}{x+1}\le 0 \quad\Leftrightarrow\quad \dfrac{x-4}{x+1}\le 0.$

Now, the expression $\dfrac{x-4}{x+1}$ may change signs at the critical values $x=-1$ and $x=4$, so the possible intervals of solution are $(-\infty,-1)$, $(-1,4]$, and $[4,\infty)$. By choosing a single test value from each interval, we see that $(-1,4]$ is the only interval that satisfies the inequality.

10. (a) False. In order for the statement to be true, it must hold for all real numbers, so, to show that the statement is false, pick $p=1$ and $q=2$ and observe that $(1+2)^2\ne 1^2+2^2$. In general, $(p+q)^2=p^2+2pq+q^2$.

(b) True as long as a and b are nonnegative real numbers. To see this, think in terms of the laws of exponents: $\sqrt{ab}=(ab)^{1/2}=a^{1/2}b^{1/2}=\sqrt{a}\,\sqrt{b}$.

(c) False. To see this, let $p=1$ and $q=2$, then $\sqrt{1^2+2^2}\ne 1+2$.

(d) False. To see this, let $T=1$ and $C=2$, then $\dfrac{1+1(2)}{2}\ne 1+1$.

(e) False. To see this, let $x=2$ and $y=3$, then $\dfrac{1}{2-3}\ne\dfrac{1}{2}-\dfrac{1}{3}$.

(f) True since $\dfrac{1/x}{a/x-b/x}\cdot\dfrac{x}{x}=\dfrac{1}{a-b}$, as long as $x\ne 0$ and $a-b\ne 0$.

Test B Analytic Geometry

1. (a) Using the point $(2,-5)$ and $m=-3$ in the point-slope equation of a line, $y-y_1=m(x-x_1)$, we get $y-(-5)=-3(x-2) \quad\Rightarrow\quad y+5=-3x+6 \quad\Rightarrow\quad y=-3x+1$.

(b) A line parallel to the x-axis must be horizontal and thus have a slope of 0. Since the line passes through the point $(2,-5)$, the y-coordinate of every point on the line is -5, so the equation is $y=-5$.

(c) A line parallel to the y-axis is vertical with undefined slope. So the x-coordinate of every point on the line is 2 and so the equation is $x=2$.

(d) Note that $2x-4y=3 \quad\Rightarrow\quad -4y=-2x+3 \quad\Rightarrow\quad y=\frac{1}{2}x-\frac{3}{4}$. Thus the slope of the given line is $m=\frac{1}{2}$. Hence, the slope of the line we're looking for is also $\frac{1}{2}$ (since the line we're looking for is required to be parallel to the given line). So the equation of the line is $y-(-5)=\frac{1}{2}(x-2) \quad\Rightarrow\quad y+5=\frac{1}{2}x-1 \quad\Rightarrow\quad y=\frac{1}{2}x-6$.

2. First we'll find the distance between the two given points in order to obtain the radius, r, of the circle: $r=\sqrt{[3-(-1)]^2+(-2-4)^2}=\sqrt{4^2+(-6)^2}=\sqrt{52}$. Next use the standard equation of a circle, $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center, to get $(x+1)^2+(y-4)^2=52$.

3. We must rewrite the equation in standard form in order to identify the center and radius. Note that $x^2 + y^2 - 6x + 10y + 9 = 0 \Rightarrow x^2 - 6x + 9 + y^2 + 10y = 0$. For the left-hand side of the latter equation, we factor the first three terms and complete the square on the last two terms as follows: $x^2 - 6x + 9 + y^2 + 10y = 0 \Rightarrow (x-3)^2 + y^2 + 10y + 25 = 25 \Rightarrow (x-3)^2 + (y+5)^2 = 25$. Thus, the center of the circle is $(3, -5)$ and the radius is 5.

4. (a) $A(-7, 4)$ and $B(5, -12) \Rightarrow m_{AB} = \dfrac{-12 - 4}{5 - (-7)} = \dfrac{-16}{12} = -\dfrac{4}{3}$

(b) $y - 4 = -\frac{4}{3}[x - (-7)] \Rightarrow y - 4 = -\frac{4}{3}x - \frac{28}{3} \Rightarrow 3y - 12 = -4x - 28 \Rightarrow 4x + 3y + 16 = 0$. Putting $y = 0$, we get $4x + 16 = 0$, so the x-intercept is -4, and substituting 0 for x results in a y-intercept of $-\frac{16}{3}$.

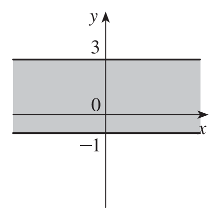
(c) The midpoint is obtained by averaging the corresponding coordinates of both points: $\left(\frac{-7+5}{2}, \frac{4+(-12)}{2}\right) = (-1, -4)$.

(d) $d = \sqrt{[5 - (-7)]^2 + (-12 - 4)^2} = \sqrt{12^2 + (-16)^2} = \sqrt{144 + 256} = \sqrt{400} = 20$

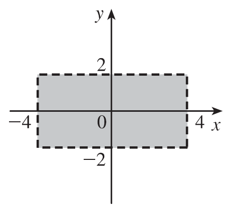
(e) The perpendicular bisector is the line that intersects the line segment $\overline{AB}$ at a right angle through its midpoint. Thus the perpendicular bisector passes through $(-1, -4)$ and has slope $\frac{3}{4}$ [the slope is obtained by taking the negative reciprocal of the answer from part (a)]. So the perpendicular bisector is given by $y + 4 = \frac{3}{4}[x - (-1)]$ or $3x - 4y = 13$.

(f) The center of the required circle is the midpoint of $\overline{AB}$, and the radius is half the length of $\overline{AB}$, which is 10. Thus, the equation is $(x+1)^2 + (y+4)^2 = 100$.

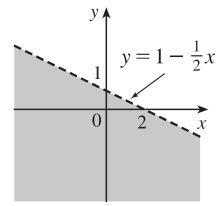
5. (a) Graph the corresponding horizontal lines (given by the equations $y = -1$ and $y = 3$) as solid lines. The inequality $y \geq -1$ describes the points (x, y) that lie on or *above* the line $y = -1$. The inequality $y \leq 3$ describes the points (x, y) that lie on or *below* the line $y = 3$. So the pair of inequalities $-1 \leq y \leq 3$ describes the points that lie on or *between* the lines $y = -1$ and $y = 3$.



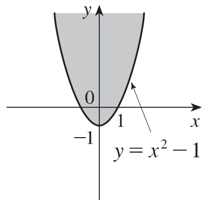
(b) Note that the given inequalities can be written as $-4 < x < 4$ and $-2 < y < 2$, respectively. So the region lies between the vertical lines $x = -4$ and $x = 4$ and between the horizontal lines $y = -2$ and $y = 2$. As shown in the graph, the region common to both graphs is a rectangle (minus its edges) centered at the origin.



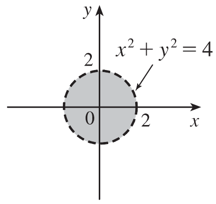
(c) We first graph $y = 1 - \frac{1}{2}x$ as a dotted line. Since $y < 1 - \frac{1}{2}x$, the points in the region lie *below* this line.



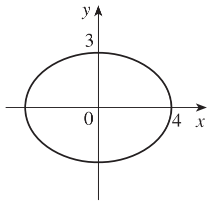
(d) We first graph the parabola $y = x^2 - 1$ using a solid curve. Since $y \geq x^2 - 1$, the points in the region lie on or *above* the parabola.



(e) We graph the circle $x^2 + y^2 = 4$ using a dotted curve. Since $\sqrt{x^2 + y^2} < 2$, the region consists of points whose distance from the origin is less than 2, that is, the points that lie *inside* the circle.



(f) The equation $9x^2 + 16y^2 = 144$ is an ellipse centered at $(0, 0)$. We put it in standard form by dividing by 144 and get $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. The x-intercepts are located at a distance of $\sqrt{16} = 4$ from the center while the y-intercepts are a distance of $\sqrt{9} = 3$ from the center (see the graph).



Test C Functions

1. (a) Locate -1 on the x-axis and then go down to the point on the graph with an x-coordinate of -1. The corresponding y-coordinate is the value of the function at $x = -1$, which is -2. So, $f(-1) = -2$.

   (b) Using the same technique as in part (a), we get $f(2) \approx 2.8$.

   (c) Locate 2 on the y-axis and then go left and right to find all points on the graph with a y-coordinate of 2. The corresponding x-coordinates are the x-values we are searching for. So $x = -3$ and $x = 1$.

   (d) Using the same technique as in part (c), we get $x \approx -2.5$ and $x \approx 0.3$.

   (e) The domain is all the x-values for which the graph exists, and the range is all the y-values for which the graph exists. Thus, the domain is $[-3, 3]$, and the range is $[-2, 3]$.

2. Note that $f(2 + h) = (2 + h)^3$ and $f(2) = 2^3 = 8$. So the difference quotient becomes

$$\frac{f(2+h) - f(2)}{h} = \frac{(2+h)^3 - 8}{h} = \frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \frac{12h + 6h^2 + h^3}{h} = \frac{h(12 + 6h + h^2)}{h} = 12 + 6h + h^2.$$

3. (a) Set the denominator equal to 0 and solve to find restrictions on the domain: $x^2 + x - 2 = 0 \quad \Rightarrow$ $(x - 1)(x + 2) = 0 \quad \Rightarrow \quad x = 1$ or $x = -2$. Thus, the domain is all real numbers except 1 or -2 or, in interval notation, $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

   (b) Note that the denominator is always greater than or equal to 1, and the numerator is defined for all real numbers. Thus, the domain is $(-\infty, \infty)$.

   (c) Note that the function h is the sum of two root functions. So h is defined on the intersection of the domains of these two root functions. The domain of a square root function is found by setting its radicand greater than or equal to 0. Now,

$4 - x \geq 0 \quad \Rightarrow \quad x \leq 4$ and $x^2 - 1 \geq 0 \quad \Rightarrow \quad (x-1)(x+1) \geq 0 \quad \Rightarrow \quad x \leq -1$ or $x \geq 1$. Thus, the domain of h is $(-\infty, -1] \cup [1, 4]$.

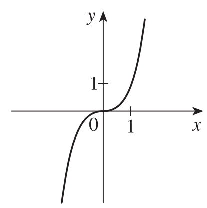
4. (a) Reflect the graph of f about the x-axis.

(b) Stretch the graph of f vertically by a factor of 2, then shift 1 unit downward.

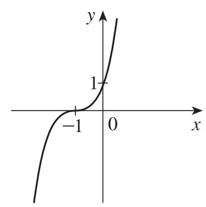
(c) Shift the graph of f right 3 units, then up 2 units.

5. (a) Make a table and then connect the points with a smooth curve:

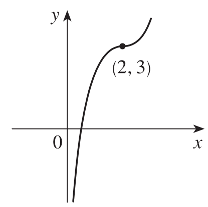
x	-2	-1	0	1	2
y	-8	-1	0	1	8



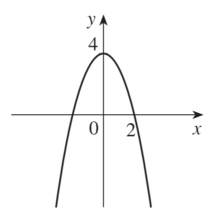
(b) Shift the graph from part (a) left 1 unit.



(c) Shift the graph from part (a) right 2 units and up 3 units.

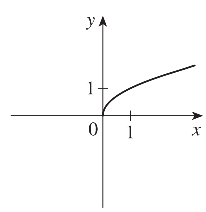


(d) First plot $y = x^2$. Next, to get the graph of $f(x) = 4 - x^2$, reflect f about the x-axis and then shift it upward 4 units.

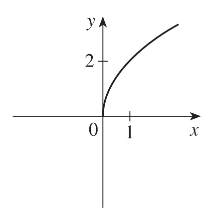


(e) Make a table and then connect the points with a smooth curve:

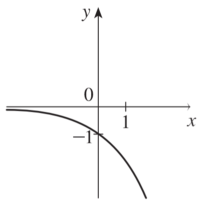
x	0	1	4	9
y	0	1	2	3



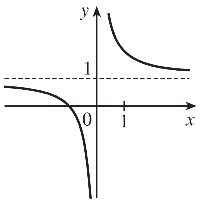
(f) Stretch the graph from part (e) vertically by a factor of two.



(g) First plot $y = 2^x$. Next, get the graph of $y = -2^x$ by reflecting the graph of $y = 2^x$ about the *x*-axis.

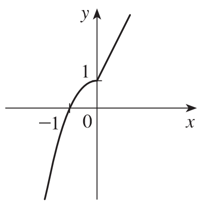


(h) Note that $y = 1 + x^{-1} = 1 + 1/x$. So first plot $y = 1/x$ and then shift it upward 1 unit.



6. (a) $f(-2) = 1 - (-2)^2 = -3$ and $f(1) = 2(1) + 1 = 3$

(b) For $x \leq 0$ plot $f(x) = 1 - x^2$ and, on the same plane, for $x > 0$ plot the graph of $f(x) = 2x + 1$.



7. (a) $(f \circ g)(x) = f(g(x)) = f(2x - 3) = (2x - 3)^2 + 2(2x - 3) - 1 = 4x^2 - 12x + 9 + 4x - 6 - 1 = 4x^2 - 8x + 2$

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 + 2x - 1) = 2(x^2 + 2x - 1) - 3 = 2x^2 + 4x - 2 - 3 = 2x^2 + 4x - 5$

(c) $(g \circ g \circ g)(x) = g(g(g(x))) = g(g(2x - 3)) = g(2(2x - 3) - 3) = g(4x - 9) = 2(4x - 9) - 3$
$= 8x - 18 - 3 = 8x - 21$

Test D Trigonometry

1. (a) $300^\circ = 300^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{300\pi}{180} = \frac{5\pi}{3}$

(b) $-18^\circ = -18^\circ\left(\frac{\pi}{180^\circ}\right) = -\frac{18\pi}{180} = -\frac{\pi}{10}$

2. (a) $\frac{5\pi}{6} = \frac{5\pi}{6}\left(\frac{180^\circ}{\pi}\right) = 150^\circ$

(b) $2 = 2\left(\frac{180^\circ}{\pi}\right) = \frac{360^\circ}{\pi} \approx 114.6^\circ$

3. We will use the arc length formula, $s = r\theta$, where s is arc length, r is the radius of the circle, and θ is the measure of the central angle in radians. First, note that $30^\circ = 30^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{6}$. So $s = (12)\left(\frac{\pi}{6}\right) = 2\pi$ cm.

4. (a) $\tan(\pi/3) = \sqrt{3}$ [You can read the value from a right triangle with sides 1, 2, and $\sqrt{3}$.]

(b) Note that $7\pi/6$ can be thought of as an angle in the third quadrant with reference angle $\pi/6$. Thus, $\sin(7\pi/6) = -\frac{1}{2}$, since the sine function is negative in the third quadrant.

(c) Note that $5\pi/3$ can be thought of as an angle in the fourth quadrant with reference angle $\pi/3$. Thus,
$\sec(5\pi/3) = \frac{1}{\cos(5\pi/3)} = \frac{1}{1/2} = 2$, since the cosine function is positive in the fourth quadrant.

5. $\sin\theta = a/24 \quad\Rightarrow\quad a = 24\sin\theta$ and $\cos\theta = b/24 \quad\Rightarrow\quad b = 24\cos\theta$

6. $\sin x = \frac{1}{3}$ and $\sin^2 x + \cos^2 x = 1 \quad\Rightarrow\quad \cos x = \sqrt{1 - \frac{1}{9}} = \dfrac{2\sqrt{2}}{3}$. Also, $\cos y = \frac{4}{5} \quad\Rightarrow\quad \sin y = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$.

So, using the sum identity for the sine, we have

$$\sin(x+y) = \sin x\,\cos y + \cos x\,\sin y = \frac{1}{3}\cdot\frac{4}{5} + \frac{2\sqrt{2}}{3}\cdot\frac{3}{5} = \frac{4+6\sqrt{2}}{15} = \frac{1}{15}\left(4+6\sqrt{2}\right)$$

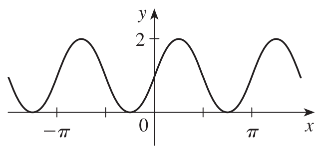
7. (a) $\tan\theta\,\sin\theta + \cos\theta = \dfrac{\sin\theta}{\cos\theta}\sin\theta + \cos\theta = \dfrac{\sin^2\theta}{\cos\theta} + \dfrac{\cos^2\theta}{\cos\theta} = \dfrac{1}{\cos\theta} = \sec\theta$

(b) $\dfrac{2\tan x}{1+\tan^2 x} = \dfrac{2\sin x/(\cos x)}{\sec^2 x} = 2\,\dfrac{\sin x}{\cos x}\,\cos^2 x = 2\sin x\,\cos x = \sin 2x$

8. $\sin 2x = \sin x \quad\Leftrightarrow\quad 2\sin x\,\cos x = \sin x \quad\Leftrightarrow\quad 2\sin x\,\cos x - \sin x = 0 \quad\Leftrightarrow\quad \sin x\,(2\cos x - 1) = 0 \quad\Leftrightarrow$

$\sin x = 0$ or $\cos x = \frac{1}{2} \quad\Rightarrow\quad x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$.

9. We first graph $y = \sin 2x$ (by compressing the graph of $\sin x$ by a factor of 2) and then shift it upward 1 unit.



1 ☐ FUNCTIONS AND LIMITS

1.1 Four Ways to Represent a Function

1. The functions $f(x) = x + \sqrt{2-x}$ and $g(u) = u + \sqrt{2-u}$ give exactly the same output values for every input value, so f and g are equal.

2. $f(x) = \dfrac{x^2 - x}{x - 1} = \dfrac{x(x-1)}{x-1} = x$ for $x - 1 \neq 0$, so f and g [where $g(x) = x$] are not equal because $f(1)$ is undefined and $g(1) = 1$.

3. (a) The point $(1, 3)$ is on the graph of f, so $f(1) = 3$.

   (b) When $x = -1$, y is about -0.2, so $f(-1) \approx -0.2$.

   (c) $f(x) = 1$ is equivalent to $y = 1$. When $y = 1$, we have $x = 0$ and $x = 3$.

   (d) A reasonable estimate for x when $y = 0$ is $x = -0.8$.

   (e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-2 \leq x \leq 4$, or $[-2, 4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-1 \leq y \leq 3$, or $[-1, 3]$.

   (f) As x increases from -2 to 1, y increases from -1 to 3. Thus, f is increasing on the interval $[-2, 1]$.

4. (a) The point $(-4, -2)$ is on the graph of f, so $f(-4) = -2$. The point $(3, 4)$ is on the graph of g, so $g(3) = 4$.

   (b) We are looking for the values of x for which the y-values are equal. The y-values for f and g are equal at the points $(-2, 1)$ and $(2, 2)$, so the desired values of x are -2 and 2.

   (c) $f(x) = -1$ is equivalent to $y = -1$. When $y = -1$, we have $x = -3$ and $x = 4$.

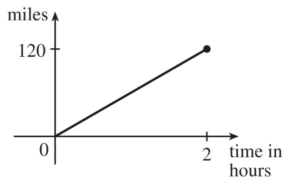
   (d) As x increases from 0 to 4, y decreases from 3 to -1. Thus, f is decreasing on the interval $[0, 4]$.

   (e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-4 \leq x \leq 4$, or $[-4, 4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \leq y \leq 3$, or $[-2, 3]$.

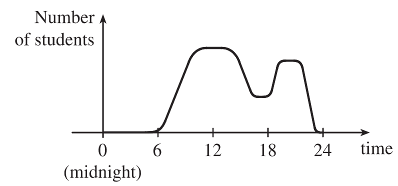
   (f) The domain of g is $[-4, 3]$ and the range is $[0.5, 4]$.

5. From Figure 1 in the text, the lowest point occurs at about $(t, a) = (12, -85)$. The highest point occurs at about $(17, 115)$. Thus, the range of the vertical ground acceleration is $-85 \leq a \leq 115$. Written in interval notation, we get $[-85, 115]$.

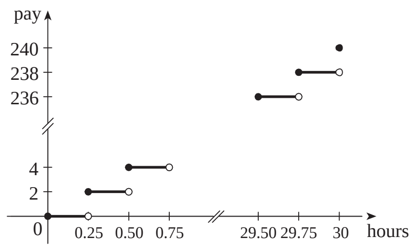
6. *Example 1:* A car is driven at 60 mi/h for 2 hours. The distance d traveled by the car is a function of the time t. The domain of the function is $\{t \mid 0 \leq t \leq 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0 \leq d \leq 120\}$, where d is measured in miles.



Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0 \leq t \leq 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0 \leq N \leq k\}$, where N is an integer and k is the largest number of students on campus at once.



Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h. The domain of the function is $[0, 30]$ and the range of the function is $\{0, 2.00, 4.00, \ldots, 238.00, 240.00\}$.



7. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

8. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2, 2]$ and the range is $[-1, 2]$.

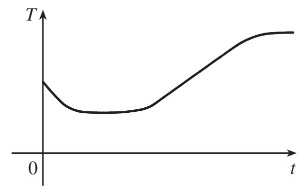
9. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.

10. No, the curve is not the graph of a function since for $x = 0, \pm 1$, and ± 2, there are infinitely many points on the curve.

11. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

12. First, the tub was filled with water to a height of 15 in. Then a person got into the tub, raising the water level to 20 in. At around 12 minutes, the person stood up in the tub but then immediately sat down. Finally, at around 17 minutes, the person got out of the tub, and then drained the water.

13. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.

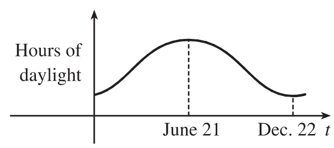


14. Runner A won the race, reaching the finish line at 100 meters in about 15 seconds, followed by runner B with a time of about 19 seconds, and then by runner C who finished in around 23 seconds. B initially led the race, followed by C, and then A. C then passed B to lead for a while. Then A passed first B, and then passed C to take the lead and finish first. Finally, B passed C to finish in second place. All three runners completed the race.

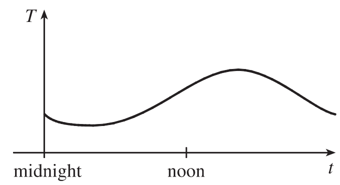
15. (a) The power consumption at 6 AM is $500\ \text{MW}$, which is obtained by reading the value of power P when $t = 6$ from the graph. At 6 PM we read the value of P when $t = 18$, obtaining approximately $730\ \text{MW}$.

(b) The minimum power consumption is determined by finding the time for the lowest point on the graph, $t = 4$, or 4 AM. The maximum power consumption corresponds to the highest point on the graph, which occurs just before $t = 12$, or right before noon. These times are reasonable, considering the power consumption schedules of most individuals and businesses.

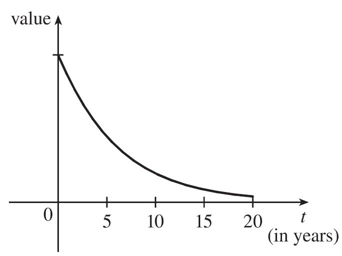
16. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22. (Exchange the dates for the southern hemisphere.)



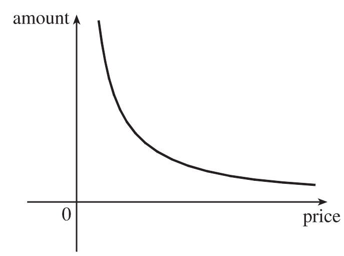
17. Of course, this graph depends strongly on the geographical location!



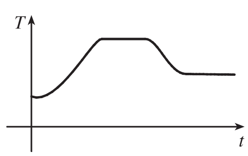
18. The value of the car decreases fairly rapidly initially, then somewhat less rapidly.



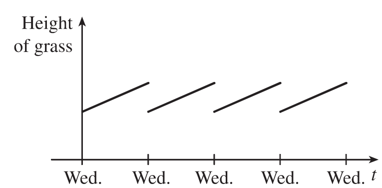
19. As the price increases, the amount sold decreases.



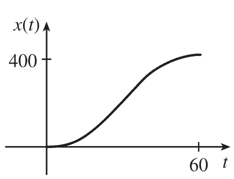
20. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.



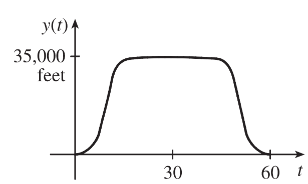
21.



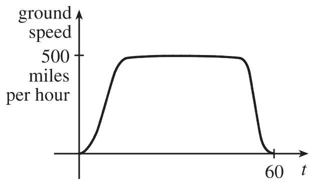
22. (a)



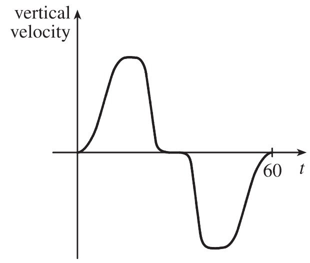
(b)



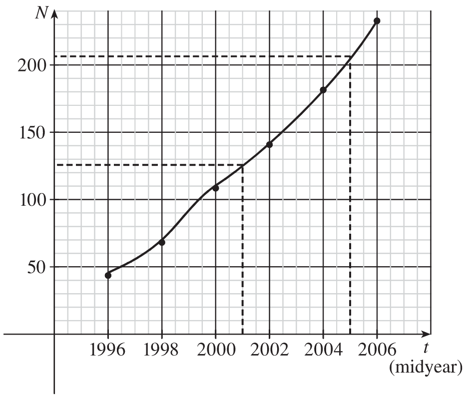
(c)



(d)

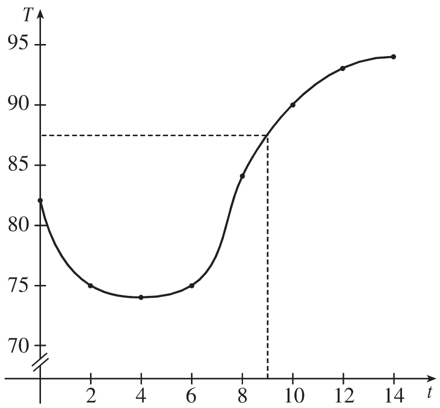


23. (a)



(b) From the graph, we estimate the number of US cell-phone subscribers to be about 126 million in 2001 and 207 million in 2005.

24. (a)



(b) From the graph in part (a), we estimate the temperature at 9:00 AM to be about 87 °F

25. $f(x) = 3x^2 - x + 2.$

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12.$

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16.$

$f(a) = 3a^2 - a + 2.$

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2.$

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4.$

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4.$

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2.$

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2.$

$$[f(a)]^2 = \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right)$$
$$= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.$$
$$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2.$$

26. A spherical balloon with radius $r+1$ has volume $V(r+1) = \frac{4}{3}\pi(r+1)^3 = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1)$. We wish to find the amount of air needed to inflate the balloon from a radius of r to $r+1$. Hence, we need to find the difference $V(r+1) - V(r) = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1) - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3r^2 + 3r + 1)$.

27. $f(x) = 4 + 3x - x^2$, so $f(3+h) = 4 + 3(3+h) - (3+h)^2 = 4 + 9 + 3h - (9 + 6h + h^2) = 4 - 3h - h^2$, and $\dfrac{f(3+h) - f(3)}{h} = \dfrac{(4 - 3h - h^2) - 4}{h} = \dfrac{h(-3-h)}{h} = -3 - h$.

28. $f(x) = x^3$, so $f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$, and $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h} = \dfrac{h(3a^2 + 3ah + h^2)}{h} = 3a^2 + 3ah + h^2$.

29. $$\frac{f(x) - f(a)}{x - a} = \frac{\dfrac{1}{x} - \dfrac{1}{a}}{x - a} = \frac{\dfrac{a - x}{xa}}{x - a} = \frac{a - x}{xa(x - a)} = \frac{-1(x - a)}{xa(x - a)} = -\frac{1}{ax}$$

30. $$\frac{f(x) - f(1)}{x - 1} = \frac{\dfrac{x+3}{x+1} - 2}{x - 1} = \frac{\dfrac{x + 3 - 2(x+1)}{x+1}}{x - 1} = \frac{x + 3 - 2x - 2}{(x+1)(x-1)}$$
$$= \frac{-x + 1}{(x+1)(x-1)} = \frac{-(x-1)}{(x+1)(x-1)} = -\frac{1}{x+1}$$

31. $f(x) = (x+4)/(x^2 - 9)$ is defined for all x except when $0 = x^2 - 9 \quad\Leftrightarrow\quad 0 = (x+3)(x-3) \quad\Leftrightarrow\quad x = -3$ or 3, so the domain is $\{x \in \mathbb{R} \mid x \neq -3, 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

32. $f(x) = (2x^3 - 5)/(x^2 + x - 6)$ is defined for all x except when $0 = x^2 + x - 6 \quad\Leftrightarrow\quad 0 = (x+3)(x-2) \quad\Leftrightarrow$ $x = -3$ or 2, so the domain is $\{x \in \mathbb{R} \mid x \neq -3, 2\} = (-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

33. $f(t) = \sqrt[3]{2t - 1}$ is defined for all real numbers. In fact $\sqrt[3]{p(t)}$, where $p(t)$ is a polynomial, is defined for all real numbers. Thus, the domain is $\mathbb{R}$, or $(-\infty, \infty)$.

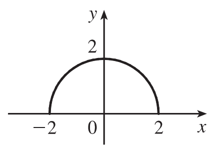
34. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$ is defined when $3 - t \geq 0 \quad\Leftrightarrow\quad t \leq 3$ and $2 + t \geq 0 \quad\Leftrightarrow\quad t \geq -2$. Thus, the domain is $-2 \leq t \leq 3$, or $[-2, 3]$.

35. $h(x) = 1\big/\sqrt[4]{x^2 - 5x}$ is defined when $x^2 - 5x > 0 \quad\Leftrightarrow\quad x(x - 5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

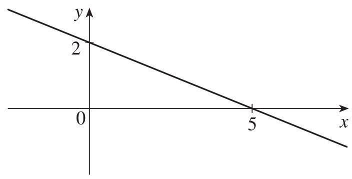
36. $f(u) = \dfrac{u+1}{1 + \dfrac{1}{u+1}}$ is defined when $u + 1 \neq 0$ $[u \neq -1]$ and $1 + \dfrac{1}{u+1} \neq 0$. Since $1 + \dfrac{1}{u+1} = 0 \quad\Rightarrow$ $\dfrac{1}{u+1} = -1 \quad\Rightarrow\quad 1 = -u - 1 \quad\Rightarrow\quad u = -2$, the domain is $\{u \mid u \neq -2, u \neq -1\} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

37. $F(p) = \sqrt{2 - \sqrt{p}}$ is defined when $p \geq 0$ and $2 - \sqrt{p} \geq 0$. Since $2 - \sqrt{p} \geq 0 \Rightarrow 2 \geq \sqrt{p} \Rightarrow \sqrt{p} \leq 2 \Rightarrow 0 \leq p \leq 4$, the domain is $[0, 4]$.

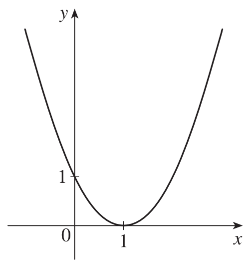
38. $h(x) = \sqrt{4 - x^2}$. Now $y = \sqrt{4 - x^2} \Rightarrow y^2 = 4 - x^2 \Leftrightarrow x^2 + y^2 = 4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\{x \mid 4 - x^2 \geq 0\} = \{x \mid 4 \geq x^2\} = \{x \mid 2 \geq |x|\} = [-2, 2]$. From the graph, the range is $0 \leq y \leq 2$, or $[0, 2]$.



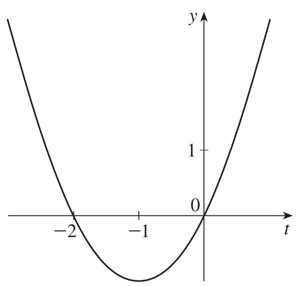
39. $f(x) = 2 - 0.4x$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a line with slope -0.4 and y-intercept 2.



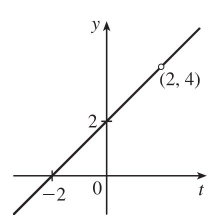
40. $F(x) = x^2 - 2x + 1 = (x - 1)^2$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of F is a parabola with vertex $(1, 0)$.



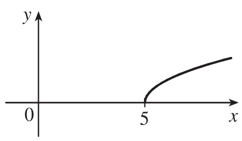
41. $f(t) = 2t + t^2$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a parabola opening upward since the coefficient of t^2 is positive. To find the t-intercepts, let $y = 0$ and solve for t. $0 = 2t + t^2 = t(2 + t) \Rightarrow t = 0$ or $t = -2$. The t-coordinate of the vertex is halfway between the t-intercepts, that is, at $t = -1$. Since $f(-1) = 2(-1) + (-1)^2 = -2 + 1 = -1$, the vertex is $(-1, -1)$.



42. $H(t) = \dfrac{4 - t^2}{2 - t} = \dfrac{(2 + t)(2 - t)}{2 - t}$, so for $t \neq 2$, $H(t) = 2 + t$. The domain is $\{t \mid t \neq 2\}$. So the graph of H is the same as the graph of the function $f(t) = t + 2$ (a line) except for the hole at $(2, 4)$.



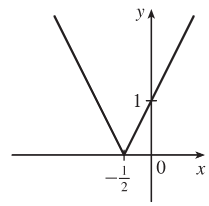
43. $g(x) = \sqrt{x - 5}$ is defined when $x - 5 \geq 0$ or $x \geq 5$, so the domain is $[5, \infty)$. Since $y = \sqrt{x - 5} \Rightarrow y^2 = x - 5 \Rightarrow x = y^2 + 5$, we see that g is the top half of a parabola.



44. $F(x) = |2x+1| = \begin{cases} 2x+1 & \text{if } 2x+1 \geq 0 \\ -(2x+1) & \text{if } 2x+1 < 1 \end{cases}$

$= \begin{cases} 2x+1 & \text{if } x \geq -\frac{1}{2} \\ -2x-1 & \text{if } x < -\frac{1}{2} \end{cases}$

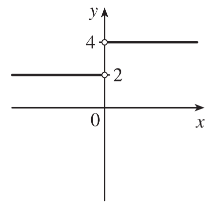
The domain is $\mathbb{R}$, or $(-\infty, \infty)$.



45. $G(x) = \dfrac{3x+|x|}{x}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

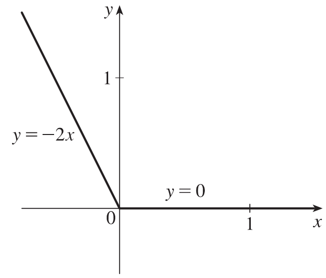
$$G(x) = \begin{cases} \dfrac{3x+x}{x} & \text{if } x > 0 \\ \dfrac{3x-x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} \dfrac{4x}{x} & \text{if } x > 0 \\ \dfrac{2x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$

Note that G is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.



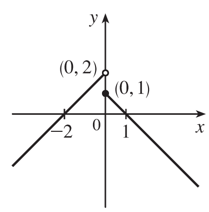
46. $g(x) = |x| - x = \begin{cases} x-x & \text{if } x \geq 0 \\ -x-x & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$.

The domain is $\mathbb{R}$, or $(-\infty, \infty)$.



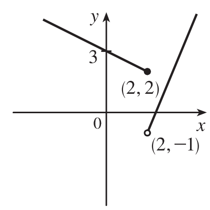
47. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$

The domain is $\mathbb{R}$.



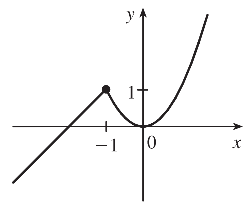
48. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x \leq 2 \\ 2x-5 & \text{if } x > 2 \end{cases}$

The domain is $\mathbb{R}$.



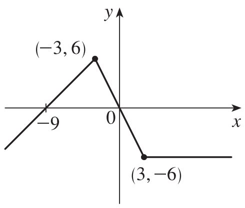
49. $f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

Note that for $x = -1$, both $x+2$ and x^2 are equal to 1. The domain is $\mathbb{R}$.



50. $f(x) = \begin{cases} x+9 & \text{if } x < -3 \\ -2x & \text{if } |x| \le 3 \\ -6 & \text{if } x > 3 \end{cases}$

Note that for $x = -3$, both $x + 9$ and $-2x$ are equal to 6; and for $x = 3$, both $-2x$ and -6 are equal to -6. The domain is $\mathbb{R}$.



51. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of the line segment joining the points $(1, -3)$ and $(5, 7)$ is $\dfrac{7-(-3)}{5-1} = \dfrac{5}{2}$, so an equation is $y - (-3) = \frac{5}{2}(x-1)$. The function is $f(x) = \frac{5}{2}x - \frac{11}{2}$, $1 \le x \le 5$.

52. The slope of the line segment joining the points $(-5, 10)$ and $(7, -10)$ is $\dfrac{-10-10}{7-(-5)} = -\dfrac{5}{3}$, so an equation is $y - 10 = -\frac{5}{3}[x - (-5)]$. The function is $f(x) = -\frac{5}{3}x + \frac{5}{3}$, $-5 \le x \le 7$.

53. We need to solve the given equation for y. $x + (y-1)^2 = 0 \Leftrightarrow (y-1)^2 = -x \Leftrightarrow y - 1 = \pm\sqrt{-x} \Leftrightarrow$ $y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$. Note that the domain is $x \le 0$.

54. $x^2 + (y-2)^2 = 4 \Leftrightarrow (y-2)^2 = 4 - x^2 \Leftrightarrow y - 2 = \pm\sqrt{4-x^2} \Leftrightarrow y = 2 \pm \sqrt{4-x^2}$. The top half is given by the function $f(x) = 2 + \sqrt{4-x^2}$, $-2 \le x \le 2$.

55. For $0 \le x \le 3$, the graph is the line with slope -1 and y-intercept 3, that is, $y = -x + 3$. For $3 < x \le 5$, the graph is the line with slope 2 passing through $(3, 0)$; that is, $y - 0 = 2(x-3)$, or $y = 2x - 6$. So the function is

$$f(x) = \begin{cases} -x+3 & \text{if } 0 \le x \le 3 \\ 2x-6 & \text{if } 3 < x \le 5 \end{cases}$$

56. For $-4 \le x \le -2$, the graph is the line with slope $-\frac{3}{2}$ passing through $(-2, 0)$; that is, $y - 0 = -\frac{3}{2}[x - (-2)]$, or $y = -\frac{3}{2}x - 3$. For $-2 < x < 2$, the graph is the top half of the circle with center $(0, 0)$ and radius 2. An equation of the circle is $x^2 + y^2 = 4$, so an equation of the top half is $y = \sqrt{4-x^2}$. For $2 \le x \le 4$, the graph is the line with slope $\frac{3}{2}$ passing through $(2, 0)$; that is, $y - 0 = \frac{3}{2}(x-2)$, or $y = \frac{3}{2}x - 3$. So the function is

$$f(x) = \begin{cases} -\frac{3}{2}x - 3 & \text{if } -4 \le x \le -2 \\ \sqrt{4-x^2} & \text{if } -2 < x < 2 \\ \frac{3}{2}x - 3 & \text{if } 2 \le x \le 4 \end{cases}$$

57. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

58. Let the length and width of the rectangle be L and W. Then the area is $LW = 16$, so that $W = 16/L$. The perimeter is $P = 2L + 2W$, so $P(L) = 2L + 2(16/L) = 2L + 32/L$, and the domain of P is $L > 0$, since lengths must be positive quantities. If we further restrict L to be larger than W, then $L > 4$ would be the domain.

59. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

60. Let the volume of the cube be V and the length of an edge be L. Then $V = L^3$ so $L = \sqrt[3]{V}$, and the surface area is $S(V) = 6L^2 = 6\left(\sqrt[3]{V}\right)^2 = 6V^{2/3}$, with domain $V > 0$.

61. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.

62. The area of the window is $A = xh + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = xh + \dfrac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window. The perimeter is $P = 2h + x + \frac{1}{2}\pi x = 30 \quad\Leftrightarrow\quad 2h = 30 - x - \frac{1}{2}\pi x \quad\Leftrightarrow\quad h = \frac{1}{4}(60 - 2x - \pi x)$. Thus,

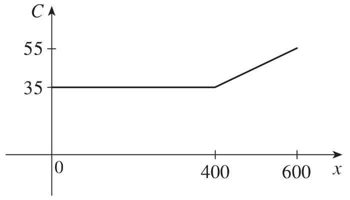
$$A(x) = x\,\frac{60 - 2x - \pi x}{4} + \frac{\pi x^2}{8} = 15x - \tfrac{1}{2}x^2 - \tfrac{\pi}{4}x^2 + \tfrac{\pi}{8}x^2 = 15x - \tfrac{4}{8}x^2 - \tfrac{\pi}{8}x^2 = 15x - x^2\left(\frac{\pi + 4}{8}\right).$$

Since the lengths x and h must be positive quantities, we have $x > 0$ and $h > 0$. For $h > 0$, we have $2h > 0 \quad\Leftrightarrow\quad 30 - x - \frac{1}{2}\pi x > 0 \quad\Leftrightarrow\quad 60 > 2x + \pi x \quad\Leftrightarrow\quad x < \dfrac{60}{2 + \pi}$. Hence, the domain of A is $0 < x < \dfrac{60}{2 + \pi}$.

63. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so $V(x) = (20 - 2x)(12 - 2x)(x) = 4(10 - x)(6 - x)(x) = 4x(60 - 16x + x^2) = 4x^3 - 64x^2 + 240x$. The sides L, W, and x must be positive. Thus, $L > 0 \quad\Leftrightarrow\quad 20 - 2x > 0 \quad\Leftrightarrow\quad x < 10$; $W > 0 \quad\Leftrightarrow\quad 12 - 2x > 0 \quad\Leftrightarrow\quad x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

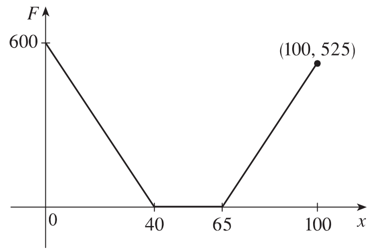
64. We can summarize the monthly cost with a piecewise defined function.

$$C(x) = \begin{cases} 35 & \text{if } 0 \le x \le 400 \\ 35 + 0.10(x - 400) & \text{if } x > 400 \end{cases}$$



65. We can summarize the amount of the fine with a piecewise defined function.

$$F(x) = \begin{cases} 15(40 - x) & \text{if } 0 \le x < 40 \\ 0 & \text{if } 40 \le x \le 65 \\ 15(x - 65) & \text{if } x > 65 \end{cases}$$



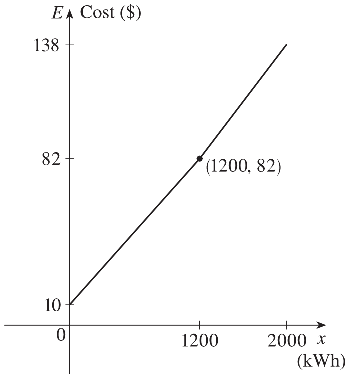
66. For the first 1200 kWh, $E(x) = 10 + 0.06x$.

For usage over 1200 kWh, the cost is

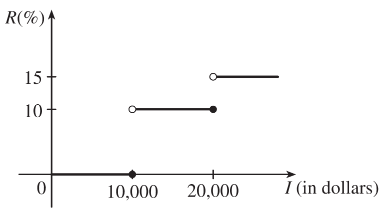
$E(x) = 10 + 0.06(1200) + 0.07(x - 1200) = 82 + 0.07(x - 1200)$.

Thus,

$$E(x) = \begin{cases} 10 + 0.06x & \text{if } 0 \le x \le 1200 \\ 82 + 0.07(x - 1200) & \text{if } x > 1200 \end{cases}$$



67. (a)

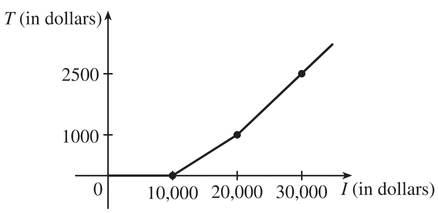


(b) On \$14,000, tax is assessed on \$4000, and $10\%(\$4000) = \$400$.

On \$26,000, tax is assessed on \$16,000, and

$10\%(\$10{,}000) + 15\%(\$6000) = \$1000 + \$900 = \$1900$.

(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from $(10{,}000, 0)$ to $(20{,}000, 1000)$. The tax on \$30,000 is \$2500, so the graph of T for $x > 20{,}000$ is the ray with initial point $(20{,}000, 1000)$ that passes through $(30{,}000, 2500)$.



68. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

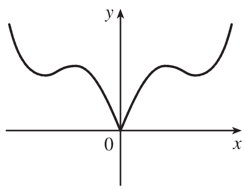
69. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y-axis.

70. f is not an even function since it is not symmetric with respect to the y-axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y-axis.

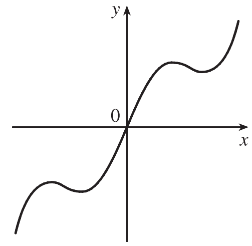
71. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

72. (a) If f is even, we get the rest of the graph by reflecting about the y-axis.



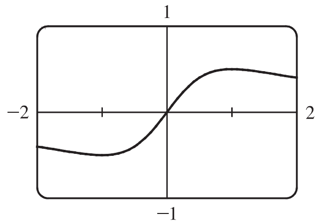
(b) If f is odd, we get the rest of the graph by rotating $180°$ about the origin.



73. $f(x) = \dfrac{x}{x^2+1}$.

$f(-x) = \dfrac{-x}{(-x)^2+1} = \dfrac{-x}{x^2+1} = -\dfrac{x}{x^2+1} = -f(x)$.

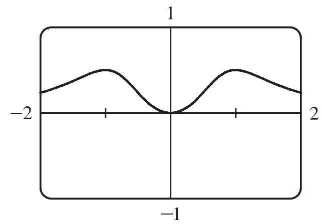
So f is an odd function.



74. $f(x) = \dfrac{x^2}{x^4+1}$.

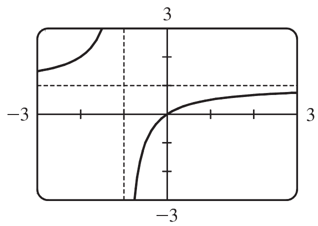
$f(-x) = \dfrac{(-x)^2}{(-x)^4+1} = \dfrac{x^2}{x^4+1} = f(x)$.

So f is an even function.



75. $f(x) = \dfrac{x}{x+1}$, so $f(-x) = \dfrac{-x}{-x+1} = \dfrac{x}{x-1}$.

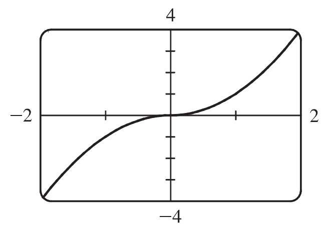
Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



76. $f(x) = x\,|x|$.

$f(-x) = (-x)\,|-x| = (-x)\,|x| = -(x\,|x|)$
$= -f(x)$

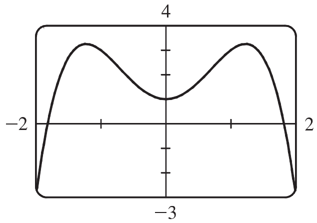
So f is an odd function.



77. $f(x) = 1 + 3x^2 - x^4$.

$f(-x) = 1+3(-x)^2-(-x)^4 = 1+3x^2-x^4 = f(x)$.

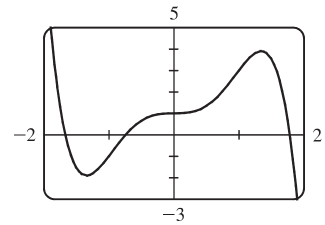
So f is an even function.



78. $f(x) = 1 + 3x^3 - x^5$, so

$$f(-x) = 1 + 3(-x)^3 - (-x)^5 = 1 + 3(-x^3) - (-x^5) = 1 - 3x^3 + x^5$$

Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



79. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f+g$ is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = -f(x) + [-g(x)] = -[f(x) + g(x)] = -(f+g)(x)$, so $f+g$ is an *odd* function.

(iii) If f is an even function and g is an odd function, then $(f+g)(-x) = f(-x) + g(-x) = f(x) + [-g(x)] = f(x) - g(x)$, which is not $(f+g)(x)$ nor $-(f+g)(x)$, so $f+g$ is *neither* even nor odd. (Exception: if f is the zero function, then $f+g$ will be *odd*. If g is the zero function, then $f+g$ will be *even*.)

80. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(iii) If f is an even function and g is an odd function, then $(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -[f(x)g(x)] = -(fg)(x)$, so fg is an *odd* function.

1.2 Mathematical Models: A Catalog of Essential Functions

1. (a) $f(x) = \log_2 x$ is a logarithmic function.

(b) $g(x) = \sqrt[4]{x}$ is a root function with $n = 4$.

(c) $h(x) = \dfrac{2x^3}{1 - x^2}$ is a rational function because it is a ratio of polynomials.

(d) $u(t) = 1 - 1.1t + 2.54t^2$ is a polynomial of degree 2 (also called a *quadratic function*).

(e) $v(t) = 5^t$ is an exponential function.

(f) $w(\theta) = \sin\theta\,\cos^2\theta$ is a trigonometric function.

2. (a) $y = \pi^x$ is an exponential function (notice that x is the *exponent*).

(b) $y = x^\pi$ is a power function (notice that x is the *base*).

(c) $y = x^2(2 - x^3) = 2x^2 - x^5$ is a polynomial of degree 5.

(d) $y = \tan t - \cos t$ is a trigonometric function.

(e) $y = s/(1 + s)$ is a rational function because it is a ratio of polynomials.

(f) $y = \sqrt{x^3 - 1}/(1 + \sqrt[3]{x})$ is an algebraic function because it involves polynomials and roots of polynomials.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y = x^5\right]$ must be f. Since g is flatter than h near the origin, we must have (c) $\left[y = x^8\right]$ matched with g and (a) $\left[y = x^2\right]$ matched with h.

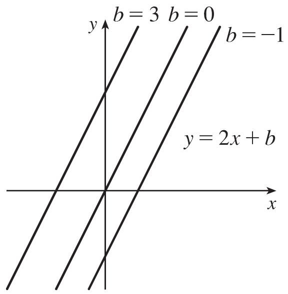
4. (a) The graph of $y = 3x$ is a line (choice G).

(b) $y = 3^x$ is an exponential function (choice f).

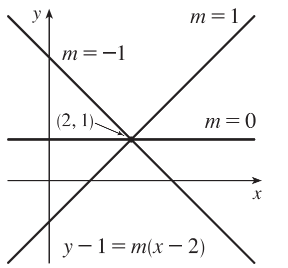
(c) $y = x^3$ is an odd polynomial function or power function (choice F).

(d) $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

5. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

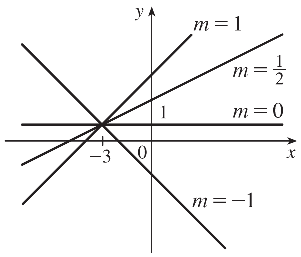


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.

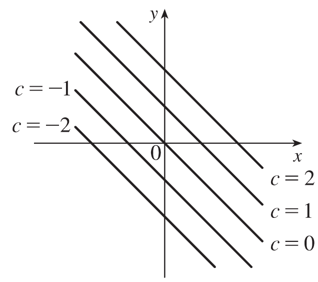


(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

6. All members of the family of linear functions $f(x) = 1 + m(x + 3)$ have graphs that are lines passing through the point $(-3, 1)$.



7. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1. The y-intercept is c.



8. The vertex of the parabola on the left is $(3, 0)$, so an equation is $y = a(x-3)^2 + 0$. Since the point $(4, 2)$ is on the parabola, we'll substitute 4 for x and 2 for y to find a. $2 = a(4-3)^2 \Rightarrow a = 2$, so an equation is $f(x) = 2(x-3)^2$.

The y-intercept of the parabola on the right is $(0, 1)$, so an equation is $y = ax^2 + bx + 1$. Since the points $(-2, 2)$ and $(1, -2.5)$ are on the parabola, we'll substitute -2 for x and 2 for y as well as 1 for x and -2.5 for y to obtain two equations with the unknowns a and b.

$(-2, 2)$: $\quad 2 = 4a - 2b + 1 \Rightarrow 4a - 2b = 1 \quad$ **(1)**

$(1, -2.5)$: $\quad -2.5 = a + b + 1 \Rightarrow a + b = -3.5 \quad$ **(2)**

$2 \cdot$ **(2)** + **(1)** gives us $6a = -6 \Rightarrow a = -1$. From **(2)**, $-1 + b = -3.5 \Rightarrow b = -2.5$, so an equation is $g(x) = -x^2 - 2.5x + 1$.

9. Since $f(-1) = f(0) = f(2) = 0$, f has zeros of -1, 0, and 2, so an equation for f is $f(x) = a[x - (-1)](x - 0)(x - 2)$, or $f(x) = ax(x+1)(x-2)$. Because $f(1) = 6$, we'll substitute 1 for x and 6 for $f(x)$.
$6 = a(1)(2)(-1) \Rightarrow -2a = 6 \Rightarrow a = -3$, so an equation for f is $f(x) = -3x(x+1)(x-2)$.

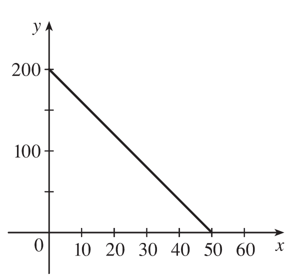
10. (a) For $T = 0.02t + 8.50$, the slope is 0.02, which means that the average surface temperature of the world is increasing at a rate of $0.02\,^\circ\text{C}$ per year. The T-intercept is 8.50, which represents the average surface temperature in $^\circ\text{C}$ in the year 1900.

(b) $t = 2100 - 1900 = 200 \Rightarrow T = 0.02(200) + 8.50 = 12.50\,^\circ\text{C}$

11. (a) $D = 200$, so $c = 0.0417D(a+1) = 0.0417(200)(a+1) = 8.34a + 8.34$. The slope is 8.34, which represents the change in mg of the dosage for a child for each change of 1 year in age.

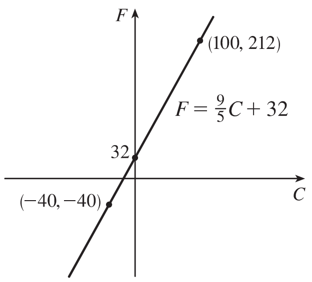
(b) For a newborn, $a = 0$, so $c = 8.34$ mg.

12. (a)



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4. The y-intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x-intercept of 50 is the smallest rental fee that results in no spaces rented.

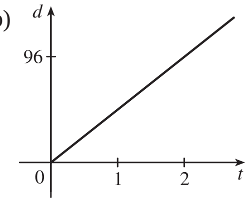
13. (a)



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of 1°C. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

14. (a) Let $d =$ distance traveled (in miles) and $t =$ time elapsed (in hours). At $t = 0$, $d = 0$ and at $t = 50$ minutes $= 50 \cdot \frac{1}{60} = \frac{5}{6}$ h, $d = 40$. Thus we have two points: $(0, 0)$ and $\left(\frac{5}{6}, 40\right)$, so $m = \dfrac{40 - 0}{\frac{5}{6} - 0} = 48$ and so $d = 48t$.

(b)



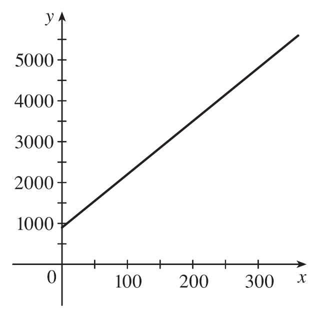
(c) The slope is 48 and represents the car's speed in mi/h.

15. (a) Using N in place of x and T in place of y, we find the slope to be $\dfrac{T_2 - T_1}{N_2 - N_1} = \dfrac{80 - 70}{173 - 113} = \dfrac{10}{60} = \dfrac{1}{6}$. So a linear equation is $T - 80 = \frac{1}{6}(N - 173) \iff T - 80 = \frac{1}{6}N - \frac{173}{6} \iff T = \frac{1}{6}N + \frac{307}{6}$ $\left[\frac{307}{6} = 51.1\overline{6}\right]$.

(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1°F.

(c) When $N = 150$, the temperature is given approximately by $T = \frac{1}{6}(150) + \frac{307}{6} = 76.1\overline{6}\,^\circ\mathrm{F} \approx 76\,^\circ\mathrm{F}$.

16. (a) Let x denote the number of chairs produced in one day and y the associated cost. Using the points $(100, 2200)$ and $(300, 4800)$, we get the slope $\frac{4800-2200}{300-100} = \frac{2600}{200} = 13$. So $y - 2200 = 13(x - 100) \iff y = 13x + 900$.



(b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.

(c) The y-intercept is 900 and it represents the fixed daily costs of operating the factory.

17. (a) We are given $\dfrac{\text{change in pressure}}{\text{10 feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.

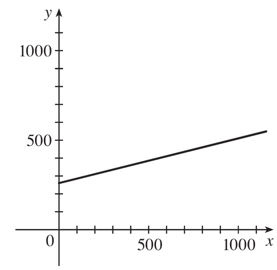
(b) When $P = 100$, then $100 = 0.434d + 15 \iff 0.434d = 85 \iff d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is $100\ \mathrm{lb/in}^2$ at a depth of approximately 196 feet.

18. (a) Using d in place of x and C in place of y, we find the slope to be $\dfrac{C_2 - C_1}{d_2 - d_1} = \dfrac{460 - 380}{800 - 480} = \dfrac{80}{320} = \dfrac{1}{4}$. So a linear equation is $C - 460 = \frac{1}{4}(d - 800) \iff C - 460 = \frac{1}{4}d - 200 \iff C = \frac{1}{4}d + 260$.

(b) Letting $d = 1500$ we get $C = \frac{1}{4}(1500) + 260 = 635$.

The cost of driving 1500 miles is \$635.

(c)



The slope of the line represents the cost per mile, \$0.25.

(d) The y-intercept represents the fixed cost, \$260.

(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

19. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

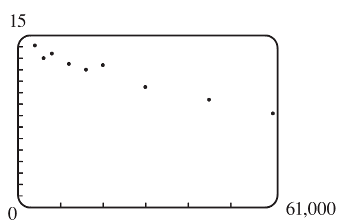
(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

20. (a) The data appear to be increasing exponentially. A model of the form $f(x) = a \cdot b^x$ or $f(x) = a \cdot b^x + c$ seems appropriate.

(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x) = a/x$ seems appropriate.

Exercises 21–24: Some values are given to many decimal places. These are the results given by several computer algebra systems — rounding is left to the reader.

21. (a)

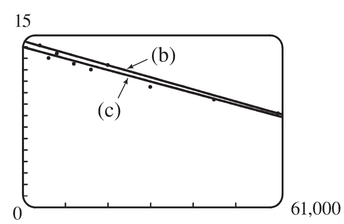


A linear model does seem appropriate.

(b) Using the points $(4000, 14.1)$ and $(60{,}000, 8.2)$, we obtain

$$y - 14.1 = \frac{8.2 - 14.1}{60{,}000 - 4000}(x - 4000) \text{ or, equivalently,}$$

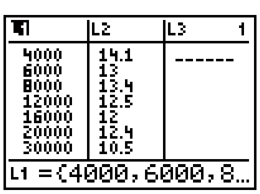
$y \approx -0.000105357x + 14.521429$.

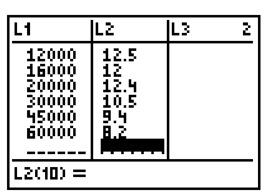


(c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$.

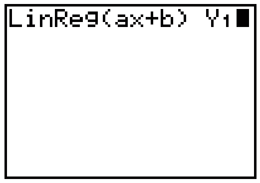
The following commands and screens illustrate how to find the least squares regression line on a TI-84 Plus.

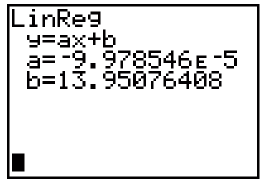
Enter the data into list one (L1) and list two (L2). Press STAT 1 to enter the editor.

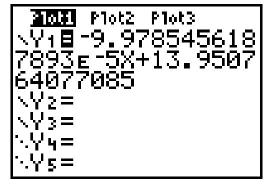




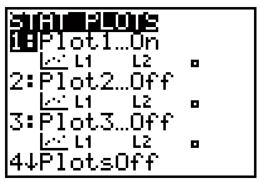
Find the regession line and store it in Y_1. Press 2nd QUIT STAT ▸ 4 VARS ▸ 1 1 ENTER.

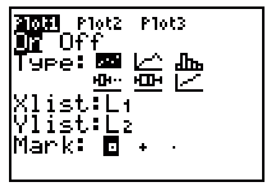




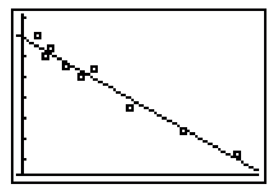


Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing ENTER or by pressing 2nd STAT PLOT 1 ENTER.





Now press ZOOM 9 to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

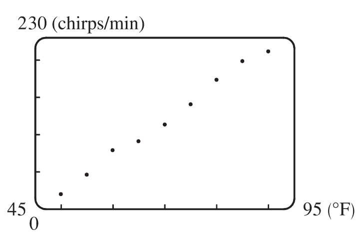


(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

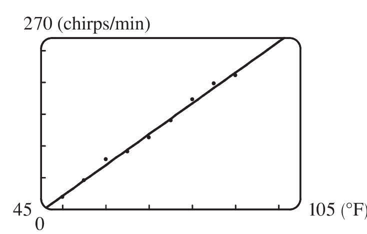
(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

(f) When $x = 200{,}000$, y is negative, so the model does not apply.

22. (a)



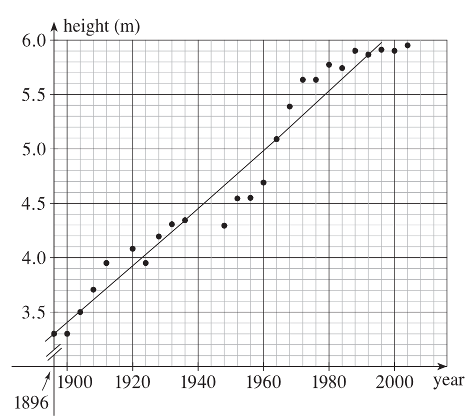
(b)



Using a computing device, we obtain the least squares regression line $y = 4.85\overline{6}x - 220.9\overline{6}$.

(c) When $x = 100^\circ\mathrm{F}$, $y = 264.7 \approx 265$ chirps/min.

23. (a) A linear model seems appropriate over the time interval considered.

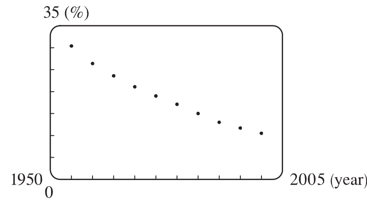


(b) Using a computing device, we obtain the regression line $y \approx 0.0265x - 46.8759$. It is plotted in the graph in part (a).

(c) For $x = 2008$, the linear model predicts a winning height of 6.27 m, considerably higher than the actual winning height of 5.96 m.

(d) It is *not* reasonable to use the model to predict the winning height at the 2100 Olympics since 2100 is too far from the 1896–2004 range on which the model is based.

24. By looking at the scatter plot of the data, we rule out the power and logarithmic models.



Scatter plot

We try various models:

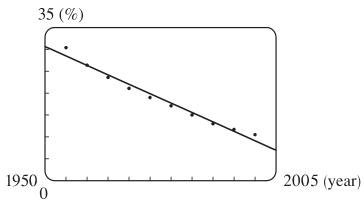
Linear $\quad y = -0.430\,545\,454\,5x + 870.183\,636\,4$

Quadratic: $\quad y = 0.004\,893\,939\,4x^2 - 19.786\,075\,76x + 20\,006.954\,85$

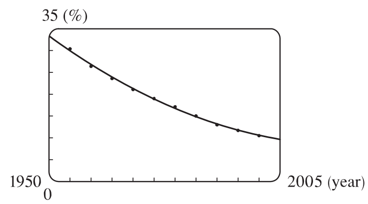
Cubic: $\quad y = -0.000\,073\,193\,47x^3 + 0.439\,114\,219\,1x^2 - 878.429\,871\,8x + 585\,960.983$

Quartic: $\quad y = 0.000\,007\,902\,097\,9x^4 - 0.062\,578\,787\,9x^3 + 185.842\,283\,8x^2 - 245\,290.9304x + 121\,409\,472.7$

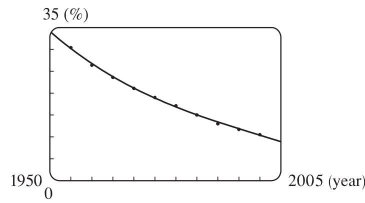
Exponential: $\quad y = 2.618\,230\,2 \times 10^{21}(0.976\,789\,309\,4)^x$



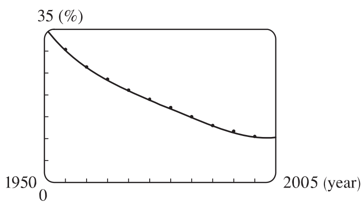
Linear model



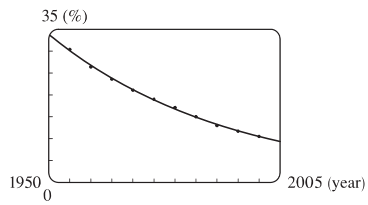
Quadratic model



Cubic model



Quartic model



Exponential model

After examining the graphs of these models, we see that all the models are good and the quartic model is the best. Using this model, we obtain estimates 13.6% and 10.2% for the rural percentages in 1988 and 2002 respectively.

25. If x is the original distance from the source, then the illumination is $f(x) = kx^{-2} = k/x^2$. Moving halfway to the lamp gives us an illumination of $f\left(\frac{1}{2}x\right) = k\left(\frac{1}{2}x\right)^{-2} = k(2/x)^2 = 4(k/x^2)$, so the light is 4 times as bright.

26. (a) If $A = 60$, then $S = 0.7A^{0.3} \approx 2.39$, so you would expect to find 2 species of bats in that cave.

(b) $S = 4 \quad \Rightarrow \quad 4 = 0.7A^{0.3} \quad \Rightarrow \quad \frac{40}{7} = A^{3/10} \quad \Rightarrow \quad A = \left(\frac{40}{7}\right)^{10/3} \approx 333.6$, so we estimate the surface area of the cave to be $334\ \text{m}^2$.

27. (a) Using a computing device, we obtain a power function $N = cA^b$, where $c \approx 3.1046$ and $b \approx 0.308$.

(b) If $A = 291$, then $N = cA^b \approx 17.8$, so you would expect to find 18 species of reptiles and amphibians on Dominica.

28. (a) $T = 1.000\,431\,227d^{\,1.499\,528\,750}$

(b) The power model in part (a) is approximately $T = d^{1.5}$. Squaring both sides gives us $T^2 = d^3$, so the model matches Kepler's Third Law, $T^2 = kd^{\,3}$.

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

2. (a) To obtain the graph of $y = f(x) + 8$ from the graph of $y = f(x)$, shift the graph 8 units upward.

(b) To obtain the graph of $y = f(x + 8)$ from the graph of $y = f(x)$, shift the graph 8 units to the left.

(c) To obtain the graph of $y = 8f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 8.

(d) To obtain the graph of $y = f(8x)$ from the graph of $y = f(x)$, shrink the graph horizontally by a factor of 8.

(e) To obtain the graph of $y = -f(x) - 1$ from the graph of $y = f(x)$, first reflect the graph about the x-axis, and then shift it 1 unit downward.

(f) To obtain the graph of $y = 8f(\frac{1}{8}x)$ from the graph of $y = f(x)$, stretch the graph horizontally and vertically by a factor of 8.

3. (a) (graph 3) The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

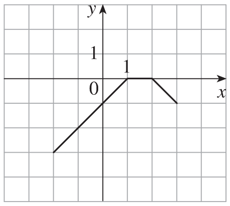
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

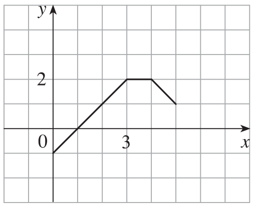
(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x + 4)$.

(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

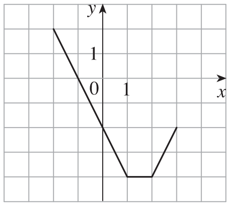
4. (a) To graph $y = f(x) - 2$, we shift the graph of f, 2 units downward.The point $(1, 2)$ on the graph of f corresponds to the point $(1, 2 - 2) = (1, 0)$.



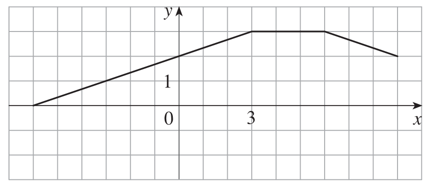
(b) To graph $y = f(x - 2)$, we shift the graph of f, 2 units to the right.The point $(1, 2)$ on the graph of f corresponds to the point $(1 + 2, 2) = (3, 2)$.



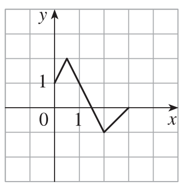
(c) To graph $y = -2f(x)$, we reflect the graph about the x-axis and stretch the graph vertically by a factor of 2. The point $(1, 2)$ on the graph of f corresponds to the point $(1, -2 \cdot 2) = (1, -4)$.



(d) To graph $y = f(\frac{1}{3}x) + 1$, we stretch the graph horizontally by a factor of 3 and shift it 1 unit upward. The point $(1, 2)$ on the graph of f corresponds to the point $(1 \cdot 3, 2 + 1) = (3, 3)$.

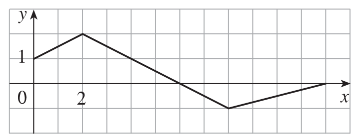


5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



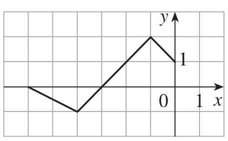
The point $(4, -1)$ on the graph of f corresponds to the point $\left(\frac{1}{2} \cdot 4, -1\right) = (2, -1)$.

(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



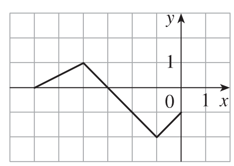
The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

6. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2. Thus, a function describing the graph is

$$y = 2f(x-2) = 2\sqrt{3(x-2) - (x-2)^2} = 2\sqrt{3x - 6 - (x^2 - 4x + 4)} = 2\sqrt{-x^2 + 7x - 10}$$

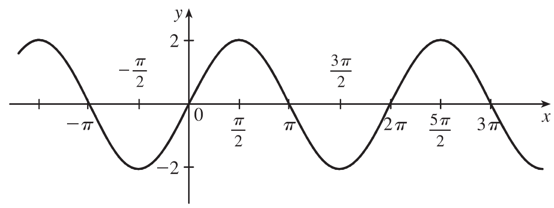
7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x-axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1 \cdot}_{\substack{\text{reflect} \\ \text{about } x\text{-axis}}} \quad f \underbrace{(x+4)}_{\substack{\text{shift} \\ \text{4 units left}}} \quad \underbrace{-\ 1}_{\substack{\text{shift} \\ \text{1 unit left}}}$$

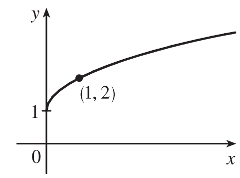
This function can be written as

$$y = -f(x+4) - 1 = -\sqrt{3(x+4) - (x+4)^2} - 1 = -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 = -\sqrt{-x^2 - 5x - 4} - 1$$

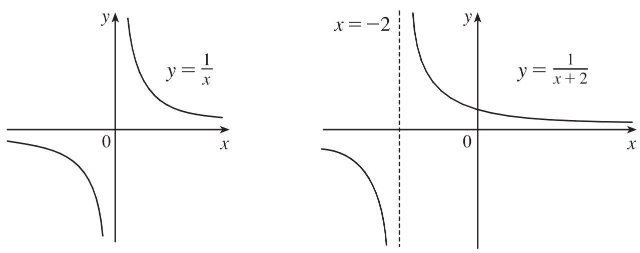
8. (a) The graph of $y = 2\sin x$ can be obtained from the graph of $y = \sin x$ by stretching it vertically by a factor of 2.



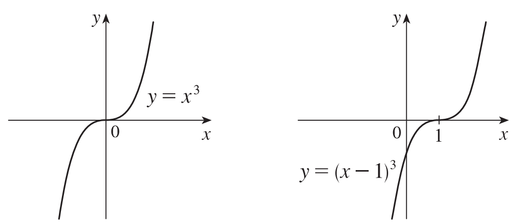
(b) The graph of $y = 1 + \sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting it upward 1 unit.



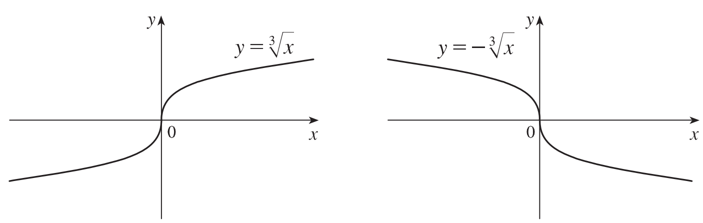
9. $y = \dfrac{1}{x+2}$: Start with the graph of the reciprocal function $y = 1/x$ and shift 2 units to the left.



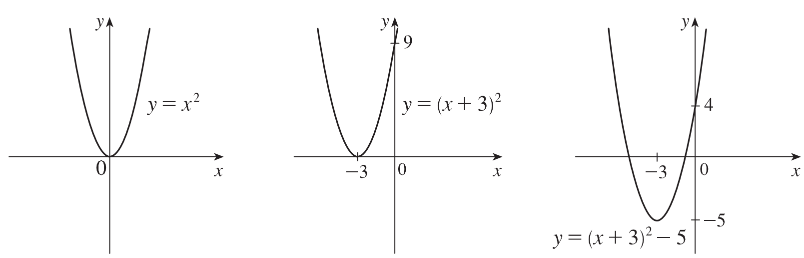
10. $y = (x-1)^3$: Start with the graph of $y = x^3$ and shift 1 unit to the right.



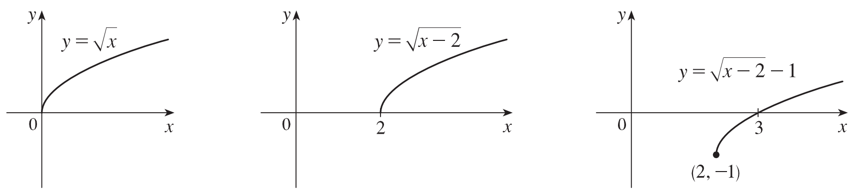
11. $y = -\sqrt[3]{x}$: Start with the graph of $y = \sqrt[3]{x}$ and reflect about the x-axis.



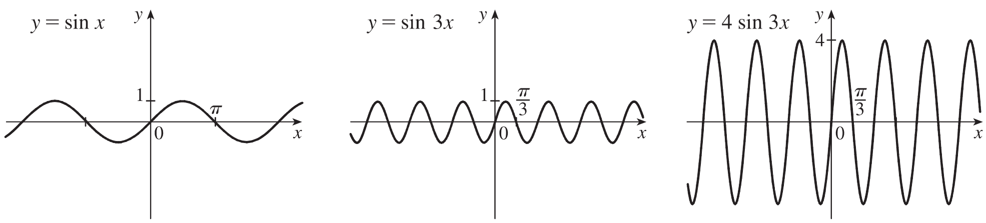
12. $y = x^2 + 6x + 4 = (x^2 + 6x + 9) - 5 = (x+3)^2 - 5$: Start with the graph of $y = x^2$, shift 3 units to the left, and then shift 5 units downward.



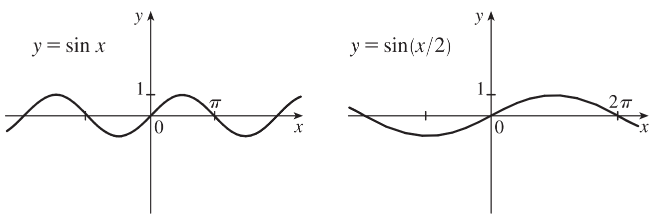
13. $y = \sqrt{x-2} - 1$: Start with the graph of $y = \sqrt{x}$, shift 2 units to the right, and then shift 1 unit downward.



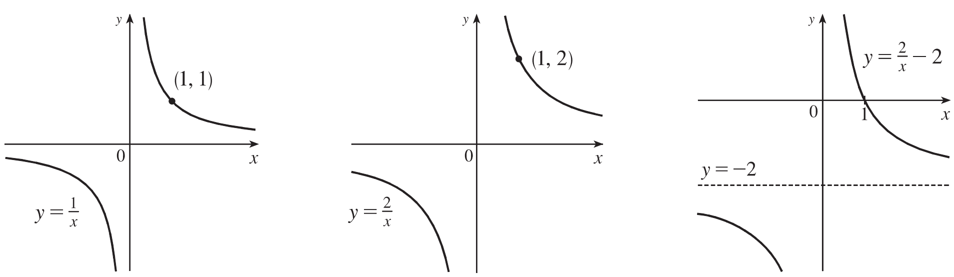
14. $y = 4\sin 3x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 3, and then stretch vertically by a factor of 4.



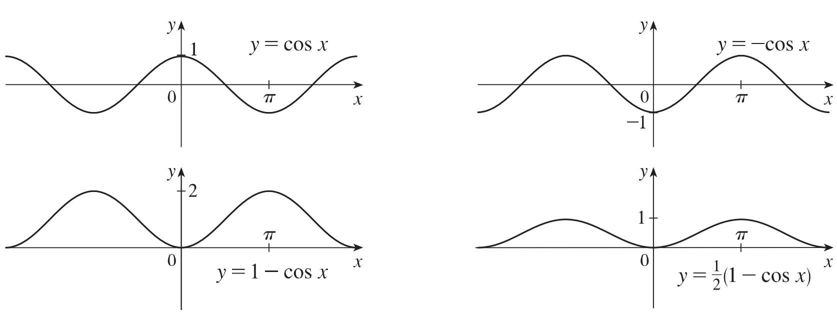
15. $y = \sin(x/2)$: Start with the graph of $y = \sin x$ and stretch horizontally by a factor of 2.



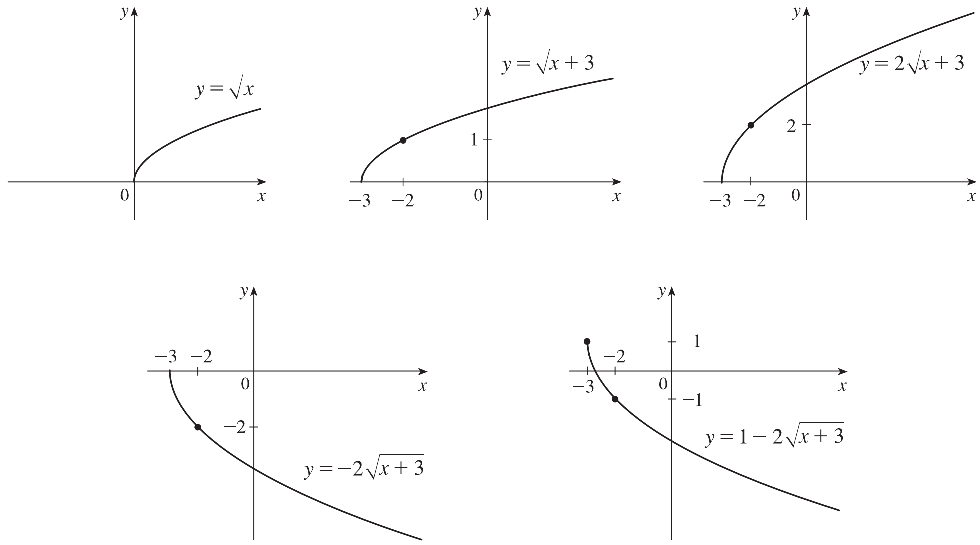
16. $y = \dfrac{2}{x} - 2$: Start with the graph of $y = \dfrac{1}{x}$, stretch vertically by a factor of 2, and then shift 2 units downward.



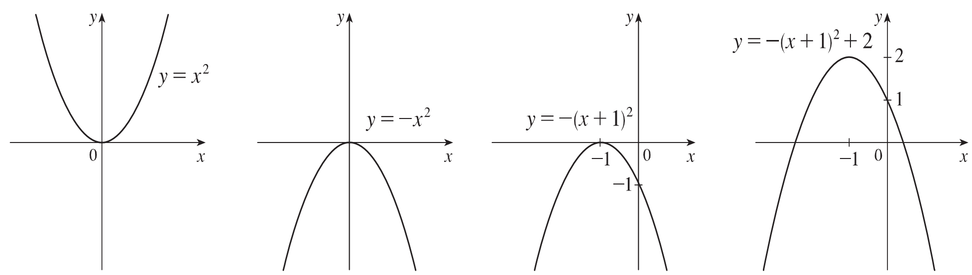
17. $y = \frac{1}{2}(1 - \cos x)$: Start with the graph of $y = \cos x$, reflect about the x-axis, shift 1 unit upward, and then shrink vertically by a factor of 2.



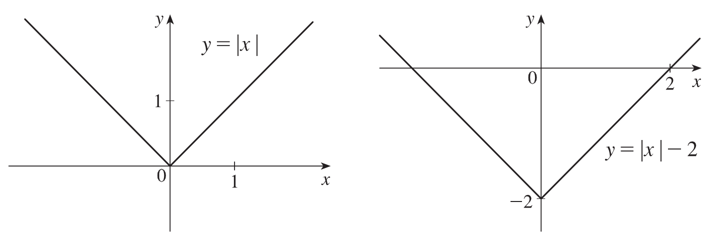
18. $y = 1 - 2\sqrt{x+3}$: Start with the graph of $y = \sqrt{x}$, shift 3 units to the left, stretch vertically by a factor of 2, reflect about the x-axis, and then shift 1 unit upward.



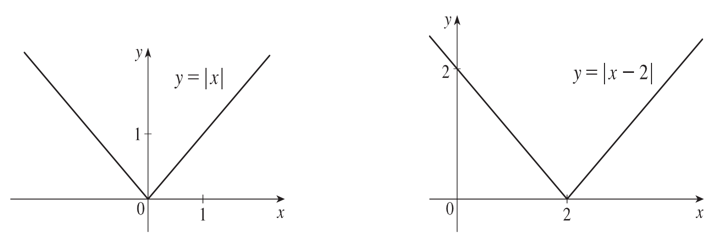
19. $y = 1 - 2x - x^2 = -(x^2 + 2x) + 1 = -(x^2 + 2x + 1) + 2 = -(x+1)^2 + 2$: Start with the graph of $y = x^2$, reflect about the x-axis, shift 1 unit to the left, and then shift 2 units upward.



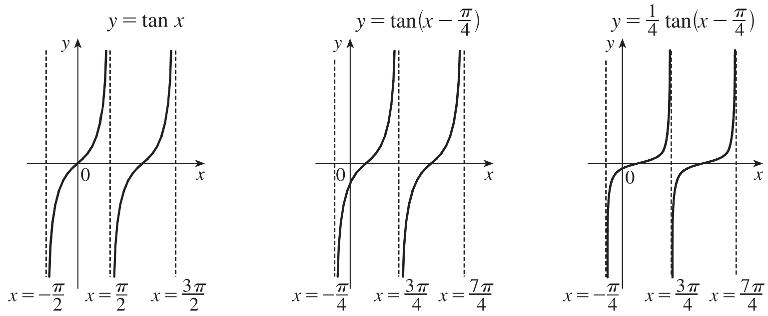
20. $y = |x| - 2$: Start with the graph of $y = |x|$ and shift 2 units downward.



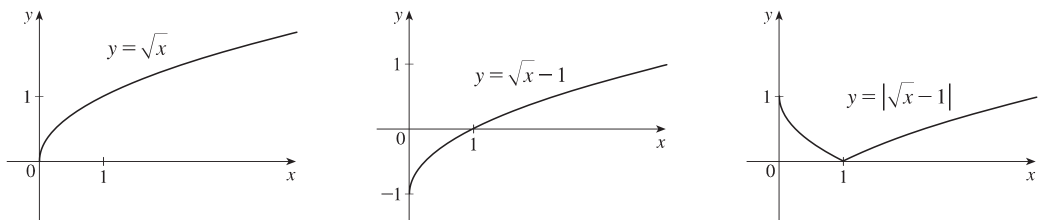
21. $y = |x - 2|$: Start with the graph of $y = |x|$ and shift 2 units to the right.



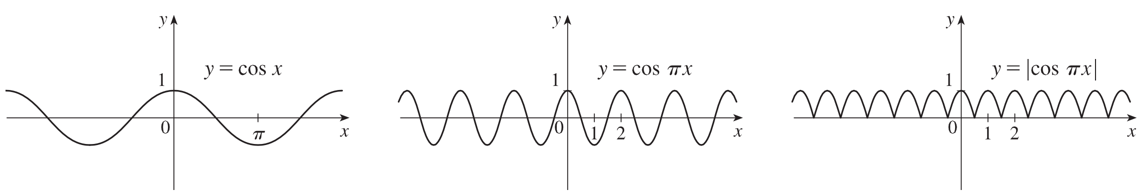
22. $y = \frac{1}{4}\tan(x - \frac{\pi}{4})$: Start with the graph of $y = \tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4.



23. $y = |\sqrt{x} - 1|$: Start with the graph of $y = \sqrt{x}$, shift it 1 unit downward, and then reflect the portion of the graph below the x-axis about the x-axis.



24. $y = |\cos \pi x|$: Start with the graph of $y = \cos x$, shrink it horizontally by a factor of π, and reflect all the parts of the graph below the x-axis about the x-axis.

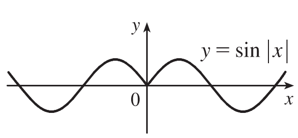


25. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 AM to 6:18 PM) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

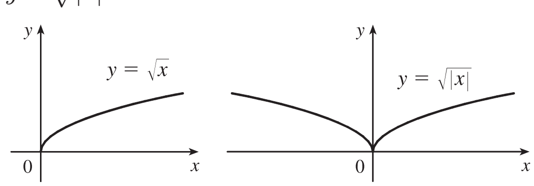
26. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0. If we take $t = 0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t) = 4.0 + 0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

27. (a) To obtain $y = f(|x|)$, the portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

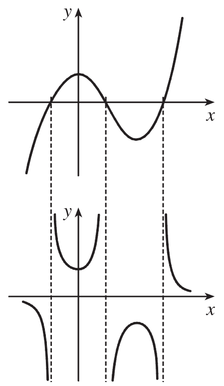
(b) $y = \sin|x|$



(c) $y = \sqrt{|x|}$



28. The most important features of the given graph are the x-intercepts and the maximum and minimum points. The graph of $y = 1/f(x)$ has vertical asymptotes at the x-values where there are x-intercepts on the graph of $y = f(x)$. The maximum of 1 on the graph of $y = f(x)$ corresponds to a minimum of $1/1 = 1$ on $y = 1/f(x)$. Similarly, the minimum on the graph of $y = f(x)$ corresponds to a maximum on the graph of $y = 1/f(x)$. As the values of y get large (positively or negatively) on the graph of $y = f(x)$, the values of y get close to zero on the graph of $y = 1/f(x)$.



29. $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$. $D = \mathbb{R}$ for both f and g.

(a) $(f + g)(x) = (x^3 + 2x^2) + (3x^2 - 1) = x^3 + 5x^2 - 1$, $D = \mathbb{R}$.

(b) $(f - g)(x) = (x^3 + 2x^2) - (3x^2 - 1) = x^3 - x^2 + 1$, $D = \mathbb{R}$.

(c) $(fg)(x) = (x^3 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2$, $D = \mathbb{R}$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1}$, $D = \left\{x \mid x \neq \pm\dfrac{1}{\sqrt{3}}\right\}$ since $3x^2 - 1 \neq 0$.

30. $f(x) = \sqrt{3 - x}$, $D = (-\infty, 3]$; $g(x) = \sqrt{x^2 - 1}$, $D = (-\infty, -1] \cup [1, \infty)$.

(a) $(f + g)(x) = \sqrt{3 - x} + \sqrt{x^2 - 1}$, $D = (-\infty, -1] \cup [1, 3]$, which is the intersection of the domains of f and g.

(b) $(f - g)(x) = \sqrt{3 - x} - \sqrt{x^2 - 1}$, $D = (-\infty, -1] \cup [1, 3]$.

(c) $(fg)(x) = \sqrt{3 - x} \cdot \sqrt{x^2 - 1}$, $D = (-\infty, -1] \cup [1, 3]$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{3 - x}}{\sqrt{x^2 - 1}}$, $D = (-\infty, -1) \cup (1, 3]$. We must exclude $x = \pm 1$ since these values would make $\dfrac{f}{g}$ undefined.

31. $f(x) = x^2 - 1$, $D = \mathbb{R}$; $g(x) = 2x + 1$, $D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 - 1 = (4x^2 + 4x + 1) - 1 = 4x^2 + 4x$, $D = \mathbb{R}$.

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 - 1) = 2(x^2 - 1) + 1 = (2x^2 - 2) + 1 = 2x^2 - 1$, $D = \mathbb{R}$.

(c) $(f \circ f)(x) = f(f(x)) = f(x^2 - 1) = (x^2 - 1)^2 - 1 = (x^4 - 2x^2 + 1) - 1 = x^4 - 2x^2$, $D = \mathbb{R}$.

(d) $(g \circ g)(x) = g(g(x)) = g(2x + 1) = 2(2x + 1) + 1 = (4x + 2) + 1 = 4x + 3$, $D = \mathbb{R}$.

32. $f(x) = x - 2$; $g(x) = x^2 + 3x + 4$. $D = \mathbb{R}$ for both f and g, and hence for their composites.

(a) $(f \circ g)(x) = f(g(x)) = f(x^2 + 3x + 4) = (x^2 + 3x + 4) - 2 = x^2 + 3x + 2$.

(b) $(g \circ f)(x) = g(f(x)) = g(x - 2) = (x - 2)^2 + 3(x - 2) + 4 = x^2 - 4x + 4 + 3x - 6 + 4 = x^2 - x + 2$.

(c) $(f \circ f)(x) = f(f(x)) = f(x - 2) = (x - 2) - 2 = x - 4$.

(d)
$$\begin{aligned}(g \circ g)(x) = g(g(x)) = g(x^2 + 3x + 4) &= (x^2 + 3x + 4)^2 + 3(x^2 + 3x + 4) + 4 \\ &= (x^4 + 9x^2 + 16 + 6x^3 + 8x^2 + 24x) + 3x^2 + 9x + 12 + 4 \\ &= x^4 + 6x^3 + 20x^2 + 33x + 32\end{aligned}$$

33. $f(x) = 1 - 3x$; $g(x) = \cos x$. $D = \mathbb{R}$ for both f and g, and hence for their composites.

(a) $(f \circ g)(x) = f(g(x)) = f(\cos x) = 1 - 3\cos x$.

(b) $(g \circ f)(x) = g(f(x)) = g(1 - 3x) = \cos(1 - 3x)$.

(c) $(f \circ f)(x) = f(f(x)) = f(1 - 3x) = 1 - 3(1 - 3x) = 1 - 3 + 9x = 9x - 2$.

(d) $(g \circ g)(x) = g(g(x)) = g(\cos x) = \cos(\cos x)$ [Note that this is *not* $\cos x \cdot \cos x$.]

34. $f(x) = \sqrt{x}$, $D = [0, \infty)$; $g(x) = \sqrt[3]{1 - x}$, $D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{1 - x}\right) = \sqrt{\sqrt[3]{1 - x}} = \sqrt[6]{1 - x}$.

The domain of $f \circ g$ is $\{x \mid \sqrt[3]{1 - x} \geq 0\} = \{x \mid 1 - x \geq 0\} = \{x \mid x \leq 1\} = (-\infty, 1]$.

(b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt[3]{1 - \sqrt{x}}$.

The domain of $g \circ f$ is $\{x \mid x$ is in the domain of f and $f(x)$ is in the domain of $g\}$. This is the domain of f, that is, $[0, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. The domain of $f \circ f$ is $\{x \mid x \geq 0$ and $\sqrt{x} \geq 0\} = [0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g\left(\sqrt[3]{1 - x}\right) = \sqrt[3]{1 - \sqrt[3]{1 - x}}$, and the domain is $(-\infty, \infty)$.

35. $f(x) = x + \dfrac{1}{x}$, $D = \{x \mid x \neq 0\}$; $g(x) = \dfrac{x + 1}{x + 2}$, $D = \{x \mid x \neq -2\}$

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\frac{x + 1}{x + 2}\right) = \frac{x + 1}{x + 2} + \frac{1}{\frac{x + 1}{x + 2}} = \frac{x + 1}{x + 2} + \frac{x + 2}{x + 1}$$

$$= \frac{(x + 1)(x + 1) + (x + 2)(x + 2)}{(x + 2)(x + 1)} = \frac{\left(x^2 + 2x + 1\right) + \left(x^2 + 4x + 4\right)}{(x + 2)(x + 1)} = \frac{2x^2 + 6x + 5}{(x + 2)(x + 1)}$$

Since $g(x)$ is not defined for $x = -2$ and $f(g(x))$ is not defined for $x = -2$ and $x = -1$, the domain of $(f \circ g)(x)$ is $D = \{x \mid x \neq -2, -1\}$.

(b) $$(g \circ f)(x) = g(f(x)) = g\left(x + \frac{1}{x}\right) = \frac{\left(x + \frac{1}{x}\right) + 1}{\left(x + \frac{1}{x}\right) + 2} = \frac{\frac{x^2 + 1 + x}{x}}{\frac{x^2 + 1 + 2x}{x}} = \frac{x^2 + x + 1}{x^2 + 2x + 1} = \frac{x^2 + x + 1}{(x + 1)^2}$$

Since $f(x)$ is not defined for $x = 0$ and $g(f(x))$ is not defined for $x = -1$, the domain of $(g \circ f)(x)$ is $D = \{x \mid x \neq -1, 0\}$.

(c) $$(f \circ f)(x) = f(f(x)) = f\left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right) + \frac{1}{x + \frac{1}{x}} = x + \frac{1}{x} + \frac{1}{\frac{x^2 + 1}{x}} = x + \frac{1}{x} + \frac{x}{x^2 + 1}$$

$$= \frac{x(x)\left(x^2 + 1\right) + 1\left(x^2 + 1\right) + x(x)}{x(x^2 + 1)} = \frac{x^4 + x^2 + x^2 + 1 + x^2}{x(x^2 + 1)}$$

$$= \frac{x^4 + 3x^2 + 1}{x(x^2 + 1)}, \quad D = \{x \mid x \neq 0\}$$

(d) $(g \circ g)(x) = g(g(x)) = g\left(\dfrac{x+1}{x+2}\right) = \dfrac{\dfrac{x+1}{x+2}+1}{\dfrac{x+1}{x+2}+2} = \dfrac{\dfrac{x+1+1(x+2)}{x+2}}{\dfrac{x+1+2(x+2)}{x+2}} = \dfrac{x+1+x+2}{x+1+2x+4} = \dfrac{2x+3}{3x+5}$

Since $g(x)$ is not defined for $x = -2$ and $g(g(x))$ is not defined for $x = -\frac{5}{3}$,

the domain of $(g \circ g)(x)$ is $D = \left\{x \mid x \neq -2, -\frac{5}{3}\right\}$.

36. $f(x) = \dfrac{x}{1+x}$, $D = \{x \mid x \neq -1\}$; $g(x) = \sin 2x$, $D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(\sin 2x) = \dfrac{\sin 2x}{1 + \sin 2x}$

Domain: $1 + \sin 2x \neq 0 \Rightarrow \sin 2x \neq -1 \Rightarrow 2x \neq \dfrac{3\pi}{2} + 2\pi n \Rightarrow x \neq \dfrac{3\pi}{4} + \pi n$ [n an integer].

(b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{x}{1+x}\right) = \sin\left(\dfrac{2x}{1+x}\right)$.

Domain: $\{x \mid x \neq -1\}$

(c) $(f \circ f)(x) = f(f(x)) = f\left(\dfrac{x}{1+x}\right) = \dfrac{\dfrac{x}{1+x}}{1+\dfrac{x}{1+x}} = \dfrac{\left(\dfrac{x}{1+x}\right)\cdot(1+x)}{\left(1+\dfrac{x}{1+x}\right)\cdot(1+x)} = \dfrac{x}{1+x+x} = \dfrac{x}{2x+1}$

Since $f(x)$ is not defined for $x = -1$, and $f(f(x))$ is not defined for $x = -\frac{1}{2}$,

the domain of $(f \circ f)(x)$ is $D = \{x \mid x \neq -1, -\frac{1}{2}\}$.

(d) $(g \circ g)(g) = g(g(x)) = g(\sin 2x) = \sin(2 \sin 2x)$.

Domain: $\mathbb{R}$

37. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f(\sin(x^2)) = 3\sin(x^2) - 2$

38. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(2^{\sqrt{x}}) = \left|2^{\sqrt{x}} - 4\right|$

39. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3+2)) = f[(x^3+2)^2]$
$= f(x^6 + 4x^3 + 4) = \sqrt{(x^6 + 4x^3 + 4) - 3} = \sqrt{x^6 + 4x^3 + 1}$

40. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt[3]{x})) = f\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right) = \tan\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right)$

41. Let $g(x) = 2x + x^2$ and $f(x) = x^4$. Then $(f \circ g)(x) = f(g(x)) = f(2x + x^2) = (2x + x^2)^4 = F(x)$.

42. Let $g(x) = \cos x$ and $f(x) = x^2$. Then $(f \circ g)(x) = f(g(x)) = f(\cos x) = (\cos x)^2 = \cos^2 x = F(x)$.

43. Let $g(x) = \sqrt[3]{x}$ and $f(x) = \dfrac{x}{1+x}$. Then $(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x}) = \dfrac{\sqrt[3]{x}}{1+\sqrt[3]{x}} = F(x)$.

44. Let $g(x) = \dfrac{x}{1+x}$ and $f(x) = \sqrt[3]{x}$. Then $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x}{1+x}\right) = \sqrt[3]{\dfrac{x}{1+x}} = G(x)$.

45. Let $g(t) = t^2$ and $f(t) = \sec t \tan t$. Then $(f \circ g)(t) = f(g(t)) = f(t^2) = \sec(t^2)\tan(t^2) = v(t)$.

46. Let $g(t) = \tan t$ and $f(t) = \dfrac{t}{1+t}$. Then $(f \circ g)(t) = f(g(t)) = f(\tan t) = \dfrac{\tan t}{1 + \tan t} = u(t)$.

47. Let $h(x) = \sqrt{x}$, $g(x) = x - 1$, and $f(x) = \sqrt{x}$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(\sqrt{x} - 1) = \sqrt{\sqrt{x} - 1} = R(x)$.

48. Let $h(x) = |x|$, $g(x) = 2 + x$, and $f(x) = \sqrt[8]{x}$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(|x|)) = f(2 + |x|) = \sqrt[8]{2 + |x|} = H(x)$.

49. Let $h(x) = \sqrt{x}$, $g(x) = \sec x$, and $f(x) = x^4$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(\sec \sqrt{x}) = (\sec \sqrt{x})^4 = \sec^4(\sqrt{x}) = H(x)$.

50. (a) $f(g(1)) = f(6) = 5$ (b) $g(f(1)) = g(3) = 2$
(c) $f(f(1)) = f(3) = 4$ (d) $g(g(1)) = g(6) = 3$
(e) $(g \circ f)(3) = g(f(3)) = g(4) = 1$ (f) $(f \circ g)(6) = f(g(6)) = f(3) = 4$

51. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of g. Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of f.

(b) $g(f(0)) = g(0) = 3$

(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$

(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

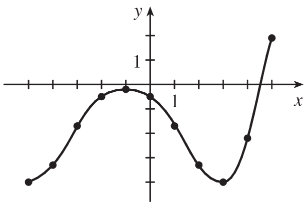
(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

52. To find a particular value of $f(g(x))$, say for $x = 0$, we note from the graph that $g(0) \approx 2.8$ and $f(2.8) \approx -0.5$. Thus, $f(g(0)) \approx f(2.8) \approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
-5	-0.2	-4
-4	1.2	-3.3
-3	2.2	-1.7
-2	2.8	-0.5
-1	3	-0.2

x	$g(x)$	$f(g(x))$
0	2.8	-0.5
1	2.2	-1.7
2	1.2	-3.3
3	-0.2	-4
4	-1.9	-2.2
5	-4.1	1.9



53. (a) Using the relationship *distance* = *rate* · *time* with the radius r as the distance, we have $r(t) = 60t$.

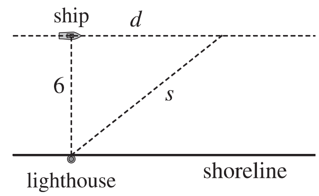
(b) $A = \pi r^2 \Rightarrow (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

54. (a) The radius r of the balloon is increasing at a rate of 2 cm/s, so $r(t) = (2 \text{ cm/s})(t \text{ s}) = 2t$ (in cm).

(b) Using $V = \frac{4}{3}\pi r^3$, we get $(V \circ r)(t) = V(r(t)) = V(2t) = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3$.

The result, $V = \frac{32}{3}\pi t^3$, gives the volume of the balloon (in cm^3) as a function of time (in s).

55. (a) From the figure, we have a right triangle with legs 6 and d, and hypotenuse s.
By the Pythagorean Theorem, $d^2 + 6^2 = s^2 \quad \Rightarrow \quad s = f(d) = \sqrt{d^2 + 36}$.



(b) Using $d = rt$, we get $d = (30 \text{ km/h})(t \text{ hours}) = 30t$ (in km). Thus, $d = g(t) = 30t$.

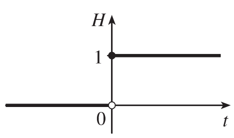
(c) $(f \circ g)(t) = f(g(t)) = f(30t) = \sqrt{(30t)^2 + 36} = \sqrt{900t^2 + 36}$. This function represents the distance between the lighthouse and the ship as a function of the time elapsed since noon.

56. (a) $d = rt \quad \Rightarrow \quad d(t) = 350t$

(b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s: $d^2 + 1^2 = s^2$. Thus, $s(d) = \sqrt{d^2 + 1}$.

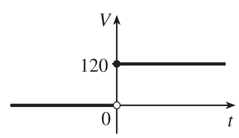
(c) $(s \circ d)(t) = s(d(t)) = s(350t) = \sqrt{(350t)^2 + 1}$

57. (a)



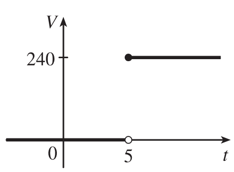
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

(b)



$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases} \quad \text{so } V(t) = 120H(t).$$

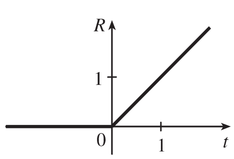
(c)



Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t - 5)$.

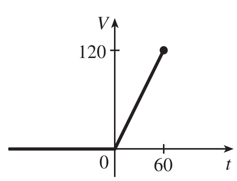
58. (a) $R(t) = tH(t)$

$$= \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$



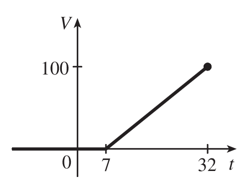
(b) $V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } 0 \leq t \leq 60 \end{cases}$

so $V(t) = 2tH(t)$, $t \leq 60$.



(c) $V(t) = \begin{cases} 0 & \text{if } t < 7 \\ 4\,(t - 7) & \text{if } 7 \leq t \leq 32 \end{cases}$

so $V(t) = 4(t - 7)H(t - 7)$, $t \leq 32$.



59. If $f(x) = m_1 x + b_1$ and $g(x) = m_2 x + b_2$, then

$(f \circ g)(x) = f(g(x)) = f(m_2 x + b_2) = m_1(m_2 x + b_2) + b_1 = m_1 m_2 x + m_1 b_2 + b_1$.

So $f \circ g$ is a linear function with slope $m_1 m_2$.

60. If $A(x) = 1.04x$, then

$(A \circ A)(x) = A(A(x)) = A(1.04x) = 1.04(1.04x) = (1.04)^2 x$,

$(A \circ A \circ A)(x) = A((A \circ A)(x)) = A((1.04)^2 x) = 1.04(1.04)^2 x = (1.04)^3 x$, and

$(A \circ A \circ A \circ A)(x) = A((A \circ A \circ A)(x)) = A((1.04)^3 x) = 1.04(1.04)^3 x, = (1.04)^4 x$.

These compositions represent the amount of the investment after 2, 3, and 4 years.

Based on this pattern, when we compose n copies of A, we get the formula $\underbrace{(A \circ A \circ \cdots \circ A)}_{n\ A's}(x) = (1.04)^n x$.

61. (a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 + c = 4x^2 + 4x + (1+c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But $h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$, so we see that $g(x) = x^2 + x - 1$.

62. We need a function g so that $g(f(x)) = g(x+4) = h(x) = 4x - 1 = 4(x+4) - 17$. So we see that the function g must be $g(x) = 4x - 17$.

63. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad [\text{because } g \text{ is even}] \quad = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

64. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

Now suppose f is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so h is odd if both f and g are odd.

Now suppose f is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so h is even if g is odd and f is even.

1.4 The Tangent and Velocity Problems

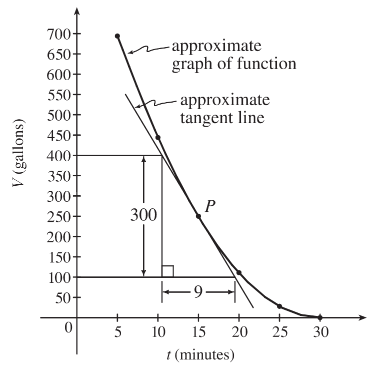
1. (a) Using $P(15, 250)$, we construct the following table:

t	Q	slope $= m_{PQ}$
5	$(5, 694)$	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	$(10, 444)$	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	$(20, 111)$	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	$(25, 28)$	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	$(30, 0)$	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9} = -33.\overline{3}$.



2. (a) Slope $= \frac{2948-2530}{42-36} = \frac{418}{6} \approx 69.67$

(b) Slope $= \frac{2948-2661}{42-38} = \frac{287}{4} = 71.75$

(c) Slope $= \frac{2948-2806}{42-40} = \frac{142}{2} = 71$

(d) Slope $= \frac{3080-2948}{44-42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats/minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. (a) $y = \dfrac{1}{1-x}$, $P(2, -1)$

	x	$Q(x, 1/(1-x))$	m_{PQ}
(i)	1.5	$(1.5, -2)$	2
(ii)	1.9	$(1.9, -1.111\,111)$	1.111 111
(iii)	1.99	$(1.99, -1.010\,101)$	1.010 101
(iv)	1.999	$(1.999, -1.001\,001)$	1.001 001
(v)	2.5	$(2.5, -0.666\,667)$	0.666 667
(vi)	2.1	$(2.1, -0.909\,091)$	0.909 091
(vii)	2.01	$(2.01, -0.990\,099)$	0.990 099
(viii)	2.001	$(2.001, -0.999\,001)$	0.999 001

(b) The slope appears to be 1.

(c) Using $m = 1$, an equation of the tangent line to the curve at $P(2, -1)$ is $y - (-1) = 1(x - 2)$, or $y = x - 3$.

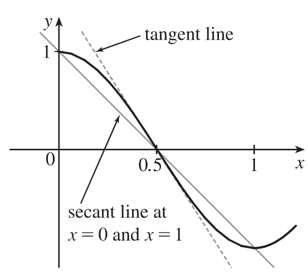
4. (a) $y = \cos \pi x$, $P(0.5, 0)$

	x	Q	m_{PQ}
(i)	0	$(0, 1)$	-2
(ii)	0.4	$(0.4, 0.309017)$	-3.090170
(iii)	0.49	$(0.49, 0.031411)$	-3.141076
(iv)	0.499	$(0.499, 0.003142)$	-3.141587
(v)	1	$(1, -1)$	-2
(vi)	0.6	$(0.6, -0.309017)$	-3.090170
(vii)	0.51	$(0.51, -0.031411)$	-3.141076
(viii)	0.501	$(0.501, -0.003142)$	-3.141587

(b) The slope appears to be $-\pi$.

(c) $y - 0 = -\pi(x - 0.5)$ or $y = -\pi x + \frac{1}{2}\pi$.

(d)



5. (a) $y = y(t) = 40t - 16t^2$. At $t = 2$, $y = 40(2) - 16(2)^2 = 16$. The average velocity between times 2 and $2 + h$ is

$$v_{\text{ave}} = \frac{y(2+h) - y(2)}{(2+h) - 2} = \frac{\left[40(2+h) - 16(2+h)^2\right] - 16}{h} = \frac{-24h - 16h^2}{h} = -24 - 16h, \text{ if } h \neq 0.$$

(i) $[2, 2.5]$: $h = 0.5$, $v_{\text{ave}} = -32$ ft/s

(ii) $[2, 2.1]$: $h = 0.1$, $v_{\text{ave}} = -25.6$ ft/s

(iii) $[2, 2.05]$: $h = 0.05$, $v_{\text{ave}} = -24.8$ ft/s

(iv) $[2, 2.01]$: $h = 0.01$, $v_{\text{ave}} = -24.16$ ft/s

(b) The instantaneous velocity when $t = 2$ (h approaches 0) is -24 ft/s.

6. (a) $y = y(t) = 10t - 1.86t^2$. At $t = 1$, $y = 10(1) - 1.86(1)^2 = 8.14$. The average velocity between times 1 and $1 + h$ is

$$v_{\text{ave}} = \frac{y(1+h) - y(1)}{(1+h) - 1} = \frac{\left[10(1+h) - 1.86(1+h)^2\right] - 8.14}{h} = \frac{6.28h - 1.86h^2}{h} = 6.28 - 1.86h, \text{ if } h \neq 0.$$

(i) $[1, 2]$: $h = 1$, $v_{\text{ave}} = 4.42$ m/s

(ii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = 5.35$ m/s

(iii) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = 6.094$ m/s

(iv) $[1, 1.01]$: $h = 0.01$, $v_{\text{ave}} = 6.2614$ m/s

(v) $[1, 1.001]$: $h = 0.001$, $v_{\text{ave}} = 6.27814$ m/s

(b) The instantaneous velocity when $t = 1$ (h approaches 0) is 6.28 m/s.

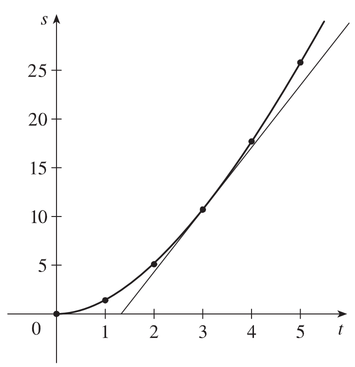
7. (a) (i) On the interval $[1, 3]$, $v_{\text{ave}} = \dfrac{s(3) - s(1)}{3 - 1} = \dfrac{10.7 - 1.4}{2} = \dfrac{9.3}{2} = 4.65$ m/s.

(ii) On the interval $[2, 3]$, $v_{\text{ave}} = \dfrac{s(3) - s(2)}{3 - 2} = \dfrac{10.7 - 5.1}{1} = 5.6$ m/s.

(iii) On the interval $[3, 5]$, $v_{\text{ave}} = \dfrac{s(5) - s(3)}{5 - 3} = \dfrac{25.8 - 10.7}{2} = \dfrac{15.1}{2} = 7.55$ m/s.

(iv) On the interval $[3, 4]$, $v_{\text{ave}} = \dfrac{s(4) - s(3)}{4 - 3} = \dfrac{17.7 - 10.7}{1} = 7$ m/s.

(b)



Using the points $(2, 4)$ and $(5, 23)$ from the approximate tangent line, the instantaneous velocity at $t = 3$ is about $\dfrac{23 - 4}{5 - 2} \approx 6.3$ m/s.

8. (a) (i) $s = s(t) = 2\sin \pi t + 3\cos \pi t$. On the interval $[1, 2]$, $v_{\text{ave}} = \dfrac{s(2) - s(1)}{2 - 1} = \dfrac{3 - (-3)}{1} = 6$ cm/s.

(ii) On the interval $[1, 1.1]$, $v_{\text{ave}} = \dfrac{s(1.1) - s(1)}{1.1 - 1} \approx \dfrac{-3.471 - (-3)}{0.1} = -4.71$ cm/s.

(iii) On the interval $[1, 1.01]$, $v_{\text{ave}} = \dfrac{s(1.01) - s(1)}{1.01 - 1} \approx \dfrac{-3.0613 - (-3)}{0.01} = -6.13$ cm/s.

(iv) On the interval $[1, 1.001]$, $v_{\text{ave}} = \dfrac{s(1.001) - s(1)}{1.001 - 1} \approx \dfrac{-3.00627 - (-3)}{1.001 - 1} = -6.27$ cm/s.

(b) The instantaneous velocity of the particle when $t = 1$ appears to be about -6.3 cm/s.

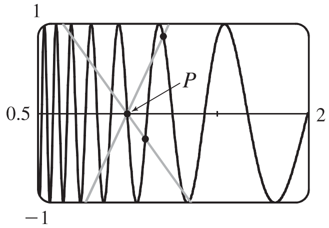
9. (a) For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

x	Q	m_{PQ}
2	$(2, 0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	-1.0847
1.3	$(1.3, -0.8230)$	-2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	-2.8173

x	Q	m_{PQ}
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	-2.1651
0.7	$(0.7, 0.7818)$	-2.6061
0.8	$(0.8, 1)$	-5
0.9	$(0.9, -0.3420)$	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108. So we estimate that the slope of the tangent line at P is about -31.4.

1.5 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $\lim\limits_{x \to -3} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to -3 (but not equal to -3).

(b) $\lim\limits_{x \to 4^+} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 4 through values larger than 4.

4. (a) As x approaches 2 from the left, the values of $f(x)$ approach 3, so $\lim_{x\to 2^-} f(x) = 3$.

   (b) As x approaches 2 from the right, the values of $f(x)$ approach 1, so $\lim_{x\to 2^+} f(x) = 1$.

   (c) $\lim_{x\to 2} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

   (d) When $x = 2$, $y = 3$, so $f(2) = 3$.

   (e) As x approaches 4, the values of $f(x)$ approach 4, so $\lim_{x\to 4} f(x) = 4$.

   (f) There is no value of $f(x)$ when $x = 4$, so $f(4)$ does not exist.

5. (a) As x approaches 1, the values of $f(x)$ approach 2, so $\lim_{x\to 1} f(x) = 2$.

   (b) As x approaches 3 from the left, the values of $f(x)$ approach 1, so $\lim_{x\to 3^-} f(x) = 1$.

   (c) As x approaches 3 from the right, the values of $f(x)$ approach 4, so $\lim_{x\to 3^+} f(x) = 4$.

   (d) $\lim_{x\to 3} f(x)$ does not exist since the left-hand limit does not equal the right-hand limit.

   (e) When $x = 3$, $y = 3$, so $f(3) = 3$.

6. (a) $h(x)$ approaches 4 as x approaches -3 from the left, so $\lim_{x\to -3^-} h(x) = 4$.

   (b) $h(x)$ approaches 4 as x approaches -3 from the right, so $\lim_{x\to -3^+} h(x) = 4$.

   (c) $\lim_{x\to -3} h(x) = 4$ because the limits in part (a) and part (b) are equal.

   (d) $h(-3)$ is not defined, so it doesn't exist.

   (e) $h(x)$ approaches 1 as x approaches 0 from the left, so $\lim_{x\to 0^-} h(x) = 1$.

   (f) $h(x)$ approaches -1 as x approaches 0 from the right, so $\lim_{x\to 0^+} h(x) = -1$.

   (g) $\lim_{x\to 0} h(x)$ does not exist because the limits in part (e) and part (f) are not equal.

   (h) $h(0) = 1$ since the point $(0, 1)$ is on the graph of h.

   (i) Since $\lim_{x\to 2^-} h(x) = 2$ and $\lim_{x\to 2^+} h(x) = 2$, we have $\lim_{x\to 2} h(x) = 2$.

   (j) $h(2)$ is not defined, so it doesn't exist.

   (k) $h(x)$ approaches 3 as x approaches 5 from the right, so $\lim_{x\to 5^+} h(x) = 3$.

   (l) $h(x)$ does not approach any one number as x approaches 5 from the left, so $\lim_{x\to 5^-} h(x)$ does not exist.

7. (a) $\lim_{t\to 0^-} g(t) = -1$ (b) $\lim_{t\to 0^+} g(t) = -2$

   (c) $\lim_{t\to 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.

   (d) $\lim_{t\to 2^-} g(t) = 2$ (e) $\lim_{t\to 2^+} g(t) = 0$

   (f) $\lim_{t\to 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.

   (g) $g(2) = 1$ (h) $\lim_{t\to 4} g(t) = 3$

8. (a) $\lim_{x\to 2} R(x) = -\infty$ (b) $\lim_{x\to 5} R(x) = \infty$

(c) $\lim_{x\to -3^-} R(x) = -\infty$ (d) $\lim_{x\to -3^+} R(x) = \infty$

(e) The equations of the vertical asymptotes are $x = -3$, $x = 2$, and $x = 5$.

9. (a) $\lim_{x\to -7} f(x) = -\infty$ (b) $\lim_{x\to -3} f(x) = \infty$ (c) $\lim_{x\to 0} f(x) = \infty$

(d) $\lim_{x\to 6^-} f(x) = -\infty$ (e) $\lim_{x\to 6^+} f(x) = \infty$

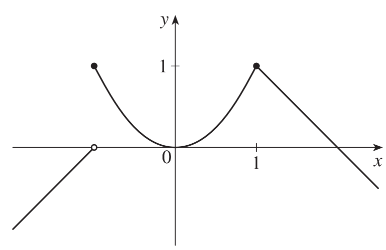
(f) The equations of the vertical asymptotes are $x = -7$, $x = -3$, $x = 0$, and $x = 6$.

10. $\lim_{t\to 12^-} f(t) = 150$ mg and $\lim_{t\to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

11. From the graph of

$$f(x) = \begin{cases} 1 + x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 1, \\ 2 - x & \text{if } x \ge 1 \end{cases}$$

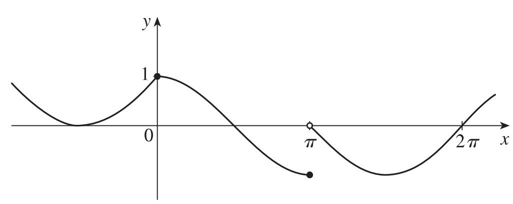
we see that $\lim_{x\to a} f(x)$ exists for all a except $a = -1$. Notice that the right and left limits are different at $a = -1$.



12. From the graph of

$$f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \le x \le \pi, \\ \sin x & \text{if } x > \pi \end{cases}$$

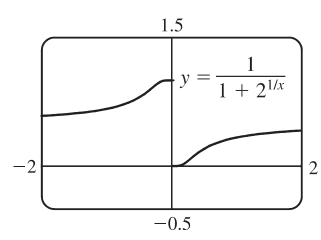
we see that $\lim_{x\to a} f(x)$ exists for all a except $a = \pi$. Notice that the right and left limits are different at $a = \pi$.



13. (a) $\lim_{x\to 0^-} f(x) = 1$

(b) $\lim_{x\to 0^+} f(x) = 0$

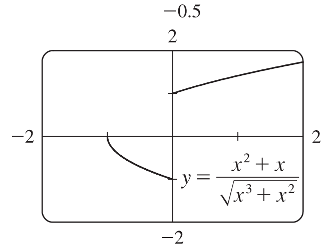
(c) $\lim_{x\to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.



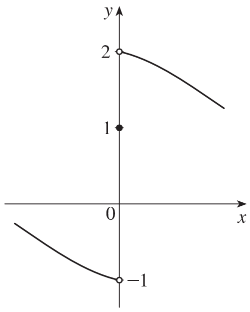
14. (a) $\lim_{x\to 0^-} f(x) = -1$

(b) $\lim_{x\to 0^+} f(x) = 1$

(c) $\lim_{x\to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

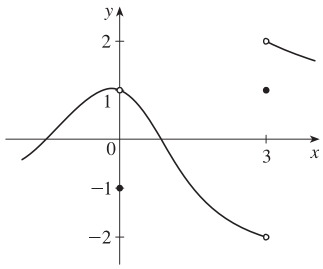


15. $\lim\limits_{x\to 0^-} f(x) = -1, \quad \lim\limits_{x\to 0^+} f(x) = 2, \quad f(0) = 1$



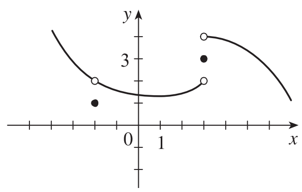
16. $\lim\limits_{x\to 0} f(x) = 1,\ \lim\limits_{x\to 3^-} f(x) = -2,\ \lim\limits_{x\to 3^+} f(x) = 2,$

$f(0) = -1,\ f(3) = 1$



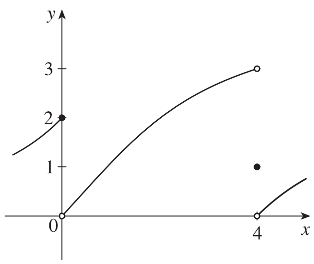
17. $\lim\limits_{x\to 3^+} f(x) = 4, \quad \lim\limits_{x\to 3^-} f(x) = 2,\ \lim\limits_{x\to -2} f(x) = 2,$

$f(3) = 3, \quad f(-2) = 1$



18. $\lim\limits_{x\to 0^-} f(x) = 2,\ \lim\limits_{x\to 0^+} f(x) = 0,\ \lim\limits_{x\to 4^-} f(x) = 3,$

$\lim\limits_{x\to 4^+} f(x) = 0,\ f(0) = 2,\ f(4) = 1$



19. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
2.5	0.714286
2.1	0.677419
2.05	0.672131
2.01	0.667774
2.005	0.667221
2.001	0.666778

x	$f(x)$
1.9	0.655172
1.95	0.661017
1.99	0.665552
1.995	0.666110
1.999	0.666556

It appears that $\lim\limits_{x\to 2} \dfrac{x^2 - 2x}{x^2 - x - 2} = 0.\bar{6} = \frac{2}{3}$.

20. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
0	0
−0.5	−1
−0.9	−9
−0.95	−19
−0.99	−99
−0.999	−999

x	$f(x)$
−2	2
−1.5	3
−1.1	11
−1.01	101
−1.001	1001

It appears that $\lim\limits_{x\to -1} \dfrac{x^2 - 2x}{x^2 - x - 2}$ *does not exist* since

$f(x) \to \infty$ as $x \to -1^-$ and $f(x) \to -\infty$ as $x \to -1^+$.

21. For $f(x) = \dfrac{\sin x}{x + \tan x}$:

x	$f(x)$
± 1	0.329033
± 0.5	0.458209
± 0.2	0.493331
± 0.1	0.498333
± 0.05	0.499583
± 0.01	0.499983

It appears that $\displaystyle\lim_{x \to 0} \frac{\sin x}{x + \tan x} = 0.5 = \frac{1}{2}$.

22. For $f(h) = \dfrac{(2+h)^5 - 32}{h}$:

h	$f(h)$
0.5	131.312500
0.1	88.410100
0.01	80.804010
0.001	80.080040
0.0001	80.008000

h	$f(h)$
-0.5	48.812500
-0.1	72.390100
-0.01	79.203990
-0.001	79.920040
-0.0001	79.992000

It appears that $\displaystyle\lim_{h \to 0} \frac{(2+h)^5 - 32}{h} = 80$.

23. For $f(x) = \dfrac{\sqrt{x+4} - 2}{x}$:

x	$f(x)$
1	0.236068
0.5	0.242641
0.1	0.248457
0.05	0.249224
0.01	0.249844

x	$f(x)$
-1	0.267949
-0.5	0.258343
-0.1	0.251582
-0.05	0.250786
-0.01	0.250156

It appears that $\displaystyle\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} = 0.25 = \tfrac{1}{4}$.

24. For $f(x) = \dfrac{\tan 3x}{\tan 5x}$:

x	$f(x)$
± 0.2	0.439279
± 0.1	0.566236
± 0.05	0.591893
± 0.01	0.599680
± 0.001	0.599997

It appears that $\displaystyle\lim_{x \to 0} \frac{\tan 3x}{\tan 5x} = 0.6 = \tfrac{3}{5}$.

25. For $f(x) = \dfrac{x^6 - 1}{x^{10} - 1}$:

x	$f(x)$
0.5	0.985337
0.9	0.719397
0.95	0.660186
0.99	0.612018
0.999	0.601200

x	$f(x)$
1.5	0.183369
1.1	0.484119
1.05	0.540783
1.01	0.588022
1.001	0.598800

It appears that $\displaystyle\lim_{x \to 1} \frac{x^6 - 1}{x^{10} - 1} = 0.6 = \tfrac{3}{5}$.

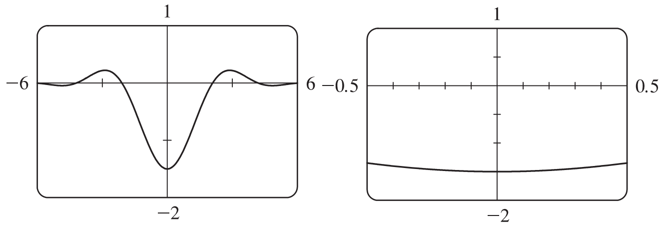
26. For $f(x) = \dfrac{9^x - 5^x}{x}$:

x	$f(x)$
0.5	1.527864
0.1	0.711120
0.05	0.646496
0.01	0.599082
0.001	0.588906

x	$f(x)$
-0.5	0.227761
-0.1	0.485984
-0.05	0.534447
-0.01	0.576706
-0.001	0.586669

It appears that $\displaystyle\lim_{x \to 0} \frac{9^x - 5^x}{x} = 0.59$. Later we will be able to show that the exact value is $\ln(9/5)$.

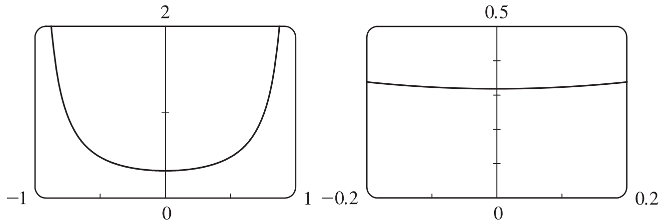
27. (a) From the graphs, it seems that $\lim_{x \to 0} \dfrac{\cos 2x - \cos x}{x^2} = -1.5$.



(b)

x	$f(x)$
± 0.1	-1.493759
± 0.01	-1.499938
± 0.001	-1.499999
± 0.0001	-1.500000

28. (a) From the graphs, it seems that $\lim_{x \to 0} \dfrac{\sin x}{\sin \pi x} = 0.32$.



(b)

x	$f(x)$
± 0.1	0.323068
± 0.01	0.318357
± 0.001	0.318310
± 0.0001	0.318310

Later we will be able to show that the exact value is $\dfrac{1}{\pi}$.

29. $\lim_{x \to -3^+} \dfrac{x+2}{x+3} = -\infty$ since the numerator is negative and the denominator approaches 0 from the positive side as $x \to -3^+$.

30. $\lim_{x \to -3^-} \dfrac{x+2}{x+3} = \infty$ since the numerator is negative and the denominator approaches 0 from the negative side as $x \to -3^-$.

31. $\lim_{x \to 1} \dfrac{2-x}{(x-1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

32. $\lim_{x \to 0} \dfrac{x-1}{x^2(x+2)} = -\infty$ since $x^2 \to 0$ as $x \to 0$ and $\dfrac{x-1}{x^2(x+2)} < 0$ for $0 < x < 1$ and for $-2 < x < 0$.

33. $\lim_{x \to -2^+} \dfrac{x-1}{x^2(x+2)} = -\infty$ since $(x+2) \to 0$ as $x \to -2^+$ and $\dfrac{x-1}{x^2(x+2)} < 0$ for $-2 < x < 0$.

34. $\lim_{x \to \pi^-} \cot x = \lim_{x \to \pi^-} \dfrac{\cos x}{\sin x} = -\infty$ since the numerator is negative and the denominator approaches 0 through positive values as $x \to \pi^-$.

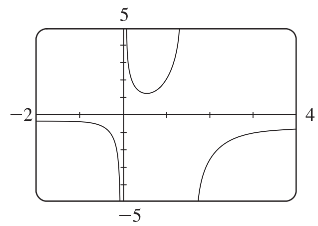
35. $\lim_{x \to 2\pi^-} x \csc x = \lim_{x \to 2\pi^-} \dfrac{x}{\sin x} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2\pi^-$.

36. $\lim_{x \to 2^-} \dfrac{x^2 - 2x}{x^2 - 4x + 4} = \lim_{x \to 2^-} \dfrac{x(x-2)}{(x-2)^2} = \lim_{x \to 2^-} \dfrac{x}{x-2} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2^-$.

37. $\lim_{x \to 2^+} \dfrac{x^2 - 2x - 8}{x^2 - 5x + 6} = \lim_{x \to 2^+} \dfrac{(x-4)(x+2)}{(x-3)(x-2)} = \infty$ since the numerator is negative and the denominator approaches 0 through negative values as $x \to 2^+$.

38. (a) The denominator of $y = \dfrac{x^2+1}{3x-2x^2} = \dfrac{x^2+1}{x(3-2x)}$ is equal to zero when $x = 0$ and $x = \frac{3}{2}$ (and the numerator is not), so $x = 0$ and $x = 1.5$ are vertical asymptotes of the function.

(b)



39. (a) $f(x) = \dfrac{1}{x^3-1}$.

From these calculations, it seems that

$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.

x	$f(x)$
0.5	−1.14
0.9	−3.69
0.99	−33.7
0.999	−333.7
0.9999	−3333.7
0.99999	−33,333.7

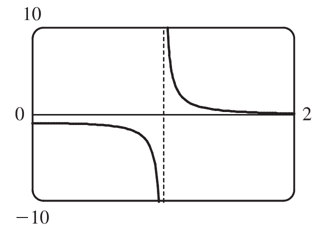
x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim\limits_{x\to 1^-} f(x) = -\infty$.

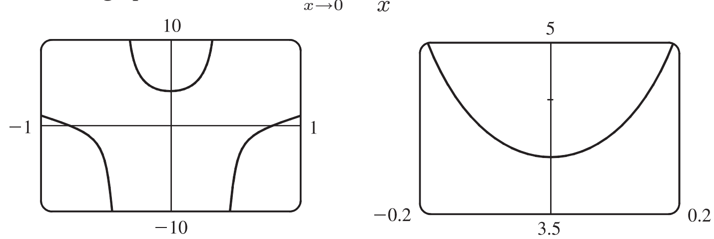
If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim\limits_{x\to 1^+} f(x) = \infty$.

(c) It appears from the graph of f that

$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.



40. (a) From the graphs, it seems that $\lim\limits_{x\to 0} \dfrac{\tan 4x}{x} = 4$.



(b)

x	$f(x)$
±0.1	4.227932
±0.01	4.002135
±0.001	4.000021
±0.0001	4.000000

41. For $f(x) = x^2 - (2^x/1000)$:

(a)

x	$f(x)$
1	0.998000
0.8	0.638259
0.6	0.358484
0.4	0.158680
0.2	0.038851
0.1	0.008928
0.05	0.001465

It appears that $\lim_{x \to 0} f(x) = 0$.

(b)

x	$f(x)$
0.04	0.000572
0.02	−0.000614
0.01	−0.000907
0.005	−0.000978
0.003	−0.000993
0.001	−0.001000

It appears that $\lim_{x \to 0} f(x) = -0.001$.

42. For $h(x) = \dfrac{\tan x - x}{x^3}$:

(a)

x	$h(x)$
1.0	0.55740773
0.5	0.37041992
0.1	0.33467209
0.05	0.33366700
0.01	0.33334667
0.005	0.33333667

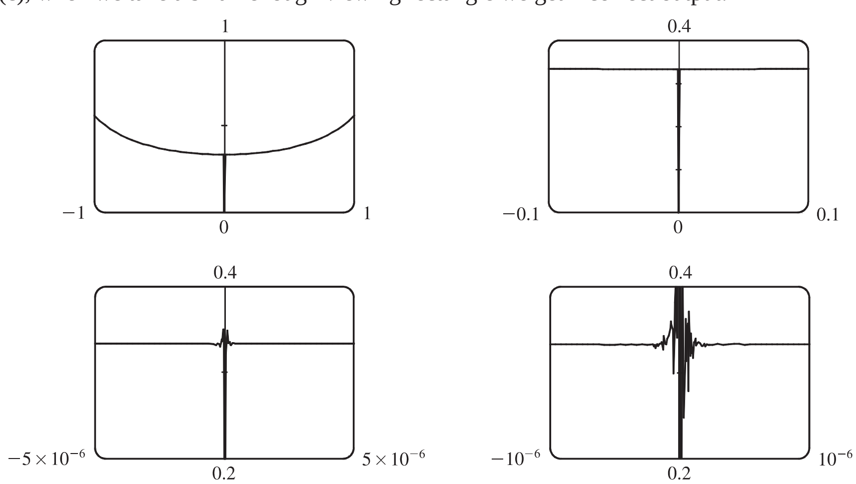
(b) It seems that $\lim_{x \to 0} h(x) = \frac{1}{3}$.

(c)

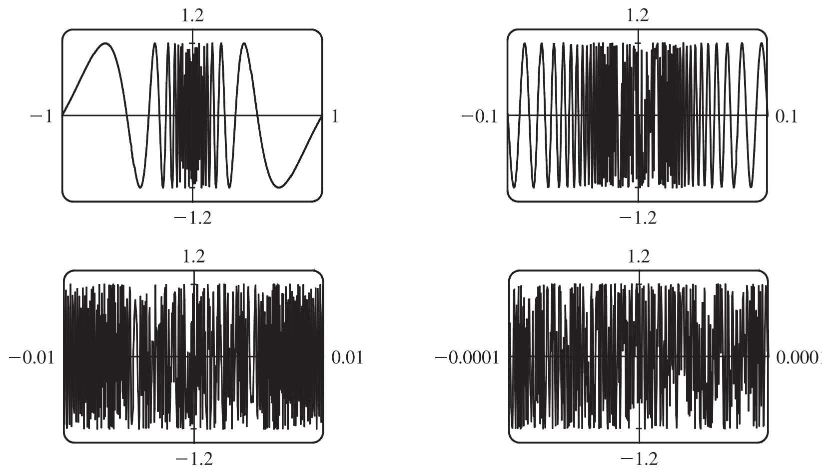
x	$h(x)$
0.001	0.33333350
0.0005	0.33333344
0.0001	0.33333000
0.00005	0.33333600
0.00001	0.33300000
0.000001	0.00000000

Here the values will vary from one calculator to another. Every calculator will eventually give *false values*.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.

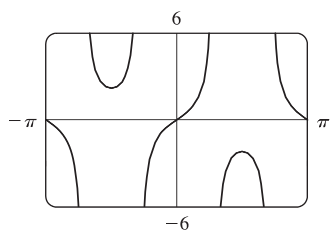


43. No matter how many times we zoom in toward the origin, the graphs of $f(x) = \sin(\pi/x)$ appear to consist of almost-vertical lines. This indicates more and more frequent oscillations as $x \to 0$.



44. $\lim\limits_{v \to c^-} m = \lim\limits_{v \to c^-} \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$. As $v \to c^-$, $\sqrt{1 - v^2/c^2} \to 0^+$, and $m \to \infty$.

45.

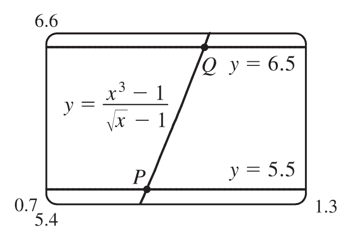


There appear to be vertical asymptotes of the curve $y = \tan(2 \sin x)$ at $x \approx \pm 0.90$ and $x \approx \pm 2.24$. To find the exact equations of these asymptotes, we note that the graph of the tangent function has vertical asymptotes at $x = \frac{\pi}{2} + \pi n$. Thus, we must have $2 \sin x = \frac{\pi}{2} + \pi n$, or equivalently, $\sin x = \frac{\pi}{4} + \frac{\pi}{2} n$. Since $-1 \leq \sin x \leq 1$, we must have $\sin x = \pm \frac{\pi}{4}$ and so $x = \pm \sin^{-1} \frac{\pi}{4}$ (corresponding to $x \approx \pm 0.90$). Just as $150°$ is the reference angle for $30°$, $\pi - \sin^{-1} \frac{\pi}{4}$ is the reference angle for $\sin^{-1} \frac{\pi}{4}$. So $x = \pm\left(\pi - \sin^{-1} \frac{\pi}{4}\right)$ are also equations of vertical asymptotes (corresponding to $x \approx \pm 2.24$).

46. (a) Let $y = \dfrac{x^3 - 1}{\sqrt{x} - 1}$.

From the table and the graph, we guess that the limit of y as x approaches 1 is 6.

x	y
0.99	5.92531
0.999	5.99250
0.9999	5.99925
1.01	6.07531
1.001	6.00750
1.0001	6.00075



(b) We need to have $5.5 < \dfrac{x^3 - 1}{\sqrt{x} - 1} < 6.5$. From the graph we obtain the approximate points of intersection $P(0.9314, 5.5)$ and $Q(1.0649, 6.5)$. Now $1 - 0.9314 = 0.0686$ and $1.0649 - 1 = 0.0649$, so by requiring that x be within 0.0649 of 1, we ensure that y is within 0.5 of 6.

1.6 Calculating Limits Using the Limit Laws

1. (a) $\lim\limits_{x\to 2}[f(x)+5g(x)] = \lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2}[5g(x)]$ [Limit Law 1]

$= \lim\limits_{x\to 2} f(x) + 5\lim\limits_{x\to 2} g(x)$ [Limit Law 3]

$= 4 + 5(-2) = -6$

(b) $\lim\limits_{x\to 2}[g(x)]^3 = \left[\lim\limits_{x\to 2} g(x)\right]^3$ [Limit Law 6]

$= (-2)^3 = -8$

(c) $\lim\limits_{x\to 2}\sqrt{f(x)} = \sqrt{\lim\limits_{x\to 2} f(x)}$ [Limit Law 11]

$= \sqrt{4} = 2$

(d) $\lim\limits_{x\to 2}\dfrac{3f(x)}{g(x)} = \dfrac{\lim\limits_{x\to 2}[3f(x)]}{\lim\limits_{x\to 2} g(x)}$ [Limit Law 5]

$= \dfrac{3\lim\limits_{x\to 2} f(x)}{\lim\limits_{x\to 2} g(x)}$ [Limit Law 3]

$= \dfrac{3(4)}{-2} = -6$

(e) Because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, $\lim\limits_{x\to 2}\dfrac{g(x)}{h(x)}$, does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(f) $\lim\limits_{x\to 2}\dfrac{g(x)\,h(x)}{f(x)} = \dfrac{\lim\limits_{x\to 2}[g(x)\,h(x)]}{\lim\limits_{x\to 2} f(x)}$ [Limit Law 5]

$= \dfrac{\lim\limits_{x\to 2} g(x)\cdot\lim\limits_{x\to 2} h(x)}{\lim\limits_{x\to 2} f(x)}$ [Limit Law 4]

$= \dfrac{-2\cdot 0}{4} = 0$

2. (a) $\lim\limits_{x\to 2}[f(x)+g(x)] = \lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} g(x) = 2 + 0 = 2$

(b) $\lim\limits_{x\to 1} g(x)$ does not exist since its left- and right-hand limits are not equal, so the given limit does not exist.

(c) $\lim\limits_{x\to 0}[f(x)g(x)] = \lim\limits_{x\to 0} f(x)\cdot\lim\limits_{x\to 0} g(x) = 0\cdot 1.3 = 0$

(d) Since $\lim\limits_{x\to -1} g(x) = 0$ and g is in the denominator, but $\lim\limits_{x\to -1} f(x) = -1 \neq 0$, the given limit does not exist.

(e) $\lim\limits_{x\to 2} x^3 f(x) = \left[\lim\limits_{x\to 2} x^3\right]\left[\lim\limits_{x\to 2} f(x)\right] = 2^3\cdot 2 = 16$

(f) $\lim\limits_{x\to 1}\sqrt{3+f(x)} = \sqrt{3+\lim\limits_{x\to 1} f(x)} = \sqrt{3+1} = 2$

3. $\lim\limits_{x\to 3}(5x^3 - 3x^2 + x - 6) = \lim\limits_{x\to 3}(5x^3) - \lim\limits_{x\to 3}(3x^2) + \lim\limits_{x\to 3} x - \lim\limits_{x\to 3} 6$ [Limit Laws 2 and 1]

$= 5\lim\limits_{x\to 3} x^3 - 3\lim\limits_{x\to 3} x^2 + \lim\limits_{x\to 3} x - \lim\limits_{x\to 3} 6$ [3]

$= 5(3^3) - 3(3^2) + 3 - 6$ [9, 8, and 7]

$= 105$

4. $$\begin{aligned}\lim_{x\to-1}(x^4-3x)(x^2+5x+3) &= \lim_{x\to-1}(x^4-3x)\lim_{x\to-1}(x^2+5x+3) && \text{[Limit Law 4]}\\ &= \left(\lim_{x\to-1}x^4-\lim_{x\to-1}3x\right)\left(\lim_{x\to-1}x^2+\lim_{x\to-1}5x+\lim_{x\to-1}3\right) && \text{[2, 1]}\\ &= \left(\lim_{x\to-1}x^4-3\lim_{x\to-1}x\right)\left(\lim_{x\to-1}x^2+5\lim_{x\to-1}x+\lim_{x\to-1}3\right) && \text{[3]}\\ &= (1+3)(1-5+3) && \text{[9, 8, and 7]}\\ &= 4(-1)=-4\end{aligned}$$

5. $$\begin{aligned}\lim_{t\to-2}\frac{t^4-2}{2t^2-3t+2} &= \frac{\lim\limits_{t\to-2}(t^4-2)}{\lim\limits_{t\to-2}(2t^2-3t+2)} && \text{[Limit Law 5]}\\ &= \frac{\lim\limits_{t\to-2}t^4-\lim\limits_{t\to-2}2}{2\lim\limits_{t\to-2}t^2-3\lim\limits_{t\to-2}t+\lim\limits_{t\to-2}2} && \text{[1, 2, and 3]}\\ &= \frac{16-2}{2(4)-3(-2)+2} && \text{[9, 7, and 8]}\\ &= \frac{14}{16}=\frac{7}{8}\end{aligned}$$

6. $$\begin{aligned}\lim_{u\to-2}\sqrt{u^4+3u+6} &= \sqrt{\lim_{u\to-2}(u^4+3u+6)} && \text{[11]}\\ &= \sqrt{\lim_{u\to-2}u^4+3\lim_{u\to-2}u+\lim_{u\to-2}6} && \text{[1, 2, and 3]}\\ &= \sqrt{(-2)^4+3(-2)+6} && \text{[9, 8, and 7]}\\ &= \sqrt{16-6+6}=\sqrt{16}=4\end{aligned}$$

7. $$\begin{aligned}\lim_{x\to8}(1+\sqrt[3]{x})(2-6x^2+x^3) &= \lim_{x\to8}(1+\sqrt[3]{x})\cdot\lim_{x\to8}(2-6x^2+x^3) && \text{[Limit Law 4]}\\ &= \left(\lim_{x\to8}1+\lim_{x\to8}\sqrt[3]{x}\right)\cdot\left(\lim_{x\to8}2-6\lim_{x\to8}x^2+\lim_{x\to8}x^3\right) && \text{[1, 2, and 3]}\\ &= \left(1+\sqrt[3]{8}\right)\cdot\left(2-6\cdot8^2+8^3\right) && \text{[7, 10, 9]}\\ &= (3)(130)=390\end{aligned}$$

8. $$\begin{aligned}\lim_{t\to2}\left(\frac{t^2-2}{t^3-3t+5}\right)^2 &= \left(\lim_{t\to2}\frac{t^2-2}{t^3-3t+5}\right)^2 && \text{[Limit Law 6]}\\ &= \left(\frac{\lim\limits_{t\to2}(t^2-2)}{\lim\limits_{t\to2}(t^3-3t+5)}\right)^2 && \text{[5]}\\ &= \left(\frac{\lim\limits_{t\to2}t^2-\lim\limits_{t\to2}2}{\lim\limits_{t\to2}t^3-3\lim\limits_{t\to2}t+\lim\limits_{t\to2}5}\right)^2 && \text{[1, 2, and 3]}\\ &= \left(\frac{4-2}{8-3(2)+5}\right)^2 && \text{[9, 7, and 8]}\\ &= \left(\frac{2}{7}\right)^2=\frac{4}{49}\end{aligned}$$

9. $$\lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}} = \sqrt{\lim_{x\to 2}\frac{2x^2+1}{3x-2}} \qquad \text{[Limit Law 11]}$$
$$= \sqrt{\frac{\lim_{x\to 2}(2x^2+1)}{\lim_{x\to 2}(3x-2)}} \qquad [5]$$
$$= \sqrt{\frac{2\lim_{x\to 2}x^2+\lim_{x\to 2}1}{3\lim_{x\to 2}x-\lim_{x\to 2}2}} \qquad \text{[1, 2, and 3]}$$
$$= \sqrt{\frac{2(2)^2+1}{3(2)-2}} = \sqrt{\frac{9}{4}} = \frac{3}{2} \qquad \text{[9, 8, and 7]}$$

10. (a) The left-hand side of the equation is not defined for $x = 2$, but the right-hand side is.

(b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \to 2$, just as in Example 3. Remember that in finding $\lim_{x\to a} f(x)$, we never consider $x = a$.

11. $$\lim_{x\to 5}\frac{x^2-6x+5}{x-5} = \lim_{x\to 5}\frac{(x-5)(x-1)}{x-5} = \lim_{x\to 5}(x-1) = 5-1 = 4$$

12. $$\lim_{x\to 4}\frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4}\frac{x(x-4)}{(x-4)(x+1)} = \lim_{x\to 4}\frac{x}{x+1} = \frac{4}{4+1} = \frac{4}{5}$$

13. $\lim_{x\to 5}\frac{x^2-5x+6}{x-5}$ does not exist since $x-5\to 0$, but $x^2-5x+6\to 6$ as $x\to 5$.

14. $\lim_{x\to -1}\frac{x^2-4x}{x^2-3x-4}$ does not exist since $x^2-3x-4\to 0$ but $x^2-4x\to 5$ as $x\to -1$.

15. $$\lim_{t\to -3}\frac{t^2-9}{2t^2+7t+3} = \lim_{t\to -3}\frac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim_{t\to -3}\frac{t-3}{2t+1} = \frac{-3-3}{2(-3)+1} = \frac{-6}{-5} = \frac{6}{5}$$

16. $$\lim_{x\to -1}\frac{2x^2+3x+1}{x^2-2x-3} = \lim_{x\to -1}\frac{(2x+1)(x+1)}{(x-3)(x+1)} = \lim_{x\to -1}\frac{2x+1}{x-3} = \frac{2(-1)+1}{-1-3} = \frac{-1}{-4} = \frac{1}{4}$$

17. $$\lim_{h\to 0}\frac{(-5+h)^2-25}{h} = \lim_{h\to 0}\frac{(25-10h+h^2)-25}{h} = \lim_{h\to 0}\frac{-10h+h^2}{h} = \lim_{h\to 0}\frac{h(-10+h)}{h} = \lim_{h\to 0}(-10+h) = -10$$

18. $$\lim_{h\to 0}\frac{(2+h)^3-8}{h} = \lim_{h\to 0}\frac{\left(8+12h+6h^2+h^3\right)-8}{h} = \lim_{h\to 0}\frac{12h+6h^2+h^3}{h}$$
$$= \lim_{h\to 0}\left(12+6h+h^2\right) = 12+0+0 = 12$$

19. By the formula for the sum of cubes, we have
$$\lim_{x\to -2}\frac{x+2}{x^3+8} = \lim_{x\to -2}\frac{x+2}{(x+2)(x^2-2x+4)} = \lim_{x\to -2}\frac{1}{x^2-2x+4} = \frac{1}{4+4+4} = \frac{1}{12}.$$

20. We use the difference of squares in the numerator and the difference of cubes in the denominator.
$$\lim_{t\to 1}\frac{t^4-1}{t^3-1} = \lim_{t\to 1}\frac{(t^2-1)(t^2+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1}\frac{(t-1)(t+1)(t^2+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1}\frac{(t+1)(t^2+1)}{t^2+t+1} = \frac{2(2)}{3} = \frac{4}{3}$$

21. $\lim\limits_{h\to 0}\dfrac{\sqrt{9+h}-3}{h}=\lim\limits_{h\to 0}\dfrac{\sqrt{9+h}-3}{h}\cdot\dfrac{\sqrt{9+h}+3}{\sqrt{9+h}+3}=\lim\limits_{h\to 0}\dfrac{\left(\sqrt{9+h}\right)^2-3^2}{h\left(\sqrt{9+h}+3\right)}=\lim\limits_{h\to 0}\dfrac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)}$

$=\lim\limits_{h\to 0}\dfrac{h}{h\left(\sqrt{9+h}+3\right)}=\lim\limits_{h\to 0}\dfrac{1}{\sqrt{9+h}+3}=\dfrac{1}{3+3}=\dfrac{1}{6}$

22. $\lim\limits_{u\to 2}\dfrac{\sqrt{4u+1}-3}{u-2}=\lim\limits_{u\to 2}\dfrac{\sqrt{4u+1}-3}{u-2}\cdot\dfrac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3}=\lim\limits_{u\to 2}\dfrac{\left(\sqrt{4u+1}\right)^2-3^2}{(u-2)\left(\sqrt{4u+1}+3\right)}$

$=\lim\limits_{u\to 2}\dfrac{4u+1-9}{(u-2)\left(\sqrt{4u+1}+3\right)}=\lim\limits_{u\to 2}\dfrac{4(u-2)}{(u-2)\left(\sqrt{4u+1}+3\right)}$

$=\lim\limits_{u\to 2}\dfrac{4}{\sqrt{4u+1}+3}=\dfrac{4}{\sqrt{9}+3}=\dfrac{2}{3}$

23. $\lim\limits_{x\to -4}\dfrac{\dfrac{1}{4}+\dfrac{1}{x}}{4+x}=\lim\limits_{x\to -4}\dfrac{\dfrac{x+4}{4x}}{4+x}=\lim\limits_{x\to -4}\dfrac{x+4}{4x(4+x)}=\lim\limits_{x\to -4}\dfrac{1}{4x}=\dfrac{1}{4(-4)}=-\dfrac{1}{16}$

24. $\lim\limits_{x\to -1}\dfrac{x^2+2x+1}{x^4-1}=\lim\limits_{x\to -1}\dfrac{(x+1)^2}{(x^2+1)(x^2-1)}=\lim\limits_{x\to -1}\dfrac{(x+1)^2}{(x^2+1)(x+1)(x-1)}$

$=\lim\limits_{x\to -1}\dfrac{x+1}{(x^2+1)(x-1)}=\dfrac{0}{2(-2)}=0$

25. $\lim\limits_{t\to 0}\dfrac{\sqrt{1+t}-\sqrt{1-t}}{t}=\lim\limits_{t\to 0}\dfrac{\sqrt{1+t}-\sqrt{1-t}}{t}\cdot\dfrac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}=\lim\limits_{t\to 0}\dfrac{\left(\sqrt{1+t}\right)^2-\left(\sqrt{1-t}\right)^2}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$

$=\lim\limits_{t\to 0}\dfrac{(1+t)-(1-t)}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}=\lim\limits_{t\to 0}\dfrac{2t}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}=\lim\limits_{t\to 0}\dfrac{2}{\sqrt{1+t}+\sqrt{1-t}}$

$=\dfrac{2}{\sqrt{1}+\sqrt{1}}=\dfrac{2}{2}=1$

26. $\lim\limits_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t^2+t}\right)=\lim\limits_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t(t+1)}\right)=\lim\limits_{t\to 0}\dfrac{t+1-1}{t(t+1)}=\lim\limits_{t\to 0}\dfrac{1}{t+1}=\dfrac{1}{0+1}=1$

27. $\lim\limits_{x\to 16}\dfrac{4-\sqrt{x}}{16x-x^2}=\lim\limits_{x\to 16}\dfrac{(4-\sqrt{x})(4+\sqrt{x})}{(16x-x^2)(4+\sqrt{x})}=\lim\limits_{x\to 16}\dfrac{16-x}{x(16-x)(4+\sqrt{x})}$

$=\lim\limits_{x\to 16}\dfrac{1}{x(4+\sqrt{x})}=\dfrac{1}{16\left(4+\sqrt{16}\right)}=\dfrac{1}{16(8)}=\dfrac{1}{128}$

28. $\lim\limits_{h\to 0}\dfrac{(3+h)^{-1}-3^{-1}}{h}=\lim\limits_{h\to 0}\dfrac{\dfrac{1}{3+h}-\dfrac{1}{3}}{h}=\lim\limits_{h\to 0}\dfrac{3-(3+h)}{h(3+h)3}=\lim\limits_{h\to 0}\dfrac{-h}{h(3+h)3}$

$=\lim\limits_{h\to 0}\left[-\dfrac{1}{3(3+h)}\right]=-\dfrac{1}{\lim\limits_{h\to 0}[3(3+h)]}=-\dfrac{1}{3(3+0)}=-\dfrac{1}{9}$

29. $\lim\limits_{t\to 0}\left(\dfrac{1}{t\sqrt{1+t}}-\dfrac{1}{t}\right)=\lim\limits_{t\to 0}\dfrac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim\limits_{t\to 0}\dfrac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\right)}=\lim\limits_{t\to 0}\dfrac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$

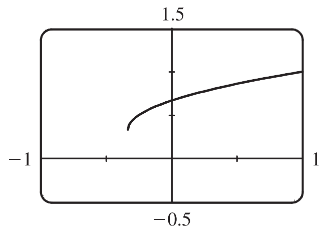
$=\lim\limits_{t\to 0}\dfrac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)}=\dfrac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\right)}=-\dfrac{1}{2}$

30. $$\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}=\lim_{x\to -4}\frac{\left(\sqrt{x^2+9}-5\right)\left(\sqrt{x^2+9}+5\right)}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x^2+9)-25}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$
$$=\lim_{x\to -4}\frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x+4)(x-4)}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$
$$=\lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5}=\frac{-4-4}{\sqrt{16+9}+5}=\frac{-8}{5+5}=-\frac{4}{5}$$

31. $$\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$
$$=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$$

32. $$\lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\dfrac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}=\lim_{h\to 0}\frac{-h(2x+h)}{hx^2(x+h)^2}$$
$$=\lim_{h\to 0}\frac{-(2x+h)}{x^2(x+h)^2}=\frac{-2x}{x^2\cdot x^2}=-\frac{2}{x^3}$$

33. (a)



$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$$

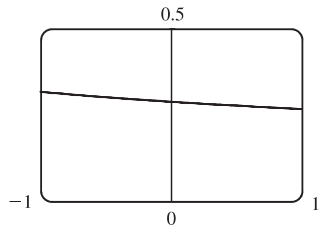
(b)

x	$f(x)$
−0.001	0.6661663
−0.0001	0.6666167
−0.00001	0.6666617
−0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

The limit appears to be $\frac{2}{3}$.

(c) $$\lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}$$
$$=\frac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right)$$ [Limit Law 3]
$$=\frac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)}+\lim_{x\to 0}1\right]$$ [1 and 11]
$$=\frac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right)$$ [1, 3, and 7]
$$=\frac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right)$$ [7 and 8]
$$=\frac{1}{3}(1+1)=\frac{2}{3}$$

34. (a)



$$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}\approx 0.29$$

(b)

x	$f(x)$
-0.001	0.2886992
-0.0001	0.2886775
-0.00001	0.2886754
-0.000001	0.2886752
0.000001	0.2886751
0.00001	0.2886749
0.0001	0.2886727
0.001	0.2886511

The limit appears to be approximately 0.2887.

(c) $$\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)=\lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$$

$$=\frac{\lim_{x\to 0}1}{\lim_{x\to 0}\sqrt{3+x}+\lim_{x\to 0}\sqrt{3}}$$ [Limit Laws 5 and 1]

$$=\frac{1}{\sqrt{\lim_{x\to 0}(3+x)}+\sqrt{3}}$$ [7 and 11]

$$=\frac{1}{\sqrt{3+0}+\sqrt{3}}$$ [1, 7, and 8]

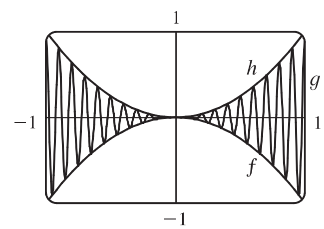
$$=\frac{1}{2\sqrt{3}}$$

35. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then

$-1\le\cos 20\pi x\le 1 \quad\Rightarrow\quad -x^2\le x^2\cos 20\pi x\le x^2 \quad\Rightarrow\quad f(x)\le g(x)\le h(x)$.

So since $\lim_{x\to 0}f(x)=\lim_{x\to 0}h(x)=0$, by the Squeeze Theorem we have

$\lim_{x\to 0}g(x)=0$.

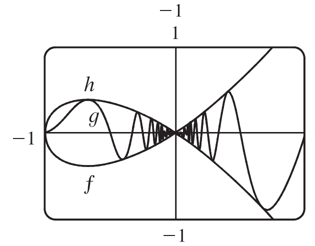


36. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\,\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then

$-1\le\sin(\pi/x)\le 1 \quad\Rightarrow\quad -\sqrt{x^3+x^2}\le\sqrt{x^3+x^2}\sin(\pi/x)\le\sqrt{x^3+x^2} \quad\Rightarrow$

$f(x)\le g(x)\le h(x)$. So since $\lim_{x\to 0}f(x)=\lim_{x\to 0}h(x)=0$, by the Squeeze Theorem

we have $\lim_{x\to 0}g(x)=0$.



37. We have $\lim_{x\to 4}(4x-9)=4(4)-9=7$ and $\lim_{x\to 4}\left(x^2-4x+7\right)=4^2-4(4)+7=7$. Since $4x-9\le f(x)\le x^2-4x+7$ for $x\ge 0$, $\lim_{x\to 4}f(x)=7$ by the Squeeze Theorem.

38. We have $\lim_{x\to 1}(2x)=2(1)=2$ and $\lim_{x\to 1}(x^4-x^2+2)=1^4-1^2+2=2$. Since $2x\le g(x)\le x^4-x^2+2$ for all x, $\lim_{x\to 1}g(x)=2$ by the Squeeze Theorem.

39. $-1\le\cos(2/x)\le 1 \quad\Rightarrow\quad -x^4\le x^4\cos(2/x)\le x^4$. Since $\lim_{x\to 0}\left(-x^4\right)=0$ and $\lim_{x\to 0}x^4=0$, we have $\lim_{x\to 0}\left[x^4\cos(2/x)\right]=0$ by the Squeeze Theorem.

40. $-1 \le \sin(2\pi/x) \le 1 \quad \Rightarrow \quad 0 \le \sin^2(2\pi/x) \le 1 \quad \Rightarrow \quad 1 \le 1 + \sin^2(2\pi/x) \le 2 \quad \Rightarrow$

$\sqrt{x} \le \sqrt{x}\left[1 + \sin^2(2\pi/x)\right] \le 2\sqrt{x}$. Since $\lim\limits_{x\to 0^+} \sqrt{x} = 0$ and $\lim\limits_{x\to 0^+} 2\sqrt{x} = 0$, we have

$\lim\limits_{x\to 0^+} \left[\sqrt{x}\,(1 + \sin^2(2\pi/x))\right] = 0$ by the Squeeze Theorem.

41. $|x-3| = \begin{cases} x-3 & \text{if } x-3 \ge 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \ge 3 \\ 3-x & \text{if } x < 3 \end{cases}$

Thus, $\lim\limits_{x\to 3^+} (2x + |x-3|) = \lim\limits_{x\to 3^+} (2x + x - 3) = \lim\limits_{x\to 3^+} (3x-3) = 3(3) - 3 = 6$ and

$\lim\limits_{x\to 3^-} (2x + |x-3|) = \lim\limits_{x\to 3^-} (2x + 3 - x) = \lim\limits_{x\to 3^-} (x+3) = 3 + 3 = 6$. Since the left and right limits are equal,

$\lim\limits_{x\to 3} (2x + |x-3|) = 6$.

42. $|x+6| = \begin{cases} x+6 & \text{if } x+6 \ge 0 \\ -(x+6) & \text{if } x+6 < 0 \end{cases} = \begin{cases} x+6 & \text{if } x \ge -6 \\ -(x+6) & \text{if } x < -6 \end{cases}$

We'll look at the one-sided limits.

$\lim\limits_{x\to -6^+} \dfrac{2x+12}{|x+6|} = \lim\limits_{x\to -6^+} \dfrac{2(x+6)}{x+6} = 2$ and $\lim\limits_{x\to -6^-} \dfrac{2x+12}{|x+6|} = \lim\limits_{x\to -6^-} \dfrac{2(x+6)}{-(x+6)} = -2$

The left and right limits are different, so $\lim\limits_{x\to -6} \dfrac{2x+12}{|x+6|}$ does not exist.

43. $\left|2x^3 - x^2\right| = \left|x^2(2x-1)\right| = \left|x^2\right| \cdot |2x-1| = x^2\,|2x-1|$

$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \ge 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \ge 0.5 \\ -(2x-1) & \text{if } x < 0.5 \end{cases}$

So $\left|2x^3 - x^2\right| = x^2[-(2x-1)]$ for $x < 0.5$.

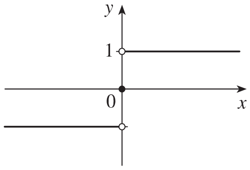
Thus, $\lim\limits_{x\to 0.5^-} \dfrac{2x-1}{|2x^3 - x^2|} = \lim\limits_{x\to 0.5^-} \dfrac{2x-1}{x^2[-(2x-1)]} = \lim\limits_{x\to 0.5^-} \dfrac{-1}{x^2} = \dfrac{-1}{(0.5)^2} = \dfrac{-1}{0.25} = -4$.

44. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to -2} \dfrac{2-|x|}{2+x} = \lim\limits_{x\to -2} \dfrac{2-(-x)}{2+x} = \lim\limits_{x\to -2} \dfrac{2+x}{2+x} = \lim\limits_{x\to -2} 1 = 1$.

45. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{-x}\right) = \lim\limits_{x\to 0^-} \dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

46. Since $|x| = x$ for $x > 0$, we have $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{x}\right) = \lim\limits_{x\to 0^+} 0 = 0$.

47. (a)



(b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\lim\limits_{x\to 0^+} \operatorname{sgn} x = \lim\limits_{x\to 0^+} 1 = 1$.

(ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\lim\limits_{x\to 0^-} \operatorname{sgn} x = \lim\limits_{x\to 0^-} -1 = -1$.

(iii) Since $\lim\limits_{x\to 0^-} \operatorname{sgn} x \ne \lim\limits_{x\to 0^+} \operatorname{sgn} x$, $\lim\limits_{x\to 0} \operatorname{sgn} x$ does not exist.

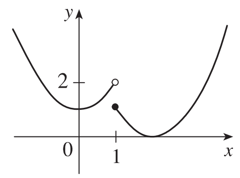
(iv) Since $|\operatorname{sgn} x| = 1$ for $x \ne 0$, $\lim\limits_{x\to 0} |\operatorname{sgn} x| = \lim\limits_{x\to 0} 1 = 1$.

48. (a) $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x-2)^2 & \text{if } x \geq 1 \end{cases}$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^2 + 1) = 1^2 + 1 = 2, \quad \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x-2)^2 = (-1)^2 = 1$

(b) Since the right-hand and left-hand limits of f at $x = 1$ are not equal, $\lim\limits_{x \to 1} f(x)$ does not exist.

(c)



49. (a) (i) $\lim\limits_{x \to 2^+} g(x) = \lim\limits_{x \to 2^+} \dfrac{x^2 + x - 6}{|x-2|} = \lim\limits_{x \to 2^+} \dfrac{(x+3)(x-2)}{|x-2|}$

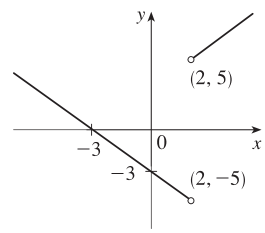
$= \lim\limits_{x \to 2^+} \dfrac{(x+3)(x-2)}{x-2}$ [since $x - 2 > 0$ if $x \to 2^+$]

$= \lim\limits_{x \to 2^+} (x+3) = 5$

(ii) The solution is similar to the solution in part (i), but now $|x-2| = 2 - x$ since $x - 2 < 0$ if $x \to 2^-$. Thus, $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^-} -(x+3) = -5$.

(b) Since the right-hand and left-hand limits of g at $x = 2$ are not equal, $\lim\limits_{x \to 2} g(x)$ does not exist.

(c)



50. (a) (i) $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^-} x = 1$

(ii) $\lim\limits_{x \to 1^+} g(x) = \lim\limits_{x \to 1^+} (2 - x^2) = 2 - 1^2 = 1$. Since $\lim\limits_{x \to 1^-} g(x) = 1$ and $\lim\limits_{x \to 1^+} g(x) = 1$, we have $\lim\limits_{x \to 1} g(x) = 1$.

Note that the fact $g(1) = 3$ does not affect the value of the limit.

(iii) When $x = 1$, $g(x) = 3$, so $g(1) = 3$.

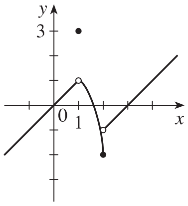
(iv) $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^-} (2 - x^2) = 2 - 2^2 = 2 - 4 = -2$

(v) $\lim\limits_{x \to 2^+} g(x) = \lim\limits_{x \to 2^+} (x - 3) = 2 - 3 = -1$

(vi) $\lim\limits_{x \to 2} g(x)$ does not exist since $\lim\limits_{x \to 2^-} g(x) \neq \lim\limits_{x \to 2^+} g(x)$.

(b)

$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$



51. (a) (i) $[\![x]\!] = -2$ for $-2 \le x < -1$, so $\lim\limits_{x \to -2^+} [\![x]\!] = \lim\limits_{x \to -2^+} (-2) = -2$

(ii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x \to -2^-} [\![x]\!] = \lim\limits_{x \to -2^-} (-3) = -3$.

The right and left limits are different, so $\lim\limits_{x \to -2} [\![x]\!]$ does not exist.

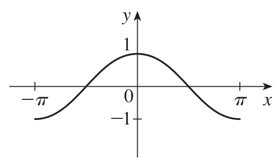
(iii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x \to -2.4} [\![x]\!] = \lim\limits_{x \to -2.4} (-3) = -3$.

(b) (i) $[\![x]\!] = n - 1$ for $n - 1 \le x < n$, so $\lim\limits_{x \to n^-} [\![x]\!] = \lim\limits_{x \to n^-} (n-1) = n - 1$.

(ii) $[\![x]\!] = n$ for $n \le x < n + 1$, so $\lim\limits_{x \to n^+} [\![x]\!] = \lim\limits_{x \to n^+} n = n$.

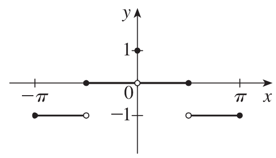
(c) $\lim\limits_{x \to a} [\![x]\!]$ exists $\Leftrightarrow$ a is not an integer.

52. (a) See the graph of $y = \cos x$.



Since $-1 \le \cos x < 0$ on $[-\pi, -\pi/2)$, we have $y = f(x) = [\![\cos x]\!] = -1$ on $[-\pi, -\pi/2)$.

Since $0 \le \cos x < 1$ on $[-\pi/2, 0) \cup (0, \pi/2]$, we have $f(x) = 0$ on $[-\pi/2, 0) \cup (0, \pi/2]$.



Since $-1 \le \cos x < 0$ on $(\pi/2, \pi]$, we have $f(x) = -1$ on $(\pi/2, \pi]$.

Note that $f(0) = 1$.

(b) (i) $\lim\limits_{x \to 0^-} f(x) = 0$ and $\lim\limits_{x \to 0^+} f(x) = 0$, so $\lim\limits_{x \to 0} f(x) = 0$.

(ii) As $x \to (\pi/2)^-$, $f(x) \to 0$, so $\lim\limits_{x \to (\pi/2)^-} f(x) = 0$.

(iii) As $x \to (\pi/2)^+$, $f(x) \to -1$, so $\lim\limits_{x \to (\pi/2)^+} f(x) = -1$.

(iv) Since the answers in parts (ii) and (iii) are not equal, $\lim\limits_{x \to \pi/2} f(x)$ does not exist.

(c) $\lim\limits_{x \to a} f(x)$ exists for all a in the open interval $(-\pi, \pi)$ except $a = -\pi/2$ and $a = \pi/2$.

53. The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\lim\limits_{x \to 2^-} f(x) = -1$ and $\lim\limits_{x \to 2^+} f(x) = -1$, so $\lim\limits_{x \to 2} f(x) = -1$. However, $f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$, so $\lim\limits_{x \to 2} f(x) \ne f(2)$.

54. $\lim\limits_{v \to c^-} \left(L_0 \sqrt{1 - \dfrac{v^2}{c^2}} \right) = L_0 \sqrt{1 - 1} = 0$. As the velocity approaches the speed of light, the length approaches 0.

A left-hand limit is necessary since L is not defined for $v > c$.

55. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Thus, by the Limit Laws,

$$\begin{aligned}\lim_{x \to a} p(x) &= \lim_{x \to a} \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \right) = a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + \cdots + a_n \lim_{x \to a} x^n \\ &= a_0 + a_1 a + a_2 a^2 + \cdots + a_n a^n = p(a)\end{aligned}$$

Thus, for any polynomial p, $\lim\limits_{x \to a} p(x) = p(a)$.

56. Let $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \neq 0$. Then

$$\lim_{x\to a} r(x) = \lim_{x\to a} \frac{p(x)}{q(x)} = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} \quad \text{[Limit Law 5]} \quad = \frac{p(a)}{q(a)} \quad \text{[Exercise 55]} \quad = r(a).$$

57. $\lim_{x\to 1} [f(x) - 8] = \lim_{x\to 1} \left[\dfrac{f(x)-8}{x-1} \cdot (x-1)\right] = \lim_{x\to 1} \dfrac{f(x)-8}{x-1} \cdot \lim_{x\to 1} (x-1) = 10 \cdot 0 = 0.$

Thus, $\lim_{x\to 1} f(x) = \lim_{x\to 1} \{[f(x)-8]+8\} = \lim_{x\to 1}[f(x)-8] + \lim_{x\to 1} 8 = 0 + 8 = 8.$

Note: The value of $\lim_{x\to 1} \dfrac{f(x)-8}{x-1}$ does not affect the answer since it's multiplied by 0. What's important is that $\lim_{x\to 1} \dfrac{f(x)-8}{x-1}$ exists.

58. (a) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \left[\dfrac{f(x)}{x^2} \cdot x^2\right] = \lim_{x\to 0} \dfrac{f(x)}{x^2} \cdot \lim_{x\to 0} x^2 = 5 \cdot 0 = 0$

(b) $\lim_{x\to 0} \dfrac{f(x)}{x} = \lim_{x\to 0} \left[\dfrac{f(x)}{x^2} \cdot x\right] = \lim_{x\to 0} \dfrac{f(x)}{x^2} \cdot \lim_{x\to 0} x = 5 \cdot 0 = 0$

59. Observe that $0 \le f(x) \le x^2$ for all x, and $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} x^2$. So, by the Squeeze Theorem, $\lim_{x\to 0} f(x) = 0$.

60. Let $f(x) = [\![x]\!]$ and $g(x) = -[\![x]\!]$. Then $\lim_{x\to 3} f(x)$ and $\lim_{x\to 3} g(x)$ do not exist [Example 10]

but $\lim_{x\to 3} [f(x) + g(x)] = \lim_{x\to 3} ([\![x]\!] - [\![x]\!]) = \lim_{x\to 3} 0 = 0.$

61. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.57.

Thus, either f or g is 0 for any value of x. Then $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist, but $\lim_{x\to 0} [f(x)g(x)] = \lim_{x\to 0} 0 = 0.$

62.
$$\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \lim_{x\to 2} \left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} \cdot \frac{\sqrt{6-x}+2}{\sqrt{6-x}+2} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)$$
$$= \lim_{x\to 2} \left[\frac{\left(\sqrt{6-x}\right)^2 - 2^2}{\left(\sqrt{3-x}\right)^2 - 1^2} \cdot \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right] = \lim_{x\to 2} \left(\frac{6-x-4}{3-x-1} \cdot \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right)$$
$$= \lim_{x\to 2} \frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)} = \lim_{x\to 2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}$$

63. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need $\lim_{x\to -2} \left(3x^2 + ax + a + 3\right) = 0 \quad \Leftrightarrow$

$3(-2)^2 + a(-2) + a + 3 = 0 \quad \Leftrightarrow \quad 12 - 2a + a + 3 = 0 \quad \Leftrightarrow \quad a = 15$. With $a = 15$, the limit becomes

$$\lim_{x\to -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim_{x\to -2} \frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x\to -2} \frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

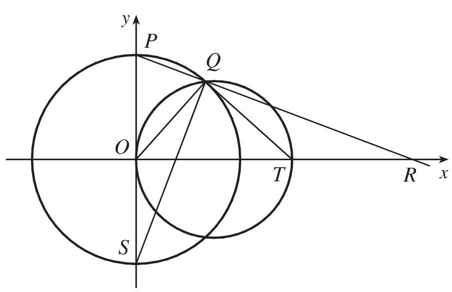
64. *Solution 1:* First, we find the coordinates of P and Q as functions of r. Then we can find the equation of the line determined by these two points, and thus find the x-intercept (the point R), and take the limit as $r \to 0$. The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \quad \Leftrightarrow \quad r^2 = 1 + 2x - 1 \quad \Leftrightarrow \quad x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the y-coordinate, we get $\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \quad \Leftrightarrow \quad y^2 = r^2\left(1 - \frac{1}{4}r^2\right) \quad \Leftrightarrow \quad y = r\sqrt{1 - \frac{1}{4}r^2}$ (the positive y-value). So the coordinates of Q are $\left(\frac{1}{2}r^2, r\sqrt{1 - \frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus $y - r = \dfrac{r\sqrt{1 - \frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0}(x - 0)$. We set $y = 0$ in order to find the x-intercept, and get

$$x = -r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1 - \frac{1}{4}r^2} - 1\right)} = \frac{-\frac{1}{2}r^2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)}{1 - \frac{1}{4}r^2 - 1} = 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)$$

Now we take the limit as $r \to 0^+$: $\displaystyle\lim_{r \to 0^+} x = \lim_{r \to 0^+} 2\left(\sqrt{1 - \tfrac{1}{4}r^2} + 1\right) = \lim_{r \to 0^+} 2\left(\sqrt{1} + 1\right) = 4$.

So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90^\circ$ (subtended by diameter PS). So $\angle SQR = 90^\circ = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90^\circ - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4, 0)$, as above.



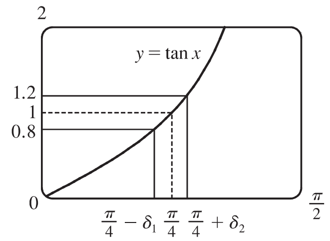
1.7 The Precise Definition of a Limit

1. If $|f(x) - 1| < 0.2$, then $-0.2 < f(x) - 1 < 0.2 \quad \Rightarrow \quad 0.8 < f(x) < 1.2$. From the graph, we see that the last inequality is true if $0.7 < x < 1.1$, so we can choose $\delta = \min\{1 - 0.7, 1.1 - 1\} = \min\{0.3, 0.1\} = 0.1$ (or any smaller positive number).

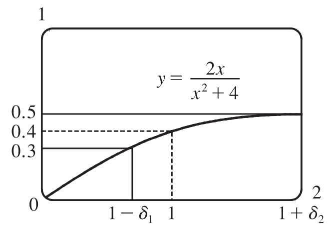
2. If $|f(x) - 2| < 0.5$, then $-0.5 < f(x) - 2 < 0.5 \quad \Rightarrow \quad 1.5 < f(x) < 2.5$. From the graph, we see that the last inequality is true if $2.6 < x < 3.8$, so we can take $\delta = \min\{3 - 2.6, 3.8 - 3\} = \min\{0.4, 0.8\} = 0.4$ (or any smaller positive number). Note that $x \neq 3$.

3. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x - 4| < |2.56 - 4| = 1.44$. On the right side, we need $|x - 4| < |5.76 - 4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold — namely, $|x - 4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

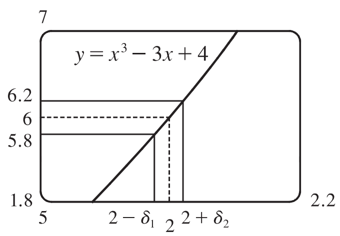
4. The leftmost question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the rightmost question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need $|x - 1| < \left|\frac{1}{\sqrt{2}} - 1\right| \approx 0.292$ (rounding down to be safe). On the right side, we need $|x - 1| < \left|\sqrt{\frac{3}{2}} - 1\right| \approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

5. 

From the graph, we find that $y = \tan x = 0.8$ when $x \approx 0.675$, so $\frac{\pi}{4} - \delta_1 \approx 0.675 \quad \Rightarrow \quad \delta_1 \approx \frac{\pi}{4} - 0.675 \approx 0.1106$. Also, $y = \tan x = 1.2$ when $x \approx 0.876$, so $\frac{\pi}{4} + \delta_2 \approx 0.876 \quad \Rightarrow \quad \delta_2 = 0.876 - \frac{\pi}{4} \approx 0.0906$. Thus, we choose $\delta = 0.0906$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

6. 

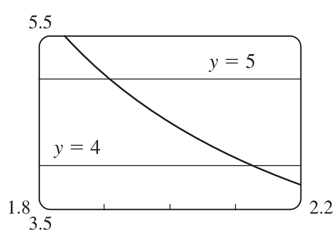
From the graph, we find that $y = 2x/(x^2 + 4) = 0.3$ when $x = \frac{2}{3}$, so $1 - \delta_1 = \frac{2}{3} \quad \Rightarrow \quad \delta_1 = \frac{1}{3}$. Also, $y = 2x/(x^2 + 4) = 0.5$ when $x = 2$, so $1 + \delta_2 = 2 \quad \Rightarrow \quad \delta_2 = 1$. Thus, we choose $\delta = \frac{1}{3}$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

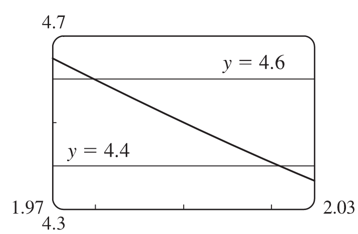
7. 

From the graph with $\varepsilon = 0.2$, we find that $y = x^3 - 3x + 4 = 5.8$ when $x \approx 1.9774$, so $2 - \delta_1 \approx 1.9774 \quad \Rightarrow \quad \delta_1 \approx 0.0226$. Also, $y = x^3 - 3x + 4 = 6.2$ when $x \approx 2.022$, so $2 + \delta_2 \approx 2.0219 \quad \Rightarrow$ $\delta_2 \approx 0.0219$. Thus, we choose $\delta = 0.0219$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

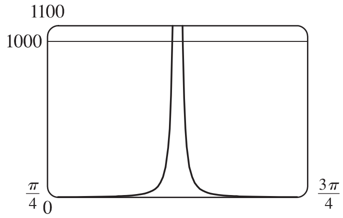
For $\varepsilon = 0.1$, we get $\delta_1 \approx 0.0112$ and $\delta_2 \approx 0.0110$, so we choose $\delta = 0.011$ (or any smaller positive number).

8. For $y = (4x + 1)/(3x - 4)$ and $\varepsilon = 0.5$, we need $1.91 \le x \le 2.125$. So since $|2 - 1.91| = 0.09$ and $|2 - 2.125| = 0.125$, we can take $0 < \delta \le 0.09$. For $\varepsilon = 0.1$, we need $1.980 \le 2.021$. So since $|2 - 1.980| = 0.02$ and $|2 - 2.021| = 0.021$, we can take $\delta = 0.02$ (or any smaller positive number).



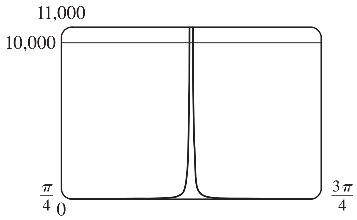


9. (a)



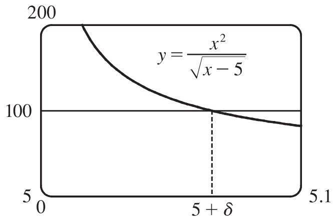
From the graph, we find that $y = \tan^2 x = 1000$ when $x \approx 1.539$ and $x \approx 1.602$ for x near $\frac{\pi}{2}$. Thus, we get $\delta \approx 1.602 - \frac{\pi}{2} \approx 0.031$ for $M = 1000$.

(b)



From the graph, we find that $y = \tan^2 x = 10{,}000$ when $x \approx 1.561$ and $x \approx 1.581$ for x near $\frac{\pi}{2}$. Thus, we get $\delta \approx 1.581 - \frac{\pi}{2} \approx 0.010$ for $M = 10{,}000$.

10.



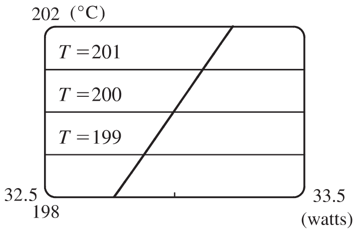
From the graph, we find that $x^2/\sqrt{x-5} = 100 \quad \Rightarrow \quad x \approx 5.066$. Thus, $5 + \delta \approx 5.0659$ and $\delta \approx 0.065$.

11. (a) $A = \pi r^2$ and $A = 1000\text{ cm}^2 \quad \Rightarrow \quad \pi r^2 = 1000 \quad \Rightarrow \quad r^2 = \frac{1000}{\pi} \quad \Rightarrow \quad r = \sqrt{\frac{1000}{\pi}} \quad (r > 0) \quad \approx 17.8412$ cm.

(b) $|A - 1000| \le 5 \quad \Rightarrow \quad -5 \le \pi r^2 - 1000 \le 5 \quad \Rightarrow \quad 1000 - 5 \le \pi r^2 \le 1000 + 5 \quad \Rightarrow$ $\sqrt{\frac{995}{\pi}} \le r \le \sqrt{\frac{1005}{\pi}} \quad \Rightarrow \quad 17.7966 \le r \le 17.8858$. $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and $\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ε is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

12. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \quad \Rightarrow$ $0.1w^2 + 2.155w + 20 = 200 \quad \Rightarrow$ [by the quadratic formula or from the graph] $w \approx 33.0$ watts $(w > 0)$



(b) From the graph, $199 \le T \le 201 \quad \Rightarrow \quad 32.89 < w < 33.11$.

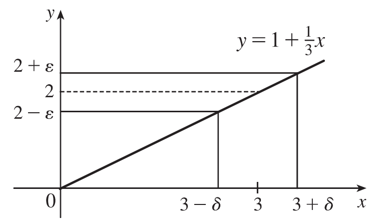
(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

13. (a) $|4x - 8| = 4\,|x - 2| < 0.1 \quad \Leftrightarrow \quad |x - 2| < \dfrac{0.1}{4}$, so $\delta = \dfrac{0.1}{4} = 0.025$.

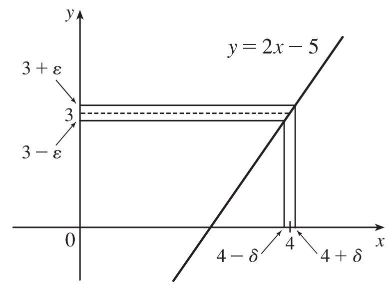
(b) $|4x - 8| = 4\,|x - 2| < 0.01 \quad \Leftrightarrow \quad |x - 2| < \dfrac{0.01}{4}$, so $\delta = \dfrac{0.01}{4} = 0.0025$.

14. $|(5x - 7) - 3| = |5x - 10| = |5(x - 2)| = 5\,|x - 2|$. We must have $|f(x) - L| < \varepsilon$, so $5\,|x - 2| < \varepsilon \quad \Leftrightarrow$ $|x - 2| < \varepsilon/5$. Thus, choose $\delta = \varepsilon/5$. For $\varepsilon = 0.1$, $\delta = 0.02$; for $\varepsilon = 0.05$, $\delta = 0.01$; for $\varepsilon = 0.01$, $\delta = 0.002$.

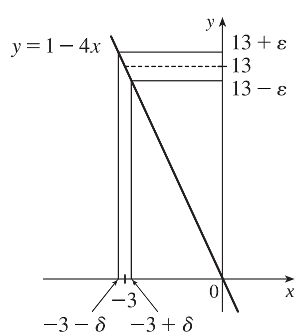
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left|(1 + \frac{1}{3}x) - 2\right| < \varepsilon$. But $\left|(1 + \frac{1}{3}x) - 2\right| < \varepsilon \quad \Leftrightarrow \quad \left|\frac{1}{3}x - 1\right| < \varepsilon \quad \Leftrightarrow$ $\left|\frac{1}{3}\right|\,|x - 3| < \varepsilon \quad \Leftrightarrow \quad |x - 3| < 3\varepsilon$. So if we choose $\delta = 3\varepsilon$, then $0 < |x - 3| < \delta \quad \Rightarrow \quad \left|(1 + \frac{1}{3}x) - 2\right| < \varepsilon$. Thus, $\lim\limits_{x \to 3}(1 + \frac{1}{3}x) = 2$ by the definition of a limit.



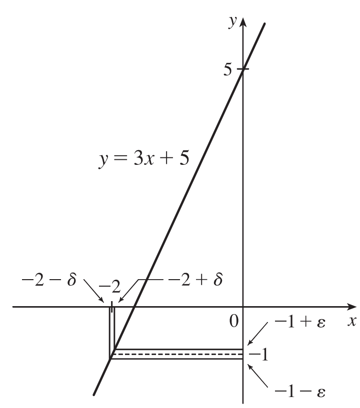
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $|(2x - 5) - 3| < \varepsilon$. But $|(2x - 5) - 3| < \varepsilon \quad \Leftrightarrow \quad |2x - 8| < \varepsilon \quad \Leftrightarrow$ $|2|\,|x - 4| < \varepsilon \quad \Leftrightarrow \quad |x - 4| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x - 4| < \delta \quad \Rightarrow \quad |(2x - 5) - 3| < \varepsilon$. Thus, $\lim\limits_{x \to 4}(2x - 5) = 3$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-3)| < \delta$, then $|(1 - 4x) - 13| < \varepsilon$. But $|(1 - 4x) - 13| < \varepsilon \quad \Leftrightarrow$ $|-4x - 12| < \varepsilon \quad \Leftrightarrow \quad |-4|\,|x + 3| < \varepsilon \quad \Leftrightarrow \quad |x - (-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x - (-3)| < \delta \quad \Rightarrow \quad |(1 - 4x) - 13| < \varepsilon$. Thus, $\lim\limits_{x \to -3}(1 - 4x) = 13$ by the definition of a limit.



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $|(3x + 5) - (-1)| < \varepsilon$. But $|(3x + 5) - (-1)| < \varepsilon \quad \Leftrightarrow$ $|3x + 6| < \varepsilon \quad \Leftrightarrow \quad |3|\,|x + 2| < \varepsilon \quad \Leftrightarrow \quad |x + 2| < \varepsilon/3$. So if we choose $\delta = \varepsilon/3$, then $0 < |x + 2| < \delta \quad \Rightarrow \quad |(3x + 5) - (-1)| < \varepsilon$. Thus, $\lim\limits_{x \to -2}(3x + 5) = -1$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $\left|\dfrac{2+4x}{3} - 2\right| < \varepsilon$. But $\left|\dfrac{2+4x}{3} - 2\right| < \varepsilon \quad\Leftrightarrow$ $\left|\dfrac{4x-4}{3}\right| < \varepsilon \quad\Leftrightarrow\quad \left|\frac{4}{3}\right| |x-1| < \varepsilon \quad\Leftrightarrow\quad |x-1| < \frac{3}{4}\varepsilon$. So if we choose $\delta = \frac{3}{4}\varepsilon$, then $0 < |x-1| < \delta \quad\Rightarrow$ $\left|\dfrac{2+4x}{3} - 2\right| < \varepsilon$. Thus, $\displaystyle\lim_{x\to 1} \frac{2+4x}{3} = 2$ by the definition of a limit.

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 10| < \delta$, then $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon$. But $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon \quad\Leftrightarrow$ $\left|8 - \frac{4}{5}x\right| < \varepsilon \quad\Leftrightarrow\quad \left|-\frac{4}{5}\right| |x-10| < \varepsilon \quad\Leftrightarrow\quad |x-10| < \frac{5}{4}\varepsilon$. So if we choose $\delta = \frac{5}{4}\varepsilon$, then $0 < |x-10| < \delta \quad\Rightarrow$ $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon$. Thus, $\displaystyle\lim_{x\to 10}(3 - \tfrac{4}{5}x) = -5$ by the definition of a limit.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|\dfrac{x^2+x-6}{x-2} - 5\right| < \varepsilon \quad\Leftrightarrow$ $\left|\dfrac{(x+3)(x-2)}{x-2} - 5\right| < \varepsilon \quad\Leftrightarrow\quad |x+3-5| < \varepsilon \quad [x \neq 2] \quad\Leftrightarrow\quad |x-2| < \varepsilon$. So choose $\delta = \varepsilon$. Then $0 < |x-2| < \delta \quad\Rightarrow\quad |x-2| < \varepsilon \quad\Rightarrow\quad |x+3-5| < \varepsilon \quad\Rightarrow\quad \left|\dfrac{(x+3)(x-2)}{x-2} - 5\right| < \varepsilon \quad [x \neq 2] \quad\Rightarrow$ $\left|\dfrac{x^2+x-6}{x-2} - 5\right| < \varepsilon$. By the definition of a limit, $\displaystyle\lim_{x\to 2} \frac{x^2+x-6}{x-2} = 5$.

22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x + 1.5| < \delta$, then $\left|\dfrac{9-4x^2}{3+2x} - 6\right| < \varepsilon \quad\Leftrightarrow$ $\left|\dfrac{(3+2x)(3-2x)}{3+2x} - 6\right| < \varepsilon \quad\Leftrightarrow\quad |3-2x-6| < \varepsilon \quad [x \neq -1.5] \quad\Leftrightarrow\quad |-2x-3| < \varepsilon \quad\Leftrightarrow\quad |-2|\,|x+1.5| < \varepsilon \quad\Leftrightarrow$ $|x+1.5| < \varepsilon/2$. So choose $\delta = \varepsilon/2$. Then $0 < |x+1.5| < \delta \quad\Rightarrow\quad |x+1.5| < \varepsilon/2 \quad\Rightarrow\quad |-2|\,|x+1.5| < \varepsilon \quad\Rightarrow$ $|-2x-3| < \varepsilon \quad\Rightarrow\quad |3-2x-6| < \varepsilon \quad\Rightarrow\quad \left|\dfrac{(3+2x)(3-2x)}{3+2x} - 6\right| < \varepsilon \quad [x \neq -1.5] \quad\Rightarrow\quad \left|\dfrac{9-4x^2}{3+2x} - 6\right| < \varepsilon$.

By the definition of a limit, $\displaystyle\lim_{x\to -1.5} \frac{9-4x^2}{3+2x} = 6$.

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \varepsilon$. So $\delta = \varepsilon$ will work.

24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|c - c| < \varepsilon$. But $|c - c| = 0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^2 - 0\right| < \varepsilon \quad\Leftrightarrow\quad x^2 < \varepsilon \quad\Leftrightarrow\quad |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $0 < |x-0| < \delta \quad\Rightarrow\quad \left|x^2 - 0\right| < \varepsilon$. Thus, $\displaystyle\lim_{x\to 0} x^2 = 0$ by the definition of a limit.

26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^3 - 0\right| < \varepsilon \quad\Leftrightarrow\quad |x|^3 < \varepsilon \quad\Leftrightarrow\quad |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$. Then $0 < |x-0| < \delta \quad\Rightarrow\quad \left|x^3 - 0\right| < \delta^3 = \varepsilon$. Thus, $\displaystyle\lim_{x\to 0} x^3 = 0$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\big||x| - 0\big| < \varepsilon$. But $\big||x|\big| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\displaystyle\lim_{x\to 0} |x| = 0$ by the definition of a limit.

28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < x - (-6) < \delta$, then $\left|\sqrt[8]{6+x} - 0\right| < \varepsilon$. But $\left|\sqrt[8]{6+x} - 0\right| < \varepsilon \;\Leftrightarrow\; \sqrt[8]{6+x} < \varepsilon \;\Leftrightarrow\; 6 + x < \varepsilon^8 \;\Leftrightarrow\; x - (-6) < \varepsilon^8$. So if we choose $\delta = \varepsilon^8$, then $0 < x - (-6) < \delta \;\Rightarrow\; \left|\sqrt[8]{6+x} - 0\right| < \varepsilon$. Thus, $\lim\limits_{x \to -6^+} \sqrt[8]{6+x} = 0$ by the definition of a right-hand limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|\left(x^2 - 4x + 5\right) - 1\right| < \varepsilon \;\Leftrightarrow\; \left|x^2 - 4x + 4\right| < \varepsilon \;\Leftrightarrow\; \left|(x-2)^2\right| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x-2| < \delta \;\Leftrightarrow\; |x-2| < \sqrt{\varepsilon} \;\Leftrightarrow\; \left|(x-2)^2\right| < \varepsilon$. Thus, $\lim\limits_{x \to 2} \left(x^2 - 4x + 5\right) = 1$ by the definition of a limit.

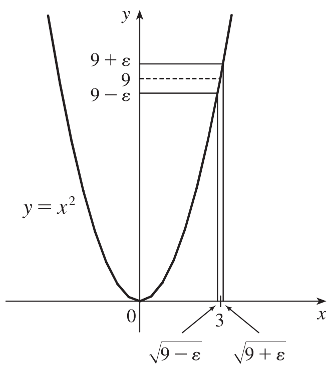
30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|(x^2 + 2x - 7) - 1\right| < \varepsilon$. But $\left|(x^2 + 2x - 7) - 1\right| < \varepsilon \;\Leftrightarrow\; \left|x^2 + 2x - 8\right| < \varepsilon \;\Leftrightarrow\; |x+4|\,|x-2| < \varepsilon$. Thus our goal is to make $|x-2|$ small enough so that its product with $|x+4|$ is less than ε. Suppose we first require that $|x-2| < 1$. Then $-1 < x - 2 < 1 \;\Rightarrow\; 1 < x < 3 \;\Rightarrow\; 5 < x + 4 < 7 \;\Rightarrow\; |x+4| < 7$, and this gives us $7\,|x-2| < \varepsilon \;\Rightarrow\; |x-2| < \varepsilon/7$. Choose $\delta = \min\{1, \varepsilon/7\}$. Then if $0 < |x-2| < \delta$, we have $|x-2| < \varepsilon/7$ and $|x+4| < 7$, so $\left|(x^2 + 2x - 7) - 1\right| = |(x+4)(x-2)| = |x+4|\,|x-2| < 7(\varepsilon/7) = \varepsilon$, as desired. Thus, $\lim\limits_{x \to 2}(x^2 + 2x - 7) = 1$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $\left|\left(x^2 - 1\right) - 3\right| < \varepsilon$ or upon simplifying we need $\left|x^2 - 4\right| < \varepsilon$ whenever $0 < |x+2| < \delta$. Notice that if $|x+2| < 1$, then $-1 < x + 2 < 1 \;\Rightarrow\; -5 < x - 2 < -3 \;\Rightarrow\; |x-2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x+2| < \delta \;\Rightarrow\; |x-2| < 5$ and $|x+2| < \varepsilon/5$, so $\left|\left(x^2 - 1\right) - 3\right| = |(x+2)(x-2)| = |x+2|\,|x-2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to -2}(x^2 - 1) = 3$.

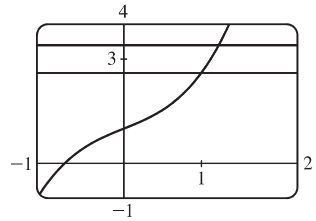
32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left|x^3 - 8\right| < \varepsilon$. Now $\left|x^3 - 8\right| = \left|(x-2)\left(x^2 + 2x + 4\right)\right|$. If $|x-2| < 1$, that is, $1 < x < 3$, then $x^2 + 2x + 4 < 3^2 + 2(3) + 4 = 19$ and so $\left|x^3 - 8\right| = |x-2|\left(x^2 + 2x + 4\right) < 19\,|x-2|$. So if we take $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$, then $0 < |x-2| < \delta \;\Rightarrow\; \left|x^3 - 8\right| = |x-2|\left(x^2 + 2x + 4\right) < \frac{\varepsilon}{19} \cdot 19 = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to 2} x^3 = 8$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x-3| < \delta$, then $|x-3| < 2 \;\Rightarrow\; -2 < x - 3 < 2 \;\Rightarrow\; 4 < x + 3 < 8 \;\Rightarrow\; |x+3| < 8$. Also $|x-3| < \frac{\varepsilon}{8}$, so $\left|x^2 - 9\right| = |x+3|\,|x-3| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim\limits_{x \to 3} x^2 = 9$.

34. From the figure, our choices for δ are $\delta_1 = 3 - \sqrt{9 - \varepsilon}$ and $\delta_2 = \sqrt{9 + \varepsilon} - 3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1, \delta_2\}$; that is, $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = \sqrt{9 + \varepsilon} - 3$.



35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\varepsilon) = \frac{\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{2/3} - 12}{6\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\right)^{1/3}}.$$ Thus, $\delta = x(\varepsilon) - 1$.

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093\,272\,342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. *1. Guessing a value for δ* Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$ whenever $0 < |x - 2| < \delta$. But $\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{|2x|} < \varepsilon$. We find a positive constant C such that $\frac{1}{|2x|} < C \Rightarrow \frac{|x-2|}{|2x|} < C\,|x-2|$ and we can make $C\,|x-2| < \varepsilon$ by taking $|x-2| < \frac{\varepsilon}{C} = \delta$. We restrict x to lie in the interval $|x-2| < 1 \Rightarrow 1 < x < 3$ so $1 > \frac{1}{x} > \frac{1}{3} \Rightarrow \frac{1}{6} < \frac{1}{2x} < \frac{1}{2} \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$. So $C = \frac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\varepsilon\}$.

2. Showing that δ works Given $\varepsilon > 0$ we let $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x-2| < \delta$, then $|x-2| < 1 \Rightarrow 1 < x < 3 \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$ (as in part 1). Also $|x-2| < 2\varepsilon$, so $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{|2x|} < \frac{1}{2} \cdot 2\varepsilon = \varepsilon$. This shows that $\lim\limits_{x\to 2}(1/x) = \frac{1}{2}$.

37. *1. Guessing a value for δ* Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x - a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then $\frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{|x-a|}{C} < \varepsilon$, and we take $|x-a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a. If $|x-a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x - a < \frac{1}{2}a \Rightarrow \frac{1}{2}a < x < \frac{3}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a} + \sqrt{a}$ is a suitable choice for the constant. So $|x-a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon\right\}$. If $0 < |x-a| < \delta$, then $|x-a| < \frac{1}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$ (as in part 1). Also $|x-a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a}\right)\varepsilon$, so

$$|\sqrt{x} - \sqrt{a}\,| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{\left(\sqrt{a/2} + \sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2} + \sqrt{a}\right)} = \varepsilon.$$ Therefore, $\lim\limits_{x\to a}\sqrt{x} = \sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim_{t\to 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \;\Rightarrow\; |H(t) - L| < \frac{1}{2} \;\Leftrightarrow$ $L - \frac{1}{2} < H(t) < L + \frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L + \frac{1}{2} \;\Rightarrow\; L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L - \frac{1}{2} < 0 \;\Rightarrow\; L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim_{t\to 0} H(t)$ does not exist.

39. Suppose that $\lim_{x\to 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \;\Rightarrow\; |f(x) - L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L \le |L| < \frac{1}{2}$. Now take any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim_{x\to 0} f(x)$ does not exist.

40. First suppose that $\lim_{x\to a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x - a| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$. Then $a - \delta < x < a \;\Rightarrow\; 0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Thus, $\lim_{x\to a^-} f(x) = L$. Also $a < x < a + \delta \;\Rightarrow$ $0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Hence, $\lim_{x\to a^+} f(x) = L$.

Now suppose $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim_{x\to a^-} f(x) = L$, there exists $\delta_1 > 0$ so that $a - \delta_1 < x < a \;\Rightarrow\; |f(x) - L| < \varepsilon$. Since $\lim_{x\to a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a + \delta_2 \;\Rightarrow$ $|f(x) - L| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x - a| < \delta \;\Rightarrow\; a - \delta_1 < x < a$ or $a < x < a + \delta_2$ so $|f(x) - L| < \varepsilon$. Hence, $\lim_{x\to a} f(x) = L$. So we have proved that $\lim_{x\to a} f(x) = L \;\Leftrightarrow\; \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$.

41. $\dfrac{1}{(x+3)^4} > 10{,}000 \;\Leftrightarrow\; (x+3)^4 < \dfrac{1}{10{,}000} \;\Leftrightarrow\; |x+3| < \dfrac{1}{\sqrt[4]{10{,}000}} \;\Leftrightarrow\; |x - (-3)| < \dfrac{1}{10}$

42. Given $M > 0$, we need $\delta > 0$ such that $0 < |x+3| < \delta \;\Rightarrow\; 1/(x+3)^4 > M$. Now $\dfrac{1}{(x+3)^4} > M \;\Leftrightarrow$ $(x+3)^4 < \dfrac{1}{M} \;\Leftrightarrow\; |x+3| < \dfrac{1}{\sqrt[4]{M}}$. So take $\delta = \dfrac{1}{\sqrt[4]{M}}$. Then $0 < |x+3| < \delta = \dfrac{1}{\sqrt[4]{M}} \;\Rightarrow\; \dfrac{1}{(x+3)^4} > M$, so $\lim_{x\to -3} \dfrac{1}{(x+3)^4} = \infty$.

43. Let $N < 0$ be given. Then, for $x < -1$, we have $\dfrac{5}{(x+1)^3} < N \;\Leftrightarrow\; \dfrac{5}{N} < (x+1)^3 \;\Leftrightarrow\; \sqrt[3]{\dfrac{5}{N}} < x + 1$.
Let $\delta = -\sqrt[3]{\dfrac{5}{N}}$. Then $-1 - \delta < x < -1 \;\Rightarrow\; \sqrt[3]{\dfrac{5}{N}} < x + 1 < 0 \;\Rightarrow\; \dfrac{5}{(x+1)^3} < N$, so $\lim_{x\to -1^-} \dfrac{5}{(x+1)^3} = -\infty$.

44. (a) Let M be given. Since $\lim_{x\to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \;\Rightarrow\; f(x) > M + 1 - c$. Since $\lim_{x\to a} g(x) = c$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \;\Rightarrow\; |g(x) - c| < 1 \;\Rightarrow\; g(x) > c - 1$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x - a| < \delta \;\Rightarrow\; f(x) + g(x) > (M + 1 - c) + (c - 1) = M$. Thus, $\lim_{x\to a} [f(x) + g(x)] = \infty$.

(b) Let $M > 0$ be given. Since $\lim_{x \to a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \quad \Rightarrow$ $|g(x) - c| < c/2 \quad \Rightarrow \quad g(x) > c/2$. Since $\lim_{x \to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \quad \Rightarrow$ $f(x) > 2M/c$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta \quad \Rightarrow \quad f(x)\,g(x) > \dfrac{2M}{c}\dfrac{c}{2} = M$, so $\lim_{x \to a} f(x)\,g(x) = \infty$.

(c) Let $N < 0$ be given. Since $\lim_{x \to a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \quad \Rightarrow$ $|g(x) - c| < -c/2 \quad \Rightarrow \quad g(x) < c/2$. Since $\lim_{x \to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \quad \Rightarrow$ $f(x) > 2N/c$. (Note that $c < 0$ and $N < 0 \quad \Rightarrow \quad 2N/c > 0$.) Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta \quad \Rightarrow$ $f(x) > 2N/c \quad \Rightarrow \quad f(x)\,g(x) < \dfrac{2N}{c} \cdot \dfrac{c}{2} = N$, so $\lim_{x \to a} f(x)\,g(x) = -\infty$.

1.8 Continuity

1. From Definition 1, $\lim_{x \to 4} f(x) = f(4)$.

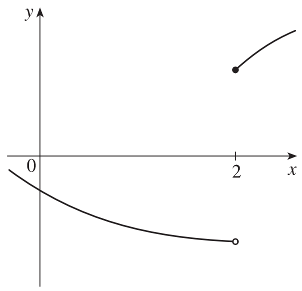
2. The graph of f has no hole, jump, or vertical asymptote.

3. (a) f is discontinuous at -4 since $f(-4)$ is not defined and at -2, 2, and 4 since the limit does not exist (the left and right limits are not the same).

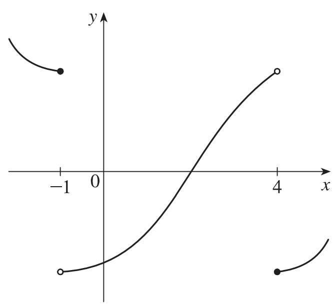
   (b) f is continuous from the left at -2 since $\lim_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim_{x \to 2^+} f(x) = f(2)$ and $\lim_{x \to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

4. g is continuous on $[-4, -2)$, $(-2, 2)$, $[2, 4)$, $(4, 6)$, and $(6, 8)$.

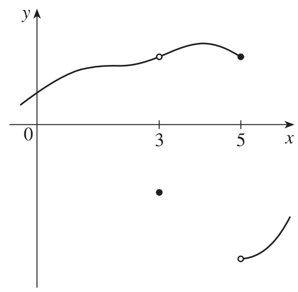
5. The graph of $y = f(x)$ must have a discontinuity at $x = 2$ and must show that $\lim_{x \to 2^+} f(x) = f(2)$.



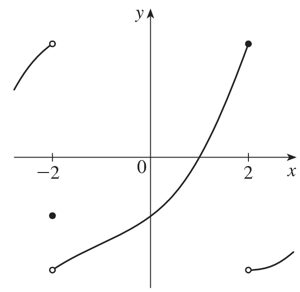
6. The graph of $y = f(x)$ must have discontinuities at $x = -1$ and $x = 4$. It must show that $\lim_{x \to -1^-} f(x) = f(-1)$ and $\lim_{x \to 4^+} f(x) = f(4)$.



7. The graph of $y = f(x)$ must have a removable discontinuity (a hole) at $x = 3$ and a jump discontinuity at $x = 5$.

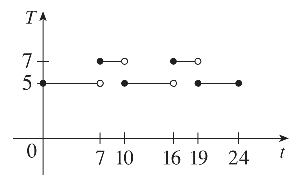


8. The graph of $y = f(x)$ must have a discontinuity at $x = -2$ with $\lim_{x\to -2^-} f(x) \neq f(-2)$ and $\lim_{x\to -2^+} f(x) \neq f(-2)$. It must also show that $\lim_{x\to 2^-} f(x) = f(2)$ and $\lim_{x\to 2^+} f(x) \neq f(2)$.



9. (a) The toll is \$7 between 7:00 AM and 10:00 AM and between 4:00 PM and 7:00 PM.

(b) The function T has jump discontinuities at $t = 7$, 10, 16, and 19. Their significance to someone who uses the road is that, because of the sudden jumps in the toll, they may want to avoid the higher rates between $t = 7$ and $t = 10$ and between $t = 16$ and $t = 19$ if feasible.



10. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

11. If f and g are continuous and $g(2) = 6$, then $\lim_{x\to 2}[3f(x) + f(x)\,g(x)] = 36 \quad\Rightarrow$

$$3\lim_{x\to 2} f(x) + \lim_{x\to 2} f(x) \cdot \lim_{x\to 2} g(x) = 36 \quad\Rightarrow\quad 3f(2) + f(2)\cdot 6 = 36 \quad\Rightarrow\quad 9f(2) = 36 \quad\Rightarrow\quad f(2) = 4.$$

12. $\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(3x^4 - 5x + \sqrt[3]{x^2+4}\right) = 3\lim_{x\to 2} x^4 - 5\lim_{x\to 2} x + \sqrt[3]{\lim_{x\to 2}(x^2+4)}$

$$= 3(2)^4 - 5(2) + \sqrt[3]{2^2+4} = 48 - 10 + 2 = 40 = f(2)$$

By the definition of continuity, f is continuous at $a = 2$.

13. $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \left(x + 2x^3\right)^4 = \left(\lim\limits_{x\to -1} x + 2 \lim\limits_{x\to -1} x^3\right)^4 = \left[-1 + 2(-1)^3\right]^4 = (-3)^4 = 81 = f(-1).$

By the definition of continuity, f is continuous at $a = -1$.

14. $\lim\limits_{t\to 1} h(t) = \lim\limits_{t\to 1} \dfrac{2t - 3t^2}{1 + t^3} = \dfrac{\lim\limits_{t\to 1}(2t - 3t^2)}{\lim\limits_{t\to 1}(1 + t^3)} = \dfrac{2\lim\limits_{t\to 1} t - 3\lim\limits_{t\to 1} t^2}{\lim\limits_{t\to 1} 1 + \lim\limits_{t\to 1} t^3} = \dfrac{2(1) - 3(1)^2}{1 + (1)^3} = \dfrac{-1}{2} = h(1).$

By the definition of continuity, h is continuous at $a = 1$.

15. For $a > 2$, we have

$$\begin{aligned} \lim_{x\to a} f(x) = \lim_{x\to a} \frac{2x+3}{x-2} &= \frac{\lim\limits_{x\to a}(2x+3)}{\lim\limits_{x\to a}(x-2)} && \text{[Limit Law 5]} \\ &= \frac{2\lim\limits_{x\to a} x + \lim\limits_{x\to a} 3}{\lim\limits_{x\to a} x - \lim\limits_{x\to a} 2} && \text{[1, 2, and 3]} \\ &= \frac{2a+3}{a-2} && \text{[7 and 8]} \\ &= f(a) \end{aligned}$$

Thus, f is continuous at $x = a$ for every a in $(2, \infty)$; that is, f is continuous on $(2, \infty)$.

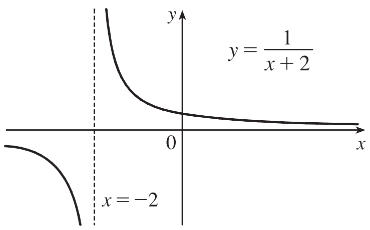
16. For $a < 3$, we have

$$\begin{aligned} \lim_{x\to a} g(x) &= \lim_{x\to a} 2\sqrt{3-x} \\ &= 2\lim_{x\to a} \sqrt{3-x} && \text{[Limit Law 3]} \\ &= 2\sqrt{\lim_{x\to a}(3-x)} && \text{[11]} \\ &= 2\sqrt{\lim_{x\to a} 3 - \lim_{x\to a} x} && \text{[2]} \\ &= 2\sqrt{3-a} && \text{[7 and 8]} \\ &= g(a) \end{aligned}$$

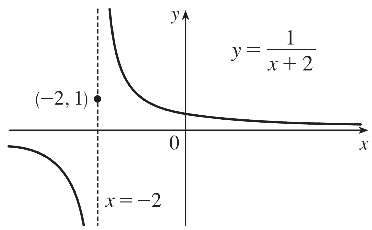
So g is continuous at $x = a$ for every a in $(-\infty, 3)$. Also, $\lim\limits_{x\to 3^-} g(x) = 0 = g(3)$, so g is continuous from the left at 3.

Thus, g is continuous on $(-\infty, 3]$.

17. $f(x) = \dfrac{1}{x+2}$ is discontinuous at $a = -2$ because $f(-2)$ is undefined.

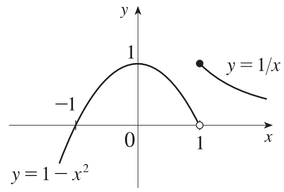


18. $f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$



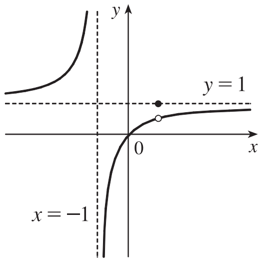
Here $f(-2) = 1$, but $\lim_{x \to -2^-} f(x) = -\infty$ and $\lim_{x \to -2^+} f(x) = \infty$, so $\lim_{x \to -2} f(x)$ does not exist and f is discontinuous at -2.

19. $f(x) = \begin{cases} 1 - x^2 & \text{if } x < 1 \\ 1/x & \text{if } x \geq 1 \end{cases}$



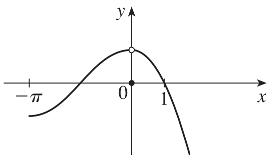
The left-hand limit of f at $a = 1$ is $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (1 - x^2) = 0$. The right-hand limit of f at $a = 1$ is $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1/x) = 1$. Since these limits are not equal, $\lim_{x \to 1} f(x)$ does not exist and f is discontinuous at 1.

20. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$



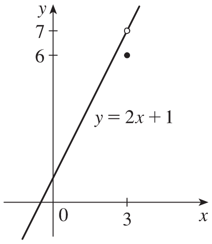
$\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \dfrac{x(x-1)}{(x+1)(x-1)} = \lim_{x \to 1} \dfrac{x}{x+1} = \dfrac{1}{2}$, but $f(1) = 1$, so f is discontinous at 1.

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$



$\lim_{x \to 0} f(x) = 1$, but $f(0) = 0 \neq 1$, so f is discontinuous at 0.

22. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$



$\lim_{x \to 3} f(x) = \lim_{x \to 3} \dfrac{2x^2 - 5x - 3}{x - 3} = \lim_{x \to 3} \dfrac{(2x+1)(x-3)}{x-3} = \lim_{x \to 3} (2x + 1) = 7$, but $f(3) = 6$, so f is discontinuous at 3.

23. $f(x) = \dfrac{x^2 - x - 2}{x - 2} = \dfrac{(x-2)(x+1)}{x-2} = x + 1$ for $x \neq 2$. Since $\lim_{x \to 2} f(x) = 2 + 1 = 3$, define $f(2) = 3$. Then f is continuous at 2.

24. $f(x) = \dfrac{x^3 - 8}{x^2 - 4} = \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \dfrac{x^2+2x+4}{x+2}$ for $x \neq 2$. Since $\lim\limits_{x \to 2} f(x) = \dfrac{4+4+4}{2+2} = 3$, define $f(2) = 3$. Then f is continuous at 2.

25. $F(x) = \dfrac{2x^2 - x - 1}{x^2 + 1}$ is a rational function, so it is continuous on its domain, $(-\infty, \infty)$, by Theorem 5(b).

26. $G(x) = \dfrac{x^2+1}{2x^2 - x - 1} = \dfrac{x^2+1}{(2x+1)(x-1)}$ is a rational function, so it is continuous on its domain, $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right) \cup (1, \infty)$, by Theorem 5(b).

27. $x^3 - 2 = 0 \quad \Rightarrow \quad x^3 = 2 \quad \Rightarrow \quad x = \sqrt[3]{2}$, so $Q(x) = \dfrac{\sqrt[3]{x-2}}{x^3 - 2}$ has domain $\left(-\infty, \sqrt[3]{2}\right) \cup \left(\sqrt[3]{2}, \infty\right)$. Now $x^3 - 2$ is continuous everywhere by Theorem 5(a) and $\sqrt[3]{x-2}$ is continuous everywhere by Theorems 5(a), 7, and 9. Thus, Q is continuous on its domain by part 5 of Theorem 4.

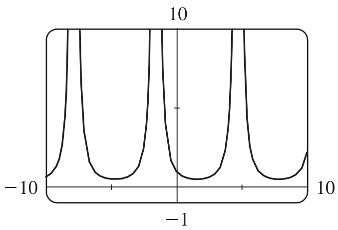
28. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x + 1$ are continuous on $\mathbb{R}$. By part 5 of Theorem 4, $h(x) = \dfrac{\sin x}{x+1}$ is continuous on its domain, $\{x \mid x \neq -1\}$.

29. By Theorem 5, the polynomial $1 - x^2$ is continuous on $(-\infty, \infty)$. By Theorem 7, $\cos$ is continuous on its domain, $\mathbb{R}$. By Theorem 9, $\cos\left(1 - x^2\right)$ is continuous on its domain, which is $\mathbb{R}$.

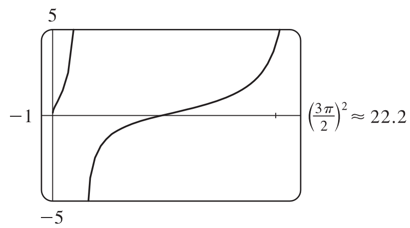
30. By Theorem 7, the trigonometric function $\tan x$ is continuous on its domain, $\left\{x \mid x \neq \frac{\pi}{2} + \pi n\right\}$. By Theorems 5(a), 7, and 9, the composite function $\sqrt{4 - x^2}$ is continuous on its domain $[-2, 2]$. By part 5 of Theorem 4, $B(x) = \dfrac{\tan x}{\sqrt{4 - x^2}}$ is continuous on its domain, $(-2, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 2)$.

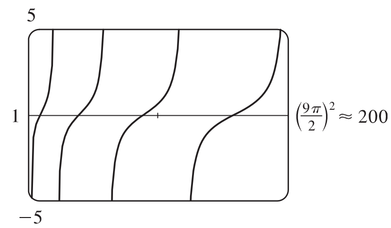
31. $M(x) = \sqrt{1 + \dfrac{1}{x}} = \sqrt{\dfrac{x+1}{x}}$ is defined when $\dfrac{x+1}{x} \geq 0 \quad \Rightarrow \quad x + 1 \geq 0$ and $x > 0$ or $x + 1 \leq 0$ and $x < 0 \quad \Rightarrow \quad x > 0$ or $x \leq -1$, so M has domain $(-\infty, -1] \cup (0, \infty)$. M is the composite of a root function and a rational function, so it is continuous at every number in its domain by Theorems 7 and 9.

32. The sine and cosine functions are continuous everywhere by Theorem 7, so $F(x) = \sin(\cos(\sin x))$, which is the composite of sine, cosine, and (once again) sine, is continuous everywhere by Theorem 9.

33. 

$y = \dfrac{1}{1 + \sin x}$ is undefined and hence discontinuous when $1 + \sin x = 0 \quad \Leftrightarrow \quad \sin x = -1 \quad \Leftrightarrow \quad x = -\frac{\pi}{2} + 2\pi n$, n an integer. The figure shows discontinuities for $n = -1$, 0, and 1; that is, $-\dfrac{5\pi}{2} \approx -7.85$, $-\dfrac{\pi}{2} \approx -1.57$, and $\dfrac{3\pi}{2} \approx 4.71$.

34. 



The function $y = f(x) = \tan\sqrt{x}$ is continuous throughout its domain because it is the composite of a trigonometric function and a root function. The square root function has domain $[0, \infty)$ and the tangent function has domain $\left\{x \mid x \neq \frac{\pi}{2} + \pi n\right\}$. So f is discontinuous when $x < 0$ and when $\sqrt{x} = \frac{\pi}{2} + \pi n \quad \Rightarrow \quad x = \left(\frac{\pi}{2} + \pi n\right)^2$, where n is a nonnegative integer. Note that as x increases, the distance between discontinuities increases.

35. Because we are dealing with root functions, $5 + \sqrt{x}$ is continuous on $[0, \infty)$, $\sqrt{x+5}$ is continuous on $[-5, \infty)$, so the quotient $f(x) = \dfrac{5+\sqrt{x}}{\sqrt{5+x}}$ is continuous on $[0, \infty)$. Since f is continuous at $x = 4$, $\lim\limits_{x\to 4} f(x) = f(4) = \frac{7}{3}$.

36. Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function $f(x) = \sin(x + \sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x\to\pi} f(x) = f(\pi) = \sin(\pi + \sin\pi) = \sin\pi = 0$.

37. Because x and $\cos x$ are continuous on $\mathbb{R}$, so is $f(x) = x\cos^2 x$. Since f is continuous at $x = \frac{\pi}{4}$,

$$\lim_{x\to\pi/4} f(x) = f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{4}\cdot\frac{1}{2} = \frac{\pi}{8}.$$

38. $x^3 - 3x + 1 = 0$ for three values of x, but 2 is not one of them. Thus, $f(x) = (x^3 - 3x + 1)^{-3}$ is continuous at $x = 2$ and $\lim\limits_{x\to 2} f(x) = f(2) = (8 - 6 + 1)^{-3} = 3^{-3} = \frac{1}{27}$.

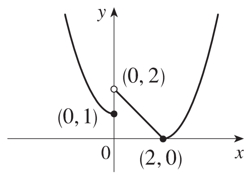
39. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty, 1)$, f is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1, \infty)$, f is continuous on $(1, \infty)$. At $x = 1$, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x^2 = 1$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \sqrt{x} = 1$. Thus, $\lim\limits_{x\to 1} f(x)$ exists and equals 1. Also, $f(1) = \sqrt{1} = 1$. Thus, f is continuous at $x = 1$. We conclude that f is continuous on $(-\infty, \infty)$.

40. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

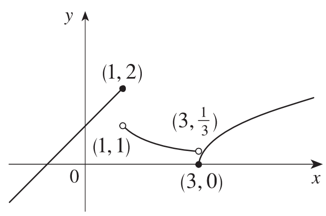
By Theorem 7, the trigonometric functions are continuous. Since $f(x) = \sin x$ on $(-\infty, \pi/4)$ and $f(x) = \cos x$ on $(\pi/4, \infty)$, f is continuous on $(-\infty, \pi/4) \cup (\pi/4, \infty)$. $\lim\limits_{x\to(\pi/4)^-} f(x) = \lim\limits_{x\to(\pi/4)^-} \sin x = \sin\frac{\pi}{4} = 1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim\limits_{x\to(\pi/4)^+} f(x) = \lim\limits_{x\to(\pi/4)^+} \cos x = 1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim\limits_{x\to(\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty, \infty)$.

41. $f(x) = \begin{cases} 1+x^2 & \text{if } x \le 0 \\ 2-x & \text{if } 0 < x \le 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$



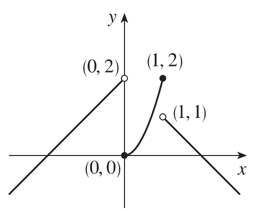
f is continuous on $(-\infty, 0)$, $(0, 2)$, and $(2, \infty)$ since it is a polynomial on each of these intervals. Now $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(1+x^2) = 1$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}(2-x) = 2$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the left at 0. Also, $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-}(2-x) = 0$, $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}(x-2)^2 = 0$, and $f(2) = 0$, so f is continuous at 2. The only number at which f is discontinuous is 0.

42. $f(x) = \begin{cases} x+1 & \text{if } x \le 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x-3} & \text{if } x \ge 3 \end{cases}$



f is continuous on $(-\infty, 1)$, $(1, 3)$, and $(3, \infty)$, where it is a polynomial, a rational function, and a composite of a root function with a polynomial, respectively. Now $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(x+1) = 2$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(1/x) = 1$, so f is discontinuous at 1. Since $f(1) = 2$, f is continuous from the left at 1. Also, $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-}(1/x) = 1/3$, and $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+}\sqrt{x-3} = 0 = f(3)$, so f is discontinuous at 3, but it is continuous from the right at 3.

43. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x > 1 \end{cases}$



f is continuous on $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$ since on each of these intervals it is a polynomial. Now $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}(x+2) = 2$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2x^2 = 0$, so f is discontinuous at 0. Since $f(0) = 0$, f is continuous from the right at 0. Also $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 2x^2 = 2$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(2-x) = 1$, so f is discontinuous at 1. Since $f(1) = 2$, f is continuous from the left at 1.

44. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r = R$.
$\lim\limits_{r\to R^-} F(r) = \lim\limits_{r\to R^-} \dfrac{GMr}{R^3} = \dfrac{GM}{R^2}$ and $\lim\limits_{r\to R^+} F(r) = \lim\limits_{r\to R^+} \dfrac{GM}{r^2} = \dfrac{GM}{R^2}$, so $\lim\limits_{r\to R} F(r) = \dfrac{GM}{R^2}$. Since $F(R) = \dfrac{GM}{R^2}$, F is continuous at R. Therefore, F is a continuous function of r.

45. $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$

f is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \left(cx^2 + 2x\right) = 4c + 4$ and

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} \left(x^3 - cx\right) = 8 - 2c$. So f is continuous $\Leftrightarrow$ $4c + 4 = 8 - 2c$ $\Leftrightarrow$ $6c = 4$ $\Leftrightarrow$ $c = \frac{2}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$.

46. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$

At $x = 2$: $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \dfrac{x^2 - 4}{x - 2} = \lim\limits_{x\to 2^-} \dfrac{(x+2)(x-2)}{x - 2} = \lim\limits_{x\to 2^-} (x + 2) = 2 + 2 = 4$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (ax^2 - bx + 3) = 4a - 2b + 3$

We must have $4a - 2b + 3 = 4$, or $\mathbf{4a - 2b = 1}$ **(1)**.

At $x = 3$: $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (2x - a + b) = 6 - a + b$

We must have $9a - 3b + 3 = 6 - a + b$, or $\mathbf{10a - 4b = 3}$ **(2)**.

Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{aligned} -8a + 4b &= -2 \\ 10a - 4b &= 3 \\ \hline 2a \qquad &= 1 \end{aligned}$$

So $a = \frac{1}{2}$. Substituting $\frac{1}{2}$ for a in **(1)** gives us $-2b = -1$, so $b = \frac{1}{2}$ as well. Thus, for f to be continuous on $(-\infty, \infty)$, $a = b = \frac{1}{2}$.

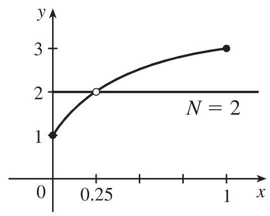
47. (a) $f(x) = \dfrac{x^4 - 1}{x - 1} = \dfrac{(x^2 + 1)(x^2 - 1)}{x - 1} = \dfrac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = (x^2 + 1)(x + 1)$ [or $x^3 + x^2 + x + 1$]

for $x \neq 1$. The discontinuity is removable and $g(x) = x^3 + x^2 + x + 1$ agrees with f for $x \neq 1$ and is continuous on $\mathbb{R}$.

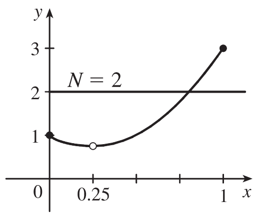
(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2} = \dfrac{x(x^2 - x - 2)}{x - 2} = \dfrac{x(x - 2)(x + 1)}{x - 2} = x(x + 1)$ [or $x^2 + x$] for $x \neq 2$. The discontinuity is removable and $g(x) = x^2 + x$ agrees with f for $x \neq 2$ and is continuous on $\mathbb{R}$.

(c) $\lim\limits_{x\to \pi^-} f(x) = \lim\limits_{x\to \pi^-} [\![\sin x]\!] = \lim\limits_{x\to \pi^-} 0 = 0$ and $\lim\limits_{x\to \pi^+} f(x) = \lim\limits_{x\to \pi^+} [\![\sin x]\!] = \lim\limits_{x\to \pi^+} (-1) = -1$, so $\lim\limits_{x\to \pi} f(x)$ does not exist. The discontinuity at $x = \pi$ is a jump discontinuity.

48.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

49. $f(x) = x^2 + 10\sin x$ is continuous on the interval $[31, 32]$, $f(31) \approx 957$, and $f(32) \approx 1030$. Since $957 < 1000 < 1030$, there is a number c in $(31, 32)$ such that $f(c) = 1000$ by the Intermediate Value Theorem. *Note:* There is also a number c in $(-32, -31)$ such that $f(c) = 1000$.

50. Suppose that $f(3) < 6$. By the Intermediate Value Theorem applied to the continuous function f on the closed interval $[2, 3]$, the fact that $f(2) = 8 > 6$ and $f(3) < 6$ implies that there is a number c in $(2, 3)$ such that $f(c) = 6$. This contradicts the fact that the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. Hence, our supposition that $f(3) < 6$ was incorrect. It follows that $f(3) \geq 6$. But $f(3) \neq 6$ because the only solutions of $f(x) = 6$ are $x = 1$ and $x = 4$. Therefore, $f(3) > 6$.

51. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

52. $f(x) = \sqrt[3]{x} + x - 1$ is continuous on the interval $[0, 1]$, $f(0) = -1$, and $f(1) = 1$. Since $-1 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sqrt[3]{x} + x - 1 = 0$, or $\sqrt[3]{x} = 1 - x$, in the interval $(0, 1)$.

53. $f(x) = \cos x - x$ is continuous on the interval $[0, 1]$, $f(0) = 1$, and $f(1) = \cos 1 - 1 \approx -0.46$. Since $-0.46 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x = 0$, or $\cos x = x$, in the interval $(0, 1)$.

54. The equation $\sin x = x^2 - x$ is equivalent to the equation $\sin x - x^2 + x = 0$. $f(x) = \sin x - x^2 + x$ is continuous on the interval $[1, 2]$, $f(1) = \sin 1 \approx 0.84$, and $f(2) = \sin 2 - 2 \approx -1.09$. Since $\sin 1 > 0 > \sin 2 - 2$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - x^2 + x = 0$, or $\sin x = x^2 - x$, in the interval $(1, 2)$.

55. (a) $f(x) = \cos x - x^3$ is continuous on the interval $[0, 1]$, $f(0) = 1 > 0$, and $f(1) = \cos 1 - 1 \approx -0.46 < 0$. Since $1 > 0 > -0.46$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x^3 = 0$, or $\cos x = x^3$, in the interval $(0, 1)$.

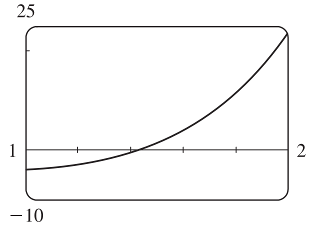
(b) $f(0.86) \approx 0.016 > 0$ and $f(0.87) \approx -0.014 < 0$, so there is a root between 0.86 and 0.87, that is, in the interval $(0.86, 0.87)$.

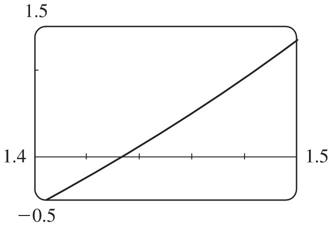
56. (a) $f(x) = x^5 - x^2 + 2x + 3$ is continuous on $[-1, 0]$, $f(-1) = -1 < 0$, and $f(0) = 3 > 0$. Since $-1 < 0 < 3$, there is a number c in $(-1, 0)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^2 + 2x + 3 = 0$ in the interval $(-1, 0)$.

(b) $f(-0.88) \approx -0.062 < 0$ and $f(-0.87) \approx 0.0047 > 0$, so there is a root between -0.88 and -0.87.

57. (a) Let $f(x) = x^5 - x^2 - 4$. Then $f(1) = 1^5 - 1^2 - 4 = -4 < 0$ and $f(2) = 2^5 - 2^2 - 4 = 24 > 0$. So by the Intermediate Value Theorem, there is a number c in $(1, 2)$ such that $f(c) = c^5 - c^2 - 4 = 0$.

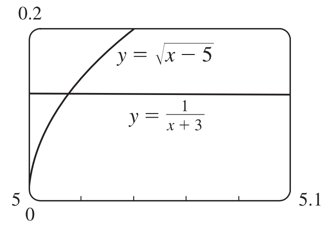
(b) We can see from the graphs that, correct to three decimal places, the root is $x \approx 1.434$.





58. (a) Let $f(x) = \sqrt{x-5} - \dfrac{1}{x+3}$. Then $f(5) = -\frac{1}{8} < 0$ and $f(6) = \frac{8}{9} > 0$, and f is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number c in $(5, 6)$ such that $f(c) = 0$. This implies that $\dfrac{1}{c+3} = \sqrt{c-5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 5.016$, correct to three decimal places.



59. $(\Rightarrow)$ If f is continuous at a, then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h\to 0} f(a+h) = f\left(\lim_{h\to 0}(a+h)\right) = f(a).$$

$(\Leftarrow)$ Let $\varepsilon > 0$. Since $\lim_{h\to 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \;\Rightarrow$ $|f(a+h) - f(a)| < \varepsilon$. So if $0 < |x - a| < \delta$, then $|f(x) - f(a)| = |f(a + (x - a)) - f(a)| < \varepsilon$. Thus, $\lim_{x\to a} f(x) = f(a)$ and so f is continuous at a.

60. $$\lim_{h\to 0} \sin(a+h) = \lim_{h\to 0} (\sin a \cos h + \cos a \sin h) = \lim_{h\to 0} (\sin a \cos h) + \lim_{h\to 0} (\cos a \sin h)$$
$$= \left(\lim_{h\to 0} \sin a\right)\left(\lim_{h\to 0} \cos h\right) + \left(\lim_{h\to 0} \cos a\right)\left(\lim_{h\to 0} \sin h\right) = (\sin a)(1) + (\cos a)(0) = \sin a$$

61. As in the previous exercise, we must show that $\lim_{h\to 0} \cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\lim_{h\to 0} \cos(a+h) = \lim_{h\to 0} (\cos a \cos h - \sin a \sin h) = \lim_{h\to 0} (\cos a \cos h) - \lim_{h\to 0} (\sin a \sin h)$$
$$= \left(\lim_{h\to 0} \cos a\right)\left(\lim_{h\to 0} \cos h\right) - \left(\lim_{h\to 0} \sin a\right)\left(\lim_{h\to 0} \sin h\right) = (\cos a)(1) - (\sin a)(0) = \cos a$$

62. (a) Since f is continuous at a, $\lim_{x \to a} f(x) = f(a)$. Thus, using the Constant Multiple Law of Limits, we have

$\lim_{x \to a} (cf)(x) = \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cf(a) = (cf)(a)$. Therefore, cf is continuous at a.

(b) Since f and g are continuous at a, $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$. Since $g(a) \neq 0$, we can use the Quotient Law

of Limits: $\lim_{x \to a} \left(\dfrac{f}{g}\right)(x) = \lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{f(a)}{g(a)} = \left(\dfrac{f}{g}\right)(a)$. Thus, $\dfrac{f}{g}$ is continuous at a.

63. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a - \delta, a + \delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1, there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x \to a} f(x) \neq f(a)$. [In fact, $\lim_{x \to a} f(x)$ does not even exist.]

64. $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$ is continuous at 0. To see why, note that $-|x| \leq g(x) \leq |x|$, so by the Squeeze Theorem $\lim_{x \to 0} g(x) = 0 = g(0)$. But g is continuous nowhere else. For if $a \neq 0$ and $\delta > 0$, the interval $(a - \delta, a + \delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a) = 0$ or a, there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|g(x) - g(a)| > |a|/2$. Thus, $\lim_{x \to a} g(x) \neq g(a)$.

65. If there is such a number, it satisfies the equation $x^3 + 1 = x \iff x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

66. $\dfrac{a}{x^3 + 2x^2 - 1} + \dfrac{b}{x^3 + x - 2} = 0 \quad \Rightarrow \quad a(x^3 + x - 2) + b(x^3 + 2x^2 - 1) = 0$. Let $p(x)$ denote the left side of the last equation. Since p is continuous on $[-1, 1]$, $p(-1) = -4a < 0$, and $p(1) = 2b > 0$, there exists a c in $(-1, 1)$ such that $p(c) = 0$ by the Intermediate Value Theorem. Note that the only root of either denominator that is in $(-1, 1)$ is $(-1 + \sqrt{5})/2 = r$, but $p(r) = (3\sqrt{5} - 9)a/2 \neq 0$. Thus, c is not a root of either denominator, so $p(c) = 0 \quad \Rightarrow$ $x = c$ is a root of the given equation.

67. $f(x) = x^4 \sin(1/x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$ since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since $-1 \leq \sin(1/x) \leq 1$, we have $-x^4 \leq x^4 \sin(1/x) \leq x^4$. Because $\lim_{x \to 0} (-x^4) = 0$ and $\lim_{x \to 0} x^4 = 0$, the Squeeze Theorem gives us $\lim_{x \to 0} (x^4 \sin(1/x)) = 0$, which equals $f(0)$. Thus, f is continuous at 0 and, hence, on $(-\infty, \infty)$.

68. (a) $\lim_{x \to 0^+} F(x) = 0$ and $\lim_{x \to 0^-} F(x) = 0$, so $\lim_{x \to 0} F(x) = 0$, which is $F(0)$, and hence F is continuous at $x = a$ if $a = 0$. For $a > 0$, $\lim_{x \to a} F(x) = \lim_{x \to a} x = a = F(a)$. For $a < 0$, $\lim_{x \to a} F(x) = \lim_{x \to a} (-x) = -a = F(a)$. Thus, F is continuous at $x = a$; that is, continuous everywhere.

(b) Assume that f is continuous on the interval I. Then for $a \in I$, $\lim_{x \to a} |f(x)| = \left| \lim_{x \to a} f(x) \right| = |f(a)|$ by Theorem 8. (If a is an endpoint of I, use the appropriate one-sided limit.) So $|f|$ is continuous on I.

(c) No, the converse is false. For example, the function $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at $x = 0$, but $|f(x)| = 1$ is continuous on $\mathbb{R}$.

69. Define $u(t)$ to be the monk's distance from the monastery, as a function of time t (in hours), on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \quad \Leftrightarrow$ $u(t_0) = d(t_0)$. So at time t_0 after 7:00 AM, the monk will be at the same place on both days.

1 REVIEW

CONCEPT CHECK

1. (a) A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

(b) If f is a function with domain A, then its **graph** is the set of ordered pairs $\{(x, f(x)) \mid x \in A\}$.

(c) Use the Vertical Line Test on page 15.

2. The four ways to represent a function are: verbally, numerically, visually, and algebraically. An example of each is given below.

Verbally: An assignment of students to chairs in a classroom (a description in words)

Numerically: A tax table that assigns an amount of tax to an income (a table of values)

Visually: A graphical history of the Dow Jones average (a graph)

Algebraically: A relationship between distance, rate, and time: $d = rt$ (an explicit formula)

3. (a) If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. If the graph of a function is symmetric with respect to the y-axis, then f is even. Examples of an even function: $f(x) = x^2$, $f(x) = x^4 + x^2$, $f(x) = |x|$, $f(x) = \cos x$.

(b) If a function f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. If the graph of a function is symmetric with respect to the origin, then f is odd. Examples of an odd function: $f(x) = x^3$, $f(x) = x^3 + x^5$, $f(x) = \sqrt[3]{x}$, $f(x) = \sin x$.

4. A function f is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon.

6. (a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

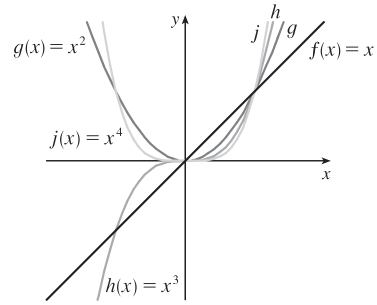
   (b) Power function: $f(x) = x^2$, $f(x) = x^a$

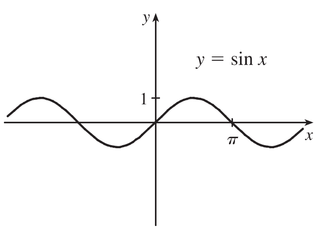
   (c) Exponential function: $f(x) = 2^x$, $f(x) = a^x$

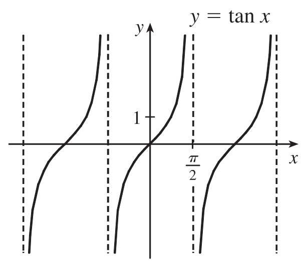
   (d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

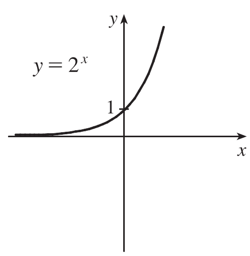
   (e) Polynomial of degree 5: $f(x) = x^5 + 2$

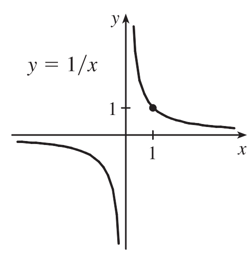
   (f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials

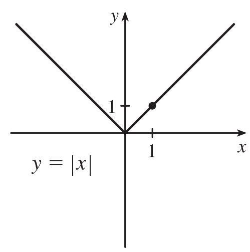
7. 

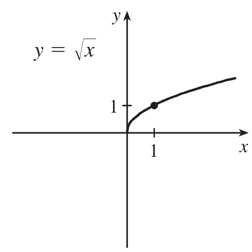
8. (a) 

   (b) 

   (c) 

   (d) 

   (e) 

   (f) 

9. (a) The domain of $f + g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

   (b) The domain of fg is also $A \cap B$.

   (c) The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. Given two functions f and g, the **composite** function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. (a) If the graph of f is shifted 2 units upward, its equation becomes $y = f(x) + 2$.

    (b) If the graph of f is shifted 2 units downward, its equation becomes $y = f(x) - 2$.

    (c) If the graph of f is shifted 2 units to the right, its equation becomes $y = f(x - 2)$.

    (d) If the graph of f is shifted 2 units to the left, its equation becomes $y = f(x + 2)$.

    (e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 2, its equation becomes $y = 2f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 2, its equation becomes $y = \frac{1}{2}f(x)$.

(i) If the graph of f is stretched horizontally by a factor of 2, its equation becomes $y = f\left(\frac{1}{2}x\right)$.

(j) If the graph of f is shrunk horizontally by a factor of 2, its equation becomes $y = f(2x)$.

12. (a) $\lim\limits_{x \to a} f(x) = L$: See Definition 1.5.1 and Figures 1 and 2 in Section 1.5.

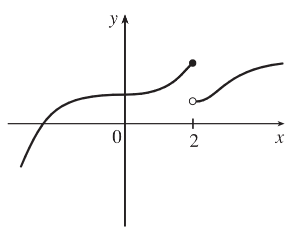
(b) $\lim\limits_{x \to a^+} f(x) = L$: See the paragraph after Definition 1.5.2 and Figure 9(b) in Section 1.5.

(c) $\lim\limits_{x \to a^-} f(x) = L$: See Definition 1.5.2 and Figure 9(a) in Section 1.5.

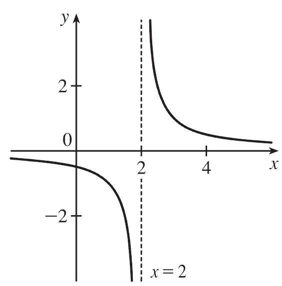
(d) $\lim\limits_{x \to a} f(x) = \infty$: See Definition 1.5.4 and Figure 12 in Section 1.5.

(e) $\lim\limits_{x \to a} f(x) = -\infty$: See Definition 1.5.5 and Figure 13 in Section 1.5.

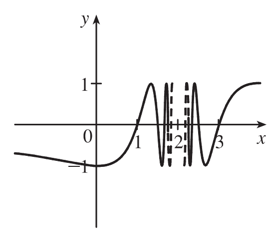
13. In general, the limit of a function fails to exist when the function does not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left- and right-hand limits are not equal.



There is an infinite discontinuity.



There are an infinite number of oscillations.

14. See Definition 1.5.6 and Figures 12–14 in Section 1.5.

15. (a) – (g) See the statements of Limit Laws 1– 6 and 11 in Section 1.6.

16. See Theorem 3 in Section 1.6.

17. (a) A function f is continuous at a number a if $f(x)$ approaches $f(a)$ as x approaches a; that is, $\lim\limits_{x \to a} f(x) = f(a)$.

(b) A function f is continuous on the interval $(-\infty, \infty)$ if f is continuous at every real number a. The graph of such a function has no breaks and every vertical line crosses it.

18. See Theorem 1.8.10.

TRUE-FALSE QUIZ

1. False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s + t) = (-1 + 1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \neq 0 = f(s + t)$.

2. False. Let $f(x) = x^2$. Then $f(-2) = 4 = f(2)$, but $-2 \neq 2$.

3. False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \neq 3f(x)$.

4. True. If $x_1 < x_2$ and f is a decreasing function, then the y-values get smaller as we move from left to right. Thus, $f(x_1) > f(x_2)$.

5. True. See the Vertical Line Test.

6. False. For example, if $x = -3$, then $\sqrt{(-3)^2} = \sqrt{9} = 3$, not -3.

7. False. Limit Law 2 applies only if the individual limits exist (these don't).

8. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

9. True. Limit Law 5 applies.

10. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as x approaches 5. (The denominator approaches 0 and the numerator doesn't.)

11. False. Consider $\lim\limits_{x\to 5} \dfrac{x(x-5)}{x-5}$ or $\lim\limits_{x\to 5} \dfrac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 3 in Section 1.5, we know that the latter limit exists (and it is equal to 1).

12. False. If $f(x) = 1/x$, $g(x) = -1/x$, and $a = 0$, then $\lim\limits_{x\to 0} f(x)$ does not exist, $\lim\limits_{x\to 0} g(x)$ does not exist, but $\lim\limits_{x\to 0} [f(x) + g(x)] = \lim\limits_{x\to 0} 0 = 0$ exists.

13. True. Suppose that $\lim\limits_{x\to a} [f(x) + g(x)]$ exists. Now $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a} g(x)$ does not exist, but $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} \{[f(x) + g(x)] - f(x)\} = \lim\limits_{x\to a} [f(x) + g(x)] - \lim\limits_{x\to a} f(x)$ [by Limit Law 2], which exists, and we have a contradiction. Thus, $\lim\limits_{x\to a} [f(x) + g(x)]$ does not exist.

14. False. Consider $\lim\limits_{x\to 6} [f(x)g(x)] = \lim\limits_{x\to 6} \left[(x-6)\,\dfrac{1}{x-6}\right]$. It exists (its value is 1) but $f(6) = 0$ and $g(6)$ does not exist, so $f(6)g(6) \neq 1$.

15. True. A polynomial is continuous everywhere, so $\lim\limits_{x\to b} p(x)$ exists and is equal to $p(b)$.

16. False. Consider $\lim\limits_{x\to 0} [f(x) - g(x)] = \lim\limits_{x\to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

17. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

18. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let $f(x) = \begin{cases} 1 & \text{if } 0 \le x < 3 \\ -1 & \text{if } x = 3 \end{cases}$ There is no number $c \in [0, 3]$ with $f(c) = 0$.

19. True. Use Theorem 1.8.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.

20. True. Use the Intermediate Value Theorem with $a = -1$, $b = 1$, and $N = \pi$, since $3 < \pi < 4$.

21. True, by the definition of a limit with $\varepsilon = 1$.

22. False. For example, let $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

Then $f(x) > 1$ for all x, but $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(x^2 + 1\right) = 1$.

23. True. $f(x) = x^{10} - 10x^2 + 5$ is continuous on the interval $[0, 2]$, $f(0) = 5$, $f(1) = -4$, and $f(2) = 989$. Since $-4 < 0 < 5$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^{10} - 10x^2 + 5 = 0$ in the interval $(0, 1)$. Similarly, there is a root in $(1, 2)$.

24. True. See Exercise 68(b) in Section 1.8.

25. False See Exercise 68(c) in Section 1.8.

EXERCISES

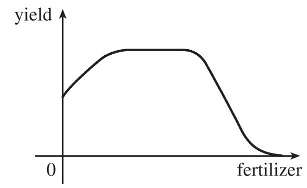
1. (a) When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.
   (b) $f(x) = 3 \Rightarrow x \approx 2.3, 5.6$
   (c) The domain of f is $-6 \leq x \leq 6$, or $[-6, 6]$.
   (d) The range of f is $-4 \leq y \leq 4$, or $[-4, 4]$.
   (e) f is increasing on $[-4, 4]$, that is, on $-4 \leq x \leq 4$.
   (f) f is odd since its graph is symmetric about the origin.

2. (a) This curve *is not* the graph of a function of x since it *fails* the Vertical Line Test.
   (b) This curve *is* the graph of a function of x since it *passes* the Vertical Line Test. The domain is $[-3, 3]$ and the range is $[-2, 3]$.

3. $f(x) = x^2 - 2x + 3$, so $f(a + h) = (a + h)^2 - 2(a + h) + 3 = a^2 + 2ah + h^2 - 2a - 2h + 3$, and

$$\frac{f(a+h) - f(a)}{h} = \frac{(a^2 + 2ah + h^2 - 2a - 2h + 3) - (a^2 - 2a + 3)}{h} = \frac{h(2a + h - 2)}{h} = 2a + h - 2.$$

4. There will be some yield with no fertilizer, increasing yields with increasing fertilizer use, a leveling-off of yields at some point, and disaster with too much fertilizer use.



5. $f(x) = 2/(3x - 1)$. Domain: $3x - 1 \neq 0 \Rightarrow 3x \neq 1 \Rightarrow x \neq \frac{1}{3}$. $D = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$
   Range: all reals except 0 ($y = 0$ is the horizontal asymptote for f.) $R = (-\infty, 0) \cup (0, \infty)$

6. $g(x) = \sqrt{16 - x^4}$. Domain: $16 - x^4 \geq 0 \Rightarrow x^4 \leq 16 \Rightarrow |x| \leq \sqrt[4]{16} \Rightarrow |x| \leq 2$. $D = [-2, 2]$
   Range: $y \geq 0$ and $y \leq \sqrt{16} \Rightarrow 0 \leq y \leq 4$. $R = [0, 4]$

7. $y = 1 + \sin x$. Domain: $\mathbb{R}$.
   Range: $-1 \leq \sin x \leq 1 \Rightarrow 0 \leq 1 + \sin x \leq 2 \Rightarrow 0 \leq y \leq 2$. $R = [0, 2]$

8. $y = F(t) = 3 + \cos 2t$. Domain: $\mathbb{R}$. $D = (-\infty, \infty)$

Range: $-1 \le \cos 2t \le 1 \;\Rightarrow\; 2 \le 3 + \cos 2t \le 4 \;\Rightarrow\; 2 \le y \le 4$. $R = [2, 4]$

9. (a) To obtain the graph of $y = f(x) + 8$, we shift the graph of $y = f(x)$ up 8 units.

(b) To obtain the graph of $y = f(x + 8)$, we shift the graph of $y = f(x)$ left 8 units.

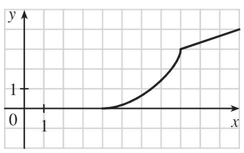
(c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

(d) To obtain the graph of $y = f(x - 2) - 2$, we shift the graph of $y = f(x)$ right 2 units (for the "-2" inside the parentheses), and then shift the resulting graph 2 units downward.

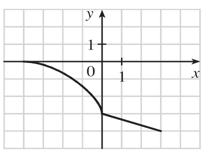
(e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.

(f) To obtain the graph of $y = 3 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 3 units upward.

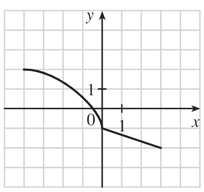
10. (a) To obtain the graph of $y = f(x - 8)$, we shift the graph of $y = f(x)$ right 8 units.



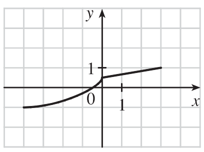
(b) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.



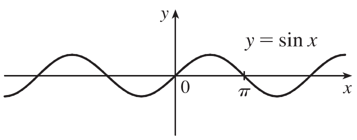
(c) To obtain the graph of $y = 2 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 2 units upward.

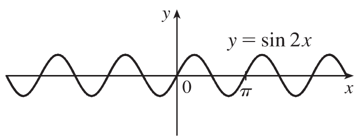


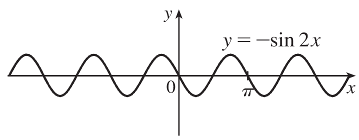
(d) To obtain the graph of $y = \frac{1}{2}f(x) - 1$, we shrink the graph of $y = f(x)$ by a factor of 2, and then shift the resulting graph 1 unit downward.



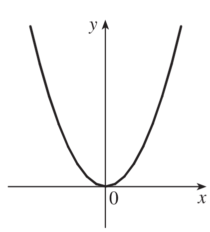
11. $y = -\sin 2x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 2, and reflect about the x-axis.



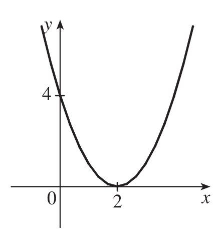




12. $y = (x - 2)^2$: Start with the graph of $y = x^2$ and shift 2 units to the right.

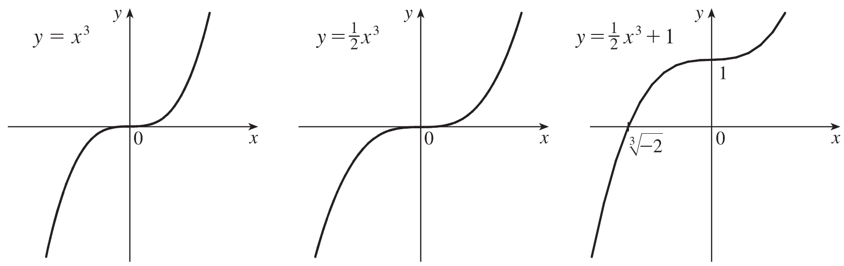


$y = x^2$

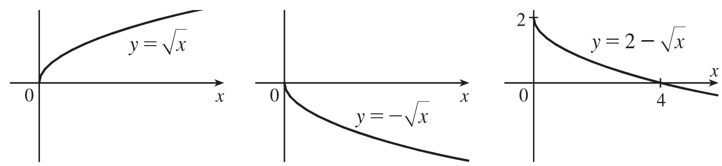


$y = (x - 2)^2$

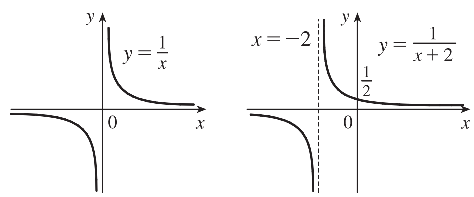
13. $y = 1 + \frac{1}{2}x^3$: Start with the graph of $y = x^3$, compress vertically by a factor of 2, and shift 1 unit upward.



14. $y = 2 - \sqrt{x}$:
Start with the graph of $y = \sqrt{x}$, reflect about the x-axis, and shift 2 units upward.



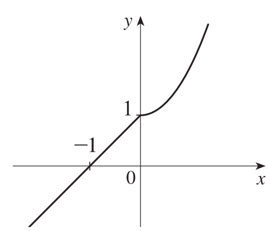
15. $f(x) = \dfrac{1}{x+2}$:
Start with the graph of $f(x) = 1/x$ and shift 2 units to the left.



16. $f(x) = \begin{cases} 1 + x & \text{if } x < 0 \\ 1 + x^2 & \text{if } x \geq 0 \end{cases}$

On $(-\infty, 0)$, graph $y = 1 + x$ (the line with slope 1 and y-intercept 1) with open endpoint $(0, 1)$.

On $[0, \infty)$, graph $y = 1 + x^2$ (the rightmost half of the parabola $y = x^2$ shifted 1 unit upward) with closed endpoint $(0, 1)$.



17. (a) The terms of f are a mixture of odd and even powers of x, so f is neither even nor odd.

(b) The terms of f are all odd powers of x, so f is odd.

(c) $f(-x) = \cos\big((-x)^2\big) = \cos(x^2) = f(x)$, so f is even.

(d) $f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

18. For the line segment from $(-2, 2)$ to $(-1, 0)$, the slope is $\dfrac{0-2}{-1+2} = -2$, and an equation is $y - 0 = -2(x+1)$ or, equivalently, $y = -2x - 2$. The circle has equation $x^2 + y^2 = 1$; the top half has equation $y = \sqrt{1 - x^2}$ (we have solved for positive y). Thus, $f(x) = \begin{cases} -2x - 2 & \text{if } -2 \leq x \leq -1 \\ \sqrt{1 - x^2} & \text{if } -1 < x \leq 1 \end{cases}$

19. $f(x) = \sqrt{x}$, $\quad D = [0, \infty)$; $\quad g(x) = \sin x$, $\quad D = \mathbb{R}$.

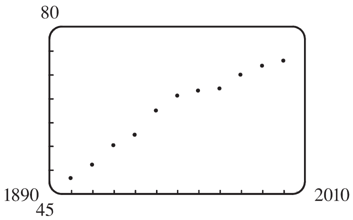
(a) $(f \circ g)(x) = f(g(x)) = f(\sin x) = \sqrt{\sin x}$. For $\sqrt{\sin x}$ to be defined, we must have $\sin x \geq 0 \quad \Leftrightarrow$
$x \in [0, \pi]$, $[2\pi, 3\pi]$, $[-2\pi, -\pi]$, $[4\pi, 5\pi]$, $[-4\pi, -3\pi]$, $\ldots$, so $D = \{x \mid x \in [2n\pi, \pi + 2n\pi]$, where n is an integer$\}$.

(b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}\,) = \sin\sqrt{x}$. x must be greater than or equal to 0 for $\sqrt{x}$ to be defined, so $D = [0, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x}\,) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. $D = [0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g(\sin x) = \sin(\sin x)$. $D = \mathbb{R}$.

20. Let $h(x) = x + \sqrt{x}$, $g(x) = \sqrt{x}$, and $f(x) = 1/x$. Then $(f \circ g \circ h)(x) = \dfrac{1}{\sqrt{x + \sqrt{x}}} = F(x)$.

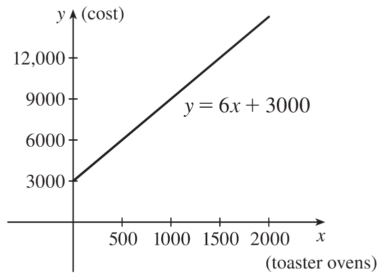
21. 

Many models appear to be plausible. Your choice depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $y = 0.2493x - 423.4818$, gives us an estimate of 77.6 years for the year 2010.

22. (a) Let x denote the number of toaster ovens produced in one week and y the associated cost. Using the points $(1000, 9000)$ and $(1500, 12{,}000)$, we get an equation of a line:

$$y - 9000 = \frac{12{,}000 - 9000}{1500 - 1000}\,(x - 1000) \quad \Rightarrow$$

$$y = 6\,(x - 1000) + 9000 \quad \Rightarrow \quad y = 6x + 3000.$$



(b) The slope of 6 means that each additional toaster oven produced adds \$6 to the weekly production cost.

(c) The y-intercept of 3000 represents the overhead cost—the cost incurred without producing anything.

23. (a) (i) $\lim\limits_{x \to 2^+} f(x) = 3$ (ii) $\lim\limits_{x \to -3^+} f(x) = 0$

(iii) $\lim\limits_{x \to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim\limits_{x \to 4} f(x) = 2$

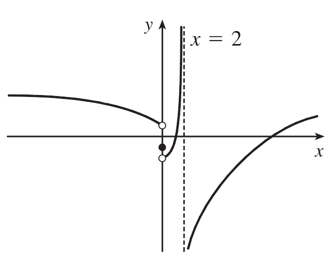
(v) $\lim\limits_{x \to 0} f(x) = \infty$ (vi) $\lim\limits_{x \to 2^-} f(x) = -\infty$

(b) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.

(c) f is discontinuous at $x = -3$, 0, 2, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

24. $\lim\limits_{x \to -0^+} f(x) = -2$, $\lim\limits_{x \to 0^-} f(x) = 1$, $f(0) = -1$,

$\lim\limits_{x \to 2^-} f(x) = \infty$, $\lim\limits_{x \to 2^+} f(x) = -\infty$



25. $\lim\limits_{x \to 0} \cos(x + \sin x) = \cos\left[\lim\limits_{x \to 0}(x + \sin x)\right]$ [by Theorem 1.8.8] $= \cos 0 = 1$

26. Since rational functions are continuous, $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x^2 + 2x - 3} = \dfrac{3^2 - 9}{3^2 + 2(3) - 3} = \dfrac{0}{12} = 0$.

27. $\displaystyle\lim_{x\to-3}\frac{x^2-9}{x^2+2x-3}=\lim_{x\to-3}\frac{(x+3)(x-3)}{(x+3)(x-1)}=\lim_{x\to-3}\frac{x-3}{x-1}=\frac{-3-3}{-3-1}=\frac{-6}{-4}=\frac{3}{2}$

28. $\displaystyle\lim_{x\to1^+}\frac{x^2-9}{x^2+2x-3}=-\infty$ since $x^2+2x-3\to0^+$ as $x\to1^+$ and $\displaystyle\frac{x^2-9}{x^2+2x-3}<0$ for $1<x<3$.

29. $\displaystyle\lim_{h\to0}\frac{(h-1)^3+1}{h}=\lim_{h\to0}\frac{\left(h^3-3h^2+3h-1\right)+1}{h}=\lim_{h\to0}\frac{h^3-3h^2+3h}{h}=\lim_{h\to0}\left(h^2-3h+3\right)=3$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\begin{aligned}\lim_{h\to0}\frac{(h-1)^3+1}{h}&=\lim_{h\to0}\frac{(h-1)^3+1^3}{h}=\lim_{h\to0}\frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h}\\&=\lim_{h\to0}\left[(h-1)^2-h+2\right]=1-0+2=3\end{aligned}$$

30. $\displaystyle\lim_{t\to2}\frac{t^2-4}{t^3-8}=\lim_{t\to2}\frac{(t+2)(t-2)}{(t-2)(t^2+2t+4)}=\lim_{t\to2}\frac{t+2}{t^2+2t+4}=\frac{2+2}{4+4+4}=\frac{4}{12}=\frac{1}{3}$

31. $\displaystyle\lim_{r\to9}\frac{\sqrt{r}}{(r-9)^4}=\infty$ since $(r-9)^4\to0^+$ as $r\to9$ and $\displaystyle\frac{\sqrt{r}}{(r-9)^4}>0$ for $r\neq9$.

32. $\displaystyle\lim_{v\to4^+}\frac{4-v}{|4-v|}=\lim_{v\to4^+}\frac{4-v}{-(4-v)}=\lim_{v\to4^+}\frac{1}{-1}=-1$

33. $\displaystyle\lim_{u\to1}\frac{u^4-1}{u^3+5u^2-6u}=\lim_{u\to1}\frac{(u^2+1)(u^2-1)}{u(u^2+5u-6)}=\lim_{u\to1}\frac{(u^2+1)(u+1)(u-1)}{u(u+6)(u-1)}=\lim_{u\to1}\frac{(u^2+1)(u+1)}{u(u+6)}=\frac{2(2)}{1(7)}=\frac{4}{7}$

34. $$\begin{aligned}\lim_{x\to3}\frac{\sqrt{x+6}-x}{x^3-3x^2}&=\lim_{x\to3}\left[\frac{\sqrt{x+6}-x}{x^2(x-3)}\cdot\frac{\sqrt{x+6}+x}{\sqrt{x+6}+x}\right]=\lim_{x\to3}\frac{\left(\sqrt{x+6}\,\right)^2-x^2}{x^2(x-3)\left(\sqrt{x+6}+x\right)}\\&=\lim_{x\to3}\frac{x+6-x^2}{x^2(x-3)\left(\sqrt{x+6}+x\right)}=\lim_{x\to3}\frac{-(x^2-x-6)}{x^2(x-3)\left(\sqrt{x+6}+x\right)}=\lim_{x\to3}\frac{-(x-3)(x+2)}{x^2(x-3)\left(\sqrt{x+6}+x\right)}\\&=\lim_{x\to3}\frac{-(x+2)}{x^2\left(\sqrt{x+6}+x\right)}=-\frac{5}{9(3+3)}=-\frac{5}{54}\end{aligned}$$

35. $\displaystyle\lim_{s\to16}\frac{4-\sqrt{s}}{s-16}=\lim_{s\to16}\frac{4-\sqrt{s}}{(\sqrt{s}+4)(\sqrt{s}-4)}=\lim_{s\to16}\frac{-1}{\sqrt{s}+4}=\frac{-1}{\sqrt{16}+4}=-\frac{1}{8}$

36. $\displaystyle\lim_{v\to2}\frac{v^2+2v-8}{v^4-16}=\lim_{v\to2}\frac{(v+4)(v-2)}{(v+2)(v-2)(v^2+4)}=\lim_{v\to2}\frac{v+4}{(v+2)(v^2+4)}=\frac{2+4}{(2+2)(2^2+4)}=\frac{3}{16}$

37. $\displaystyle\lim_{x\to0}\frac{1-\sqrt{1-x^2}}{x}\cdot\frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}=\lim_{x\to0}\frac{1-(1-x^2)}{x\left(1+\sqrt{1-x^2}\right)}=\lim_{x\to0}\frac{x^2}{x\left(1+\sqrt{1-x^2}\right)}=\lim_{x\to0}\frac{x}{1+\sqrt{1-x^2}}=0$

38. $$\begin{aligned}\lim_{x\to1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)&=\lim_{x\to1}\left[\frac{1}{x-1}+\frac{1}{(x-1)(x-2)}\right]=\lim_{x\to1}\left[\frac{x-2}{(x-1)(x-2)}+\frac{1}{(x-1)(x-2)}\right]\\&=\lim_{x\to1}\left[\frac{x-1}{(x-1)(x-2)}\right]=\lim_{x\to1}\frac{1}{x-2}=\frac{1}{1-2}=-1\end{aligned}$$

39. Since $2x-1\le f(x)\le x^2$ for $0<x<3$ and $\displaystyle\lim_{x\to1}(2x-1)=1=\lim_{x\to1}x^2$, we have $\displaystyle\lim_{x\to1}f(x)=1$ by the Squeeze Theorem.

40. Let $f(x) = -x^2$, $g(x) = x^2 \cos(1/x^2)$ and $h(x) = x^2$. Then since $\left|\cos(1/x^2)\right| \le 1$ for $x \ne 0$, we have $f(x) \le g(x) \le h(x)$ for $x \ne 0$, and so $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0 \;\Rightarrow\; \lim_{x\to 0} g(x) = 0$ by the Squeeze Theorem.

41. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $|(14-5x)-4| < \varepsilon$. But $|(14-5x)-4| < \varepsilon \;\Leftrightarrow\; |-5x+10| < \varepsilon \;\Leftrightarrow\; |-5|\,|x-2| < \varepsilon \;\Leftrightarrow\; |x-2| < \varepsilon/5$. So if we choose $\delta = \varepsilon/5$, then $0 < |x-2| < \delta \;\Rightarrow\; |(14-5x)-4| < \varepsilon$. Thus, $\lim_{x\to 2}(14-5x) = 4$ by the definition of a limit.

42. Given $\varepsilon > 0$ we must find $\delta > 0$ so that if $0 < |x-0| < \delta$, then $|\sqrt[3]{x} - 0| < \varepsilon$. Now $|\sqrt[3]{x} - 0| = |\sqrt[3]{x}| < \varepsilon \;\Rightarrow\; |x| = |\sqrt[3]{x}|^3 < \varepsilon^3$. So take $\delta = \varepsilon^3$. Then $0 < |x-0| = |x| < \varepsilon^3 \;\Rightarrow\; |\sqrt[3]{x} - 0| = |\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\varepsilon^3} = \varepsilon$. Therefore, by the definition of a limit, $\lim_{x\to 0} \sqrt[3]{x} = 0$.

43. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x-2| < \delta$, then $\left|x^2 - 3x - (-2)\right| < \varepsilon$. First, note that if $|x-2| < 1$, then $-1 < x-2 < 1$, so $0 < x-1 < 2 \;\Rightarrow\; |x-1| < 2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $0 < |x-2| < \delta \;\Rightarrow\; \left|x^2-3x-(-2)\right| = |(x-2)(x-1)| = |x-2|\,|x-1| < (\varepsilon/2)(2) = \varepsilon$.
Thus, $\lim_{x\to 2}(x^2 - 3x) = -2$ by the definition of a limit.

44. Given $M > 0$, we need $\delta > 0$ such that if $0 < x-4 < \delta$, then $2/\sqrt{x-4} > M$. This is true $\;\Leftrightarrow\; \sqrt{x-4} < 2/M \;\Leftrightarrow\; x-4 < 4/M^2$. So if we choose $\delta = 4/M^2$, then $0 < x-4 < \delta \;\Rightarrow\; 2/\sqrt{x-4} > M$. So by the definition of a limit, $\lim_{x\to 4^+}\left(2/\sqrt{x-4}\right) = \infty$.

45. (a) $f(x) = \sqrt{-x}$ if $x < 0$, $f(x) = 3-x$ if $0 \le x < 3$, $f(x) = (x-3)^2$ if $x > 3$.

(i) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(3-x) = 3$

(ii) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\sqrt{-x} = 0$

(iii) Because of (i) and (ii), $\lim_{x\to 0} f(x)$ does not exist.

(iv) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(3-x) = 0$

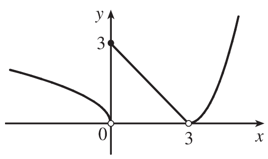
(v) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x-3)^2 = 0$

(vi) Because of (iv) and (v), $\lim_{x\to 3} f(x) = 0$.

(b) f is discontinuous at 0 since $\lim_{x\to 0} f(x)$ does not exist.

f is discontinuous at 3 since $f(3)$ does not exist.

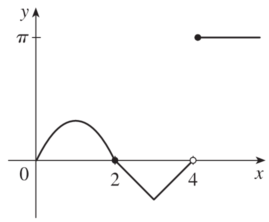
(c)



46. (a) $g(x) = 2x - x^2$ if $0 \le x \le 2$, $g(x) = 2-x$ if $2 < x \le 3$, $g(x) = x-4$ if $3 < x < 4$, $g(x) = \pi$ if $x \ge 4$.

Therefore, $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-}\left(2x - x^2\right) = 0$ and $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+}(2-x) = 0$. Thus, $\lim_{x\to 2} g(x) = 0 = g(2)$, so g is continuous at 2. $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-}(2-x) = -1$ and $\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+}(x-4) = -1$. Thus, $\lim_{x\to 3} g(x) = -1 = g(3)$, so g is continuous at 3. $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-}(x-4) = 0$ and $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+}\pi = \pi$. Thus, $\lim_{x\to 4} g(x)$ does not exist, so g is discontinuous at 4. But $\lim_{x\to 4^+} g(x) = \pi = g(4)$, so g is continuous from the right at 4.

(b)



47. x^3 is continuous on $\mathbb{R}$ since it is a polynomial and $\cos x$ is also continuous on $\mathbb{R}$, so the product $x^3 \cos x$ is continuous on $\mathbb{R}$. The root function $\sqrt[4]{x}$ is continuous on its domain, $[0, \infty)$, and so the sum, $h(x) = \sqrt[4]{x} + x^3 \cos x$, is continuous on its domain, $[0, \infty)$.

48. $x^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0, \infty)$ by Theorem 7 in Section 1.8, so the composition $\sqrt{x^2 - 9}$ is continuous on $\{x \mid x^2 - 9 \geq 0\} = (-\infty, -3] \cup [3, \infty)$ by Theorem 9. Note that $x^2 - 2 \neq 0$ on this set and so the quotient function $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$ is continuous on its domain, $(-\infty, -3] \cup [3, \infty)$ by Theorem 4.

49. $f(x) = x^5 - x^3 + 3x - 5$ is continuous on the interval $[1, 2]$, $f(1) = -2$, and $f(2) = 25$. Since $-2 < 0 < 25$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^3 + 3x - 5 = 0$ in the interval $(1, 2)$.

50. Let $f(x) = 2 \sin x - 3 + 2x$. Now f is continuous on $[0, 1]$ and $f(0) = -3 < 0$ and $f(1) = 2 \sin 1 - 1 \approx 0.68 > 0$. So by the Intermediate Value Theorem there is a number c in $(0, 1)$ such that $f(c) = 0$, that is, the equation $2 \sin x = 3 - 2x$ has a root in $(0, 1)$.

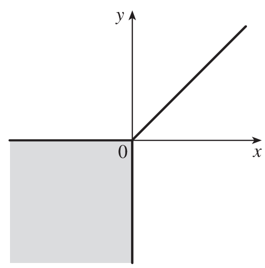
PRINCIPLES OF PROBLEM SOLVING

1. Remember that $|a| = a$ if $a \geq 0$ and that $|a| = -a$ if $a < 0$. Thus,

$$x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad y + |y| = \begin{cases} 2y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

We will consider the equation $x + |x| = y + |y|$ in four cases.

(1) $x \geq 0, y \geq 0$	(2) $x \geq 0, y < 0$	(3) $x < 0, y \geq 0$	(4) $x < 0, y < 0$
$2x = 2y$	$2x = 0$	$0 = 2y$	$0 = 0$
$x = y$	$x = 0$	$0 = y$	



Case 1 gives us the line $y = x$ with nonnegative x and y.

Case 2 gives us the portion of the y-axis with y negative.

Case 3 gives us the portion of the x-axis with x negative.

Case 4 gives us the entire third quadrant.

2. $|x - y| + |x| - |y| \leq 2$ [call this inequality ($\star$)]

Case (i): $x \geq y \geq 0$. Then ($\star$) $\Leftrightarrow$ $x - y + x - y \leq 2$ $\Leftrightarrow$ $x - y \leq 1$ $\Leftrightarrow$ $y \geq x - 1$.

Case (ii): $y \geq x \geq 0$. Then ($\star$) $\Leftrightarrow$ $y - x + x - y \leq 2$ $\Leftrightarrow$ $0 \leq 2$ (true).

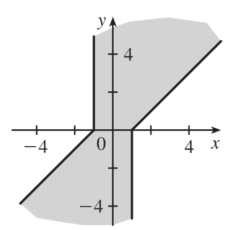
Case (iii): $x \geq 0$ and $y \leq 0$. Then ($\star$) $\Leftrightarrow$ $x - y + x + y \leq 2$ $\Leftrightarrow$ $2x \leq 2$ $\Leftrightarrow$ $x \leq 1$.

Case (iv): $x \leq 0$ and $y \geq 0$. Then ($\star$) $\Leftrightarrow$ $y - x - x - y \leq 2$ $\Leftrightarrow$ $-2x \leq 2$ $\Leftrightarrow$ $x \geq -1$.

Case (v): $y \leq x \leq 0$. Then ($\star$) $\Leftrightarrow$ $x - y - x + y \leq 2$ $\Leftrightarrow$ $0 \leq 2$ (true).

Case (vi): $x \leq y \leq 0$. Then ($\star$) $\Leftrightarrow$ $y - x - x + y \leq 2$ $\Leftrightarrow$ $y - x \leq 1$ $\Leftrightarrow$ $y \leq x + 1$.

Note: Instead of considering cases (iv), (v), and (vi), we could have noted that the region is unchanged if x and y are replaced by $-x$ and $-y$, so the region is symmetric about the origin. Therefore, we need only draw cases (i), (ii), and (iii), and rotate through $180°$ about the origin.



3. $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$.

$f_1(x) = f_0(f_0(x)) = f_0\left(x^2\right) = \left(x^2\right)^2 = x^4$, $f_2(x) = f_0(f_1(x)) = f_0(x^4) = (x^4)^2 = x^8$,

$f_3(x) = f_0(f_2(x)) = f_0(x^8) = (x^8)^2 = x^{16}, \ldots$. Thus, a general formula is $f_n(x) = x^{2^{n+1}}$.

4. (a) $f_0(x) = 1/(2 - x)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$.

$$f_1(x) = f_0\left(\frac{1}{2-x}\right) = \frac{1}{2 - \dfrac{1}{2-x}} = \frac{2-x}{2(2-x)-1} = \frac{2-x}{3-2x},$$

$$f_2(x) = f_0\left(\frac{2-x}{3-2x}\right) = \frac{1}{2 - \dfrac{2-x}{3-2x}} = \frac{3-2x}{2(3-2x)-(2-x)} = \frac{3-2x}{4-3x},$$

$$f_3(x) = f_0\left(\frac{3-2x}{4-3x}\right) = \frac{1}{2-\dfrac{3-2x}{4-3x}} = \frac{4-3x}{2(4-3x)-(3-2x)} = \frac{4-3x}{5-4x}, \ldots$$

Thus, we conjecture that the general formula is $f_n(x) = \dfrac{n+1-nx}{n+2-(n+1)x}$.

To prove this, we use the Principle of Mathematical Induction. We have already verified that f_n is true for $n = 1$.

Assume that the formula is true for $n = k$; that is, $f_k(x) = \dfrac{k+1-kx}{k+2-(k+1)x}$. Then

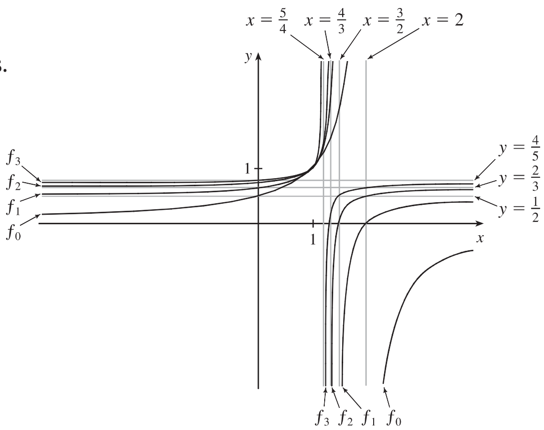
$$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{k+1-kx}{k+2-(k+1)x}\right) = \frac{1}{2-\dfrac{k+1-kx}{k+2-(k+1)x}}$$

$$= \frac{k+2-(k+1)x}{2\,[k+2-(k+1)x]-(k+1-kx)} = \frac{k+2-(k+1)x}{k+3-(k+2)x}$$

This shows that the formula for f_n is true for $n = k+1$. Therefore, by mathematical induction, the formula is true for all positive integers n.

(b) From the graph, we can make several observations:

- The values at each fixed $x = a$ keep increasing as n increases.
- The vertical asymptote gets closer to $x = 1$ as n increases.
- The horizontal asymptote gets closer to $y = 1$ as n increases.
- The x-intercept for f_{n+1} is the value of the vertical asymptote for f_n.
- The y-intercept for f_n is the value of the horizontal asymptote for f_{n+1}.



5. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to1}\frac{t^2-1}{t^3-1} = \lim_{t\to1}\frac{(t-1)(t+1)}{(t-1)\,(t^2+t+1)} = \lim_{t\to1}\frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x}+1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$.

6. First rationalize the numerator: $\displaystyle\lim_{x\to0}\frac{\sqrt{ax+b}-2}{x}\cdot\frac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim_{x\to0}\frac{ax+b-4}{x\left(\sqrt{ax+b}+2\right)}$. Now since the denominator approaches 0 as $x \to 0$, the limit will exist only if the numerator also approaches 0 as $x \to 0$. So we require that $a(0)+b-4=0 \;\Rightarrow\; b=4$. So the equation becomes $\displaystyle\lim_{x\to0}\frac{a}{\sqrt{ax+4}+2} = 1 \;\Rightarrow\; \frac{a}{\sqrt{4}+2} = 1 \;\Rightarrow\; a = 4$.

Therefore, $a = b = 4$.

7. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$.

Therefore, $\displaystyle\lim_{x\to0}\frac{|2x-1|-|2x+1|}{x} = \lim_{x\to0}\frac{-(2x-1)-(2x+1)}{x} = \lim_{x\to0}\frac{-4x}{x} = \lim_{x\to0}(-4) = -4.$

8. Let R be the midpoint of OP, so the coordinates of R are $\left(\frac{1}{2}x, \frac{1}{2}x^2\right)$ since the coordinates of P are $\left(x, x^2\right)$. Let $Q = (0, a)$. Since the slope $m_{OP} = \dfrac{x^2}{x} = x$, $m_{QR} = -\dfrac{1}{x}$ (negative reciprocal). But $m_{QR} = \dfrac{\frac{1}{2}x^2 - a}{\frac{1}{2}x - 0} = \dfrac{x^2 - 2a}{x}$, so we conclude that $-1 = x^2 - 2a \quad \Rightarrow \quad 2a = x^2 + 1 \quad \Rightarrow \quad a = \frac{1}{2}x^2 + \frac{1}{2}$. As $x \to 0$, $a \to \frac{1}{2}$, and the limiting position of Q is $\left(0, \frac{1}{2}\right)$.

9. (a) For $0 < x < 1$, $[\![x]\!] = 0$, so $\dfrac{[\![x]\!]}{x} = 0$, and $\displaystyle\lim_{x\to 0^+} \frac{[\![x]\!]}{x} = 0$. For $-1 < x < 0$, $[\![x]\!] = -1$, so $\dfrac{[\![x]\!]}{x} = \dfrac{-1}{x}$, and $\displaystyle\lim_{x\to 0^-} \frac{[\![x]\!]}{x} = \lim_{x\to 0^-} \left(\frac{-1}{x}\right) = \infty$. Since the one-sided limits are not equal, $\displaystyle\lim_{x\to 0} \frac{[\![x]\!]}{x}$ does not exist.

(b) For $x > 0$, $1/x - 1 \le [\![1/x]\!] \le 1/x \quad \Rightarrow \quad x(1/x - 1) \le x[\![1/x]\!] \le x(1/x) \quad \Rightarrow \quad 1 - x \le x[\![1/x]\!] \le 1$.
As $x \to 0^+$, $1 - x \to 1$, so by the Squeeze Theorem, $\displaystyle\lim_{x\to 0^+} x[\![1/x]\!] = 1$.
For $x < 0$, $1/x - 1 \le [\![1/x]\!] \le 1/x \quad \Rightarrow \quad x(1/x - 1) \ge x[\![1/x]\!] \ge x(1/x) \quad \Rightarrow \quad 1 - x \ge x[\![1/x]\!] \ge 1$.
As $x \to 0^-$, $1 - x \to 1$, so by the Squeeze Theorem, $\displaystyle\lim_{x\to 0^-} x[\![1/x]\!] = 1$.
Since the one-sided limits are equal, $\displaystyle\lim_{x\to 0} x[\![1/x]\!] = 1$.

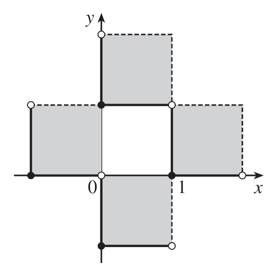
10. (a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$. Since $[\![x]\!]^2$ and $[\![y]\!]^2$ are positive integers or 0, there are only 4 cases:

Case (i): $[\![x]\!] = 1$, $[\![y]\!] = 0 \quad \Rightarrow 1 \le x < 2$ and $0 \le y < 1$

Case (ii): $[\![x]\!] = -1$, $[\![y]\!] = 0 \Rightarrow -1 \le x < 0$ and $0 \le y < 1$

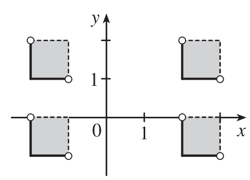
Case (iii): $[\![x]\!] = 0$, $[\![y]\!] = 1 \quad \Rightarrow 0 \le x < 1$ and $1 \le y < 2$

Case (iv): $[\![x]\!] = 0$, $[\![y]\!] = -1 \Rightarrow 0 \le x < 1$ and $-1 \le y < 0$

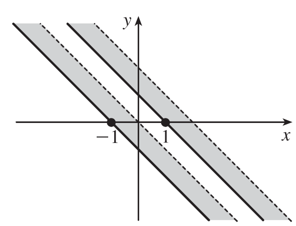


(b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$. The only integral solution of $n^2 - m^2 = 3$ is $n = \pm 2$ and $m = \pm 1$. So the graph is

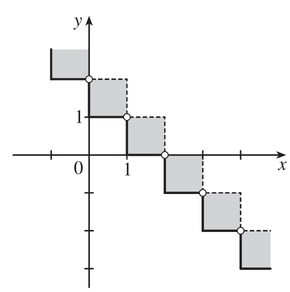
$$\{(x, y) \mid [\![x]\!] = \pm 2,\ [\![y]\!] = \pm 1\} = \left\{ (x, y) \;\middle|\; \begin{array}{l} 2 \le x \le 3 \text{ or } -2 \le x < 1, \\ 1 \le y < 2 \text{ or } -1 \le y < 0 \end{array} \right\}.$$



(c) $[\![x + y]\!]^2 = 1 \quad \Rightarrow \quad [\![x + y]\!] = \pm 1 \quad \Rightarrow \quad 1 \le x + y < 2$ or $-1 \le x + y < 0$



(d) For $n \le x < n + 1$, $[\![x]\!] = n$. Then $[\![x]\!] + [\![y]\!] = 1 \quad \Rightarrow \quad [\![y]\!] = 1 - n \quad \Rightarrow \quad 1 - n \le y < 2 - n$. Choosing integer values for n produces the graph.

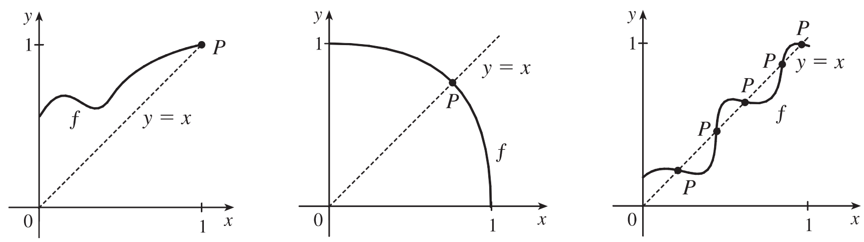


11. f is continuous on $(-\infty, a)$ and (a, ∞). To make f continuous on $\mathbb{R}$, we must have continuity at a. Thus,

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \quad\Rightarrow\quad \lim_{x\to a^+} x^2 = \lim_{x\to a^-} (x+1) \quad\Rightarrow\quad a^2 = a+1 \quad\Rightarrow\quad a^2 - a - 1 = 0 \quad\Rightarrow$$

[by the quadratic formula] $\quad a = \left(1 \pm \sqrt{5}\right)/2 \approx 1.618$ or -0.618.

12. (a) Here are a few possibilities:



(b) The "obstacle" is the line $x = y$ (see diagram). Any intersection of the graph of f with the line $y = x$ constitutes a fixed point, and if the graph of the function does not cross the line somewhere in $(0, 1)$, then it must either start at $(0, 0)$ (in which case 0 is a fixed point) or finish at $(1, 1)$ (in which case 1 is a fixed point).

(c) Consider the function $F(x) = f(x) - x$, where f is any continuous function with domain $[0, 1]$ and range in $[0, 1]$. We shall prove that f has a fixed point. Now if $f(0) = 0$ then we are done: f has a fixed point (the number 0), which is what we are trying to prove. So assume $f(0) \neq 0$. For the same reason we can assume that $f(1) \neq 1$. Then $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there exists some number c in the interval $(0, 1)$ such that $F(c) = f(c) - c = 0$. So $f(c) = c$, and therefore f has a fixed point.

13. $$\lim_{x\to a} f(x) = \lim_{x\to a} \left(\tfrac{1}{2}\left[f(x) + g(x)\right] + \tfrac{1}{2}\left[f(x) - g(x)\right]\right) = \tfrac{1}{2}\lim_{x\to a}\left[f(x) + g(x)\right] + \tfrac{1}{2}\lim_{x\to a}\left[f(x) - g(x)\right]$$
$$= \tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot 1 = \tfrac{3}{2},$$

and $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a}\Big(\left[f(x) + g(x)\right] - f(x)\Big) = \lim\limits_{x\to a}\left[f(x) + g(x)\right] - \lim\limits_{x\to a} f(x) = 2 - \tfrac{3}{2} = \tfrac{1}{2}$.

So $\lim\limits_{x\to a}\left[f(x)g(x)\right] = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right] = \tfrac{3}{2}\cdot\tfrac{1}{2} = \tfrac{3}{4}$.

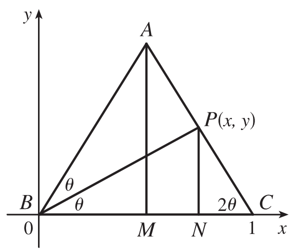
Another solution: Since $\lim\limits_{x\to a}\left[f(x) + g(x)\right]$ and $\lim\limits_{x\to a}\left[f(x) - g(x)\right]$ exist, we must have

$\lim\limits_{x\to a}\left[f(x) + g(x)\right]^2 = \left(\lim\limits_{x\to a}\left[f(x) + g(x)\right]\right)^2$ and $\lim\limits_{x\to a}\left[f(x) - g(x)\right]^2 = \left(\lim\limits_{x\to a}\left[f(x) - g(x)\right]\right)^2$, so

$\lim\limits_{x\to a}\left[f(x)\,g(x)\right] = \lim\limits_{x\to a}\tfrac{1}{4}\left(\left[f(x) + g(x)\right]^2 - \left[f(x) - g(x)\right]^2\right)$ [because all of the f^2 and g^2 cancel]

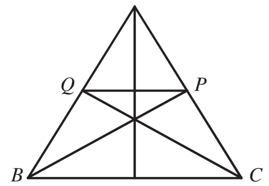
$$= \tfrac{1}{4}\left(\lim_{x\to a}\left[f(x) + g(x)\right]^2 - \lim_{x\to a}\left[f(x) - g(x)\right]^2\right) = \tfrac{1}{4}\left(2^2 - 1^2\right) = \tfrac{3}{4}.$$

14. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular from P, as shown. We see from $\angle NCP$ that $\tan 2\theta = \dfrac{y}{1-x}$, and from $\angle NBP$ that $\tan\theta = y/x$. Using the double-angle formula for tangents, we get $\dfrac{y}{1-x} = \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2(y/x)}{1-(y/x)^2}$. After a bit of simplification, this becomes $\dfrac{1}{1-x} = \dfrac{2x}{x^2-y^2} \iff y^2 = x(3x-2)$.

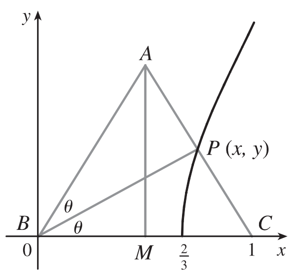


As the altitude AM decreases in length, the point P will approach the x-axis, that is, $y \to 0$, so the limiting location of P must be one of the roots of the equation $x(3x-2) = 0$. Obviously it is not $x = 0$ (the point P can never be to the left of the altitude AM, which it would have to be in order to approach 0) so it must be $3x - 2 = 0$, that is, $x = \frac{2}{3}$.

Solution 2: We add a few lines to the original diagram, as shown. Now note that $\angle BPQ = \angle PBC$ (alternate angles; $QP \parallel BC$ by symmetry) and similarly $\angle CQP = \angle QCB$. So ΔBPQ and ΔCQP are isosceles, and the line segments BQ, QP and PC are all of equal length. As $|AM| \to 0$, P and Q approach points on the base, and the point P is seen to approach a position two-thirds of the way between B and C, as above.



(b) The equation $y^2 = x(3x-2)$ calculated in part (a) is the equation of the curve traced out by P. Now as $|AM| \to \infty$, $2\theta \to \frac{\pi}{2}$, $\theta \to \frac{\pi}{4}$, $x \to 1$, and since $\tan\theta = y/x$, $y \to 1$. Thus, P only traces out the part of the curve with $0 \le y < 1$.



15. (a) Consider $G(x) = T(x + 180°) - T(x)$. Fix any number a. If $G(a) = 0$, we are done: Temperature at $a =$ Temperature at $a + 180°$. If $G(a) > 0$, then $G(a + 180°) = T(a + 360°) - T(a + 180°) = T(a) - T(a + 180°) = -G(a) < 0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a + 180°]$. If $G(a) < 0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

Complete Solutions Manual

for

MULTIVARIABLE CALCULUS

SEVENTH EDITION

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PREFACE

This *Complete Solutions Manual* contains detailed solutions to all exercises in the text *Multivariable Calculus,* Seventh Edition (Chapters 10–17 of *Calculus,* Seventh Edition, and *Calculus: Early Transcendentals,* Seventh Edition) by James Stewart. A *Student Solutions Manual* is also available, which contains solutions to the odd-numbered exercises in each chapter section, review section, True-False Quiz, and Problems Plus section as well as all solutions to the Concept Check questions. (It does not, however, include solutions to any of the projects.)

Because of differences between the regular version and the *Early Transcendentals* version of the text, some references are given in a dual format. In these cases, users of the *Early Transcendentals* text should use the references denoted by "ET."

While we have extended every effort to ensure the accuracy of the solutions presented, we would appreciate correspondence regarding any errors that may exist. Other suggestions or comments are also welcome, and can be sent to dan clegg at `dclegg@palomar.edu` or in care of the publisher: Brooks/Cole, Cengage Learning, 20 Davis Drive, Belmont CA 94002-3098.

We would like to thank James Stewart for entrusting us with the writing of this manual and offering suggestions and Kathi Townes of TECH-arts for typesetting and producing this manual as well as creating the illustrations. We also thank Richard Stratton, Liz Covello, and Elizabeth Neustaetter of Brooks/Cole, Cengage Learning, for their trust, assistance, and patience.

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ABBREVIATIONS AND SYMBOLS

CD	concave downward
CU	concave upward
D	the domain of f
FDT	First Derivative Test
HA	horizontal asymptote(s)
I	interval of convergence
I/D	Increasing/Decreasing Test
IP	inflection point(s)
R	radius of convergence
VA	vertical asymptote(s)
$\overset{\text{CAS}}{=}$	indicates the use of a computer algebra system.
$\overset{\text{H}}{=}$	indicates the use of l'Hospital's Rule.
$\overset{j}{=}$	indicates the use of Formula j in the Table of Integrals in the back endpapers.
$\overset{\text{s}}{=}$	indicates the use of the substitution $\{u = \sin x, du = \cos x\,dx\}$.
$\overset{\text{c}}{=}$	indicates the use of the substitution $\{u = \cos x, du = -\sin x\,dx\}$.

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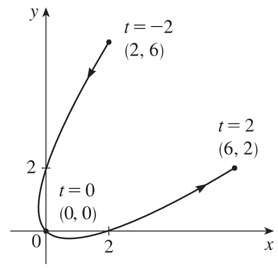
10 ☐ PARAMETRIC EQUATIONS AND POLAR COORDINATES

10.1 Curves Defined by Parametric Equations

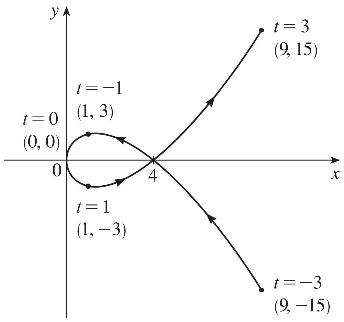
1. $x = t^2 + t, \quad y = t^2 - t, \quad -2 \le t \le 2$

t	-2	-1	0	1	2
x	2	0	0	2	6
y	6	2	0	0	2



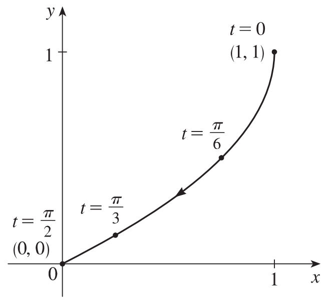
2. $x = t^2, \quad y = t^3 - 4t, \quad -3 \le t \le 3$

t	± 3	± 2	± 1	0
x	9	4	1	0
y	± 15	0	∓ 3	0



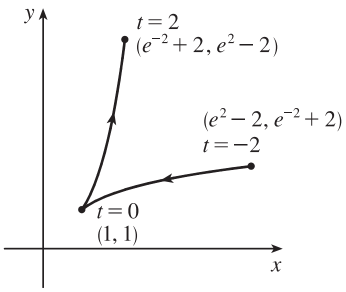
3. $x = \cos^2 t, \quad y = 1 - \sin t, \quad 0 \le t \le \pi/2$

t	0	$\pi/6$	$\pi/3$	$\pi/2$
x	1	$3/4$	$1/4$	0
y	1	$1/2$	$1 - \frac{\sqrt{3}}{2} \approx 0.13$	0



4. $x = e^{-t} + t, \quad y = e^t - t, \quad -2 \le t \le 2$

t	-2	-1	0	1	2
x	$e^2 - 2$ 5.39	$e - 1$ 1.72	1	$e^{-1} + 1$ 1.37	$e^{-2} + 2$ 2.14
y	$e^{-2} + 2$ 2.14	$e^{-1} + 1$ 1.37	1	$e - 1$ 1.72	$e^2 - 2$ 5.39



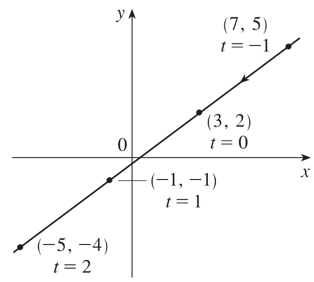
5. $x = 3 - 4t,\ y = 2 - 3t$

(a)

t	-1	0	1	2
x	7	3	-1	-5
y	5	2	-1	-4

(b) $x = 3 - 4t \ \Rightarrow\ 4t = -x + 3 \ \Rightarrow\ t = -\frac{1}{4}x + \frac{3}{4}$, so

$y = 2 - 3t = 2 - 3\left(-\frac{1}{4}x + \frac{3}{4}\right) = 2 + \frac{3}{4}x - \frac{9}{4} \ \Rightarrow\ y = \frac{3}{4}x - \frac{1}{4}$



6. $x = 1 - 2t,\ y = \frac{1}{2}t - 1,\ -2 \le t \le 4$

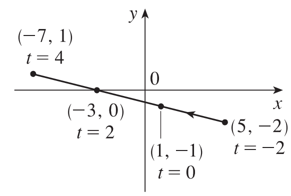
(a)

t	-2	0	2	4
x	5	1	-3	-7
y	-2	-1	0	1

(b) $x = 1 - 2t \ \Rightarrow\ 2t = -x + 1 \ \Rightarrow\ t = -\frac{1}{2}x + \frac{1}{2}$, so

$y = \frac{1}{2}t - 1 = \frac{1}{2}\left(-\frac{1}{2}x + \frac{1}{2}\right) - 1 = -\frac{1}{4}x + \frac{1}{4} - 1 \ \Rightarrow\ y = -\frac{1}{4}x - \frac{3}{4}$,

with $-7 \le x \le 5$



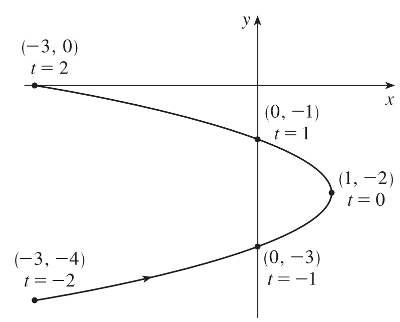
7. $x = 1 - t^2,\ y = t - 2,\ -2 \le t \le 2$

(a)

t	-2	-1	0	1	2
x	-3	0	1	0	-3
y	-4	-3	-2	-1	0

(b) $y = t - 2 \ \Rightarrow\ t = y + 2$, so $x = 1 - t^2 = 1 - (y + 2)^2 \ \Rightarrow$

$x = -(y + 2)^2 + 1$, or $x = -y^2 - 4y - 3$, with $-4 \le y \le 0$



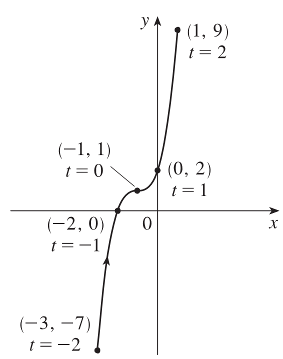
8. $x = t - 1,\ y = t^3 + 1,\ -2 \le t \le 2$

(a)

t	-2	-1	0	1	2
x	-3	-2	-1	0	1
y	-7	0	1	2	9

(b) $x = t - 1 \ \Rightarrow\ t = x + 1$, so $y = t^3 + 1 \ \Rightarrow\ y = (x + 1)^3 + 1$,

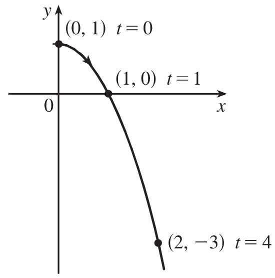
or $y = x^3 + 3x^2 + 3x + 2$, with $-3 \le x \le 1$



9. $x = \sqrt{t},\ y = 1 - t$

(a)

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	−1	−2	−3



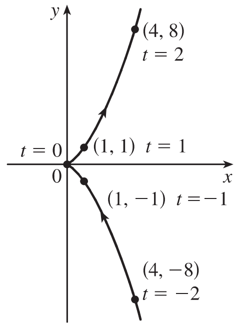
(b) $x = \sqrt{t} \Rightarrow t = x^2 \Rightarrow y = 1 - t = 1 - x^2$. Since $t \geq 0$, $x \geq 0$.

So the curve is the right half of the parabola $y = 1 - x^2$.

10. $x = t^2, y = t^3$

(a)

t	−2	−1	0	1	2
x	4	1	0	1	4
y	−8	−1	0	1	8

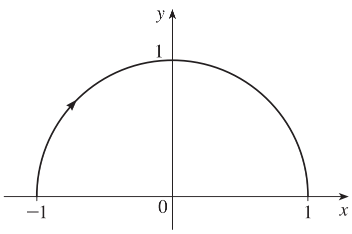


(b) $y = t^3 \Rightarrow t = \sqrt[3]{y} \Rightarrow x = t^2 = \left(\sqrt[3]{y}\right)^2 = y^{2/3}$. $t \in \mathbb{R}, y \in \mathbb{R}, x \geq 0$.

11. (a) $x = \sin \frac{1}{2}\theta, y = \cos \frac{1}{2}\theta, -\pi \leq \theta \leq \pi$.

$x^2 + y^2 = \sin^2 \frac{1}{2}\theta + \cos^2 \frac{1}{2}\theta = 1$. For $-\pi \leq \theta \leq 0$, we have $-1 \leq x \leq 0$ and $0 \leq y \leq 1$. For $0 < \theta \leq \pi$, we have $0 < x \leq 1$ and $1 > y \geq 0$. The graph is a semicircle.

(b)

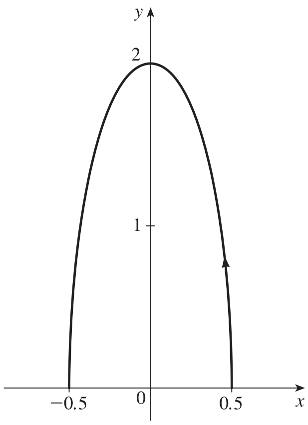


12. (a) $x = \frac{1}{2}\cos\theta, y = 2\sin\theta, 0 \leq \theta \leq \pi$.

$(2x)^2 + \left(\frac{1}{2}y\right)^2 = \cos^2\theta + \sin^2\theta = 1 \Rightarrow 4x^2 + \frac{1}{4}y^2 = 1 \Rightarrow$

$\dfrac{x^2}{(1/2)^2} + \dfrac{y^2}{2^2} = 1$, which is an equation of an ellipse with x-intercepts $\pm\frac{1}{2}$ and y-intercepts ± 2. For $0 \leq \theta \leq \pi/2$, we have $\frac{1}{2} \geq x \geq 0$ and $0 \leq y \leq 2$. For $\pi/2 < \theta \leq \pi$, we have $0 > x \geq -\frac{1}{2}$ and $2 > y \geq 0$. So the graph is the top half of the ellipse.

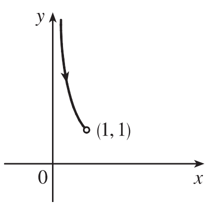
(b)



13. (a) $x = \sin t, y = \csc t, 0 < t < \frac{\pi}{2}$. $y = \csc t = \dfrac{1}{\sin t} = \dfrac{1}{x}$.

For $0 < t < \frac{\pi}{2}$, we have $0 < x < 1$ and $y > 1$. Thus, the curve is the portion of the hyperbola $y = 1/x$ with $y > 1$.

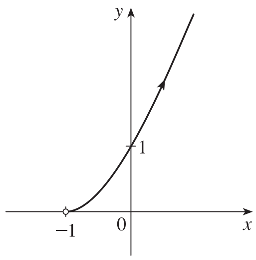
(b)



14. (a) $x = e^t - 1$, $y = e^{2t}$.

$y = (e^t)^2 = (x+1)^2$ and since $x > -1$, we have the right side of the parabola $y = (x+1)^2$.

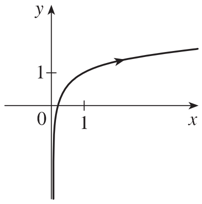
(b)



15. (a) $x = e^{2t} \;\Rightarrow\; 2t = \ln x \;\Rightarrow\; t = \frac{1}{2}\ln x$.

$y = t + 1 = \frac{1}{2}\ln x + 1$.

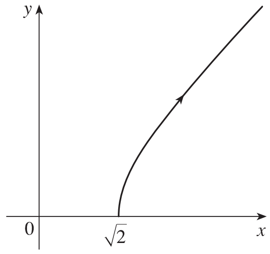
(b)



16. (a) $x = \sqrt{t+1} \;\Rightarrow\; x^2 = t + 1 \;\Rightarrow\; t = x^2 - 1$.

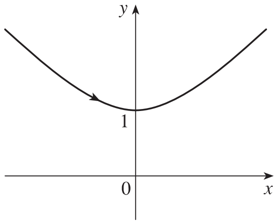
$y = \sqrt{t-1} = \sqrt{(x^2-1)-1} = \sqrt{x^2-2}$. The curve is the part of the hyperbola $x^2 - y^2 = 2$ with $x \geq \sqrt{2}$ and $y \geq 0$.

(b)



17. (a) $x = \sinh t$, $y = \cosh t \;\Rightarrow\; y^2 - x^2 = \cosh^2 t - \sinh^2 t = 1$. Since $y = \cosh t \geq 1$, we have the upper branch of the hyperbola $y^2 - x^2 = 1$.

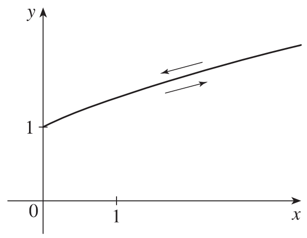
(b)



18. (a) $x = \tan^2\theta$, $y = \sec\theta$, $-\pi/2 < \theta < \pi/2$.

$1 + \tan^2\theta = \sec^2\theta \;\Rightarrow\; 1 + x = y^2 \;\Rightarrow\; x = y^2 - 1$. For $-\pi/2 < \theta \leq 0$, we have $x \geq 0$ and $y \geq 1$. For $0 < \theta < \pi/2$, we have $0 < x$ and $1 < y$. Thus, the curve is the portion of the parabola $x = y^2 - 1$ in the first quadrant. As θ increases from $-\pi/2$ to 0, the point (x, y) approaches $(0, 1)$ along the parabola. As θ increases from 0 to $\pi/2$, the point (x, y) retreats from $(0, 1)$ along the parabola.

(b)



19. $x = 3 + 2\cos t$, $y = 1 + 2\sin t$, $\pi/2 \leq t \leq 3\pi/2$. By Example 4 with $r = 2$, $h = 3$, and $k = 1$, the motion of the particle takes place on a circle centered at $(3, 1)$ with a radius of 2. As t goes from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, the particle starts at the point $(3, 3)$ and moves counterclockwise along the circle $(x-3)^2 + (y-1)^2 = 4$ to $(3, -1)$ [one-half of a circle].

20. $x = 2\sin t$, $y = 4 + \cos t \;\Rightarrow\; \sin t = \dfrac{x}{2}$, $\cos t = y - 4$. $\sin^2 t + \cos^2 t = 1 \;\Rightarrow\; \left(\dfrac{x}{2}\right)^2 + (y-4)^2 = 1$. The motion of the particle takes place on an ellipse centered at $(0, 4)$. As t goes from 0 to $\frac{3\pi}{2}$, the particle starts at the point $(0, 5)$ and moves clockwise to $(-2, 4)$ [three-quarters of an ellipse].

21. $x = 5\sin t$, $y = 2\cos t$ $\Rightarrow$ $\sin t = \dfrac{x}{5}$, $\cos t = \dfrac{y}{2}$. $\sin^2 t + \cos^2 t = 1$ $\Rightarrow$ $\left(\dfrac{x}{5}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1$. The motion of the particle takes place on an ellipse centered at $(0,0)$. As t goes from $-\pi$ to 5π, the particle starts at the point $(0,-2)$ and moves clockwise around the ellipse 3 times.

22. $y = \cos^2 t = 1 - \sin^2 t = 1 - x^2$. The motion of the particle takes place on the parabola $y = 1 - x^2$. As t goes from -2π to $-\pi$, the particle starts at the point $(0,1)$, moves to $(1,0)$, and goes back to $(0,1)$. As t goes from $-\pi$ to 0, the particle moves to $(-1,0)$ and goes back to $(0,1)$. The particle repeats this motion as t goes from 0 to 2π.

23. We must have $1 \le x \le 4$ and $2 \le y \le 3$. So the graph of the curve must be contained in the rectangle $[1,4]$ by $[2,3]$.

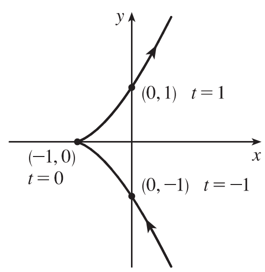
24. (a) From the first graph, we have $1 \le x \le 2$. From the second graph, we have $-1 \le y \le 1$. The only choice that satisfies either of those conditions is III.

(b) From the first graph, the values of x cycle through the values from -2 to 2 four times. From the second graph, the values of y cycle through the values from -2 to 2 six times. Choice I satisfies these conditions.

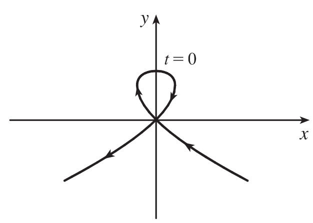
(c) From the first graph, the values of x cycle through the values from -2 to 2 three times. From the second graph, we have $0 \le y \le 2$. Choice IV satisfies these conditions.

(d) From the first graph, the values of x cycle through the values from -2 to 2 two times. From the second graph, the values of y do the same thing. Choice II satisfies these conditions.

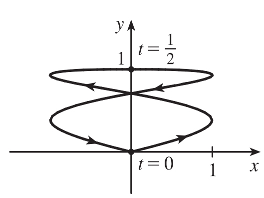
25. When $t = -1$, $(x,y) = (0,-1)$. As t increases to 0, x decreases to -1 and y increases to 0. As t increases from 0 to 1, x increases to 0 and y increases to 1. As t increases beyond 1, both x and y increase. For $t < -1$, x is positive and decreasing and y is negative and increasing. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



26. For $t < -1$, x is positive and decreasing, while y is negative and increasing (these points are in Quadrant IV). When $t = -1$, $(x,y) = (0,0)$ and, as t increases from -1 to 0, x becomes negative and y increases from 0 to 1. At $t = 0$, $(x,y) = (0,1)$ and, as t increases from 0 to 1, y decreases from 1 to 0 and x is positive. At $t = 1$, $(x,y) = (0,0)$ again, so the loop is completed. For $t > 1$, x and y both become large negative. This enables us to draw a rough sketch. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



27. When $t = 0$ we see that $x = 0$ and $y = 0$, so the curve starts at the origin. As t increases from 0 to $\frac{1}{2}$, the graphs show that y increases from 0 to 1 while x increases from 0 to 1, decreases to 0 and to -1, then increases back to 0, so we arrive at the point $(0,1)$. Similarly, as t increases from $\frac{1}{2}$ to 1, y decreases from 1 to 0 while x repeats its pattern, and we arrive back at the origin. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



28. (a) $x = t^4 - t + 1 = (t^4 + 1) - t > 0$ [think of the graphs of $y = t^4 + 1$ and $y = t$] and $y = t^2 \geq 0$, so these equations are matched with graph V.

(b) $y = \sqrt{t} \geq 0$. $x = t^2 - 2t = t(t - 2)$ is negative for $0 < t < 2$, so these equations are matched with graph I.

(c) $x = \sin 2t$ has period $2\pi/2 = \pi$. Note that

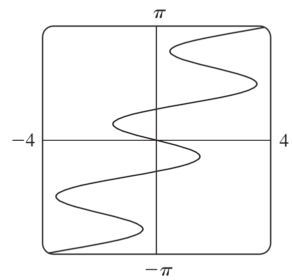
$y(t + 2\pi) = \sin[t + 2\pi + \sin 2(t + 2\pi)] = \sin(t + 2\pi + \sin 2t) = \sin(t + \sin 2t) = y(t)$, so y has period 2π.
These equations match graph II since x cycles through the values -1 to 1 twice as y cycles through those values once.

(d) $x = \cos 5t$ has period $2\pi/5$ and $y = \sin 2t$ has period π, so x will take on the values -1 to 1, and then 1 to -1, before y takes on the values -1 to 1. Note that when $t = 0$, $(x, y) = (1, 0)$. These equations are matched with graph VI.

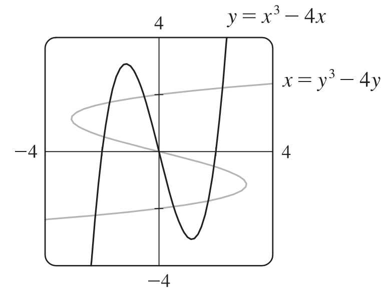
(e) $x = t + \sin 4t$, $y = t^2 + \cos 3t$. As t becomes large, t and t^2 become the dominant terms in the expressions for x and y, so the graph will look like the graph of $y = x^2$, but with oscillations. These equations are matched with graph IV.

(f) $x = \dfrac{\sin 2t}{4 + t^2}$, $y = \dfrac{\cos 2t}{4 + t^2}$. As $t \to \infty$, x and y both approach 0. These equations are matched with graph III.

29. Use $y = t$ and $x = t - 2\sin \pi t$ with a t-interval of $[-\pi, \pi]$.



30. Use $x_1 = t$, $y_1 = t^3 - 4t$ and $x_2 = t^3 - 4t$, $y_2 = t$ with a t-interval of $[-3, 3]$. There are 9 points of intersection; $(0, 0)$ is fairly obvious. The point in quadrant I is approximately $(2.2, 2.2)$, and by symmetry, the point in quadrant III is approximately $(-2.2, -2.2)$. The other six points are approximately $(\mp 1.9, \pm 0.5)$, $(\mp 1.7, \pm 1.7)$, and $(\mp 0.5, \pm 1.9)$.



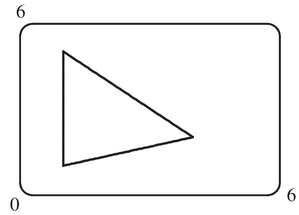
31. (a) $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \leq t \leq 1$. Clearly the curve passes through $P_1(x_1, y_1)$ when $t = 0$ and through $P_2(x_2, y_2)$ when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2. For every value of t, x and y satisfy the relation $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is the equation of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Finally, any point (x, y) on that line satisfies $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$; if we call that common value t, then the given parametric equations yield the point (x, y); and any (x, y) on the line between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$.

(b) $x = -2 + [3 - (-2)]t = -2 + 5t$ and $y = 7 + (-1 - 7)t = 7 - 8t$ for $0 \leq t \leq 1$.

32. For the side of the triangle from A to B, use $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$. Hence, the equations are

$$x = x_1 + (x_2 - x_1)\,t = 1 + (4 - 1)\,t = 1 + 3t,$$
$$y = y_1 + (y_2 - y_1)\,t = 1 + (2 - 1)\,t = 1 + t.$$



Graphing $x = 1 + 3t$ and $y = 1 + t$ with $0 \le t \le 1$ gives us the side of the triangle from A to B. Similarly, for the side BC we use $x = 4 - 3t$ and $y = 2 + 3t$, and for the side AC we use $x = 1$ and $y = 1 + 4t$.

33. The circle $x^2 + (y - 1)^2 = 4$ has center $(0, 1)$ and radius 2, so by Example 4 it can be represented by $x = 2\cos t$, $y = 1 + 2\sin t$, $0 \le t \le 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.

(a) To get a clockwise orientation, we could change the equations to $x = 2\cos t$, $y = 1 - 2\sin t$, $0 \le t \le 2\pi$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x = 2\cos t$, $y = 1 + 2\sin t$ with the domain expanded to $0 \le t \le 6\pi$.

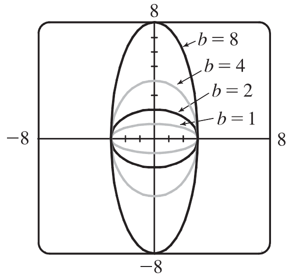
(c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2\cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use $x = 2\cos t$, $y = 1 + 2\sin t$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x = -2\sin t$, $y = 1 + 2\cos t$, $0 \le t \le \pi$.

34. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a\sin t$ and $y = b\cos t$ with $0 \le t \le 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

(b) The equations are $x = 3\sin t$ and $y = b\cos t$ for $b \in \{1, 2, 4, 8\}$.

(c) As b increases, the ellipse stretches vertically.



35. *Big circle:* It's centered at $(2, 2)$ with a radius of 2, so by Example 4, parametric equations are

$$x = 2 + 2\cos t, \qquad y = 2 + 2\sin t, \qquad 0 \le t \le 2\pi$$

Small circles: They are centered at $(1, 3)$ and $(3, 3)$ with a radius of 0.1. By Example 4, parametric equations are

(left) $\quad x = 1 + 0.1\cos t, \qquad y = 3 + 0.1\sin t, \qquad 0 \le t \le 2\pi$

and *(right)* $\quad x = 3 + 0.1\cos t, \qquad y = 3 + 0.1\sin t, \qquad 0 \le t \le 2\pi$

Semicircle: It's the lower half of a circle centered at $(2, 2)$ with radius 1. By Example 4, parametric equations are

$$x = 2 + 1\cos t, \qquad y = 2 + 1\sin t, \qquad \pi \le t \le 2\pi$$

To get all four graphs on the same screen with a typical graphing calculator, we need to change the last t-interval to $[0, 2\pi]$ in order to match the others. We can do this by changing t to $0.5t$. This change gives us the upper half. There are several ways to get the lower half—one is to change the "+" to a "−" in the y-assignment, giving us

$$x = 2 + 1\cos(0.5t), \qquad y = 2 - 1\sin(0.5t), \qquad 0 \le t \le 2\pi$$

36. If you are using a calculator or computer that can overlay graphs (using multiple t-intervals), the following is appropriate.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \qquad y = t, \qquad 1.5 \le t \le 4$$

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \qquad y = t, \qquad 1.5 \le t \le 4$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = t, \qquad y = 1.5, \qquad 1 \le t \le 10$$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + t, \qquad y = 4 + t, \qquad 0 \le t \le 3$$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about $30°$ above the horizontal, so use

$$x = 3 + 1\cos t, \qquad y = 1 + 1\sin t, \qquad \tfrac{5\pi}{6} \le t \le \tfrac{13\pi}{6}$$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

$$x = 8 + 1\cos t, \qquad y = 1 + 1\sin t, \qquad \tfrac{5\pi}{6} \le t \le \tfrac{13\pi}{6}$$

If you are using a calculator or computer that cannot overlay graphs (using one t-interval), the following is appropriate. We'll start by picking the t-interval $[0, 2.5]$ since it easily matches the t-values for the two sides. We now need to find parametric equations for all graphs with $0 \le t \le 2.5$.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \qquad y = 1.5 + t, \qquad 0 \le t \le 2.5$$

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \qquad y = 1.5 + t, \qquad 0 \le t \le 2.5$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = 1 + 3.6t, \qquad y = 1.5, \qquad 0 \le t \le 2.5$$

To get the *x*-assignment, think of creating a linear function such that when $t = 0$, $x = 1$ and when $t = 2.5$, $x = 10$. We can use the point-slope form of a line with $(t_1, x_1) = (0, 1)$ and $(t_2, x_2) = (2.5, 10)$.

$$x - 1 = \frac{10 - 1}{2.5 - 0}(t - 0) \quad \Rightarrow \quad x = 1 + 3.6t.$$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + 1.2t, \qquad y = 4 + 1.2t, \qquad 0 \le t \le 2.5$$

$(t_1, x_1) = (0, 10)$ and $(t_2, x_2) = (2.5, 13)$ gives us $x - 10 = \dfrac{13 - 10}{2.5 - 0}(t - 0) \quad \Rightarrow \quad x = 10 + 1.2t.$

$(t_1, y_1) = (0, 4)$ and $(t_2, y_2) = (2.5, 7)$ gives us $y - 4 = \dfrac{7 - 4}{2.5 - 0}(t - 0) \quad \Rightarrow \quad y = 4 + 1.2t.$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about 30° above the horizontal, so use

$$x = 3 + 1\cos\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad y = 1 + 1\sin\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad 0 \le t \le 2.5$$

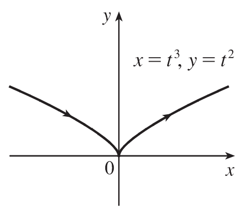
$(t_1, \theta_1) = \left(0, \frac{5\pi}{6}\right)$ and $(t_2, \theta_2) = \left(\frac{5}{2}, \frac{13\pi}{6}\right)$ gives us $\theta - \frac{5\pi}{6} = \dfrac{\frac{13\pi}{6} - \frac{5\pi}{6}}{\frac{5}{2} - 0}(t - 0) \;\Rightarrow\; \theta = \frac{5\pi}{6} + \frac{8\pi}{15}t.$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

$$x = 8 + 1\cos\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad y = 1 + 1\sin\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad 0 \le t \le 2.5$$

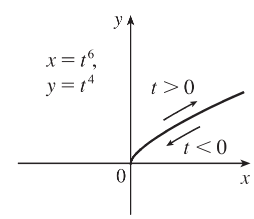
37. (a) $x = t^3 \;\Rightarrow\; t = x^{1/3}$, so $y = t^2 = x^{2/3}$.

We get the entire curve $y = x^{2/3}$ traversed in a left to right direction.



(b) $x = t^6 \;\Rightarrow\; t = x^{1/6}$, so $y = t^4 = x^{4/6} = x^{2/3}$.

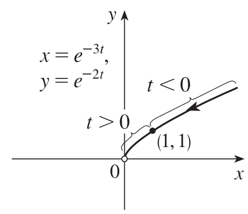
Since $x = t^6 \ge 0$, we only get the right half of the curve $y = x^{2/3}$.



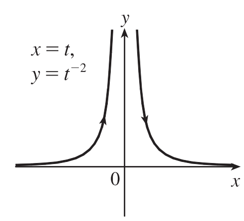
(c) $x = e^{-3t} = (e^{-t})^3$ [so $e^{-t} = x^{1/3}$],

$y = e^{-2t} = (e^{-t})^2 = (x^{1/3})^2 = x^{2/3}$.

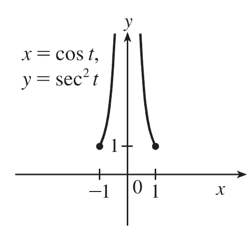
If $t < 0$, then x and y are both larger than 1. If $t > 0$, then x and y are between 0 and 1. Since $x > 0$ and $y > 0$, the curve never quite reaches the origin.



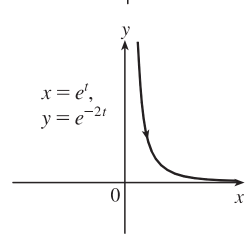
38. (a) $x = t$, so $y = t^{-2} = x^{-2}$. We get the entire curve $y = 1/x^2$ traversed in a left-to-right direction.



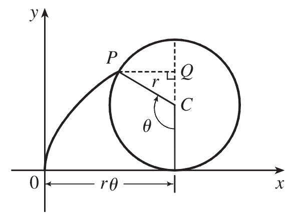
(b) $x = \cos t$, $y = \sec^2 t = \dfrac{1}{\cos^2 t} = \dfrac{1}{x^2}$. Since $\sec t \ge 1$, we only get the parts of the curve $y = 1/x^2$ with $y \ge 1$. We get the first quadrant portion of the curve when $x > 0$, that is, $\cos t > 0$, and we get the second quadrant portion of the curve when $x < 0$, that is, $\cos t < 0$.



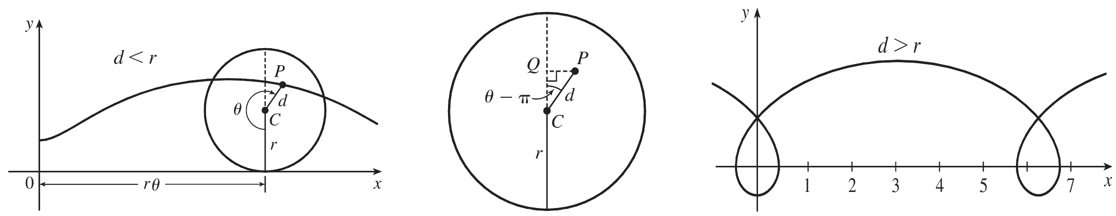
(c) $x = e^t$, $y = e^{-2t} = (e^t)^{-2} = x^{-2}$. Since e^t and e^{-2t} are both positive, we only get the first quadrant portion of the curve $y = 1/x^2$.



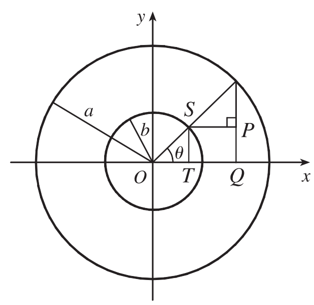
39. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 7, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$ [since $\cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha = -\cos\alpha$], so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\theta))$ [since $\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha = \sin\alpha$]. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



40. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d < r$. As in Example 7, C has coordinates $(r\theta, r)$. Now Q (in the second diagram) has coordinates $(r\theta, r + d\cos(\theta - \pi)) = (r\theta, r - d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta + d\sin(\theta - \pi), r - d\cos\theta)$. That is, P has coordinates (x, y), where $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$. When $d = r$, these equations agree with those of the cycloid.

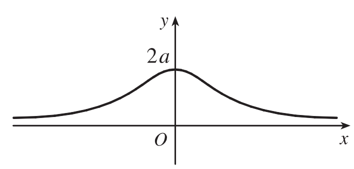


41. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a\cos\theta$ and $y = |ST| = b\sin\theta$. Thus, the parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$. To eliminate θ we rearrange: $\sin\theta = y/b \Rightarrow \sin^2\theta = (y/b)^2$ and $\cos\theta = x/a \Rightarrow \cos^2\theta = (x/a)^2$. Adding the two equations: $\sin^2\theta + \cos^2\theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



42. A has coordinates $(a\cos\theta, a\sin\theta)$. Since OA is perpendicular to AB, ΔOAB is a right triangle and B has coordinates $(a\sec\theta, 0)$. It follows that P has coordinates $(a\sec\theta, b\sin\theta)$. Thus, the parametric equations are $x = a\sec\theta$, $y = b\sin\theta$.

43. $C = (2a\cot\theta, 2a)$, so the x-coordinate of P is $x = 2a\cot\theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a\sin\theta$ and $A = ((2a\sin\theta)\cos\theta, (2a\sin\theta)\sin\theta)$. Thus, the y-coordinate of P is $y = 2a\sin^2\theta$.



44. (a) Let θ be the angle of inclination of segment OP. Then $|OB| = \dfrac{2a}{\cos\theta}$.

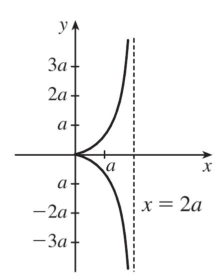
Let $C = (2a, 0)$. Then by use of right triangle OAC we see that $|OA| = 2a\cos\theta$.

Now

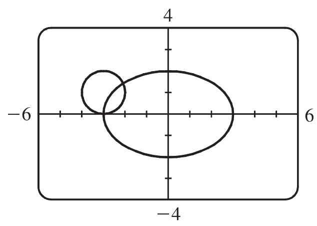
$$|OP| = |AB| = |OB| - |OA|$$
$$= 2a\left(\frac{1}{\cos\theta} - \cos\theta\right) = 2a\,\frac{1 - \cos^2\theta}{\cos\theta} = 2a\,\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\,\tan\theta$$

So P has coordinates $x = 2a\sin\theta\,\tan\theta\cdot\cos\theta = 2a\sin^2\theta$ and $y = 2a\sin\theta\,\tan\theta\cdot\sin\theta = 2a\sin^2\theta\,\tan\theta$.

(b)



45. (a)



There are 2 points of intersection:

$(-3, 0)$ and approximately $(-2.1, 1.4)$.

(b) A collision point occurs when $x_1 = x_2$ and $y_1 = y_2$ for the same t. So solve the equations:

$$3\sin t = -3 + \cos t \quad \textbf{(1)}$$
$$2\cos t = 1 + \sin t \quad \textbf{(2)}$$

From **(2)**, $\sin t = 2\cos t - 1$. Substituting into **(1)**, we get $3(2\cos t - 1) = -3 + \cos t \quad\Rightarrow\quad 5\cos t = 0 \;\; (\star) \quad\Rightarrow$ $\cos t = 0 \quad\Rightarrow\quad t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. We check that $t = \frac{3\pi}{2}$ satisfies **(1)** and **(2)** but $t = \frac{\pi}{2}$ does not. So the only collision point occurs when $t = \frac{3\pi}{2}$, and this gives the point $(-3, 0)$. [We could check our work by graphing x_1 and x_2 together as functions of t and, on another plot, y_1 and y_2 as functions of t. If we do so, we see that the only value of t for which *both* pairs of graphs intersect is $t = \frac{3\pi}{2}$.]

(c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but there are no collision points, since $(\star)$ in part (b) becomes $5\cos t = 6 \quad\Rightarrow\quad \cos t = \frac{6}{5} > 1$.

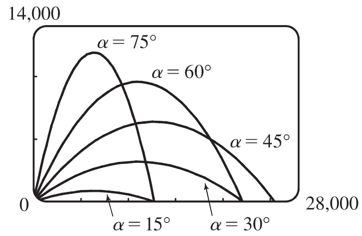
46. (a) If $\alpha = 30°$ and $v_0 = 500$ m/s, then the equations become $x = (500\cos 30°)t = 250\sqrt{3}t$ and $y = (500\sin 30°)t - \frac{1}{2}(9.8)t^2 = 250t - 4.9t^2$. $y = 0$ when $t = 0$ (when the gun is fired) and again when $t = \frac{250}{4.9} \approx 51$ s. Then $x = \left(250\sqrt{3}\right)\left(\frac{250}{4.9}\right) \approx 22{,}092$ m, so the bullet hits the ground about 22 km from the gun.

The formula for y is quadratic in t. To find the maximum y-value, we will complete the square:

$$y = -4.9\left(t^2 - \tfrac{250}{4.9}t\right) = -4.9\left[t^2 - \tfrac{250}{4.9}t + \left(\tfrac{125}{4.9}\right)^2\right] + \tfrac{125^2}{4.9} = -4.9\left(t - \tfrac{125}{4.9}\right)^2 + \tfrac{125^2}{4.9} \le \tfrac{125^2}{4.9}$$

with equality when $t = \frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9} \approx 3189$ m.

(b)



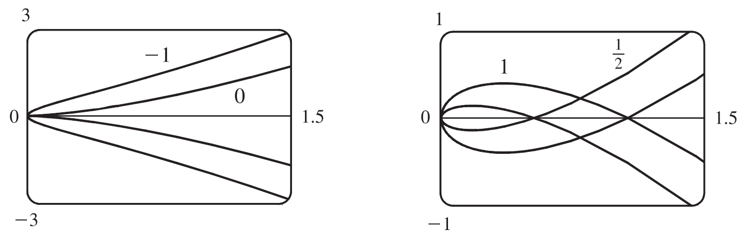
As α $(0° < \alpha < 90°)$ increases up to $45°$, the projectile attains a greater height and a greater range. As α increases past $45°$, the projectile attains a greater height, but its range decreases.

(c) $x = (v_0\cos\alpha)t \quad\Rightarrow\quad t = \dfrac{x}{v_0\cos\alpha}$.

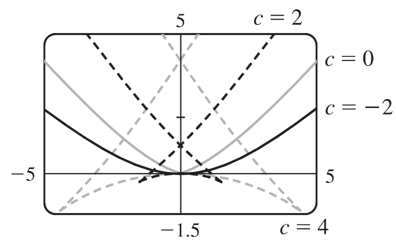
$$y = (v_0\sin\alpha)t - \tfrac{1}{2}gt^2 \quad\Rightarrow\quad y = (v_0\sin\alpha)\frac{x}{v_0\cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)x - \left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2,$$

which is the equation of a parabola (quadratic in x).

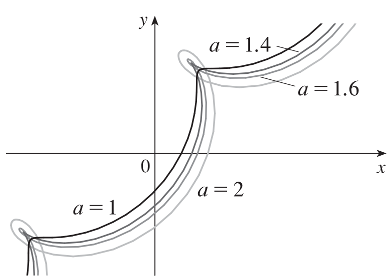
47. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the x-axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.



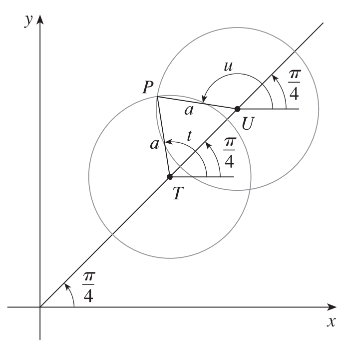
48. $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the y-axis. When $c < 0$, the graph resembles that of a polynomial of even degree, but when $c = 0$ there is a corner at the origin, and when $c > 0$, the graph crosses itself at the origin, and has two cusps below the x-axis. The size of the "swallowtail" increases as c increases.



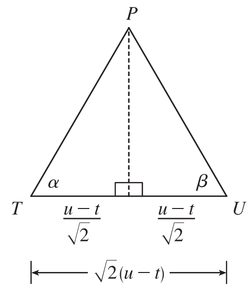
49. $x = t + a\cos t, y = t + a\sin t, a > 0$. From the first figure, we see that curves roughly follow the line $y = x$, and they start having loops when a is between 1.4 and 1.6. The loops increase in size as a increases.



While not required, the following is a solution to determine the *exact* values for which the curve has a loop, that is, we seek the values of a for which there exist parameter values t and u such that $t < u$ and $(t + a\cos t, t + a\sin t) = (u + a\cos u, u + a\sin u)$.



In the diagram at the left, T denotes the point (t, t), U the point (u, u), and P the point $(t + a\cos t, t + a\sin t) = (u + a\cos u, u + a\sin u)$. Since $\overline{PT} = \overline{PU} = a$, the triangle PTU is isosceles. Therefore its base angles, $\alpha = \angle PTU$ and $\beta = \angle PUT$ are equal. Since $\alpha = t - \frac{\pi}{4}$ and $\beta = 2\pi - \frac{3\pi}{4} - u = \frac{5\pi}{4} - u$, the relation $\alpha = \beta$ implies that $u + t = \frac{3\pi}{2}$ **(1)**.



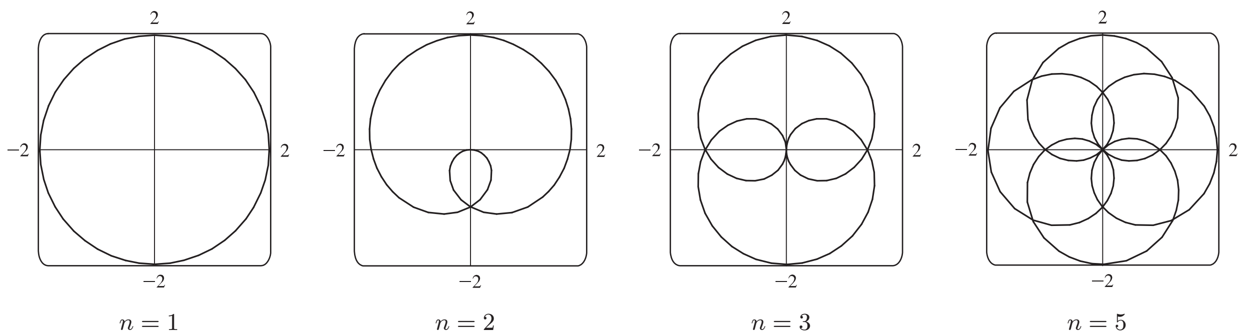
Since $\overline{TU} = \text{distance}((t,t),(u,u)) = \sqrt{2(u-t)^2} = \sqrt{2}\,(u-t)$, we see that $\cos\alpha = \dfrac{\frac{1}{2}\overline{TU}}{\overline{PT}} = \dfrac{(u-t)/\sqrt{2}}{a}$, so $u - t = \sqrt{2}\,a\cos\alpha$, that is, $u - t = \sqrt{2}\,a\cos\left(t - \frac{\pi}{4}\right)$ **(2)**. Now $\cos\left(t-\frac{\pi}{4}\right) = \sin\left[\frac{\pi}{2} - \left(t - \frac{\pi}{4}\right)\right] = \sin\left(\frac{3\pi}{4} - t\right)$, so we can rewrite **(2)** as $u - t = \sqrt{2}\,a\sin\left(\frac{3\pi}{4} - t\right)$ **(2′)**. Subtracting **(2′)** from **(1)** and dividing by 2, we obtain $t = \frac{3\pi}{4} - \frac{\sqrt{2}}{2}a\sin\left(\frac{3\pi}{4} - t\right)$, or $\frac{3\pi}{4} - t = \frac{a}{\sqrt{2}}\sin\left(\frac{3\pi}{4} - t\right)$ **(3)**. Since $a > 0$ and $t < u$, it follows from **(2′)** that $\sin\left(\frac{3\pi}{4} - t\right) > 0$. Thus from **(3)** we see that $t < \frac{3\pi}{4}$. [We have implicitly assumed that $0 < t < \pi$ by the way we drew our diagram, but we lost no generality by doing so since replacing t by $t + 2\pi$ merely increases x and y by 2π. The curve's basic shape repeats every time we change t by 2π.] Solving for a in **(3)**, we get $a = \dfrac{\sqrt{2}\left(\frac{3\pi}{4} - t\right)}{\sin\left(\frac{3\pi}{4} - t\right)}$. Write $z = \frac{3\pi}{4} - t$. Then $a = \dfrac{\sqrt{2}\,z}{\sin z}$, where $z > 0$. Now $\sin z < z$ for $z > 0$, so $a > \sqrt{2}$. $\left[\text{As } z \to 0^+, \text{ that is, as } t \to \left(\frac{3\pi}{4}\right)^-, a \to \sqrt{2}\right]$.

50. Consider the curves $x = \sin t + \sin nt$, $y = \cos t + \cos nt$, where n is a positive integer. For $n = 1$, we get a circle of radius 2 centered at the origin. For $n > 1$, we get a curve lying on or inside that circle that traces out $n - 1$ loops as t ranges from 0 to 2π.

Note:

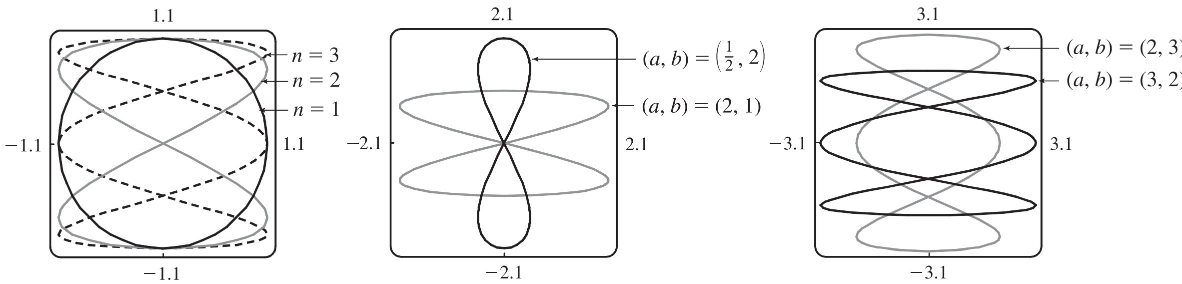
$$\begin{aligned} x^2 + y^2 &= (\sin t + \sin nt)^2 + (\cos t + \cos nt)^2 \\ &= \sin^2 t + 2\sin t\,\sin nt + \sin^2 nt + \cos^2 t + 2\cos t\,\cos nt + \cos^2 nt \\ &= (\sin^2 t + \cos^2 t) + (\sin^2 nt + \cos^2 nt) + 2(\cos t\,\cos nt + \sin t\,\sin nt) \\ &= 1 + 1 + 2\cos(t - nt) = 2 + 2\cos((1-n)t) \le 4 = 2^2, \end{aligned}$$

with equality for $n = 1$. This shows that each curve lies on or inside the curve for $n = 1$, which is a circle of radius 2 centered at the origin.



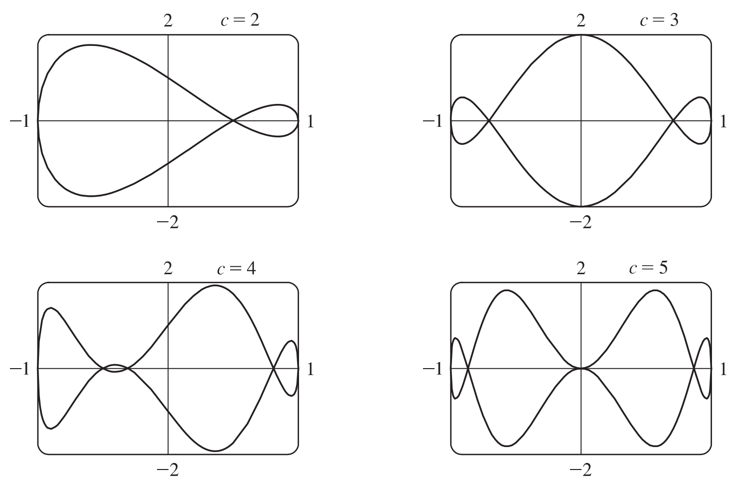
51. Note that all the Lissajous figures are symmetric about the x-axis. The parameters a and b simply stretch the graph in the x- and y-directions respectively. For $a = b = n = 1$ the graph is simply a circle with radius 1. For $n = 2$ the graph crosses

itself at the origin and there are loops above and below the x-axis. In general, the figures have $n-1$ points of intersection, all of which are on the y-axis, and a total of n closed loops.

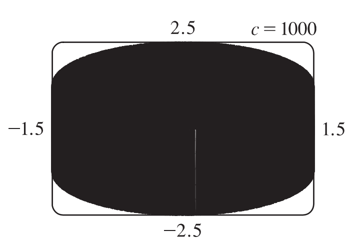


$a=b=1$ $\qquad$ $n=2$ $\qquad$ $n=3$

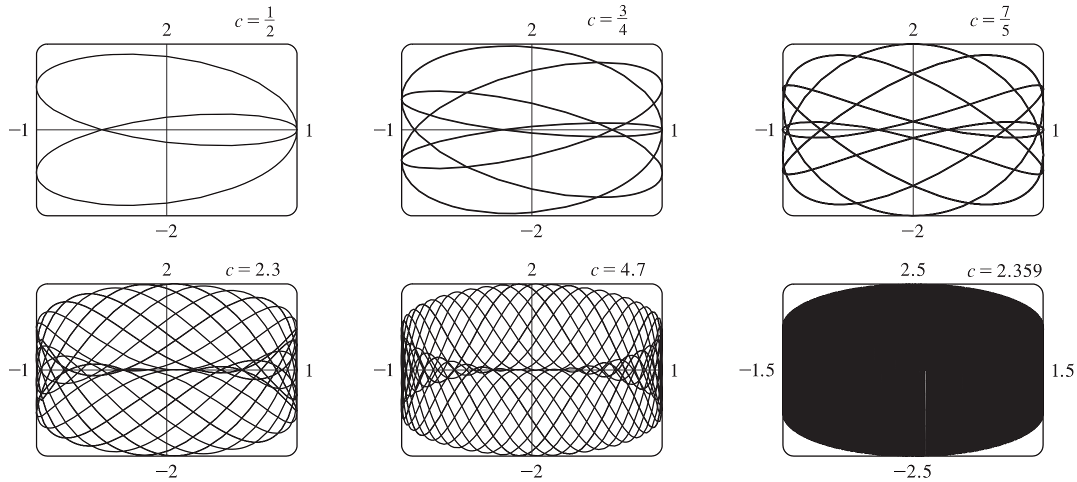
52. $x=\cos t$, $y=\sin t-\sin ct$. If $c=1$, then $y=0$, and the curve is simply the line segment from $(-1,0)$ to $(1,0)$. The graphs are shown for $c=2,3,4$ and 5.



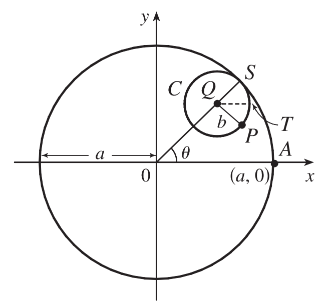
It is easy to see that all the curves lie in the rectangle $[-1,1]$ by $[-2,2]$. When c is an integer, $x(t+2\pi)=x(t)$ and $y(t+2\pi)=y(t)$, so the curve is closed. When c is a positive integer greater than 1, the curve intersects the x-axis $c+1$ times and has c loops (one of which degenerates to a tangency at the origin when c is an odd integer of the form $4k+1$). As c increases, the curve's loops become thinner, but stay in the region bounded by the semicircles $y=\pm\left(1+\sqrt{1-x^2}\,\right)$ and the line segments from $(-1,-1)$ to $(-1,1)$ and from $(1,-1)$ to $(1,1)$. This is true because $|y|=|\sin t-\sin ct|\le|\sin t|+|\sin ct|\le\sqrt{1-x^2}+1$. This curve appears to fill the entire region when c is very large, as shown in the figure for $c=1000$.



When c is a fraction, we get a variety of shapes with multiple loops, but always within the same region. For some fractional values, such as $c = 2.359$, the curve again appears to fill the region.



LABORATORY PROJECT Running Circles Around Circles

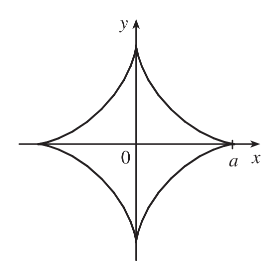


1. The center Q of the smaller circle has coordinates $((a-b)\cos\theta, (a-b)\sin\theta)$. Arc PS on circle C has length $a\theta$ since it is equal in length to arc AS (the smaller circle rolls without slipping against the larger.)

   Thus, $\angle PQS = \frac{a}{b}\theta$ and $\angle PQT = \frac{a}{b}\theta - \theta$, so P has coordinates

$$x = (a-b)\cos\theta + b\cos(\angle PQT) = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

and $$y = (a-b)\sin\theta - b\sin(\angle PQT) = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right).$$

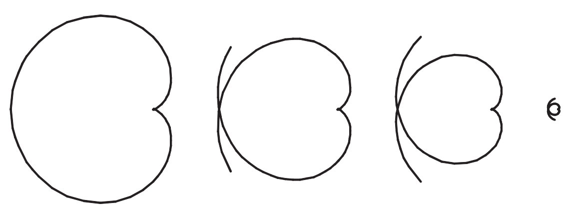


2. With $b = 1$ and a a positive integer greater than 2, we obtain a hypocycloid of a cusps. Shown in the figure is the graph for $a = 4$. Let $a = 4$ and $b = 1$. Using the sum identities to expand $\cos 3\theta$ and $\sin 3\theta$, we obtain

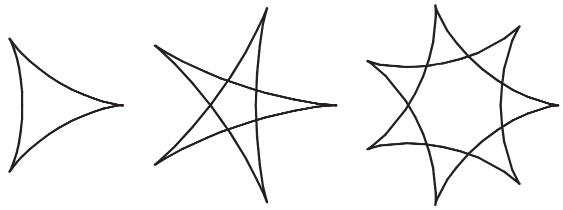
$$x = 3\cos\theta + \cos 3\theta = 3\cos\theta + \left(4\cos^3\theta - 3\cos\theta\right) = 4\cos^3\theta$$

and $$y = 3\sin\theta - \sin 3\theta = 3\sin\theta - \left(3\sin\theta - 4\sin^3\theta\right) = 4\sin^3\theta.$$

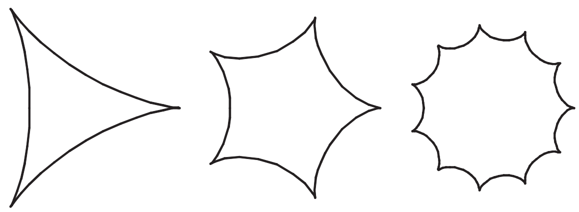
3. The graphs at the right are obtained with $b = 1$ and $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{10}$ with $-2\pi \le \theta \le 2\pi$. We conclude that as the denominator d increases, the graph gets smaller, but maintains the basic shape shown.



Letting $d = 2$ and $n = 3$, 5, and 7 with $-2\pi \leq \theta \leq 2\pi$ gives us the following:



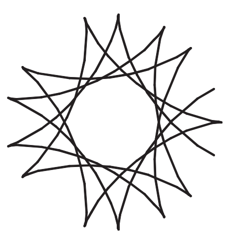
So if d is held constant and n varies, we get a graph with n cusps (assuming n/d is in lowest form). When $n = d + 1$, we obtain a hypocycloid of n cusps. As n increases, we must expand the range of θ in order to get a closed curve. The following graphs have $a = \frac{3}{2}, \frac{5}{4}$, and $\frac{11}{10}$.



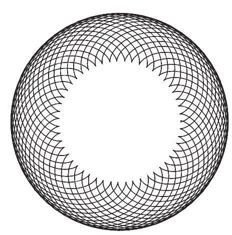
4. If $b = 1$, the equations for the hypocycloid are

$$x = (a-1)\cos\theta + \cos\left((a-1)\,\theta\right) \qquad\qquad y = (a-1)\sin\theta - \sin\left((a-1)\,\theta\right)$$

which is a hypocycloid of a cusps (from Problem 2). In general, if $a > 1$, we get a figure with cusps on the "outside ring" and if $a < 1$, the cusps are on the "inside ring". In any case, as the values of θ get larger, we get a figure that looks more and more like a washer. If we were to graph the hypocycloid for all values of θ, every point on the washer would eventually be arbitrarily close to a point on the curve.

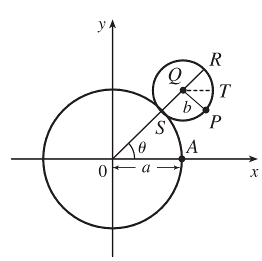


$a = \sqrt{2}, \quad -10\pi \leq \theta \leq 10\pi$



$a = e - 2, \quad 0 \leq \theta \leq 446$

5. The center Q of the smaller circle has coordinates $((a+b)\cos\theta, (a+b)\sin\theta)$.



Arc PS has length $a\theta$ (as in Problem 1), so that $\angle PQS = \dfrac{a\theta}{b}$, $\angle PQR = \pi - \dfrac{a\theta}{b}$,

and $\angle PQT = \pi - \dfrac{a\theta}{b} - \theta = \pi - \left(\dfrac{a+b}{b}\right)\theta$ since $\angle RQT = \theta$.

Thus, the coordinates of P are

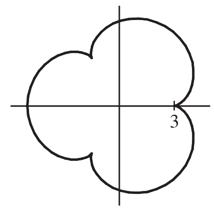
$$x = (a+b)\cos\theta + b\cos\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

and $$y = (a+b)\sin\theta - b\sin\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right).$$

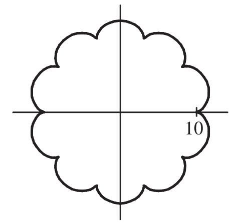
6. Let $b = 1$ and the equations become

$$x = (a+1)\cos\theta - \cos((a+1)\theta) \qquad y = (a+1)\sin\theta - \sin((a+1)\theta)$$

If $a = 1$, we have a cardioid. If a is a positive integer greater than 1, we get the graph of an "a-leafed clover", with cusps that are a units from the origin. (Some of the pairs of figures are not to scale.)

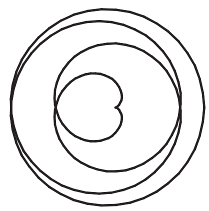


$a = 3, -2\pi \le \theta \le 2\pi$

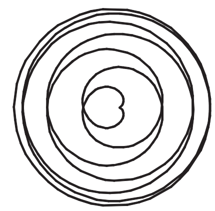


$a = 10, -2\pi \le \theta \le 2\pi$

If $a = n/d$ with $n = 1$, we obtain a figure that does not increase in size and requires $-d\pi \le \theta \le d\pi$ to be a closed curve traced exactly once.

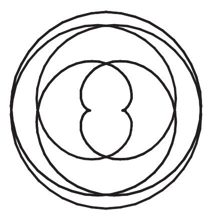


$a = \frac{1}{4}, -4\pi \le \theta \le 4\pi$

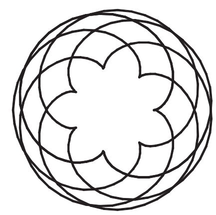


$a = \frac{1}{7}, -7\pi \le \theta \le 7\pi$

Next, we keep d constant and let n vary. As n increases, so does the size of the figure. There is an n-pointed star in the middle.

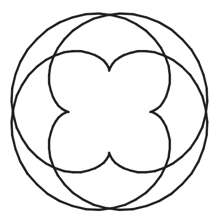


$a = \frac{2}{5}, -5\pi \le \theta \le 5\pi$

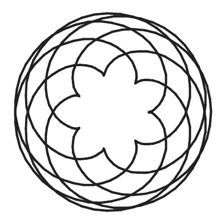


$a = \frac{7}{5}, -5\pi \le \theta \le 5\pi$

Now if $n = d + 1$ we obtain figures similar to the previous ones, but the size of the figure does not increase.

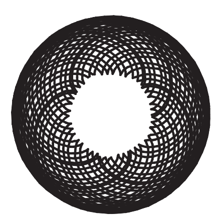


$a = \frac{4}{3}, -3\pi \le \theta \le 3\pi$

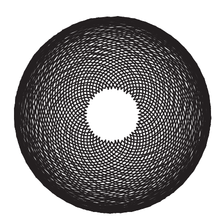


$a = \frac{7}{6}, -6\pi \le \theta \le 6\pi$

If a is irrational, we get washers that increase in size as a increases.



$a = \sqrt{2}, 0 \le \theta \le 200$



$a = e - 2, 0 \le \theta \le 446$

10.2 Calculus with Parametric Curves

1. $x = t\sin t,\ y = t^2 + t \quad\Rightarrow\quad \dfrac{dy}{dt} = 2t + 1$, $\dfrac{dx}{dt} = t\cos t + \sin t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{t\cos t + \sin t}$.

2. $x = \dfrac{1}{t},\ y = \sqrt{t}\,e^{-t} \quad\Rightarrow\quad \dfrac{dy}{dt} = t^{1/2}(-e^{-t}) + e^{-t}\left(\frac{1}{2}t^{-1/2}\right) = \frac{1}{2}t^{-1/2}e^{-t}(-2t+1) = \dfrac{-2t+1}{2t^{1/2}e^t}$, $\dfrac{dx}{dt} = -\dfrac{1}{t^2}$, and

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-2t+1}{2t^{1/2}e^t}\left(-\dfrac{t^2}{1}\right) = \dfrac{(2t-1)t^{3/2}}{2e^t}$.

3. $x = 1 + 4t - t^2,\ y = 2 - t^3;\ t = 1.\quad \dfrac{dy}{dt} = -3t^2$, $\dfrac{dx}{dt} = 4 - 2t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-3t^2}{4-2t}$. When $t = 1$,

$(x, y) = (4, 1)$ and $dy/dx = -\frac{3}{2}$, so an equation of the tangent to the curve at the point corresponding to $t = 1$ is

$y - 1 = -\frac{3}{2}(x - 4)$, or $y = -\frac{3}{2}x + 7$.

4. $x = t - t^{-1},\ y = 1 + t^2;\ t = 1.\quad \dfrac{dy}{dt} = 2t$, $\dfrac{dx}{dt} = 1 + t^{-2} = \dfrac{t^2+1}{t^2}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = 2t\left(\dfrac{t^2}{t^2+1}\right) = \dfrac{2t^3}{t^2+1}$.

When $t = 1$, $(x, y) = (0, 2)$ and $dy/dx = \frac{2}{2} = 1$, so an equation of the tangent to the curve at the point corresponding to $t = 1$ is $y - 2 = 1(x - 0)$, or $y = x + 2$.

5. $x = t\cos t,\ y = t\sin t;\ t = \pi.\quad \dfrac{dy}{dt} = t\cos t + \sin t$, $\dfrac{dx}{dt} = t(-\sin t) + \cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t\cos t + \sin t}{-t\sin t + \cos t}$.

When $t = \pi$, $(x, y) = (-\pi, 0)$ and $dy/dx = -\pi/(-1) = \pi$, so an equation of the tangent to the curve at the point corresponding to $t = \pi$ is $y - 0 = \pi[x - (-\pi)]$, or $y = \pi x + \pi^2$.

6. $x = \sin^3\theta,\ y = \cos^3\theta,\ \theta = \pi/6.\quad \dfrac{dy}{d\theta} = 3\cos^2\theta\,(-\sin\theta)$, $\dfrac{dx}{d\theta} = 3\sin^2\theta\,\cos\theta$, and

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{-3\cos^2\theta\,\sin\theta}{3\sin^2\theta\,\cos\theta} = -\cot\theta$. When $\theta = \pi/6$, $(x, y) = \left(\frac{1}{8}, \frac{3}{8}\sqrt{3}\right)$ and $dy/dx = -\cot(\pi/6) = -\sqrt{3}$,

so an equation of the tangent line to the curve at the point corresponding to $\theta = \pi/6$ is $y - \frac{3}{8}\sqrt{3} = -\sqrt{3}\left(x - \frac{1}{8}\right)$,

or $y = -\sqrt{3}\,x + \frac{1}{2}\sqrt{3}$.

7. (a) $x = 1 + \ln t,\ y = t^2 + 2;\ (1, 3).\quad \dfrac{dy}{dt} = 2t$, $\dfrac{dx}{dt} = \dfrac{1}{t}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$. At $(1, 3)$,

$x = 1 + \ln t = 1 \quad\Rightarrow\quad \ln t = 0 \quad\Rightarrow\quad t = 1$ and $\dfrac{dy}{dx} = 2$, so an equation of the tangent is $y - 3 = 2(x - 1)$,

or $y = 2x + 1$.

(b) $x = 1 + \ln t \quad\Rightarrow\quad \ln t = x - 1 \quad\Rightarrow\quad t = e^{x-1}$, so $y = t^2 + 2 = (e^{x-1})^2 + 2 = e^{2x-2} + 2$, and $y' = e^{2x-2}\cdot 2$.

At $(1, 3)$, $y' = e^{2(1)-2}\cdot 2 = 2$, so an equation of the tangent is $y - 3 = 2(x - 1)$, or $y = 2x + 1$.

8. (a) $x = 1 + \sqrt{t},\ y = e^{t^2};\ (2, e).\quad \dfrac{dy}{dt} = e^{t^2}\cdot 2t$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2te^{t^2}}{1/(2\sqrt{t})} = 4t^{3/2}e^{t^2}$. At $(2, e)$,

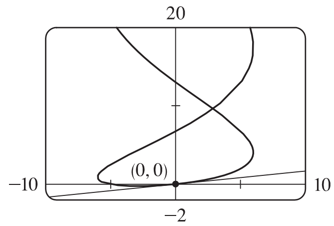
$x = 1 + \sqrt{t} = 2 \quad\Rightarrow\quad \sqrt{t} = 1 \quad\Rightarrow\quad t = 1$ and $\dfrac{dy}{dx} = 4e$, so an equation of the tangent is $y - e = 4e(x - 2)$,

or $y = 4ex - 7e$.

(b) $x = 1 + \sqrt{t} \quad \Rightarrow \quad \sqrt{t} = x - 1 \quad \Rightarrow \quad t = (x-1)^2$, so $y = e^{t^2} = e^{(x-1)^4}$, and $y' = e^{(x-1)^4} \cdot 4(x-1)^3$.

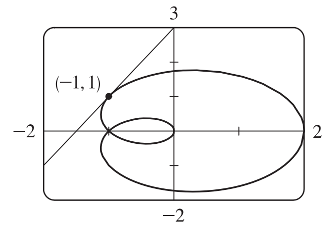
At $(2, e)$, $y' = e \cdot 4 = 4e$, so an equation of the tangent is $y - e = 4e(x - 2)$, or $y = 4ex - 7e$.

9. $x = 6\sin t,\ y = t^2 + t;\ (0,0)$.



$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{6\cos t}$. The point $(0,0)$ corresponds to $t = 0$, so the slope of the tangent at that point is $\frac{1}{6}$. An equation of the tangent is therefore $y - 0 = \frac{1}{6}(x - 0)$, or $y = \frac{1}{6}x$.

10. $x = \cos t + \cos 2t,\ y = \sin t + \sin 2t;\ (-1, 1)$.



$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t + 2\cos 2t}{-\sin t - 2\sin 2t}$. To find the value of t corresponding to the point $(-1, 1)$, solve $x = -1 \quad \Rightarrow \quad \cos t + \cos 2t = -1 \quad \Rightarrow$ $\cos t + 2\cos^2 t - 1 = -1 \quad \Rightarrow \quad \cos t\ (1 + 2\cos t) = 0 \quad \Rightarrow \quad \cos t = 0$ or $\cos t = -\frac{1}{2}$. The interval $[0, 2\pi]$ gives the complete curve, so we need only find the values of t in this interval. Thus, $t = \frac{\pi}{2}$ or $t = \frac{2\pi}{3}$ or $t = \frac{4\pi}{3}$. Checking $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}$, and $\frac{4\pi}{3}$ in the equation for y, we find that $t = \frac{\pi}{2}$ corresponds to $(-1, 1)$. The slope of the tangent at $(-1, 1)$ with $t = \frac{\pi}{2}$ is $\dfrac{0-2}{-1-0} = 2$. An equation of the tangent is therefore $y - 1 = 2(x + 1)$, or $y = 2x + 3$.

11. $x = t^2 + 1,\ y = t^2 + t \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+1}{2t} = 1 + \dfrac{1}{2t} \quad \Rightarrow \quad \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{-1/(2t^2)}{2t} = -\dfrac{1}{4t^3}$.

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t < 0$.

12. $x = t^3 + 1,\ y = t^2 - t \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t-1}{3t^2} = \dfrac{2}{3t} - \dfrac{1}{3t^2} \quad \Rightarrow$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{-\frac{2}{3t^2} + \frac{2}{3t^3}}{3t^2} = \dfrac{\frac{2-2t}{3t^3}}{3t^2} = \dfrac{2(1-t)}{9t^5}$. The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $0 < t < 1$.

13. $x = e^t,\ y = te^{-t} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-te^{-t} + e^{-t}}{e^t} = \dfrac{e^{-t}(1-t)}{e^t} = e^{-2t}(1-t) \quad \Rightarrow$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{e^{-2t}(-1) + (1-t)(-2e^{-2t})}{e^t} = \dfrac{e^{-2t}(-1-2+2t)}{e^t} = e^{-3t}(2t-3)$. The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t > \frac{3}{2}$.

14. $x = t^2 + 1,\ y = e^t - 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{e^t}{2t} \quad \Rightarrow \quad \dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \dfrac{\frac{2te^t - e^t \cdot 2}{(2t)^2}}{2t} = \dfrac{2e^t(t-1)}{(2t)^3} = \dfrac{e^t(t-1)}{4t^3}$.

The curve is CU when $\dfrac{d^2y}{dx^2} > 0$, that is, when $t < 0$ or $t > 1$.

15. $x = 2\sin t,\ y = 3\cos t,\ 0 < t < 2\pi.$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3\sin t}{2\cos t} = -\frac{3}{2}\tan t, \text{ so } \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{3}{2}\sec^2 t}{2\cos t} = -\frac{3}{4}\sec^3 t.$$

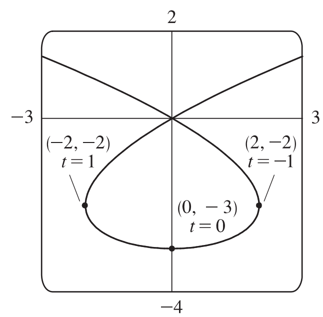
The curve is CU when $\sec^3 t < 0 \quad\Rightarrow\quad \sec t < 0 \quad\Rightarrow\quad \cos t < 0 \quad\Rightarrow\quad \frac{\pi}{2} < t < \frac{3\pi}{2}$.

16. $x = \cos 2t,\ \ y = \cos t,\ \ 0 < t < \pi.$

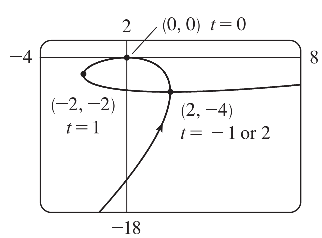
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{-2\sin 2t} = \frac{\sin t}{2\cdot 2\sin t\,\cos t} = \frac{1}{4\cos t} = \frac{1}{4}\sec t, \text{ so } \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{1}{4}\sec t\,\tan t}{-4\sin t\,\cos t} = -\frac{1}{16}\sec^3 t.$$

The curve is CU when $\sec^3 t < 0 \quad\Rightarrow\quad \sec t < 0 \quad\Rightarrow\quad \cos t < 0 \quad\Rightarrow\quad \frac{\pi}{2} < t < \pi$.

17. $x = t^3 - 3t,\ y = t^2 - 3.\quad \frac{dy}{dt} = 2t$, so $\frac{dy}{dt} = 0 \quad\Leftrightarrow\quad t = 0 \quad\Leftrightarrow$ $(x, y) = (0, -3).\quad \frac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$, so $\frac{dx}{dt} = 0 \quad\Leftrightarrow$ $t = -1$ or $1 \quad\Leftrightarrow\quad (x, y) = (2, -2)$ or $(-2, -2)$. The curve has a horizontal tangent at $(0, -3)$ and vertical tangents at $(2, -2)$ and $(-2, -2)$.



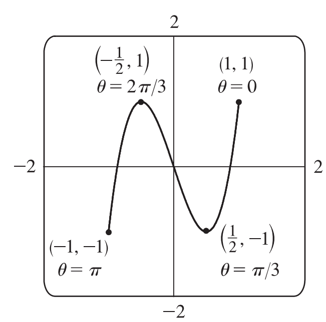
18. $x = t^3 - 3t,\ y = t^3 - 3t^2.\quad \frac{dy}{dt} = 3t^2 - 6t = 3t(t-2)$, so $\frac{dy}{dt} = 0 \quad\Leftrightarrow$ $t = 0$ or $2 \quad\Leftrightarrow\quad (x, y) = (0, 0)$ or $(2, -4).\quad \frac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1)$, so $\frac{dx}{dt} = 0 \quad\Leftrightarrow\quad t = -1$ or $1 \quad\Leftrightarrow\quad (x, y) = (2, -4)$ or $(-2, -2)$. The curve has horizontal tangents at $(0, 0)$ and $(2, -4)$, and vertical tangents at $(2, -4)$ and $(-2, -2)$.



19. $x = \cos\theta,\ y = \cos 3\theta$. The whole curve is traced out for $0 \le \theta \le \pi$.

$\frac{dy}{d\theta} = -3\sin 3\theta$, so $\frac{dy}{d\theta} = 0 \quad\Leftrightarrow\quad \sin 3\theta = 0 \quad\Leftrightarrow\quad 3\theta = 0, \pi, 2\pi$, or $3\pi \quad\Leftrightarrow$ $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$, or $\pi \quad\Leftrightarrow\quad (x, y) = (1, 1), \left(\frac{1}{2}, -1\right), \left(-\frac{1}{2}, 1\right)$, or $(-1, -1)$.

$\frac{dx}{d\theta} = -\sin\theta$, so $\frac{dx}{d\theta} = 0 \quad\Leftrightarrow\quad \sin\theta = 0 \quad\Leftrightarrow\quad \theta = 0$ or $\pi \quad\Leftrightarrow$ $(x, y) = (1, 1)$ or $(-1, -1)$. Both $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ equal 0 when $\theta = 0$ and π.

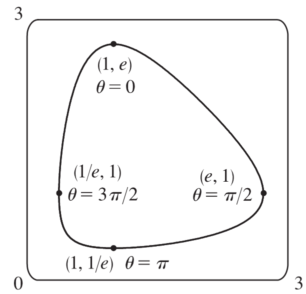


To find the slope when $\theta = 0$, we find $\lim\limits_{\theta\to 0}\frac{dy}{dx} = \lim\limits_{\theta\to 0}\frac{-3\sin 3\theta}{-\sin\theta} \overset{\text{H}}{=} \lim\limits_{\theta\to 0}\frac{-9\cos 3\theta}{-\cos\theta} = 9$, which is the same slope when $\theta = \pi$.

Thus, the curve has horizontal tangents at $\left(\frac{1}{2}, -1\right)$ and $\left(-\frac{1}{2}, 1\right)$, and there are no vertical tangents.

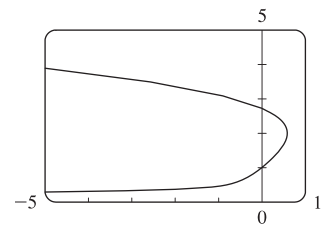
20. $x = e^{\sin\theta}$, $y = e^{\cos\theta}$. The whole curve is traced out for $0 \le \theta < 2\pi$.

$\dfrac{dy}{d\theta} = -\sin\theta\, e^{\cos\theta}$, so $\dfrac{dy}{d\theta} = \theta \iff \sin\theta = 0 \iff \theta = 0$ or $\pi \iff$ $(x, y) = (1, e)$ or $(1, 1/e)$. $\dfrac{dx}{d\theta} = \cos\theta\, e^{\sin\theta}$, so $\dfrac{dx}{d\theta} = 0 \iff \cos\theta = 0 \iff$ $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \iff (x, y) = (e, 1)$ or $(1/e, 1)$. The curve has horizontal tangents at $(1, e)$ and $(1, 1/e)$, and vertical tangents at $(e, 1)$ and $(1/e, 1)$.



21. From the graph, it appears that the rightmost point on the curve $x = t - t^6$, $y = e^t$ is about $(0.6, 2)$. To find the exact coordinates, we find the value of t for which the graph has a vertical tangent, that is, $0 = dx/dt = 1 - 6t^5 \iff t = 1/\sqrt[5]{6}$. Hence, the rightmost point is

$$\left(1/\sqrt[5]{6} - 1/\left(6\sqrt[5]{6}\right), e^{1/\sqrt[5]{6}}\right) = \left(5 \cdot 6^{-6/5}, e^{6^{-1/5}}\right) \approx (0.58, 2.01).$$



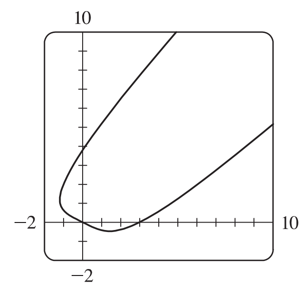
22. From the graph, it appears that the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$ are $(1.5, -0.5)$ and $(-1.2, 1.2)$, respectively. To find the exact coordinates, we solve $dy/dt = 0$ (horizontal tangents) and $dx/dt = 0$ (vertical tangents).

$\dfrac{dy}{dt} = 0 \iff 1 + 4t^3 = 0 \iff t = -\dfrac{1}{\sqrt[3]{4}}$, so the lowest point is

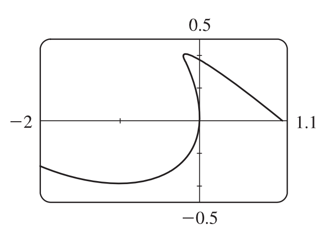
$$\left(\frac{1}{\sqrt[3]{256}} + \frac{2}{\sqrt[3]{4}}, -\frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{256}}\right) = \left(\frac{9}{\sqrt[3]{256}}, -\frac{3}{\sqrt[3]{256}}\right) \approx (1.42, -0.47).$$

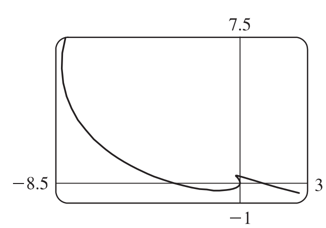
$\dfrac{dx}{dt} = 0 \iff 4t^3 - 2 = 0 \iff t = \dfrac{1}{\sqrt[3]{2}}$, so the leftmost point is

$$\left(\frac{1}{\sqrt[3]{16}} - \frac{2}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{16}}\right) = \left(-\frac{3}{\sqrt[3]{16}}, \frac{3}{\sqrt[3]{16}}\right) \approx (-1.19, 1.19).$$



23. We graph the curve $x = t^4 - 2t^3 - 2t^2$, $y = t^3 - t$ in the viewing rectangle $[-2, 1.1]$ by $[-0.5, 0.5]$. This rectangle corresponds approximately to $t \in [-1, 0.8]$.

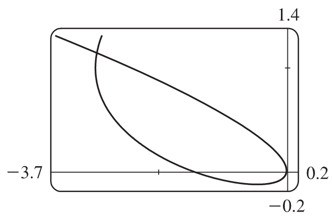


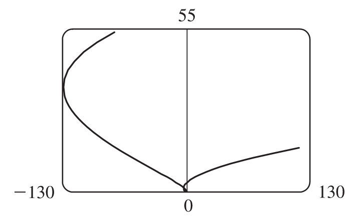


We estimate that the curve has horizontal tangents at about $(-1, -0.4)$ and $(-0.17, 0.39)$ and vertical tangents at about $(0, 0)$ and $(-0.19, 0.37)$. We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 1}{4t^3 - 6t^2 - 4t}$. The horizontal tangents occur when $dy/dt = 3t^2 - 1 = 0 \iff t = \pm\frac{1}{\sqrt{3}}$, so both horizontal tangents are shown in our graph. The vertical tangents occur when

$dx/dt = 2t(2t^2 - 3t - 2) = 0 \quad \Leftrightarrow \quad 2t(2t+1)(t-2) = 0 \quad \Leftrightarrow \quad t = 0, -\frac{1}{2}$ or 2. It seems that we have missed one vertical tangent, and indeed if we plot the curve on the t-interval $[-1.2, 2.2]$ we see that there is another vertical tangent at $(-8, 6)$.

24. We graph the curve $x = t^4 + 4t^3 - 8t^2$, $y = 2t^2 - t$ in the viewing rectangle $[-3.7, 0.2]$ by $[-0.2, 1.4]$. It appears that there is a horizontal tangent at about $(-0.4, -0.1)$, and vertical tangents at about $(-3, 1)$ and $(0, 0)$.

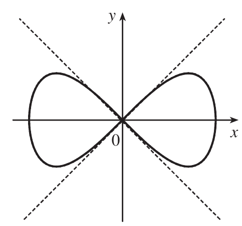




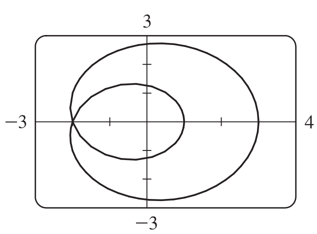
We calculate $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t-1}{4t^3 + 12t^2 - 16t}$, so there is a horizontal tangent where $dy/dt = 4t - 1 = 0 \quad \Leftrightarrow \quad t = \frac{1}{4}$.

This point (the lowest point) is shown in the first graph. There are vertical tangents where $dx/dt = 4t^3 + 12t^2 - 16t = 0 \quad \Leftrightarrow$ $4t(t^2 + 3t - 4) = 0 \quad \Leftrightarrow \quad 4t(t+4)(t-1) = 0$. We have missed one vertical tangent corresponding to $t = -4$, and if we plot the graph for $t \in [-5, 3]$, we see that the curve has another vertical tangent line at approximately $(-128, 36)$.

25. $x = \cos t$, $y = \sin t \cos t$. $\quad dx/dt = -\sin t$, $dy/dt = -\sin^2 t + \cos^2 t = \cos 2t$. $(x, y) = (0, 0) \quad \Leftrightarrow \quad \cos t = 0 \quad \Leftrightarrow \quad t$ is an odd multiple of $\frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $dx/dt = -1$ and $dy/dt = -1$, so $dy/dx = 1$. When $t = \frac{3\pi}{2}$, $dx/dt = 1$ and $dy/dt = -1$. So $dy/dx = -1$. Thus, $y = x$ and $y = -x$ are both tangent to the curve at $(0, 0)$.



26.



From the graph, we discover that the graph of the curve $x = \cos t + 2\cos 2t$, $y = \sin t + 2\sin 2t$ crosses itself at the point $(-2, 0)$. To find t at $(-2, 0)$, solve $y = 0 \quad \Leftrightarrow \quad \sin t + 2\sin 2t = 0 \quad \Leftrightarrow \quad \sin t + 4\sin t\,\cos t = 0 \quad \Leftrightarrow$ $\sin t\,(1 + 4\cos t) = 0 \quad \Leftrightarrow \quad \sin t = 0$ or $\cos t = -\frac{1}{4}$. We find that $t = \pm\arccos\left(-\frac{1}{4}\right)$ corresponds to $(-2, 0)$.

Now $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t + 4\cos 2t}{-\sin t - 4\sin 2t} = -\dfrac{\cos t + 8\cos^2 t - 4}{\sin t + 8\sin t\,\cos t}$. When $t = \arccos\left(-\frac{1}{4}\right)$, $\cos t = -\frac{1}{4}$, $\sin t = \dfrac{\sqrt{15}}{4}$,

and $\dfrac{dy}{dx} = -\dfrac{-\frac{1}{4} + \frac{1}{2} - 4}{\frac{\sqrt{15}}{4} - \frac{\sqrt{15}}{2}} = -\dfrac{-\frac{15}{4}}{-\frac{\sqrt{15}}{4}} = -\sqrt{15}$. By symmetry, $t = -\arccos\left(-\frac{1}{4}\right) \quad \Rightarrow \quad \dfrac{dy}{dx} = \sqrt{15}$.

The tangent lines are $y - 0 = \pm\sqrt{15}\,(x + 2)$, or $y = \sqrt{15}\,x + 2\sqrt{15}$ and $y = -\sqrt{15}\,x - 2\sqrt{15}$.

27. $x = r\theta - d\sin\theta,\; y = r - d\cos\theta$.

(a) $\dfrac{dx}{d\theta} = r - d\cos\theta$, $\dfrac{dy}{d\theta} = d\sin\theta$, so $\dfrac{dy}{dx} = \dfrac{d\sin\theta}{r - d\cos\theta}$.

(b) If $0 < d < r$, then $|d\cos\theta| \le d < r$, so $r - d\cos\theta \ge r - d > 0$. This shows that $dx/d\theta$ never vanishes, so the trochoid can have no vertical tangent if $d < r$.

28. $x = a\cos^3\theta,\ y = a\sin^3\theta$.

(a) $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$, so $\dfrac{dy}{dx} = -\dfrac{\sin\theta}{\cos\theta} = -\tan\theta$.

(b) The tangent is horizontal $\Leftrightarrow$ $dy/dx = 0$ $\Leftrightarrow$ $\tan\theta = 0$ $\Leftrightarrow$ $\theta = n\pi$ $\Leftrightarrow$ $(x, y) = (\pm a, 0)$.

The tangent is vertical $\Leftrightarrow$ $\cos\theta = 0$ $\Leftrightarrow$ θ is an odd multiple of $\frac{\pi}{2}$ $\Leftrightarrow$ $(x, y) = (0, \pm a)$.

(c) $dy/dx = \pm 1$ $\Leftrightarrow$ $\tan\theta = \pm 1$ $\Leftrightarrow$ θ is an odd multiple of $\frac{\pi}{4}$ $\Leftrightarrow$ $(x, y) = \left(\pm\frac{\sqrt{2}}{4}a, \pm\frac{\sqrt{2}}{4}a\right)$

[All sign choices are valid.]

29. $x = 2t^3,\ y = 1 + 4t - t^2$ $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4 - 2t}{6t^2}$. Now solve $\dfrac{dy}{dx} = 1$ $\Leftrightarrow$ $\dfrac{4 - 2t}{6t^2} = 1$ $\Leftrightarrow$ $6t^2 + 2t - 4 = 0$ $\Leftrightarrow$ $2(3t - 2)(t + 1) = 0$ $\Leftrightarrow$ $t = \frac{2}{3}$ or $t = -1$. If $t = \frac{2}{3}$, the point is $\left(\frac{16}{27}, \frac{29}{9}\right)$, and if $t = -1$, the point is $(-2, -4)$.

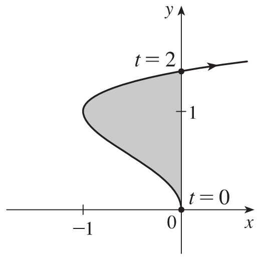
30. $x = 3t^2 + 1, y = 2t^3 + 1, \dfrac{dx}{dt} = 6t, \dfrac{dy}{dt} = 6t^2$, so $\dfrac{dy}{dx} = \dfrac{6t^2}{6t} = t$ [even where $t = 0$].

So at the point corresponding to parameter value t, an equation of the tangent line is $y - (2t^3 + 1) = t[x - (3t^2 + 1)]$. If this line is to pass through $(4, 3)$, we must have $3 - (2t^3 + 1) = t[4 - (3t^2 + 1)]$ $\Leftrightarrow$ $2t^3 - 2 = 3t^3 - 3t$ $\Leftrightarrow$ $t^3 - 3t + 2 = 0$ $\Leftrightarrow$ $(t - 1)^2(t + 2) = 0$ $\Leftrightarrow$ $t = 1$ or -2. Hence, the desired equations are $y - 3 = x - 4$, or $y = x - 1$, tangent to the curve at $(4, 3)$, and $y - (-15) = -2(x - 13)$, or $y = -2x + 11$, tangent to the curve at $(13, -15)$.

31. By symmetry of the ellipse about the x- and y-axes,

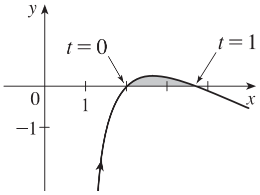
$$A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 b\sin\theta\,(-a\sin\theta)\,d\theta = 4ab\int_0^{\pi/2}\sin^2\theta\,d\theta = 4ab\int_0^{\pi/2}\tfrac{1}{2}(1 - \cos 2\theta)\,d\theta$$
$$= 2ab\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left(\tfrac{\pi}{2}\right) = \pi ab$$

32. The curve $x = t^2 - 2t = t(t - 2)$, $y = \sqrt{t}$ intersects the y-axis when $x = 0$, that is, when $t = 0$ and $t = 2$. The corresponding values of y are 0 and $\sqrt{2}$. The shaded area is given by



$$\int_{y=0}^{y=\sqrt{2}}(x_R - x_L)\,dy = \int_{t=0}^{t=2}[0 - x(t)]\,y'(t)\,dt = -\int_0^2 (t^2 - 2t)\left(\frac{1}{2\sqrt{t}}\,dt\right)$$
$$= -\int_0^2\left(\tfrac{1}{2}t^{3/2} - t^{1/2}\right)dt = -\left[\tfrac{1}{5}t^{5/2} - \tfrac{2}{3}t^{3/2}\right]_0^2$$
$$= -\left(\tfrac{1}{5}\cdot 2^{5/2} - \tfrac{2}{3}\cdot 2^{3/2}\right) = -2^{1/2}\left(\tfrac{4}{5} - \tfrac{4}{3}\right)$$
$$= -\sqrt{2}\left(-\tfrac{8}{15}\right) = \tfrac{8}{15}\sqrt{2}$$

33. The curve $x = 1 + e^t$, $y = t - t^2 = t(1 - t)$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = 1$. The corresponding values of x are 2 and $1 + e$. The shaded area is given by



$$\int_{x=2}^{x=1+e} (y_T - y_B)\,dx = \int_{t=0}^{t=1} [y(t) - 0]\,x'(t)\,dt = \int_0^1 (t - t^2)e^t\,dt$$

$$= \int_0^1 te^t\,dt - \int_0^1 t^2 e^t\,dt = \int_0^1 te^t\,dt - \left[t^2 e^t\right]_0^1 + 2\int_0^1 te^t\,dt \qquad \text{[Formula 97 or parts]}$$

$$= 3\int_0^1 te^t\,dt - (e - 0) = 3\left[(t-1)e^t\right]_0^1 - e \qquad \text{[Formula 96 or parts]}$$

$$= 3[0 - (-1)] - e = 3 - e$$

34. By symmetry, $A = 4\int_0^a y\,dx = 4\int_{\pi/2}^0 a\sin^3\theta(-3a\cos^2\theta\,\sin\theta)\,d\theta = 12a^2\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta$. Now

$$\int \sin^4\theta\,\cos^2\theta\,d\theta = \int \sin^2\theta\left(\tfrac{1}{4}\sin^2 2\theta\right)d\theta = \tfrac{1}{8}\int (1 - \cos 2\theta)\sin^2 2\theta\,d\theta$$

$$= \tfrac{1}{8}\int\left[\tfrac{1}{2}(1 - \cos 4\theta) - \sin^2 2\theta\,\cos 2\theta\right]d\theta = \tfrac{1}{16}\theta - \tfrac{1}{64}\sin 4\theta - \tfrac{1}{48}\sin^3 2\theta + C$$

so $\int_0^{\pi/2}\sin^4\theta\,\cos^2\theta\,d\theta = \left[\frac{1}{16}\theta - \frac{1}{64}\sin 4\theta - \frac{1}{48}\sin^3 2\theta\right]_0^{\pi/2} = \frac{\pi}{32}$. Thus, $A = 12a^2\left(\frac{\pi}{32}\right) = \frac{3}{8}\pi a^2$.

35. $x = r\theta - d\sin\theta,\ y = r - d\cos\theta$.

$$A = \int_0^{2\pi r} y\,dx = \int_0^{2\pi}(r - d\cos\theta)(r - d\cos\theta)\,d\theta = \int_0^{2\pi}(r^2 - 2dr\cos\theta + d^2\cos^2\theta)\,d\theta$$

$$= \left[r^2\theta - 2dr\sin\theta + \tfrac{1}{2}d^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = 2\pi r^2 + \pi d^2$$

36. (a) By symmetry, the area of $\mathcal{R}$ is twice the area inside $\mathcal{R}$ above the x-axis. The top half of the loop is described by $x = t^2$, $y = t^3 - 3t$, $-\sqrt{3} \le t \le 0$, so, using the Substitution Rule with $y = t^3 - 3t$ and $dx = 2t\,dt$, we find that

$$\text{area} = 2\int_0^3 y\,dx = 2\int_0^{-\sqrt{3}}(t^3 - 3t)2t\,dt = 2\int_0^{-\sqrt{3}}(2t^4 - 6t^2)\,dt = 2\left[\tfrac{2}{5}t^5 - 2t^3\right]_0^{-\sqrt{3}}$$

$$= 2\left[\tfrac{2}{5}(-3^{1/2})^5 - 2(-3^{1/2})^3\right] = 2\left[\tfrac{2}{5}\left(-9\sqrt{3}\right) - 2\left(-3\sqrt{3}\right)\right] = \tfrac{24}{5}\sqrt{3}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

$$\text{volume} = \pi\int_0^3 y^2\,dx = \pi\int_0^{-\sqrt{3}}(t^3 - 3t)^2 2t\,dt = 2\pi\int_0^{-\sqrt{3}}(t^6 - 6t^4 + 9t^2)t\,dt = 2\pi\left[\tfrac{1}{8}t^8 - t^6 + \tfrac{9}{4}t^4\right]_0^{-\sqrt{3}}$$

$$= 2\pi\left[\tfrac{1}{8}(-3^{1/2})^8 - (-3^{1/2})^6 + \tfrac{9}{4}(-3^{1/2})^4\right] = 2\pi\left[\tfrac{81}{8} - 27 + \tfrac{81}{4}\right] = \tfrac{27}{4}\pi$$

(c) By symmetry, the y-coordinate of the centroid is 0. To find the x-coordinate, we note that it is the same as the x-coordinate of the centroid of the top half of $\mathcal{R}$, the area of which is $\frac{1}{2}\cdot\frac{24}{5}\sqrt{3} = \frac{12}{5}\sqrt{3}$. So, using Formula 8.3.8 with $A = \frac{12}{5}\sqrt{3}$, we get

$$\overline{x} = \tfrac{5}{12\sqrt{3}}\int_0^3 xy\,dx = \tfrac{5}{12\sqrt{3}}\int_0^{-\sqrt{3}} t^2(t^3 - 3t)2t\,dt = \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}t^7 - \tfrac{3}{5}t^5\right]_0^{-\sqrt{3}}$$

$$= \tfrac{5}{6\sqrt{3}}\left[\tfrac{1}{7}(-3^{1/2})^7 - \tfrac{3}{5}(-3^{1/2})^5\right] = \tfrac{5}{6\sqrt{3}}\left[-\tfrac{27}{7}\sqrt{3} + \tfrac{27}{5}\sqrt{3}\right] = \tfrac{9}{7}$$

So the coordinates of the centroid of $\mathcal{R}$ are $(x, y) = \left(\frac{9}{7}, 0\right)$.

37. $x = t + e^{-t},\ y = t - e^{-t},\ 0 \le t \le 2.\quad dx/dt = 1 - e^{-t}$ and $dy/dt = 1 + e^{-t}$, so $(dx/dt)^2 + (dy/dt)^2 = (1 - e^{-t})^2 + (1 + e^{-t})^2 = 1 - 2e^{-t} + e^{-2t} + 1 + 2e^{-t} + e^{-2t} = 2 + 2e^{-2t}$.

Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^2 \sqrt{2 + 2e^{-2t}}\,dt \approx 3.1416$.

38. $x = t^2 - t,\ y = t^4,\ 1 \le t \le 4.\quad dx/dt = 2t - 1$ and $dy/dt = 4t^3$, so

$(dx/dt)^2 + (dy/dt)^2 = (2t-1)^2 + (4t^3)^2 = 4t^2 - 4t + 1 + 16t^6.$

Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_1^4 \sqrt{16t^6 + 4t^2 - 4t + 1}\,dt \approx 255.3756.$

39. $x = t - 2\sin t,\ y = 1 - 2\cos t,\ 0 \le t \le 4\pi.\quad dx/dt = 1 - 2\cos t$ and $dy/dt = 2\sin t$, so

$(dx/dt)^2 + (dy/dt)^2 = (1 - 2\cos t)^2 + (2\sin t)^2 = 1 - 4\cos t + 4\cos^2 t + 4\sin^2 t = 5 - 4\cos t.$

Thus, $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{4\pi} \sqrt{5 - 4\cos t}\,dt \approx 26.7298.$

40. $x = t + \sqrt{t},\ y = t - \sqrt{t},\ 0 \le t \le 1.\quad \dfrac{dx}{dt} = 1 + \dfrac{1}{2\sqrt{t}}$ and $\dfrac{dy}{dt} = 1 - \dfrac{1}{2\sqrt{t}}$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 + \frac{1}{2\sqrt{t}}\right)^2 + \left(1 - \frac{1}{2\sqrt{t}}\right)^2 = 1 + \frac{1}{\sqrt{t}} + \frac{1}{4t} + 1 - \frac{1}{\sqrt{t}} + \frac{1}{4t} = 2 + \frac{1}{2t}.$$

Thus, $L = \displaystyle\int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^1 \sqrt{2 + \frac{1}{2t}}\,dt = \lim_{t\to 0^+} \int_t^1 \sqrt{2 + \frac{1}{2t}}\,dt \approx 2.0915.$

41. $x = 1 + 3t^2,\ \ y = 4 + 2t^3,\ \ 0 \le t \le 1.$ $dx/dt = 6t$ and $dy/dt = 6t^2$, so $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$

$$\text{Thus,}\quad L = \int_0^1 \sqrt{36t^2 + 36t^4}\,dt = \int_0^1 6t\sqrt{1 + t^2}\,dt = 6\int_1^2 \sqrt{u}\,\left(\tfrac{1}{2}du\right) \quad [u = 1 + t^2,\ du = 2t\,dt]$$

$$= 3\left[\tfrac{2}{3}u^{3/2}\right]_1^2 = 2(2^{3/2} - 1) = 2\left(2\sqrt{2} - 1\right)$$

42. $x = e^t + e^{-t},\ y = 5 - 2t,\ 0 \le t \le 3.\quad dx/dt = e^t - e^{-t}$ and $dy/dt = -2$, so

$(dx/dt)^2 + (dy/dt)^2 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2.$

Thus, $L = \int_0^3 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^3 = e^3 - e^{-3} - (1 - 1) = e^3 - e^{-3}.$

43. $x = t\sin t,\ y = t\cos t,\ 0 \le t \le 1.\quad \dfrac{dx}{dt} = t\cos t + \sin t$ and $\dfrac{dy}{dt} = -t\sin t + \cos t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2\cos^2 t + 2t\sin t\,\cos t + \sin^2 t + t^2\sin^2 t - 2t\sin t\,\cos t + \cos^2 t$$
$$= t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1.$$

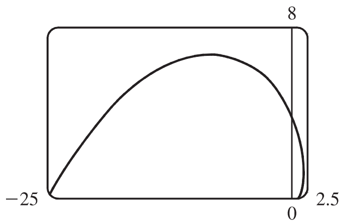
Thus, $L = \int_0^1 \sqrt{t^2 + 1}\,dt \overset{21}{=} \left[\tfrac{1}{2}t\sqrt{t^2 + 1} + \tfrac{1}{2}\ln\left(t + \sqrt{t^2 + 1}\,\right)\right]_0^1 = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\ln\left(1 + \sqrt{2}\,\right).$

44. $x = 3\cos t - \cos 3t,\ y = 3\sin t - \sin 3t,\ 0 \le t \le \pi.\quad \dfrac{dx}{dt} = -3\sin t + 3\sin 3t$ and $\dfrac{dy}{dt} = 3\cos t - 3\cos 3t$, so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^2 t - 18\sin t\,\sin 3t + 9\sin^2(3t) + 9\cos^2 t - 18\cos t\,\cos 3t + 9\cos^2(3t)$$
$$= 9(\cos^2 t + \sin^2 t) - 18(\cos t\,\cos 3t + \sin t\,\sin 3t) + 9[\cos^2(3t) + \sin^2(3t)]$$
$$= 9(1) - 18\cos(t - 3t) + 9(1) = 18 - 18\cos(-2t) = 18(1 - \cos 2t)$$
$$= 18[1 - (1 - 2\sin^2 t)] = 36\sin^2 t.$$

Thus, $L = \int_0^\pi \sqrt{36\sin^2 t}\,dt = 6\int_0^\pi |\sin t|\,dt = 6\int_0^\pi \sin t\,dt = -6\left[\cos t\right]_0^\pi = -6\,(-1 - 1) = 12.$

45.



$x = e^t \cos t,\ y = e^t \sin t,\ 0 \le t \le \pi.$

$$\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 = [e^t(\cos t - \sin t)]^2 + [e^t(\sin t + \cos t)]^2$$
$$= (e^t)^2(\cos^2 t - 2\cos t\,\sin t + \sin^2 t)$$
$$+ (e^t)^2(\sin^2 t + 2\sin t\,\cos t + \cos^2 t$$
$$= e^{2t}(2\cos^2 t + 2\sin^2 t) = 2e^{2t}$$

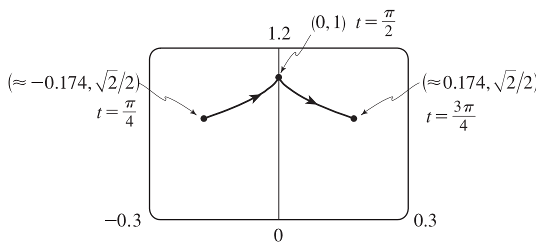
Thus, $L = \int_0^\pi \sqrt{2e^{2t}}\,dt = \int_0^\pi \sqrt{2}\,e^t\,dt = \sqrt{2}\,\left[\,e^t\,\right]_0^\pi = \sqrt{2}\,(e^\pi - 1).$

46. $x = \cos t + \ln(\tan \frac{1}{2}t), \quad y = \sin t, \quad \pi/4 \le t \le 3\pi/4.$

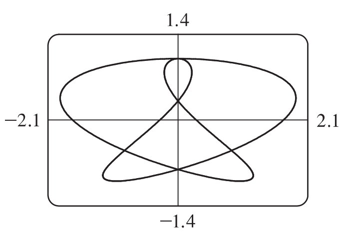
$\dfrac{dx}{dt} = -\sin t + \dfrac{\frac{1}{2}\sec^2(t/2)}{\tan(t/2)} = -\sin t + \dfrac{1}{2\sin(t/2)\,\cos(t/2)} = -\sin t + \dfrac{1}{\sin t}$ and $\dfrac{dy}{dt} = \cos t$, so

$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = \sin^2 t - 2 + \dfrac{1}{\sin^2 t} + \cos^2 t = 1 - 2 + \csc^2 t = \cot^2 t$. Thus,

$$L = \int_{\pi/4}^{3\pi/4} |\cot t|\,dt = 2\int_{\pi/4}^{\pi/2} \cot t\,dt$$
$$= 2\Big[\ln|\sin t|\Big]_{\pi/4}^{\pi/2} = 2\left(\ln 1 - \ln \frac{1}{\sqrt{2}}\right)$$
$$= 2\big(0 + \ln\sqrt{2}\,\big) = 2\big(\tfrac{1}{2}\ln 2\big) = \ln 2.$$



47.



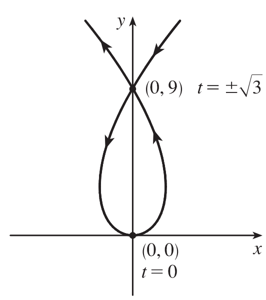
The figure shows the curve $x = \sin t + \sin 1.5t$, $y = \cos t$ for $0 \le t \le 4\pi$.
$dx/dt = \cos t + 1.5\cos 1.5t$ and $dy/dt = -\sin t$, so
$(dx/dt)^2 + (dy/dt)^2 = \cos^2 t + 3\cos t\,\cos 1.5t + 2.25\cos^2 1.5t + \sin^2 t.$
Thus, $L = \int_0^{4\pi} \sqrt{1 + 3\cos t\,\cos 1.5t + 2.25\cos^2 1.5t}\,dt \approx 16.7102.$

48. $x = 3t - t^3$, $y = 3t^2$. $\quad dx/dt = 3 - 3t^2$ and $dy/dt = 6t$, so

$$\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = (3 + 3t^2)^2$$

and the length of the loop is given by

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2)\,dt = 2\int_0^{\sqrt{3}} (3 + 3t^2)\,dt = 2[3t + t^3]_0^{\sqrt{3}}$$
$$= 2\big(3\sqrt{3} + 3\sqrt{3}\,\big) = 12\sqrt{3}.$$



49. $x = t - e^t,\ y = t + e^t,\ -6 \le t \le 6.$

$\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 = (1 - e^t)^2 + (1 + e^t)^2 = (1 - 2e^t + e^{2t}) + (1 + 2e^t + e^{2t}) = 2 + 2e^{2t}$, so $L = \int_{-6}^{6} \sqrt{2 + 2e^{2t}}\,dt$.

Set $f(t) = \sqrt{2 + 2e^{2t}}$. Then by Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-(-6)}{6} = 2$, we get

$L \approx \frac{2}{3}[f(-6) + 4f(-4) + 2f(-2) + 4f(0) + 2f(2) + 4f(4) + f(6)] \approx 612.3053.$

50. $x = 2a\cot\theta \;\Rightarrow\; dx/dt = -2a\csc^2\theta$ and $y = 2a\sin^2\theta \;\Rightarrow\; dy/dt = 4a\sin\theta\,\cos\theta = 2a\sin 2\theta$.

So $L = \int_{\pi/4}^{\pi/2}\sqrt{4a^2\csc^4\theta + 4a^2\sin^2 2\theta}\,d\theta = 2a\int_{\pi/4}^{\pi/2}\sqrt{\csc^4\theta + \sin^2 2\theta}\,d\theta$. Using Simpson's Rule with

$n = 4$, $\Delta\theta = \frac{\pi/2 - \pi/4}{4} = \frac{\pi}{16}$, and $f(\theta) = \sqrt{\csc^4\theta + \sin^2 2\theta}$, we get

$L \approx 2a \cdot S_4 = (2a)\,\frac{\pi}{16\cdot 3}\left[f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 2.2605a$.

51. $x = \sin^2 t,\; y = \cos^2 t,\; 0 \le t \le 3\pi$.

$(dx/dt)^2 + (dy/dt)^2 = (2\sin t\,\cos t)^2 + (-2\cos t\sin t)^2 = 8\sin^2 t\cos^2 t = 2\sin^2 2t \;\Rightarrow$

Distance $= \int_0^{3\pi}\sqrt{2}\,|\sin 2t|\;dt = 6\sqrt{2}\int_0^{\pi/2}\sin 2t\,dt$ [by symmetry] $= -3\sqrt{2}\left[\cos 2t\right]_0^{\pi/2} = -3\sqrt{2}\,(-1-1) = 6\sqrt{2}$.

The full curve is traversed as t goes from 0 to $\frac{\pi}{2}$, because the curve is the segment of $x + y = 1$ that lies in the first quadrant (since $x, y \ge 0$), and this segment is completely traversed as t goes from 0 to $\frac{\pi}{2}$. Thus, $L = \int_0^{\pi/2}\sin 2t\,dt = \sqrt{2}$, as above.

52. $x = \cos^2 t, y = \cos t,\; 0 \le t \le 4\pi$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-2\cos t\sin t)^2 + (-\sin t)^2 = \sin^2 t\,(4\cos^2 t + 1)$

$$\begin{aligned}\text{Distance} &= \int_0^{4\pi}|\sin t|\sqrt{4\cos^2 t + 1}\,dt = 4\int_0^{\pi}\sin t\sqrt{4\cos^2 t + 1}\,dt\\ &= -4\int_1^{-1}\sqrt{4u^2+1}\,du \quad [u = \cos t,\, du = -\sin t\,dt] \;= 4\int_{-1}^{1}\sqrt{4u^2+1}\,du\\ &= 8\int_0^1\sqrt{4u^2+1}\,du = 8\int_0^{\tan^{-1}2}\sec\theta\cdot\tfrac{1}{2}\sec^2\theta\,d\theta \qquad [2u = \tan\theta,\; 2\,du = \sec^2\theta\,d\theta]\\ &= 4\int_0^{\tan^{-1}2}\sec^3\theta\,d\theta \overset{71}{=} \Big[2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta|\Big]_0^{\tan^{-1}2} = 4\sqrt{5} + 2\ln\left(\sqrt{5}+2\right)\end{aligned}$$

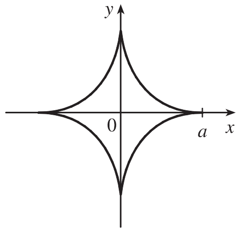
Thus, $L = \int_0^{\pi}|\sin t|\sqrt{4\cos^2 t + 1}\,dt = \sqrt{5} + \frac{1}{2}\ln\left(\sqrt{5}+2\right)$.

53. $x = a\sin\theta,\; y = b\cos\theta,\; 0 \le \theta \le 2\pi$.

$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= (a\cos\theta)^2 + (-b\sin\theta)^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2(1-\sin^2\theta) + b^2\sin^2\theta\\ &= a^2 - (a^2 - b^2)\,\sin^2\theta = a^2 - c^2\sin^2\theta = a^2\left(1 - \frac{c^2}{a^2}\sin^2\theta\right) = a^2(1 - e^2\sin^2\theta)\end{aligned}$$

So $L = 4\int_0^{\pi/2}\sqrt{a^2\left(1 - e^2\sin^2\theta\right)}\,d\theta$ [by symmetry] $= 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta$.

54. $x = a\cos^3\theta, y = a\sin^3\theta$.



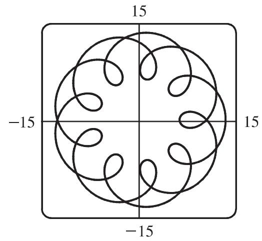
$$\begin{aligned}\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 &= (-3a\cos^2\theta\,\sin\theta)^2 + (3a\sin^2\theta\,\cos\theta)^2\\ &= 9a^2\cos^4\theta\,\sin^2\theta + 9a^2\sin^4\theta\,\cos^2\theta\\ &= 9a^2\sin^2\theta\,\cos^2\theta(\cos^2\theta + \sin^2\theta) = 9a^2\sin^2\theta\,\cos^2\theta.\end{aligned}$$

The graph has four-fold symmetry and the curve in the first quadrant corresponds to $0 \le \theta \le \pi/2$. Thus,

$$\begin{aligned}L &= 4\int_0^{\pi/2}3a\sin\theta\,\cos\theta\,d\theta \qquad [\text{since } a > 0 \text{ and } \sin\theta \text{ and } \cos\theta \text{ are positive for } 0 \le \theta \le \pi/2]\\ &= 12a\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2} = 12a\left(\tfrac{1}{2} - 0\right) = 6a\end{aligned}$$

55. (a) $x = 11\cos t - 4\cos(11t/2),\ y = 11\sin t - 4\sin(11t/2)$.

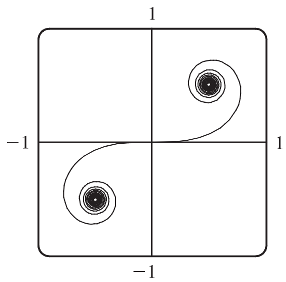
Notice that $0 \le t \le 2\pi$ does not give the complete curve because $x(0) \ne x(2\pi)$. In fact, we must take $t \in [0, 4\pi]$ in order to obtain the complete curve, since the first term in each of the parametric equations has period 2π and the second has period $\frac{2\pi}{11/2} = \frac{4\pi}{11}$, and the least common integer multiple of these two numbers is 4π.



(b) We use the CAS to find the derivatives dx/dt and dy/dt, and then use Theorem 6 to find the arc length. Recent versions of Maple express the integral $\int_0^{4\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ as $88E\left(2\sqrt{2}\, i\right)$, where $E(x)$ is the elliptic integral $\displaystyle\int_0^1 \frac{\sqrt{1 - x^2t^2}}{\sqrt{1 - t^2}}\, dt$ and i is the imaginary number $\sqrt{-1}$.

Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi));` to estimate the length, and find that the arc length is approximately 294.03. Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to $11\int_0^{4\pi} \sqrt{-4\cos t\, \cos\left(\frac{11t}{2}\right) - 4\sin t\, \sin\left(\frac{11t}{2}\right) + 5}\, dt$.

56. (a) It appears that as $t \to \infty$, $(x, y) \to \left(\frac{1}{2}, \frac{1}{2}\right)$, and as $t \to -\infty$, $(x, y) \to \left(-\frac{1}{2}, -\frac{1}{2}\right)$.



(b) By the Fundamental Theorem of Calculus, $dx/dt = \cos\left(\frac{\pi}{2}t^2\right)$ and $dy/dt = \sin\left(\frac{\pi}{2}t^2\right)$, so by Formula 4, the length of the curve from the origin to the point with parameter value t is

$$L = \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du = \int_0^t \sqrt{\cos^2\left(\frac{\pi}{2}u^2\right) + \sin^2\left(\frac{\pi}{2}u^2\right)}\, du$$
$$= \int_0^t 1\, du = t \qquad [\text{or } -t \text{ if } t < 0]$$

We have used u as the dummy variable so as not to confuse it with the upper limit of integration.

57. $x = t\sin t,\ y = t\cos t,\ 0 \le t \le \pi/2$. $dx/dt = t\cos t + \sin t$ and $dy/dt = -t\sin t + \cos t$, so

$$(dx/dt)^2 + (dy/dt)^2 = t^2\cos^2 t + 2t\sin t\, \cos t + \sin^2 t + t^2\sin^2 t - 2t\sin t\, \cos t + \cos^2 t$$
$$= t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1$$

$S = \int 2\pi y\, ds = \int_0^{\pi/2} 2\pi t\cos t\sqrt{t^2 + 1}\, dt \approx 4.7394$.

58. $x = \sin t,\ y = \sin 2t,\ 0 \le t \le \pi/2$. $dx/dt = \cos t$ and $dy/dt = 2\cos 2t$, so $(dx/dt)^2 + (dy/dt)^2 = \cos^2 t + 4\cos^2 2t$.

$S = \int 2\pi y\, ds = \int_0^{\pi/2} 2\pi\sin 2t\sqrt{\cos^2 t + 4\cos^2 2t}\, dt \approx 8.0285$.

59. $x = 1 + te^t,\ y = (t^2 + 1)e^t,\ 0 \le t \le 1$.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (te^t + e^t)^2 + [(t^2 + 1)e^t + e^t(2t)]^2 = [e^t(t + 1)]^2 + [e^t(t^2 + 2t + 1)]^2$$
$$= e^{2t}(t + 1)^2 + e^{2t}(t + 1)^4 = e^{2t}(t + 1)^2[1 + (t + 1)^2], \quad \text{so}$$

$S = \int 2\pi y\, ds = \int_0^1 2\pi(t^2 + 1)e^t\, \sqrt{e^{2t}(t + 1)^2(t^2 + 2t + 2)}\, dt = \int_0^1 2\pi(t^2 + 1)e^{2t}(t + 1)\, \sqrt{t^2 + 2t + 2}\, dt \approx 103.5999$.

60. $x = t^2 - t^3,\ y = t + t^4,\ 0 \le t \le 1.$

$(dx/dt)^2 + (dy/dt)^2 = (2t - 3t^2)^2 + (1 + 4t^3)^2 = 4t^2 - 12t^3 + 9t^4 + 1 + 8t^3 + 16t^6$, so

$S = \int 2\pi y\, ds = \int_0^1 2\pi(t + t^4)\sqrt{16t^6 + 9t^4 - 4t^3 + 4t^2 + 1}\, dt \approx 12.7176.$

61. $x = t^3,\ y = t^2,\ 0 \le t \le 1.\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(3t^2\right)^2 + (2t)^2 = 9t^4 + 4t^2.$

$$S = \int_0^1 2\pi y\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi t^2\sqrt{9t^4 + 4t^2}\, dt = 2\pi\int_0^1 t^2\sqrt{t^2(9t^2 + 4)}\, dt$$

$$= 2\pi\int_4^{13}\left(\frac{u-4}{9}\right)\sqrt{u}\left(\tfrac{1}{18}\,du\right) \quad \begin{bmatrix} u = 9t^2 + 4,\ t^2 = (u-4)/9, \\ du = 18t\,dt, \text{ so } t\,dt = \frac{1}{18}\,du \end{bmatrix} \quad = \frac{2\pi}{9\cdot 18}\int_4^{13}(u^{3/2} - 4u^{1/2})\,du$$

$$= \tfrac{\pi}{81}\left[\tfrac{2}{5}u^{5/2} - \tfrac{8}{3}u^{3/2}\right]_4^{13} = \tfrac{\pi}{81}\cdot\tfrac{2}{15}\left[3u^{5/2} - 20u^{3/2}\right]_4^{13}$$

$$= \tfrac{2\pi}{1215}\left[\left(3\cdot 13^2\sqrt{13} - 20\cdot 13\sqrt{13}\right) - (3\cdot 32 - 20\cdot 8)\right] = \tfrac{2\pi}{1215}\left(247\sqrt{13} + 64\right)$$

62. $x = 3t - t^3,\ y = 3t^2,\ 0 \le t \le 1.\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = 9(1 + 2t^2 + t^4) = [3(1 + t^2)]^2.$

$S = \int_0^1 2\pi\cdot 3t^2\cdot 3(1 + t^2)\,dt = 18\pi\int_0^1(t^2 + t^4)\,dt = 18\pi\left[\frac{1}{3}t^3 + \frac{1}{5}t^5\right]_0^1 = \frac{48}{5}\pi$

63. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le \frac{\pi}{2}.\quad \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (-3a\cos^2\theta\,\sin\theta)^2 + (3a\sin^2\theta\,\cos\theta)^2 = 9a^2\sin^2\theta\,\cos^2\theta.$

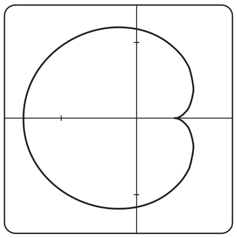
$S = \int_0^{\pi/2} 2\pi\cdot a\sin^3\theta\cdot 3a\sin\theta\,\cos\theta\,d\theta = 6\pi a^2\int_0^{\pi/2}\sin^4\theta\,\cos\theta\,d\theta = \frac{6}{5}\pi a^2\left[\sin^5\theta\right]_0^{\pi/2} = \frac{6}{5}\pi a^2$

64. $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (-2\sin\theta + 2\sin 2\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2$

$$= 4[(\sin^2\theta - 2\sin\theta\,\sin 2\theta + \sin^2 2\theta) + (\cos^2\theta - 2\cos\theta\,\cos 2\theta + \cos^2 2\theta)]$$

$$= 4[1 + 1 - 2(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta)] = 8[1 - \cos(2\theta - \theta)] = 8(1 - \cos\theta)$$

We plot the graph with parameter interval $[0, 2\pi]$, and see that we should only integrate between 0 and π. (If the interval $[0, 2\pi]$ were taken, the surface of revolution would be generated twice.) Also note that $y = 2\sin\theta - \sin 2\theta = 2\sin\theta(1 - \cos\theta)$. So



$$S = \int_0^\pi 2\pi\cdot 2\sin\theta(1 - \cos\theta)\,2\sqrt{2}\sqrt{1 - \cos\theta}\,d\theta$$

$$= 8\sqrt{2}\pi\int_0^\pi(1 - \cos\theta)^{3/2}\sin\theta\,d\theta = 8\sqrt{2}\pi\int_0^2\sqrt{u^3}\,du \quad \begin{bmatrix} u = 1 - \cos\theta, \\ du = \sin\theta\,d\theta \end{bmatrix}$$

$$= 8\sqrt{2}\pi\left[\left(\tfrac{2}{5}\right)u^{5/2}\right]_0^2 = \tfrac{16}{5}\sqrt{2}\pi(2^{5/2}) = \tfrac{128}{5}\pi$$

65. $x = 3t^2,\ y = 2t^3,\ 0 \le t \le 5 \quad\Rightarrow\quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (6t)^2 + (6t^2)^2 = 36t^2(1 + t^2) \quad\Rightarrow$

$$S = \int_0^5 2\pi x\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^5 2\pi(3t^2)6t\sqrt{1 + t^2}\,dt = 18\pi\int_0^5 t^2\sqrt{1 + t^2}\,2t\,dt$$

$$= 18\pi\int_1^{26}(u - 1)\sqrt{u}\,du \quad \begin{bmatrix} u = 1 + t^2, \\ du = 2t\,dt \end{bmatrix} \quad = 18\pi\int_1^{26}(u^{3/2} - u^{1/2})\,du = 18\pi\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right]_1^{26}$$

$$= 18\pi\left[\left(\tfrac{2}{5}\cdot 676\sqrt{26} - \tfrac{2}{3}\cdot 26\sqrt{26}\right) - \left(\tfrac{2}{5} - \tfrac{2}{3}\right)\right] = \tfrac{24}{5}\pi\left(949\sqrt{26} + 1\right)$$

66. $x = e^t - t$, $y = 4e^{t/2}$, $0 \le t \le 1$. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2$.

$$S = \int_0^1 2\pi(e^t - t)\sqrt{(e^t-1)^2 + (2e^{t/2})^2}\,dt = \int_0^1 2\pi(e^t - t)(e^t + 1)d$$
$$= 2\pi\left[\tfrac{1}{2}e^{2t} + e^t - (t-1)e^t - \tfrac{1}{2}t^2\right]_0^1 = \pi(e^2 + 2e - 6)$$

67. If f' is continuous and $f'(t) \ne 0$ for $a \le t \le b$, then either $f'(t) > 0$ for all t in $[a, b]$ or $f'(t) < 0$ for all t in $[a, b]$. Thus, f is monotonic (in fact, strictly increasing or strictly decreasing) on $[a, b]$. It follows that f has an inverse. Set $F = g \circ f^{-1}$, that is, define F by $F(x) = g(f^{-1}(x))$. Then $x = f(t) \Rightarrow f^{-1}(x) = t$, so $y = g(t) = g(f^{-1}(x)) = F(x)$.

68. By Formula 8.2.5 with $y = F(x)$, $S = \int_a^b 2\pi F(x)\sqrt{1 + [F'(x)]^2}\,dx$. But by Formula 10.2.1,

$1 + [F'(x)]^2 = 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \left(\dfrac{dy/dt}{dx/dt}\right)^2 = \dfrac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}$. Using the Substitution Rule with $x = x(t)$, where $a = x(\alpha)$ and $b = x(\beta)$, we have $\left[\text{since } dx = \dfrac{dx}{dt}\,dt\right]$

$S = \displaystyle\int_\alpha^\beta 2\pi\, F(x(t))\sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\,\frac{dx}{dt}\,dt = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$, which is Formula 10.2.6.

69. (a) $\phi = \tan^{-1}\left(\dfrac{dy}{dx}\right) \Rightarrow \dfrac{d\phi}{dt} = \dfrac{d}{dt}\tan^{-1}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{1 + (dy/dx)^2}\left[\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\right]$. But $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\dot{y}}{\dot{x}} \Rightarrow$

$\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{\dot{y}}{\dot{x}}\right) = \dfrac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2} \Rightarrow \dfrac{d\phi}{dt} = \dfrac{1}{1 + (\dot{y}/\dot{x})^2}\left(\dfrac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}\right) = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$. Using the Chain Rule, and the fact that $s = \displaystyle\int_0^t \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt \Rightarrow \tfrac{ds}{dt} = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2} = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2}$, we have that

$\dfrac{d\phi}{ds} = \dfrac{d\phi/dt}{ds/dt} = \left(\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}\right)\dfrac{1}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = \dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$. So $\kappa = \left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right| = \dfrac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

(b) $x = x$ and $y = f(x) \Rightarrow \dot{x} = 1$, $\ddot{x} = 0$ and $\dot{y} = \dfrac{dy}{dx}$, $\ddot{y} = \dfrac{d^2y}{dx^2}$.

So $\kappa = \dfrac{\left|1 \cdot (d^2y/dx^2) - 0 \cdot (dy/dx)\right|}{[1 + (dy/dx)^2]^{3/2}} = \dfrac{\left|d^2y/dx^2\right|}{[1 + (dy/dx)^2]^{3/2}}$.

70. (a) $y = x^2 \Rightarrow \dfrac{dy}{dx} = 2x \Rightarrow \dfrac{d^2y}{dx^2} = 2$. So $\kappa = \dfrac{\left|d^2y/dx^2\right|}{[1 + (dy/dx)^2]^{3/2}} = \dfrac{2}{(1 + 4x^2)^{3/2}}$, and at $(1, 1)$,

$\kappa = \dfrac{2}{5^{3/2}} = \dfrac{2}{5\sqrt{5}}$.

(b) $\kappa' = \dfrac{d\kappa}{dx} = -3(1 + 4x^2)^{-5/2}(8x) = 0 \Leftrightarrow x = 0 \Rightarrow y = 0$. This is a maximum since $\kappa' > 0$ for $x < 0$ and $\kappa' < 0$ for $x > 0$. So the parabola $y = x^2$ has maximum curvature at the origin.

71. $x = \theta - \sin\theta \;\Rightarrow\; \dot{x} = 1 - \cos\theta \;\Rightarrow\; \ddot{x} = \sin\theta$, and $y = 1 - \cos\theta \;\Rightarrow\; \dot{y} = \sin\theta \;\Rightarrow\; \ddot{y} = \cos\theta$. Therefore,

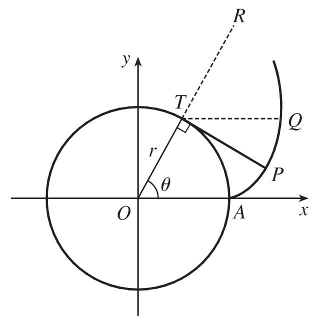
$\kappa = \dfrac{\left|\cos\theta - \cos^2\theta - \sin^2\theta\right|}{[(1-\cos\theta)^2 + \sin^2\theta]^{3/2}} = \dfrac{\left|\cos\theta - (\cos^2\theta + \sin^2\theta)\right|}{(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)^{3/2}} = \dfrac{|\cos\theta - 1|}{(2 - 2\cos\theta)^{3/2}}$. The top of the arch is characterized by a horizontal tangent, and from Example 2(b) in Section 10.2, the tangent is horizontal when $\theta = (2n-1)\pi$, so take $n = 1$ and substitute $\theta = \pi$ into the expression for κ: $\kappa = \dfrac{|\cos\pi - 1|}{(2 - 2\cos\pi)^{3/2}} = \dfrac{|-1-1|}{[2 - 2(-1)]^{3/2}} = \dfrac{1}{4}$.

72. (a) Every straight line has parametrizations of the form $x = a + vt$, $y = b + wt$, where a, b are arbitrary and $v, w \neq 0$. For example, a straight line passing through distinct points (a, b) and (c, d) can be described as the parametrized curve $x = a + (c - a)t$, $y = b + (d - b)t$. Starting with $x = a + vt$, $y = b + wt$, we compute $\dot{x} = v$, $\dot{y} = w$, $\ddot{x} = \ddot{y} = 0$, and $\kappa = \dfrac{|v \cdot 0 - w \cdot 0|}{(v^2 + w^2)^{3/2}} = 0$.

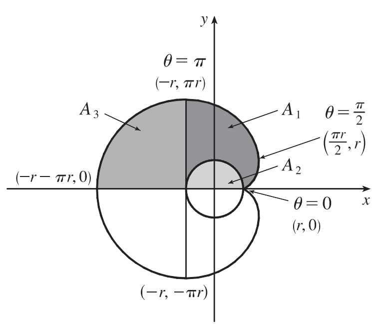
(b) Parametric equations for a circle of radius r are $x = r\cos\theta$ and $y = r\sin\theta$. We can take the center to be the origin. So $\dot{x} = -r\sin\theta \;\Rightarrow\; \ddot{x} = -r\cos\theta$ and $\dot{y} = r\cos\theta \;\Rightarrow\; \ddot{y} = -r\sin\theta$. Therefore,

$\kappa = \dfrac{\left|r^2\sin^2\theta + r^2\cos^2\theta\right|}{(r^2\sin^2\theta + r^2\cos^2\theta)^{3/2}} = \dfrac{r^2}{r^3} = \dfrac{1}{r}$. And so for any θ (and thus any point), $\kappa = \dfrac{1}{r}$.

73. The coordinates of T are $(r\cos\theta, r\sin\theta)$. Since TP was unwound from arc TA, TP has length $r\theta$. Also $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$, so P has coordinates $x = r\cos\theta + r\theta\cos\left(\frac{1}{2}\pi - \theta\right) = r(\cos\theta + \theta\sin\theta)$, $y = r\sin\theta - r\theta\sin\left(\frac{1}{2}\pi - \theta\right) = r(\sin\theta - \theta\cos\theta)$.



74. If the cow walks with the rope taut, it traces out the portion of the involute in Exercise 73 corresponding to the range $0 \le \theta \le \pi$, arriving at the point $(-r, \pi r)$ when $\theta = \pi$. With the rope now fully extended, the cow walks in a semicircle of radius πr, arriving at $(-r, -\pi r)$. Finally, the cow traces out another portion of the involute, namely the reflection about the x-axis of the initial involute path. (This corresponds to the range $-\pi \le \theta \le 0$.) Referring to the figure, we see that the total grazing area is $2(A_1 + A_3)$. A_3 is one-quarter of the area of a circle of radius πr, so $A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2$. We will compute $A_1 + A_2$ and then subtract $A_2 = \frac{1}{2}\pi r^2$ to obtain A_1.



To find $A_1 + A_2$, first note that the rightmost point of the involute is $\left(\frac{\pi}{2}r, r\right)$. [To see this, note that $dx/d\theta = 0$ when $\theta = 0$ or $\frac{\pi}{2}$. $\theta = 0$ corresponds to the cusp at $(r, 0)$ and $\theta = \frac{\pi}{2}$ corresponds to $\left(\frac{\pi}{2}r, r\right)$.] The leftmost point of the involute is

$(-r, \pi r)$. Thus, $A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y\,dx - \int_{\theta=0}^{\pi/2} y\,dx = \int_{\theta=\pi}^{0} y\,dx$.

Now $y\,dx = r(\sin\theta - \theta\cos\theta)\,r\theta\cos\theta\,d\theta = r^2(\theta\sin\theta\,\cos\theta - \theta^2\cos^2\theta)d\theta$. Integrate:

$(1/r^2)\int y\,dx = -\theta\cos^2\theta - \frac{1}{2}\left(\theta^2 - 1\right)\sin\theta\,\cos\theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + C$. This enables us to compute

$$A_1 + A_2 = r^2\left[-\theta\cos^2\theta - \tfrac{1}{2}(\theta^2 - 1)\,\sin\theta\,\cos\theta - \tfrac{1}{6}\theta^3 + \tfrac{1}{2}\theta\right]_\pi^0 = r^2\left[0 - \left(-\pi - \frac{\pi^3}{6} + \frac{\pi}{2}\right)\right] = r^2\left(\frac{\pi}{2} + \frac{\pi^3}{6}\right)$$

Therefore, $A_1 = (A_1 + A_2) - A_2 = \frac{1}{6}\pi^3 r^2$, so the grazing area is $2(A_1 + A_3) = 2\left(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2\right) = \frac{5}{6}\pi^3 r^2$.

LABORATORY PROJECT Bézier Curves

1. The parametric equations for a cubic Bézier curve are

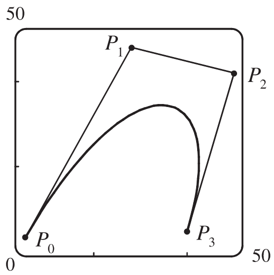
$$x = x_0(1-t)^3 + 3x_1 t(1-t)^2 + 3x_2 t^2(1-t) + x_3 t^3$$
$$y = y_0(1-t)^3 + 3y_1 t(1-t)^2 + 3y_2 t^2(1-t) + y_3 t^3$$

where $0 \le t \le 1$. We are given the points $P_0(x_0, y_0) = (4, 1)$, $P_1(x_1, y_1) = (28, 48)$, $P_2(x_2, y_2) = (50, 42)$, and $P_3(x_3, y_3) = (40, 5)$. The curve is then given by

$$x(t) = 4(1-t)^3 + 3\cdot 28t(1-t)^2 + 3\cdot 50t^2(1-t) + 40t^3$$
$$y(t) = 1(1-t)^3 + 3\cdot 48t(1-t)^2 + 3\cdot 42t^2(1-t) + 5t^3$$

where $0 \le t \le 1$. The line segments are of the form $x = x_0 + (x_1 - x_0)t$, $y = y_0 + (y_1 - y_0)t$:

P_0P_1	$x = 4 + 24t,$	$y = 1 + 47t$
P_1P_2	$x = 28 + 22t,$	$y = 48 - 6t$
P_2P_3	$x = 50 - 10t,$	$y = 42 - 37t$



2. It suffices to show that the slope of the tangent at P_0 is the same as that of line segment P_0P_1, namely $\dfrac{y_1 - y_0}{x_1 - x_0}$.

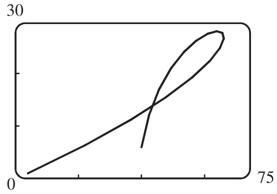
We calculate the slope of the tangent to the Bézier curve:

$$\frac{dy/dt}{dx/dt} = \frac{-3y_0(1-t)^2 + 3y_1\left[-2t(1-t) + (1-t)^2\right] + 3y_2\left[-t^2 + (2t)(1-t)\right] + 3y_3t^2}{-3x_0^2(1-t) + 3x_1[-2t(1-t) + (1-t)^2] + 3x_2[-t^2 + (2t)(1-t)] + 3x_3t^2}$$

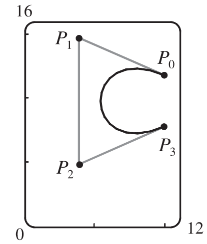
At point P_0, $t = 0$, so the slope of the tangent is $\dfrac{-3y_0 + 3y_1}{-3x_0 + 3x_1} = \dfrac{y_1 - y_0}{x_1 - x_0}$. So the tangent to the curve at P_0 passes through P_1. Similarly, the slope of the tangent at point P_3 [where $t = 1$] is $\dfrac{-3y_2 + 3y_3}{-3x_2 + 3x_3} = \dfrac{y_3 - y_2}{x_3 - x_2}$, which is also the slope of line P_2P_3.

3. It seems that if P_1 were to the right of P_2, a loop would appear.

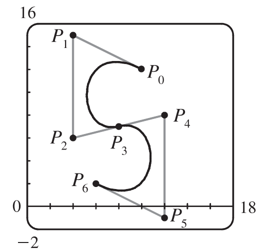
We try setting $P_1 = (110, 30)$, and the resulting curve does indeed have a loop.



4. Based on the behavior of the Bézier curve in Problems 1–3, we suspect that the four control points should be in an exaggerated C shape. We try $P_0(10,12)$, $P_1(4,15)$, $P_2(4,5)$, and $P_3(10,8)$, and these produce a decent C. If you are using a CAS, it may be necessary to instruct it to make the x- and y-scales the same so as not to distort the figure (this is called a "constrained projection" in Maple.)

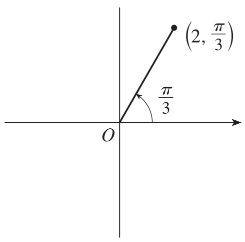


5. We use the same P_0 and P_1 as in Problem 4, and use part of our C as the top of an S. To prevent the center line from slanting up too much, we move P_2 up to $(4,6)$ and P_3 down and to the left, to $(8,7)$. In order to have a smooth joint between the top and bottom halves of the S (and a symmetric S), we determine points P_4, P_5, and P_6 by rotating points P_2, P_1, and P_0 about the center of the letter (point P_3). The points are therefore $P_4(12,8)$, $P_5(12,-1)$, and $P_6(6,2)$.



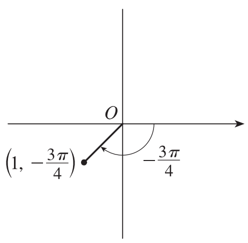
10.3 Polar Coordinates

1. (a) $\left(2, \frac{\pi}{3}\right)$



By adding 2π to $\frac{\pi}{3}$, we obtain the point $\left(2, \frac{7\pi}{3}\right)$. The direction opposite $\frac{\pi}{3}$ is $\frac{4\pi}{3}$, so $\left(-2, \frac{4\pi}{3}\right)$ is a point that satisfies the $r < 0$ requirement.

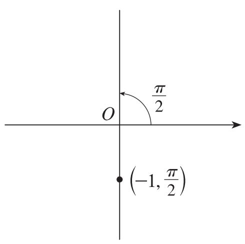
(b) $\left(1, -\frac{3\pi}{4}\right)$



$\boldsymbol{r > 0}$: $\left(1, -\frac{3\pi}{4} + 2\pi\right) = \left(1, \frac{5\pi}{4}\right)$

$\boldsymbol{r < 0}$: $\left(-1, -\frac{3\pi}{4} + \pi\right) = \left(-1, \frac{\pi}{4}\right)$

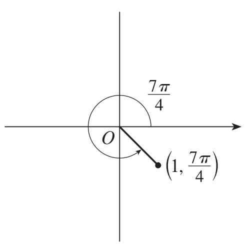
(c) $\left(-1, \frac{\pi}{2}\right)$



$\boldsymbol{r > 0}$: $\left(-(-1), \frac{\pi}{2} + \pi\right) = \left(1, \frac{3\pi}{2}\right)$

$\boldsymbol{r < 0}$: $\left(-1, \frac{\pi}{2} + 2\pi\right) = \left(-1, \frac{5\pi}{2}\right)$

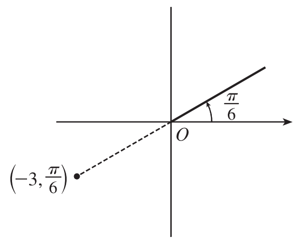
2. (a) $\left(1, \frac{7\pi}{4}\right)$



$\boldsymbol{r > 0}$: $\left(1, \frac{7\pi}{4} - 2\pi\right) = \left(1, -\frac{\pi}{4}\right)$

$\boldsymbol{r < 0}$: $\left(-1, \frac{7\pi}{4} - \pi\right) = \left(-1, \frac{3\pi}{4}\right)$

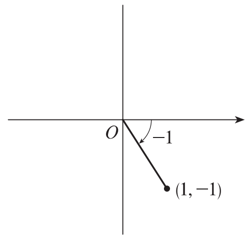
(b) $\left(-3, \frac{\pi}{6}\right)$



$\boldsymbol{r > 0}$**:** $\left(-(-3), \frac{\pi}{6} + \pi\right) = \left(3, \frac{7\pi}{6}\right)$

$\boldsymbol{r < 0}$**:** $\left(-3, \frac{\pi}{6} + 2\pi\right) = \left(-3, \frac{13\pi}{6}\right)$

(c) $(1, -1)$

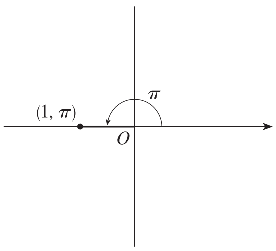


$\theta = -1$ radian $\approx -57.3°$

$\boldsymbol{r > 0}$**:** $(1, -1 + 2\pi)$

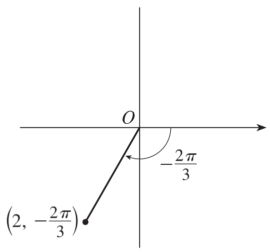
$\boldsymbol{r < 0}$**:** $(-1, -1 + \pi)$

3. (a)



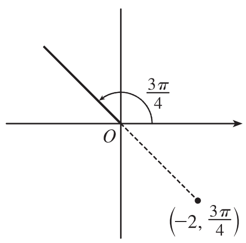
$x = 1\cos\pi = 1(-1) = -1$ and
$y = 1\sin\pi = 1(0) = 0$ give us
the Cartesian coordinates $(-1, 0)$.

(b)



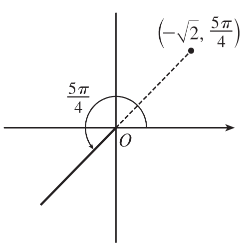
$x = 2\cos\left(-\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1$ and
$y = 2\sin\left(-\frac{2\pi}{3}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$
give us $\left(-1, -\sqrt{3}\right)$.

(c)



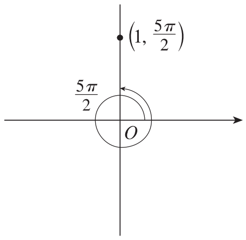
$x = -2\cos\frac{3\pi}{4} = -2\left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2}$ and
$y = -2\sin\frac{3\pi}{4} = -2\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$
gives us $\left(\sqrt{2}, -\sqrt{2}\right)$.

4. (a)



$x = -\sqrt{2}\cos\frac{5\pi}{4} = -\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = 1$ and
$y = -\sqrt{2}\sin\frac{5\pi}{4} = -\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = 1$
gives us $(1, 1)$.

(b)

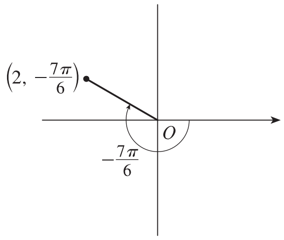


$x = 1\cos\frac{5\pi}{2} = 1(0) = 0$ and

$y = 1\sin\frac{5\pi}{2} = 1(1) = 1$

gives us $(0, 1)$.

(c)



$x = 2\cos\left(-\frac{7\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$ and

$y = 2\sin\left(-\frac{7\pi}{6}\right) = 2\left(\frac{1}{2}\right) = 1$

give us $\left(-\sqrt{3}, 1\right)$.

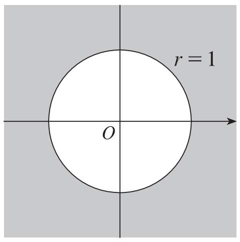
5. (a) $x = 2$ and $y = -2 \quad \Rightarrow \quad r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{-2}{2}\right) = -\frac{\pi}{4}$. Since $(2, -2)$ is in the fourth quadrant, the polar coordinates are (i) $\left(2\sqrt{2}, \frac{7\pi}{4}\right)$ and (ii) $\left(-2\sqrt{2}, \frac{3\pi}{4}\right)$.

(b) $x = -1$ and $y = \sqrt{3} \quad \Rightarrow \quad r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2$ and $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \frac{2\pi}{3}$. Since $\left(-1, \sqrt{3}\right)$ is in the second quadrant, the polar coordinates are (i) $\left(2, \frac{2\pi}{3}\right)$ and (ii) $\left(-2, \frac{5\pi}{3}\right)$.

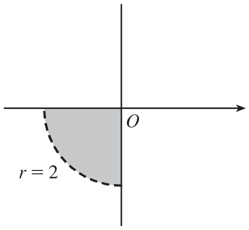
6. (a) $x = 3\sqrt{3}$ and $y = 3 \quad \Rightarrow \quad r = \sqrt{\left(3\sqrt{3}\right)^2 + 3^2} = \sqrt{27 + 9} = 6$ and $\theta = \tan^{-1}\left(\frac{3}{3\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. Since $\left(3\sqrt{3}, 3\right)$ is in the first quadrant, the polar coordinates are (i) $\left(6, \frac{\pi}{6}\right)$ and (ii) $\left(-6, \frac{7\pi}{6}\right)$.

(b) $x = 1$ and $y = -2 \quad \Rightarrow \quad r = \sqrt{1^2 + (-2)^2} = \sqrt{5}$ and $\theta = \tan^{-1}\left(\frac{-2}{1}\right) = -\tan^{-1} 2$. Since $(1, -2)$ is in the fourth quadrant, the polar coordinates are (i) $\left(\sqrt{5}, 2\pi - \tan^{-1} 2\right)$ and (ii) $\left(-\sqrt{5}, \pi - \tan^{-1} 2\right)$.

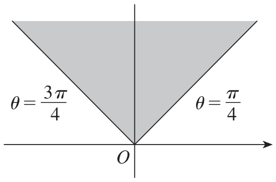
7. $r \geq 1$. The curve $r = 1$ represents a circle with center O and radius 1. So $r \geq 1$ represents the region on or outside the circle. Note that θ can take on any value.



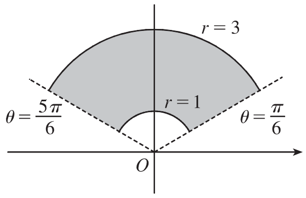
8. $0 \leq r < 2, \ \pi \leq \theta \leq 3\pi/2$. This is the region inside the circle $r = 2$ in the third quadrant.



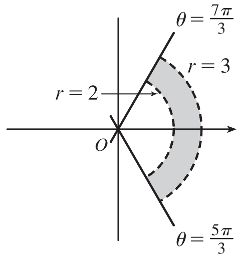
9. $r \geq 0, \ \pi/4 \leq \theta \leq 3\pi/4$.
$\theta = k$ represents a line through O.



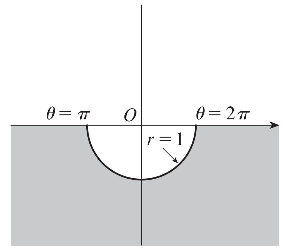
10. $1 \leq r \leq 3, \ \pi/6 < \theta < 5\pi/6$



11. $2 < r < 3,\ \frac{5\pi}{3} \leq \theta \leq \frac{7\pi}{3}$



12. $r \geq 1,\ \pi \leq \theta \leq 2\pi$



13. Converting the polar coordinates $(2, \pi/3)$ and $(4, 2\pi/3)$ to Cartesian coordinates gives us $\left(2\cos\frac{\pi}{3}, 2\sin\frac{\pi}{3}\right) = \left(1, \sqrt{3}\right)$ and $\left(4\cos\frac{2\pi}{3}, 4\sin\frac{2\pi}{3}\right) = \left(-2, 2\sqrt{3}\right)$. Now use the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2-1)^2 + \left(2\sqrt{3} - \sqrt{3}\right)^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

14. The points (r_1, θ_1) and (r_2, θ_2) in Cartesian coordinates are $(r_1\cos\theta_1, r_1\sin\theta_1)$ and $(r_2\cos\theta_2, r_2\sin\theta_2)$, respectively. The *square* of the distance between them is

$$\begin{aligned}
&(r_2\cos\theta_2 - r_1\cos\theta_1)^2 + (r_2\sin\theta_2 - r_1\sin\theta_1)^2 \\
&\qquad = \left(r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\cos^2\theta_1\right) + \left(r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_1^2\sin^2\theta_1\right) \\
&\qquad = r_1^2\left(\sin^2\theta_1 + \cos^2\theta_1\right) + r_2^2\left(\sin^2\theta_2 + \cos^2\theta_2\right) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\
&\qquad = r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2,
\end{aligned}$$

so the distance between them is $\sqrt{r_1^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + r_2^2}$.

15. $r^2 = 5 \quad\Leftrightarrow\quad x^2 + y^2 = 5$, a circle of radius $\sqrt{5}$ centered at the origin.

16. $r = 4\sec\theta \quad\Leftrightarrow\quad \dfrac{r}{\sec\theta} = 4 \quad\Leftrightarrow\quad r\cos\theta = 4 \quad\Leftrightarrow\quad x = 4$, a vertical line.

17. $r = 2\cos\theta \quad\Rightarrow\quad r^2 = 2r\cos\theta \quad\Leftrightarrow\quad x^2 + y^2 = 2x \quad\Leftrightarrow\quad x^2 - 2x + 1 + y^2 = 1 \quad\Leftrightarrow\quad (x-1)^2 + y^2 = 1$, a circle of radius 1 centered at $(1, 0)$. The first two equations are actually equivalent since $r^2 = 2r\cos\theta \quad\Rightarrow\quad r(r - 2\cos\theta) = 0 \quad\Rightarrow$ $r = 0$ or $r = 2\cos\theta$. But $r = 2\cos\theta$ gives the point $r = 0$ (the pole) when $\theta = 0$. Thus, the equation $r = 2\cos\theta$ is equivalent to the compound condition ($r = 0$ or $r = 2\cos\theta$).

18. $\theta = \dfrac{\pi}{3} \quad\Rightarrow\quad \tan\theta = \tan\dfrac{\pi}{3} \quad\Rightarrow\quad \dfrac{y}{x} = \sqrt{3} \quad\Leftrightarrow\quad y = \sqrt{3}\,x$, a line through the origin.

19. $r^2\cos 2\theta = 1 \quad\Leftrightarrow\quad r^2(\cos^2\theta - \sin^2\theta) = 1 \quad\Leftrightarrow\quad (r\cos\theta)^2 - (r\sin\theta)^2 = 1 \quad\Leftrightarrow\quad x^2 - y^2 = 1$, a hyperbola centered at the origin with foci on the x-axis.

20. $r = \tan\theta\sec\theta = \dfrac{\sin\theta}{\cos^2\theta} \quad\Rightarrow\quad r\cos^2\theta = \sin\theta \quad\Leftrightarrow\quad (r\cos\theta)^2 = r\sin\theta \quad\Leftrightarrow\quad x^2 = y$, a parabola with vertex at the origin opening upward. The first implication is reversible since $\cos\theta = 0$ would imply $\sin\theta = r\cos^2\theta = 0$, contradicting the fact that $\cos^2\theta + \sin^2\theta = 1$.

21. $y = 2 \quad \Leftrightarrow \quad r\sin\theta = 2 \quad \Leftrightarrow \quad r = \dfrac{2}{\sin\theta} \quad \Leftrightarrow \quad r = 2\csc\theta$

22. $y = x \quad \Rightarrow \quad \dfrac{y}{x} = 1\ [x \neq 0] \quad \Rightarrow \quad \tan\theta = 1 \quad \Rightarrow \quad \theta = \tan^{-1} 1 \quad \Rightarrow \quad \theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{5\pi}{4}$ [either includes the pole]

23. $y = 1 + 3x \quad \Leftrightarrow \quad r\sin\theta = 1 + 3r\cos\theta \quad \Leftrightarrow \quad r\sin\theta - 3r\cos\theta = 1 \quad \Leftrightarrow \quad r(\sin\theta - 3\cos\theta) = 1 \quad \Leftrightarrow$

$r = \dfrac{1}{\sin\theta - 3\cos\theta}$

24. $4y^2 = x \quad \Leftrightarrow \quad 4(r\sin\theta)^2 = r\cos\theta \quad \Leftrightarrow \quad 4r^2\sin^2\theta - r\cos\theta = 0 \quad \Leftrightarrow \quad r(4r\sin^2\theta - \cos\theta) = 0 \quad \Leftrightarrow \quad r = 0$ or

$r = \dfrac{\cos\theta}{4\sin^2\theta} \quad \Leftrightarrow \quad r = 0$ or $r = \frac{1}{4}\cot\theta\,\csc\theta$. $r = 0$ is included in $r = \frac{1}{4}\cot\theta\,\csc\theta$ when $\theta = \frac{\pi}{2}$, so the curve is represented by the single equation $r = \frac{1}{4}\cot\theta\,\csc\theta$.

25. $x^2 + y^2 = 2cx \quad \Leftrightarrow \quad r^2 = 2cr\cos\theta \quad \Leftrightarrow \quad r^2 - 2cr\cos\theta = 0 \quad \Leftrightarrow \quad r(r - 2c\cos\theta) = 0 \quad \Leftrightarrow \quad r = 0$ or $r = 2c\cos\theta$.
$r = 0$ is included in $r = 2c\cos\theta$ when $\theta = \frac{\pi}{2} + n\pi$, so the curve is represented by the single equation $r = 2c\cos\theta$.

26. $xy = 4 \quad \Leftrightarrow \quad (r\cos\theta)(r\sin\theta) = 4 \quad \Leftrightarrow \quad r^2\left(\frac{1}{2}\cdot 2\sin\theta\,\cos\theta\right) = 4 \quad \Leftrightarrow \quad r^2\sin 2\theta = 8 \quad \Rightarrow \quad r^2 = 8\csc 2\theta$

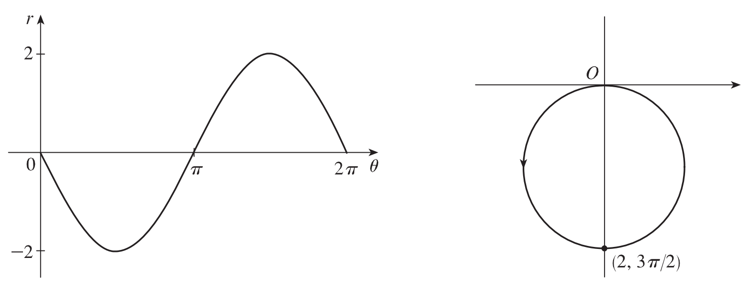
27. (a) The description leads immediately to the polar equation $\theta = \frac{\pi}{6}$, and the Cartesian equation $y = \tan\left(\frac{\pi}{6}\right)x = \frac{1}{\sqrt{3}}x$ is slightly more difficult to derive.

(b) The easier description here is the Cartesian equation $x = 3$.

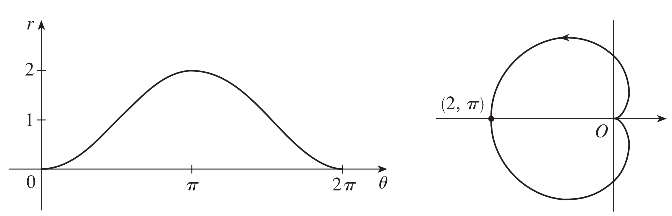
28. (a) Because its center is not at the origin, it is more easily described by its Cartesian equation, $(x-2)^2 + (y-3)^2 = 5^2$.

(b) This circle is more easily given in polar coordinates: $r = 4$. The Cartesian equation is also simple: $x^2 + y^2 = 16$.

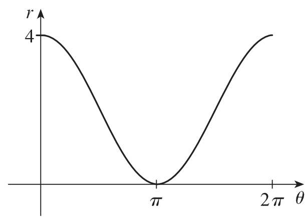
29. $r = -2\sin\theta$

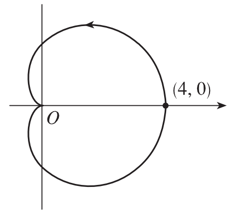


30. $r = 1 - \cos\theta$

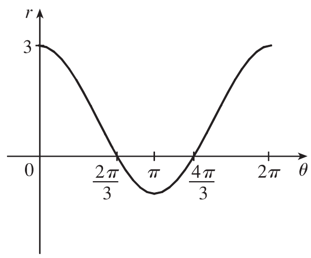


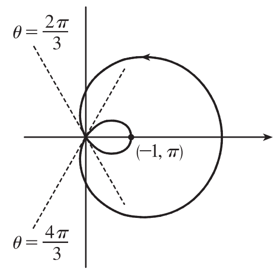
31. $r = 2(1 + \cos\theta)$



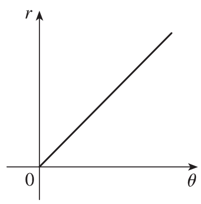


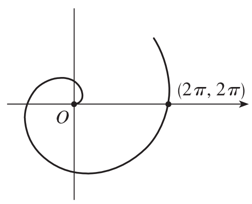
32. $r = 1 + 2\cos\theta$



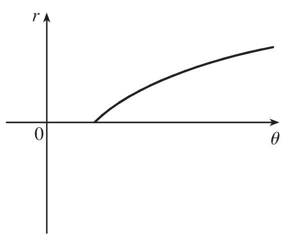


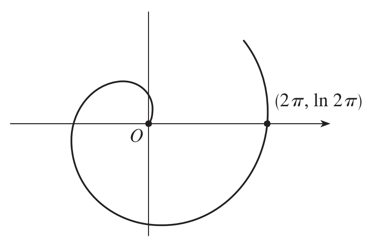
33. $r = \theta, \quad \theta \geq 0$



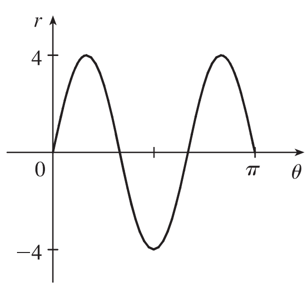


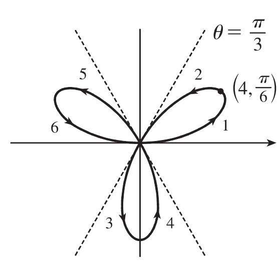
34. $r = \ln\theta, \ \theta \geq 1$



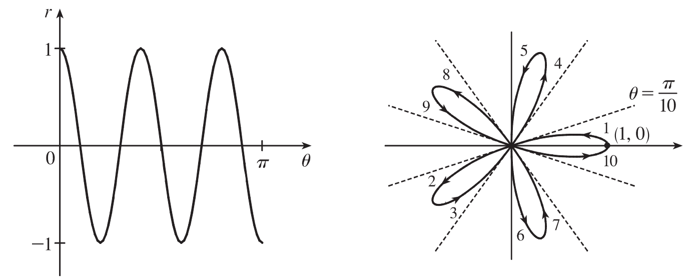


35. $r = 4\sin 3\theta$

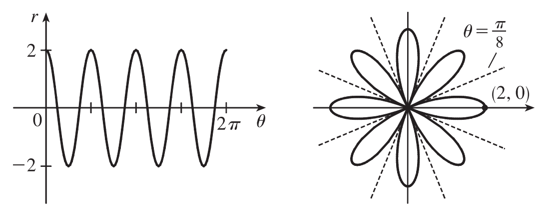




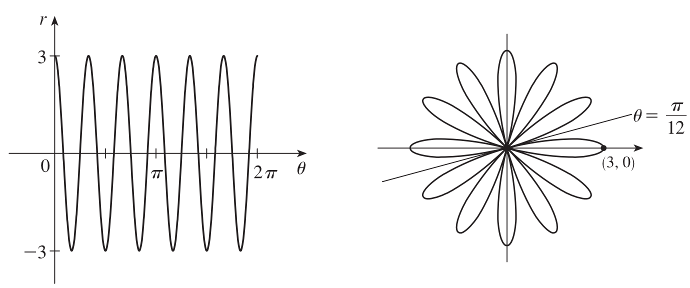
36. $r = \cos 5\theta$



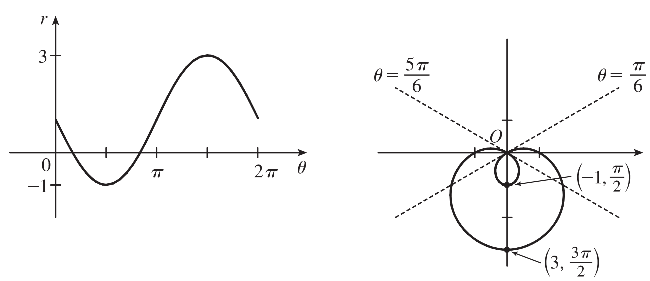
37. $r = 2\cos 4\theta$



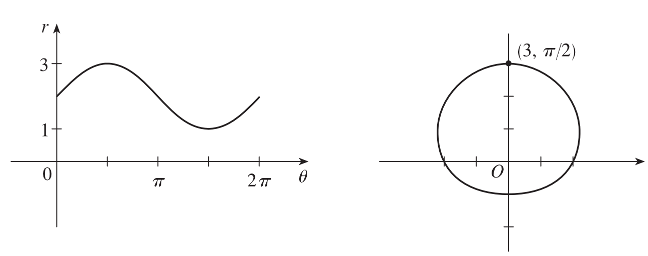
38. $r = 3\cos 6\theta$



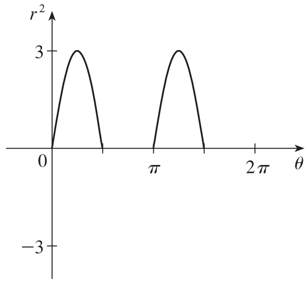
39. $r = 1 - 2\sin\theta$

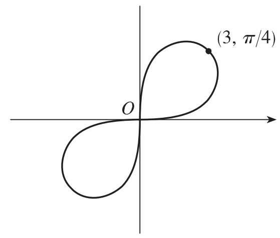


40. $r = 2 + \sin\theta$

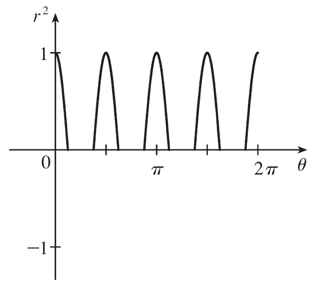


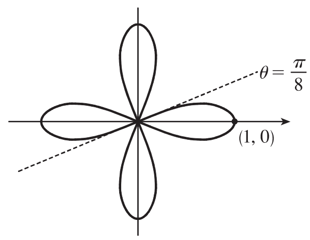
41. $r^2 = 9\sin 2\theta$



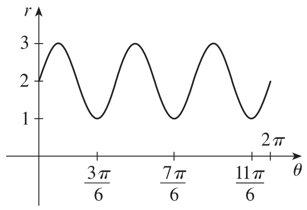


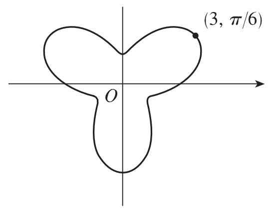
42. $r^2 = \cos 4\theta$



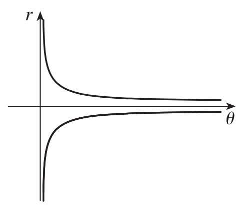


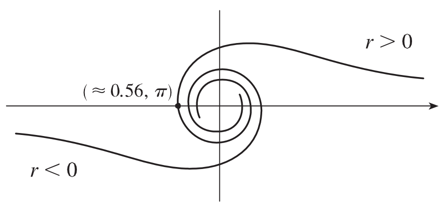
43. $r = 2 + \sin 3\theta$



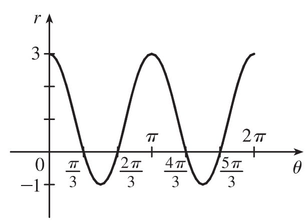


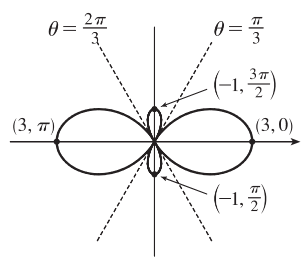
44. $r^2\theta = 1 \quad \Leftrightarrow \quad r = \pm 1/\sqrt{\theta}$ for $\theta > 0$



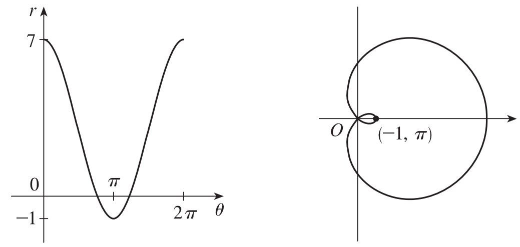


45. $r = 1 + 2\cos 2\theta$

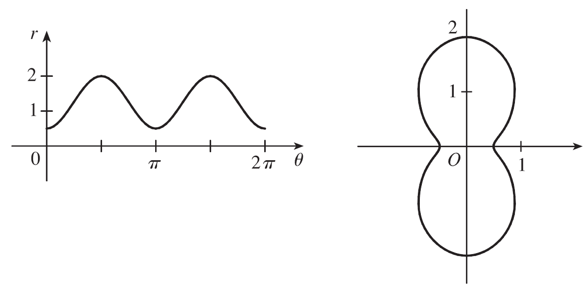




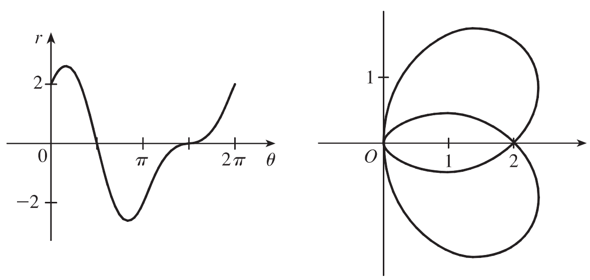
46. $r = 3 + 4\cos\theta$



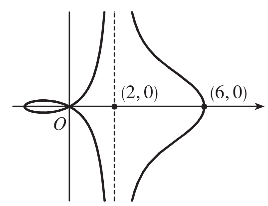
47. For $\theta = 0$, π, and 2π, r has its minimum value of about 0.5. For $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, r attains its maximum value of 2. We see that the graph has a similar shape for $0 \le \theta \le \pi$ and $\pi \le \theta \le 2\pi$.



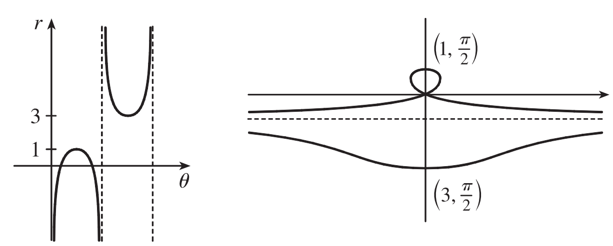
48.



49. $x = r\cos\theta = (4 + 2\sec\theta)\cos\theta = 4\cos\theta + 2$. Now, $r \to \infty \;\Rightarrow\; (4 + 2\sec\theta) \to \infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^{-}$ or $\theta \to \left(\frac{3\pi}{2}\right)^{+}$ [since we need only consider $0 \le \theta < 2\pi$], so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} (4\cos\theta + 2) = 2$. Also, $r \to -\infty \;\Rightarrow\; (4 + 2\sec\theta) \to -\infty \;\Rightarrow\; \theta \to \left(\frac{\pi}{2}\right)^{+}$ or $\theta \to \left(\frac{3\pi}{2}\right)^{-}$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} (4\cos\theta + 2) = 2$. Therefore, $\lim\limits_{r\to\pm\infty} x = 2 \;\Rightarrow\; x = 2$ is a vertical asymptote.



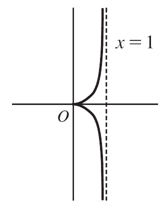
50. $y = r\sin\theta = 2\sin\theta - \csc\theta\sin\theta = 2\sin\theta - 1$.
$r \to \infty \;\Rightarrow\; (2 - \csc\theta) \to \infty \;\Rightarrow\; \csc\theta \to -\infty \;\Rightarrow\; \theta \to \pi^{+}$ [since we need only consider $0 \le \theta < 2\pi$] and so $\lim\limits_{r\to\infty} y = \lim\limits_{\theta\to\pi^+} 2\sin\theta - 1 = -1$.



Also $r \to -\infty \;\Rightarrow\; (2 - \csc\theta) \to -\infty \;\Rightarrow\; \csc\theta \to \infty \;\Rightarrow\; \theta \to \pi^{-}$ and so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi^-} 2\sin\theta - 1 = -1$.

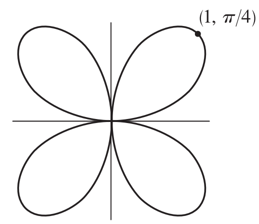
Therefore $\lim\limits_{r\to\pm\infty} y = -1 \;\Rightarrow\; y = -1$ is a horizontal asymptote.

51. To show that $x = 1$ is an asymptote we must prove $\lim\limits_{r\to\pm\infty} x = 1$.



$x = (r)\cos\theta = (\sin\theta\,\tan\theta)\cos\theta = \sin^2\theta$. Now, $r \to \infty \Rightarrow \sin\theta\,\tan\theta \to \infty \Rightarrow$ $\theta \to \left(\frac{\pi}{2}\right)^-$, so $\lim\limits_{r\to\infty} x = \lim\limits_{\theta\to\pi/2^-} \sin^2\theta = 1$. Also, $r \to -\infty \Rightarrow \sin\theta\,\tan\theta \to -\infty \Rightarrow$ $\theta \to \left(\frac{\pi}{2}\right)^+$, so $\lim\limits_{r\to-\infty} x = \lim\limits_{\theta\to\pi/2^+} \sin^2\theta = 1$. Therefore, $\lim\limits_{r\to\pm\infty} x = 1 \Rightarrow x = 1$ is a vertical asymptote. Also notice that $x = \sin^2\theta \geq 0$ for all θ, and $x = \sin^2\theta \leq 1$ for all θ. And $x \neq 1$, since the curve is not defined at odd multiples of $\frac{\pi}{2}$. Therefore, the curve lies entirely within the vertical strip $0 \leq x < 1$.

52. The equation is $(x^2 + y^2)^3 = 4x^2y^2$, but using polar coordinates we know that $x^2 + y^2 = r^2$ and $x = r\cos\theta$ and $y = r\sin\theta$. Substituting into the given equation: $r^6 = 4r^2\cos^2\theta\, r^2\sin^2\theta \Rightarrow r^2 = 4\cos^2\theta\,\sin^2\theta \Rightarrow$ $r = \pm 2\cos\theta\,\sin\theta = \pm\sin 2\theta$. $r = \pm\sin 2\theta$ is sketched at right.



53. (a) We see that the curve $r = 1 + c\sin\theta$ crosses itself at the origin, where $r = 0$ (in fact the inner loop corresponds to negative r-values,) so we solve the equation of the limaçon for $r = 0 \Leftrightarrow c\sin\theta = -1 \Leftrightarrow \sin\theta = -1/c$. Now if $|c| < 1$, then this equation has no solution and hence there is no inner loop. But if $c < -1$, then on the interval $(0, 2\pi)$ the equation has the two solutions $\theta = \sin^{-1}(-1/c)$ and $\theta = \pi - \sin^{-1}(-1/c)$, and if $c > 1$, the solutions are $\theta = \pi + \sin^{-1}(1/c)$ and $\theta = 2\pi - \sin^{-1}(1/c)$. In each case, $r < 0$ for θ between the two solutions, indicating a loop.

(b) For $0 < c < 1$, the dimple (if it exists) is characterized by the fact that y has a local maximum at $\theta = \frac{3\pi}{2}$. So we determine for what c-values $\dfrac{d^2y}{d\theta^2}$ is negative at $\theta = \frac{3\pi}{2}$, since by the Second Derivative Test this indicates a maximum:

$y = r\sin\theta = \sin\theta + c\sin^2\theta \Rightarrow \dfrac{dy}{d\theta} = \cos\theta + 2c\sin\theta\,\cos\theta = \cos\theta + c\sin 2\theta \Rightarrow \dfrac{d^2y}{d\theta^2} = -\sin\theta + 2c\cos 2\theta.$

At $\theta = \frac{3\pi}{2}$, this is equal to $-(-1) + 2c(-1) = 1 - 2c$, which is negative only for $c > \frac{1}{2}$. A similar argument shows that for $-1 < c < 0$, y only has a local minimum at $\theta = \frac{\pi}{2}$ (indicating a dimple) for $c < -\frac{1}{2}$.

54. (a) $r = \sqrt{\theta},\ 0 \leq \theta \leq 16\pi$. r increases as θ increases and there are eight full revolutions. The graph must be either II or V. When $\theta = 2\pi$, $r = \sqrt{2\pi} \approx 2.5$ and when $\theta = 16\pi$, $r = \sqrt{16\pi} \approx 7$, so the last revolution intersects the polar axis at approximately 3 times the distance that the first revolution intersects the polar axis, which is depicted in graph V.

(b) $r = \theta^2,\ 0 \leq \theta \leq 16\pi$. See part (a). This is graph II.

(c) $r = \cos(\theta/3)$. $0 \leq \frac{\theta}{3} \leq 2\pi \Rightarrow 0 \leq \theta \leq 6\pi$, so this curve will repeat itself every 6π radians.
$\cos\left(\frac{\theta}{3}\right) = 0 \Rightarrow \frac{\theta}{3} = \frac{\pi}{2} + \pi n \Rightarrow \theta = \frac{3\pi}{2} + 3\pi n$, so there will be two "pole" values, $\frac{3\pi}{2}$ and $\frac{9\pi}{2}$.
This is graph VI.

(d) $r = 1 + 2\cos\theta$ is a limaçon [see Exercise 53(a)] with $c = 2$. This is graph III.

(e) Since $-1 \leq \sin 3\theta \leq 1$, $1 \leq 2 + \sin 3\theta \leq 3$, so $r = 2 + \sin 3\theta$ is never 0; that is, the curve never intersects the pole. This is graph I.

(f) $r = 1 + 2\sin 3\theta$. Solving $r = 0$ will give us many "pole" values, so this is graph IV.

55. $r = 2\sin\theta \quad \Rightarrow \quad x = r\cos\theta = 2\sin\theta\cos\theta = \sin 2\theta$, $y = r\sin\theta = 2\sin^2\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \cdot 2\sin\theta\,\cos\theta}{\cos 2\theta \cdot 2} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

When $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = \tan\left(2 \cdot \frac{\pi}{6}\right) = \tan\frac{\pi}{3} = \sqrt{3}$. [*Another method:* Use Equation 3.]

56. $r = 2 - \sin\theta \quad \Rightarrow \quad x = r\cos\theta = (2 - \sin\theta)\cos\theta$, $y = r\sin\theta = (2 - \sin\theta)\sin\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2 - \sin\theta)\cos\theta + \sin\theta(-\cos\theta)}{(2 - \sin\theta)(-\sin\theta) + \cos\theta(-\cos\theta)} = \frac{2\cos\theta - 2\sin\theta\,\cos\theta}{-2\sin\theta + \sin^2\theta - \cos^2\theta} = \frac{2\cos\theta - \sin 2\theta}{-2\sin\theta - \cos 2\theta}$$

When $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{2(1/2) - \left(\sqrt{3}/2\right)}{-2\left(\sqrt{3}/2\right) - (-1/2)} = \frac{1 - \sqrt{3}/2}{-\sqrt{3} + 1/2} \cdot \frac{2}{2} = \frac{2 - \sqrt{3}}{1 - 2\sqrt{3}}$.

57. $r = 1/\theta \quad \Rightarrow \quad x = r\cos\theta = (\cos\theta)/\theta$, $y = r\sin\theta = (\sin\theta)/\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-1/\theta^2) + (1/\theta)\cos\theta}{\cos\theta(-1/\theta^2) - (1/\theta)\sin\theta} \cdot \frac{\theta^2}{\theta^2} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}$$

When $\theta = \pi$, $\frac{dy}{dx} = \frac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \frac{-\pi}{1} = -\pi$.

58. $r = \cos(\theta/3) \quad \Rightarrow \quad x = r\cos\theta = \cos(\theta/3)\cos\theta$, $y = r\sin\theta = \cos(\theta/3)\sin\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos(\theta/3)\cos\theta + \sin\theta\,\left(-\frac{1}{3}\sin(\theta/3)\right)}{\cos(\theta/3)\,(-\sin\theta) + \cos\theta\,\left(-\frac{1}{3}\sin(\theta/3)\right)}$$

When $\theta = \pi$, $\frac{dy}{dx} = \frac{\frac{1}{2}\,(-1) + (0)\,\left(-\sqrt{3}/6\right)}{\frac{1}{2}\,(0) + (-1)\,\left(-\sqrt{3}/6\right)} = \frac{-1/2}{\sqrt{3}/6} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$.

59. $r = \cos 2\theta \quad \Rightarrow \quad x = r\cos\theta = \cos 2\theta\,\cos\theta$, $y = r\sin\theta = \cos 2\theta\,\sin\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos 2\theta\,\cos\theta + \sin\theta\,(-2\sin 2\theta)}{\cos 2\theta\,(-\sin\theta) + \cos\theta\,(-2\sin 2\theta)}$$

When $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{0\left(\sqrt{2}/2\right) + \left(\sqrt{2}/2\right)(-2)}{0\left(-\sqrt{2}/2\right) + \left(\sqrt{2}/2\right)(-2)} = \frac{-\sqrt{2}}{-\sqrt{2}} = 1$.

60. $r = 1 + 2\cos\theta \quad \Rightarrow \quad x = r\cos\theta = (1 + 2\cos\theta)\cos\theta$, $y = r\sin\theta = (1 + 2\cos\theta)\sin\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1 + 2\cos\theta)\cos\theta + \sin\theta\,(-2\sin\theta)}{(1 + 2\cos\theta)(-\sin\theta) + \cos\theta\,(-2\sin\theta)}$$

When $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{2\left(\frac{1}{2}\right) + \left(\sqrt{3}/2\right)\left(-\sqrt{3}\right)}{2\left(-\sqrt{3}/2\right) + \frac{1}{2}\left(-\sqrt{3}\right)} \cdot \frac{2}{2} = \frac{2 - 3}{-2\sqrt{3} - \sqrt{3}} = \frac{-1}{-3\sqrt{3}} = \frac{\sqrt{3}}{9}$.

61. $r = 3\cos\theta \quad\Rightarrow\quad x = r\cos\theta = 3\cos\theta\,\cos\theta,\; y = r\sin\theta = 3\cos\theta\,\sin\theta \quad\Rightarrow$

$\frac{dy}{d\theta} = -3\sin^2\theta + 3\cos^2\theta = 3\cos 2\theta = 0 \quad\Rightarrow\quad 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2} \quad\Leftrightarrow\quad \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

So the tangent is horizontal at $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$ and $\left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)$ $\left[\text{same as } \left(\frac{3}{\sqrt{2}}, -\frac{\pi}{4}\right)\right]$.

$\frac{dx}{d\theta} = -6\sin\theta\cos\theta = -3\sin 2\theta = 0 \quad\Rightarrow\quad 2\theta = 0$ or $\pi \quad\Leftrightarrow\quad \theta = 0$ or $\frac{\pi}{2}$. So the tangent is vertical at $(3, 0)$ and $\left(0, \frac{\pi}{2}\right)$.

62. $r = 1 - \sin\theta \quad\Rightarrow\quad x = r\cos\theta = \cos\theta\,(1 - \sin\theta),\; y = r\sin\theta = \sin\theta\,(1 - \sin\theta) \quad\Rightarrow$

$\frac{dy}{d\theta} = \sin\theta\,(-\cos\theta) + (1 - \sin\theta)\,\cos\theta = \cos\theta\,(1 - 2\sin\theta) = 0 \quad\Rightarrow\quad \cos\theta = 0$ or $\sin\theta = \frac{1}{2} \quad\Rightarrow$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, or $\frac{3\pi}{2} \quad\Rightarrow\quad$ horizontal tangent at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$, and $\left(2, \frac{3\pi}{2}\right)$.

$$\begin{aligned}\frac{dx}{d\theta} &= \cos\theta\,(-\cos\theta) + (1 - \sin\theta)(-\sin\theta) = -\cos^2\theta - \sin\theta + \sin^2\theta = 2\sin^2\theta - \sin\theta - 1 \\ &= (2\sin\theta + 1)(\sin\theta - 1) = 0 \quad\Rightarrow\end{aligned}$$

$\sin\theta = -\frac{1}{2}$ or $1 \quad\Rightarrow\quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$, or $\frac{\pi}{2} \quad\Rightarrow\quad$ vertical tangent at $\left(\frac{3}{2}, \frac{7\pi}{6}\right)$, $\left(\frac{3}{2}, \frac{11\pi}{6}\right)$, and $\left(0, \frac{\pi}{2}\right)$.

Note that the tangent is vertical, not horizontal, when $\theta = \frac{\pi}{2}$, since

$$\lim_{\theta\to(\pi/2)^-} \frac{dy/d\theta}{dx/d\theta} = \lim_{\theta\to(\pi/2)^-} \frac{\cos\theta\,(1 - 2\sin\theta)}{(2\sin\theta + 1)(\sin\theta - 1)} = \infty \text{ and } \lim_{\theta\to(\pi/2)^+} \frac{dy/d\theta}{dx/d\theta} = -\infty.$$

63. $r = 1 + \cos\theta \quad\Rightarrow\quad x = r\cos\theta = \cos\theta\,(1 + \cos\theta),\; y = r\sin\theta = \sin\theta\,(1 + \cos\theta) \quad\Rightarrow$

$\frac{dy}{d\theta} = (1 + \cos\theta)\,\cos\theta - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0 \quad\Rightarrow\quad \cos\theta = \frac{1}{2}$ or $-1 \quad\Rightarrow$

$\theta = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3} \quad\Rightarrow\quad$ horizontal tangent at $\left(\frac{3}{2}, \frac{\pi}{3}\right)$, $(0, \pi)$, and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.

$\frac{dx}{d\theta} = -(1 + \cos\theta)\sin\theta - \cos\theta\sin\theta = -\sin\theta\,(1 + 2\cos\theta) = 0 \quad\Rightarrow\quad \sin\theta = 0$ or $\cos\theta = -\frac{1}{2} \quad\Rightarrow$

$\theta = 0, \pi, \frac{2\pi}{3}$, or $\frac{4\pi}{3} \quad\Rightarrow\quad$ vertical tangent at $(2, 0)$, $\left(\frac{1}{2}, \frac{2\pi}{3}\right)$, and $\left(\frac{1}{2}, \frac{4\pi}{3}\right)$.

Note that the tangent is horizontal, not vertical when $\theta = \pi$, since $\lim\limits_{\theta\to\pi} \dfrac{dy/d\theta}{dx/d\theta} = 0$.

64. $r = e^\theta \quad\Rightarrow\quad x = r\cos\theta = e^\theta\cos\theta,\; y = r\sin\theta = e^\theta\sin\theta \quad\Rightarrow$

$\frac{dy}{d\theta} = e^\theta\sin\theta + e^\theta\cos\theta = e^\theta(\sin\theta + \cos\theta) = 0 \quad\Rightarrow\quad \sin\theta = -\cos\theta \quad\Rightarrow\quad \tan\theta = -1 \quad\Rightarrow$

$\theta = -\frac{1}{4}\pi + n\pi$ [n any integer] $\quad\Rightarrow\quad$ horizontal tangents at $\left(e^{\pi(n-1/4)}, \pi\left(n - \frac{1}{4}\right)\right)$.

$\frac{dx}{d\theta} = e^\theta\cos\theta - e^\theta\sin\theta = e^\theta\,(\cos\theta - \sin\theta) = 0 \quad\Rightarrow\quad \sin\theta = \cos\theta \quad\Rightarrow\quad \tan\theta = 1 \quad\Rightarrow$

$\theta = \frac{1}{4}\pi + n\pi$ [n any integer] $\quad\Rightarrow\quad$ vertical tangents at $\left(e^{\pi(n+1/4)}, \pi\left(n + \frac{1}{4}\right)\right)$.

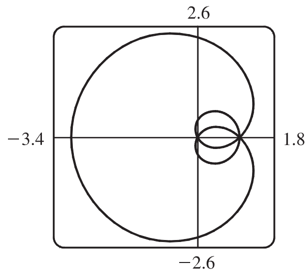
65. $r = a\sin\theta + b\cos\theta \quad\Rightarrow\quad r^2 = ar\sin\theta + br\cos\theta \quad\Rightarrow\quad x^2 + y^2 = ay + bx \quad\Rightarrow$

$x^2 - bx + \left(\frac{1}{2}b\right)^2 + y^2 - ay + \left(\frac{1}{2}a\right)^2 = \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}a\right)^2 \quad\Rightarrow\quad \left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}(a^2 + b^2)$, and this is a circle with center $\left(\frac{1}{2}b, \frac{1}{2}a\right)$ and radius $\frac{1}{2}\sqrt{a^2 + b^2}$.

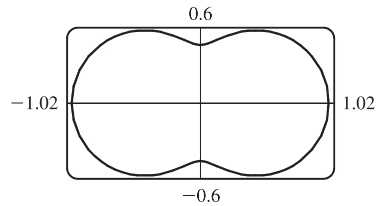
66. These curves are circles which intersect at the origin and at $\left(\frac{1}{\sqrt{2}}\,a, \frac{\pi}{4}\right)$. At the origin, the first circle has a horizontal tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle [$r = a\sin\theta$], $dy/d\theta = a\cos\theta\,\sin\theta + a\sin\theta\,\cos\theta = a\sin 2\theta = a$ at $\theta = \frac{\pi}{4}$ and $dx/d\theta = a\cos^2\theta - a\sin^2\theta = a\cos 2\theta = 0$

at $\theta = \frac{\pi}{4}$, so the tangent here is vertical. Similarly, for the second circle $[r = a\cos\theta]$, $dy/d\theta = a\cos 2\theta = 0$ and $dx/d\theta = -a\sin 2\theta = -a$ at $\theta = \frac{\pi}{4}$, so the tangent is horizontal, and again the tangents are perpendicular.

67. $r = 1 + 2\sin(\theta/2)$. The parameter interval is $[0, 4\pi]$.

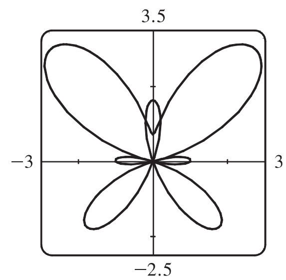


68. $r = \sqrt{1 - 0.8\sin^2\theta}$. The parameter interval is $[0, 2\pi]$.



69. $r = e^{\sin\theta} - 2\cos(4\theta)$.

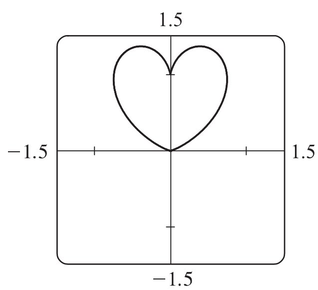
The parameter interval is $[0, 2\pi]$.

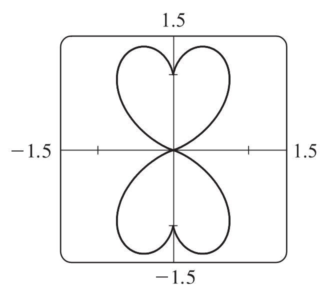


70. $r = |\tan\theta|^{|\cot\theta|}$.

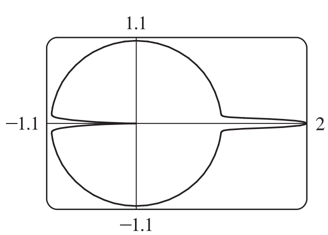
The parameter interval $[0, \pi]$ produces the heart-shaped valentine curve shown in the first window.

The complete curve, including the reflected heart, is produced by the parameter interval $[0, 2\pi]$, but perhaps you'll agree that the first curve is more appropriate.

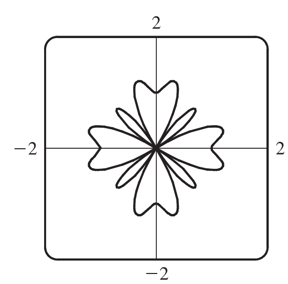




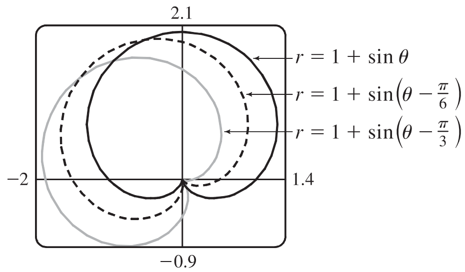
71. $r = 1 + \cos^{999}\theta$. The parameter interval is $[0, 2\pi]$.



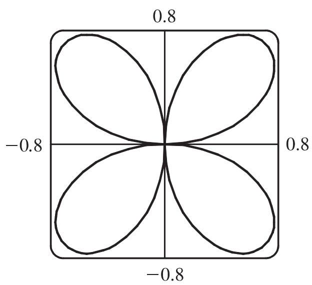
72. $r = \sin^2(4\theta) + \cos(4\theta)$. The parameter interval is $[0, 2\pi]$.



73. It appears that the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{6}\right)$ is the same shape as the graph of $r = 1 + \sin\theta$, but rotated counterclockwise about the origin by $\frac{\pi}{6}$. Similarly, the graph of $r = 1 + \sin\left(\theta - \frac{\pi}{3}\right)$ is rotated by $\frac{\pi}{3}$. In general, the graph of $r = f(\theta - \alpha)$ is the same shape as that of $r = f(\theta)$, but rotated counterclockwise through α about the origin. That is, for any point (r_0, θ_0) on the curve $r = f(\theta)$, the point $(r_0, \theta_0 + \alpha)$ is on the curve $r = f(\theta - \alpha)$, since $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$.



74.



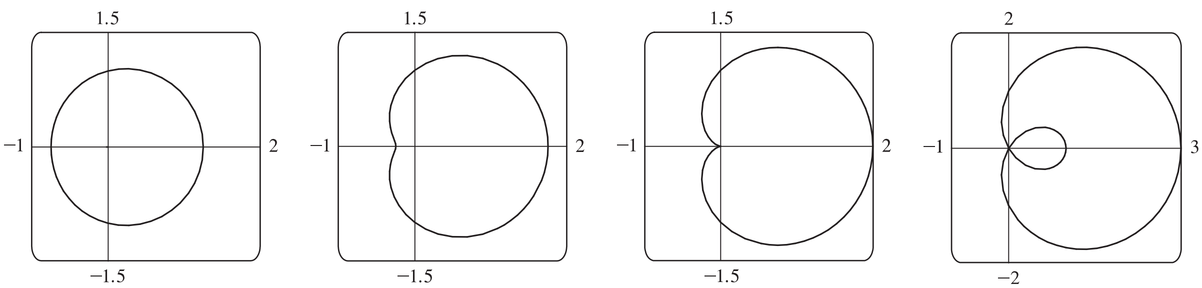
From the graph, the highest points seem to have $y \approx 0.77$. To find the exact value, we solve $dy/d\theta = 0$. $y = r\sin\theta = \sin\theta\,\sin 2\theta \quad\Rightarrow$

$$\begin{aligned} dy/d\theta &= 2\sin\theta\,\cos 2\theta + \cos\theta\,\sin 2\theta \\ &= 2\sin\theta\,(2\cos^2\theta - 1) + \cos\theta\,(2\sin\theta\,\cos\theta) \\ &= 2\sin\theta\,(3\cos^2\theta - 1) \end{aligned}$$

In the first quadrant, this is 0 when $\cos\theta = \frac{1}{\sqrt{3}} \quad\Leftrightarrow\quad \sin\theta = \sqrt{\frac{2}{3}} \quad\Leftrightarrow$

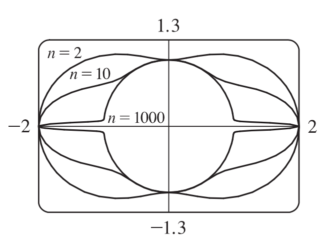
$y = 2\sin^2\theta\cos\theta = 2 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4}{9}\sqrt{3} \approx 0.77$.

75. Consider curves with polar equation $r = 1 + c\cos\theta$, where c is a real number. If $c = 0$, we get a circle of radius 1 centered at the pole. For $0 < c \leq 0.5$, the curve gets slightly larger, moves right, and flattens out a bit on the left side. For $0.5 < c < 1$, the left side has a dimple shape. For $c = 1$, the dimple becomes a cusp. For $c > 1$, there is an internal loop. For $c \geq 0$, the rightmost point on the curve is $(1 + c, 0)$. For $c < 0$, the curves are reflections through the vertical axis of the curves with $c > 0$.

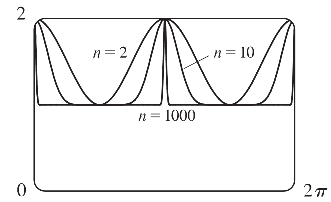


$c = 0.25$ $c = 0.75$ $c = 1$ $c = 2$

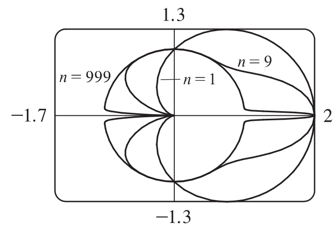
76. Consider the polar curves $r = 1 + \cos^n\theta$, where n is a positive integer. First, let **n be an even positive integer**. The first figure shows that the curve has a peanut shape for $n = 2$, but as n increases, the ends are squeezed. As n becomes large, the curves look more and more like the unit circle, but with spikes to the points $(2, 0)$ and $(2, \pi)$.



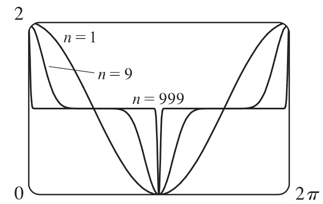
The second figure shows r as a function of θ in Cartesian coordinates for the same values of n. We can see that for large n, the graph is similar to the graph of $y = 1$, but with spikes to $y = 2$ for $x = 0$, π, and 2π. (Note that when $0 < \cos\theta < 1$, $\cos^{1000}\theta$ is very small.)



Next, let ***n* be an odd positive integer**. The third figure shows that the curve is a cardioid for $n = 1$, but as n increases, the heart shape becomes more pronounced. As n becomes large, the curves again look more like the unit circle, but with an outward spike to $(2, 0)$ and an inward spike to $(0, \pi)$.



The fourth figure shows r as a function of θ in Cartesian coordinates for the same values of n. We can see that for large n, the graph is similar to the graph of $y = 1$, but spikes to $y = 2$ for $x = 0$ and π, and to $y = 0$ for $x = \pi$.

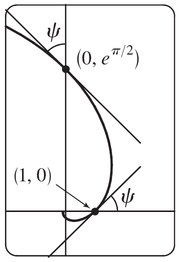


77. $\tan\psi = \tan(\phi - \theta) = \dfrac{\tan\phi - \tan\theta}{1 + \tan\phi\,\tan\theta} = \dfrac{\dfrac{dy}{dx} - \tan\theta}{1 + \dfrac{dy}{dx}\tan\theta} = \dfrac{\dfrac{dy/d\theta}{dx/d\theta} - \tan\theta}{1 + \dfrac{dy/d\theta}{dx/d\theta}\tan\theta}$

$$= \frac{\dfrac{dy}{d\theta} - \dfrac{dx}{d\theta}\tan\theta}{\dfrac{dx}{d\theta} + \dfrac{dy}{d\theta}\tan\theta} = \frac{\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right) - \tan\theta\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right)}{\left(\dfrac{dr}{d\theta}\cos\theta - r\sin\theta\right) + \tan\theta\left(\dfrac{dr}{d\theta}\sin\theta + r\cos\theta\right)} = \frac{r\cos\theta + r\cdot\dfrac{\sin^2\theta}{\cos\theta}}{\dfrac{dr}{d\theta}\cos\theta + \dfrac{dr}{d\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}}$$

$$= \frac{r\cos^2\theta + r\sin^2\theta}{\dfrac{dr}{d\theta}\cos^2\theta + \dfrac{dr}{d\theta}\sin^2\theta} = \frac{r}{dr/d\theta}$$

78. (a) $r = e^\theta \quad\Rightarrow\quad dr/d\theta = e^\theta$, so by Exercise 77, $\tan\psi = r/e^\theta = 1 \quad\Rightarrow$ $\psi = \arctan 1 = \frac{\pi}{4}$.

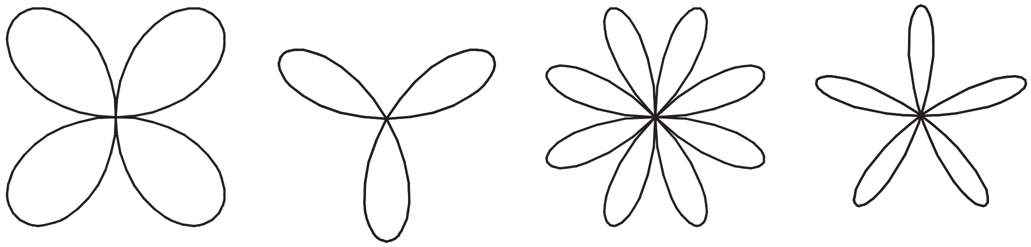


(b) The Cartesian equation of the tangent line at $(1, 0)$ is $y = x - 1$, and that of the tangent line at $(0, e^{\pi/2})$ is $y = e^{\pi/2} - x$.

(c) Let a be the tangent of the angle between the tangent and radial lines, that is, $a = \tan\psi$. Then, by Exercise 77, $a = \dfrac{r}{dr/d\theta} \quad\Rightarrow\quad \dfrac{dr}{d\theta} = \dfrac{1}{a}r \quad\Rightarrow$ $r = Ce^{\theta/a}$ (by Theorem 9.4.2).

LABORATORY PROJECT Families of Polar Curves

1. (a) $r = \sin n\theta$.

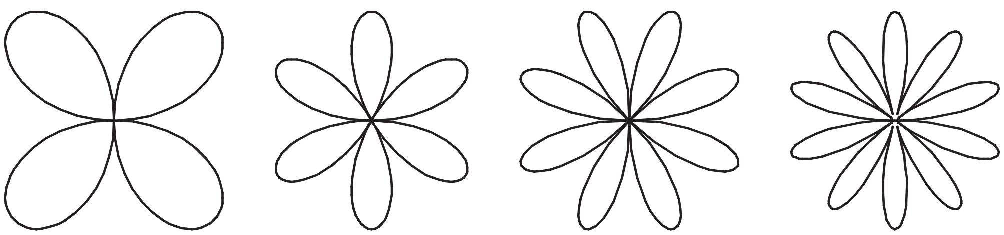


$n = 2$ $n = 3$ $n = 4$ $n = 5$

From the graphs, it seems that when n is even, the number of loops in the curve (called a rose) is $2n$, and when n is odd, the number of loops is simply n. This is because in the case of n odd, every point on the graph is traversed twice, due to the fact that

$$r(\theta + \pi) = \sin[n(\theta + \pi)] = \sin n\theta \, \cos n\pi + \cos n\theta \, \sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even} \\ -\sin n\theta & \text{if } n \text{ is odd} \end{cases}$$

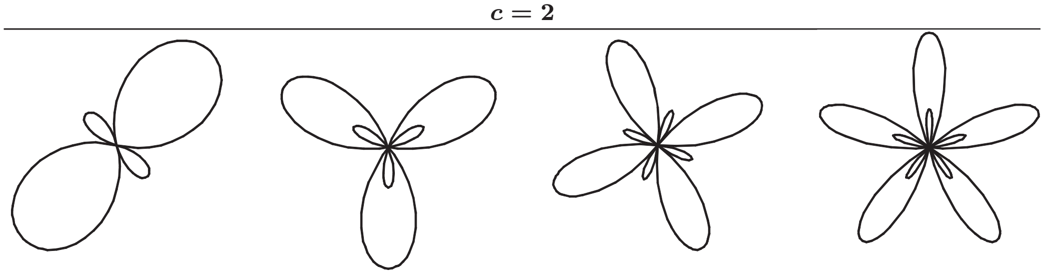
(b) The graph of $r = |\sin n\theta|$ has $2n$ loops whether n is odd or even, since $r(\theta + \pi) = r(\theta)$.



$n = 2$ $n = 3$ $n = 4$ $n = 5$

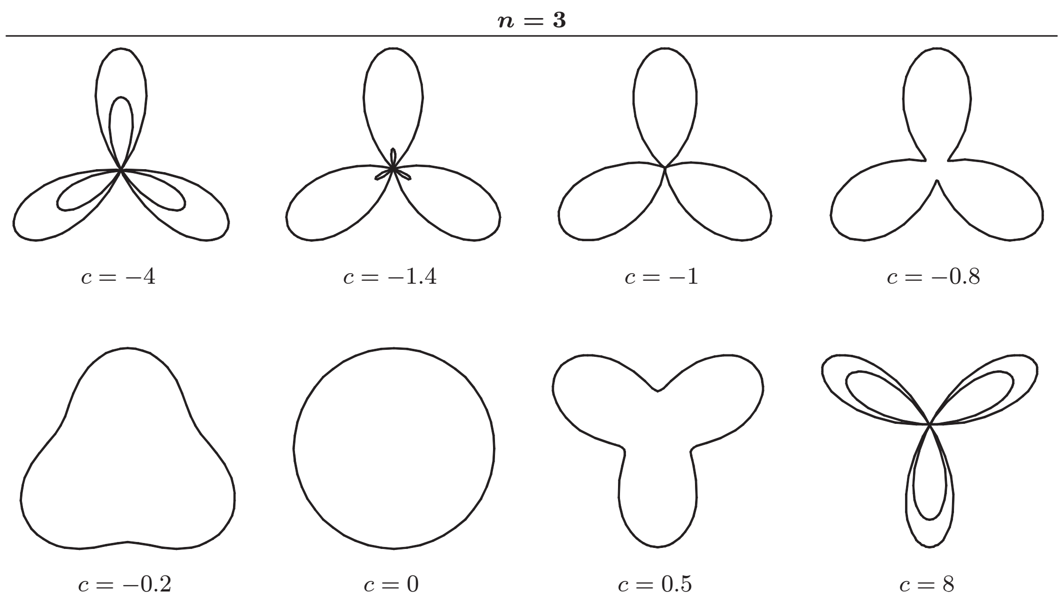
2. $r = 1 + c \sin n\theta$. We vary n while keeping c constant at 2. As n changes, the curves change in the same way as those in Exercise 1: the number of loops increases. Note that if n is even, the smaller loops are outside the larger ones; if n is odd, they are inside.

$\boldsymbol{c = 2}$



$n = 2$ $n = 3$ $n = 4$ $n = 5$

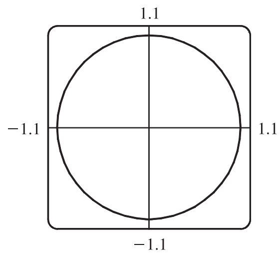
Now we vary c while keeping $n = 3$. As c increases toward 0, the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At $c = -1$, the small loops disappear entirely, and for $-1 < c < 1$, the graph is a simple, closed curve (at $c = 0$ it is a circle). As c continues to increase, the same changes are seen, but in reverse order, since $1 + (-c)\sin n\theta = 1 + c\sin n(\theta + \pi)$, so the graph for $c = c_0$ is the same as that for $c = -c_0$, with a rotation through π. As $c \to \infty$, the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2. Maple's `animate` command (or Mathematica's `Animate`) is very useful for seeing the changes that occur as c varies.



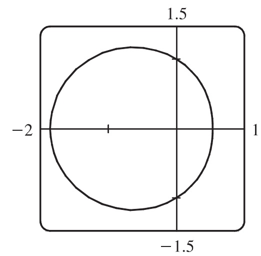
3. $r = \dfrac{1 - a\cos\theta}{1 + a\cos\theta}$. We start with $a = 0$, since in this case the curve is simply the circle $r = 1$.

As a increases, the graph moves to the left, and its right side becomes flattened. As a increases through about 0.4, the right side seems to grow a dimple, which upon closer investigation (with narrower θ-ranges) seems to appear at $a \approx 0.42$ [the actual value is $\sqrt{2} - 1$]. As $a \to 1$, this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at $a = 1$ the denominator vanishes at $\theta = \pi$, and the dimple becomes an actual cusp. For $a > 1$ we must choose our parameter interval carefully, since $r \to \infty$ as $1 + a\cos\theta \to 0 \quad\Leftrightarrow\quad \theta \to \pm\cos^{-1}(-1/a)$. As a increases from 1, the curve splits into two parts. The left part has a loop, which grows larger as a increases, and the right part grows broader vertically, and its left tip develops a dimple when $a \approx 2.42$ [actually, $\sqrt{2} + 1$]. As a increases, the dimple grows more and more pronounced. If $a < 0$, we get the same graph as we do for the corresponding positive a-value, but with a rotation through π about the pole, as happened when c was replaced with $-c$ in Exercise 2.

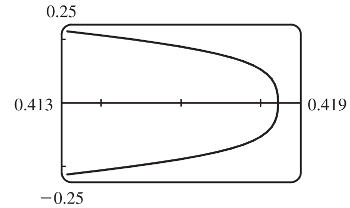
[continued]



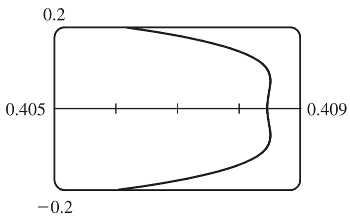
$a = 0$



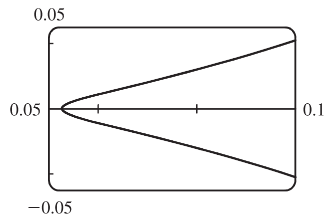
$a = 0.3$



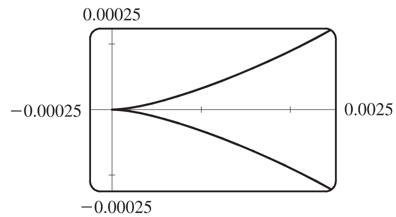
$a = 0.41, |\theta| \leq 0.5$



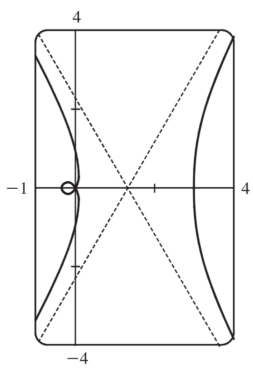
$a = 0.42, |\theta| \leq 0.5$



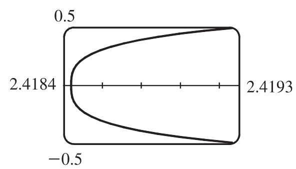
$a = 0.9, |\theta| \leq 0.5$



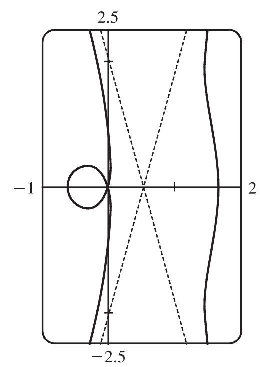
$a = 1, |\theta| \leq 0.1$



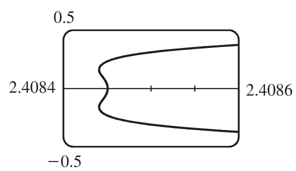
$a = 2$



$a = 2.41, |\theta - \pi| \leq 0.2$



$a = 4$



$a = 2.42, |\theta - \pi| \leq 0.2$

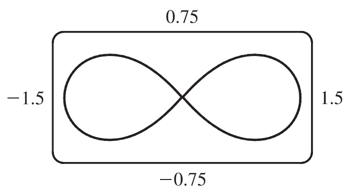
4. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for r in terms of θ, a, and c. We note that the given equation, $r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$, is a quadratic in r^2, so we use the quadratic formula and find that

$$r^2 = \frac{2c^2\cos 2\theta \pm \sqrt{4c^4\cos^2 2\theta - 4(c^4 - a^4)}}{2} = c^2\cos 2\theta \pm \sqrt{a^4 - c^4\sin^2 2\theta}$$

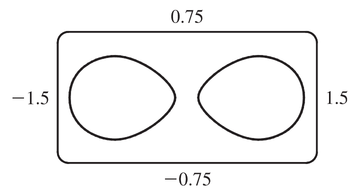
so $r = \pm\sqrt{c^2\cos 2\theta \pm \sqrt{a^4 - c^4\sin^2 2\theta}}$. So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all four functions have period π.

We start with the case $a = c = 1$, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decreases further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1, the two lobes of the curve join together, and as a increases further they continue to merge, until at

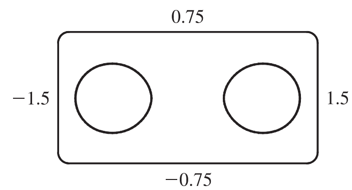
$a \approx 1.4$, the graph no longer has dimples, and has an oval shape. As $a \to \infty$, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance. Note that the shape of the graph seems to depend only on the ratio c/a, while the size of the graph varies as c and a jointly increase.



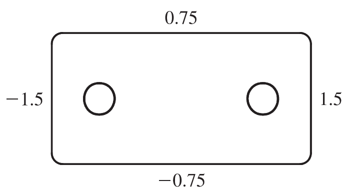
$(a, c) = (1, 1)$



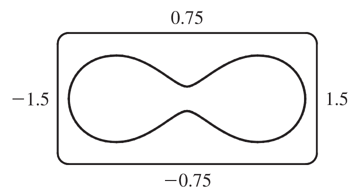
$(a, c) = (0.99, 1)$



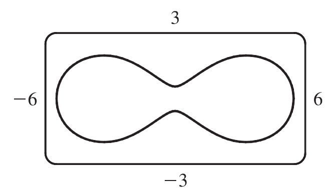
$(a, c) = (0.9, 1)$



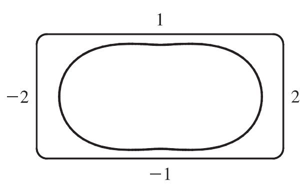
$(a, c) = (0.6, 1)$



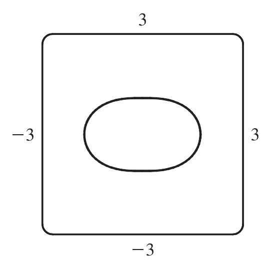
$(a, c) = (1.01, 1)$



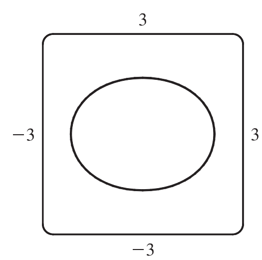
$(a, c) = (4.04, 4)$



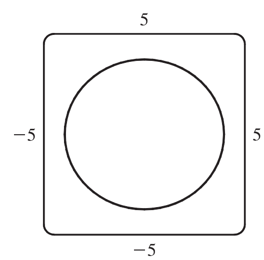
$(a, c) = (1.3, 1)$



$(a, c) = (1.5, 1)$



$(a, c) = (2, 1)$



$(a, c) = (4, 1)$

10.4 Areas and Lengths in Polar Coordinates

1. $r = e^{-\theta/4}$, $\pi/2 \le \theta \le \pi$.

$$A = \int_{\pi/2}^{\pi} \tfrac{1}{2} r^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2} (e^{-\theta/4})^2\, d\theta = \int_{\pi/2}^{\pi} \tfrac{1}{2} e^{-\theta/2}\, d\theta = \tfrac{1}{2} \Big[-2e^{-\theta/2} \Big]_{\pi/2}^{\pi} = -1(e^{-\pi/2} - e^{-\pi/4}) = e^{-\pi/4} - e^{-\pi/2}$$

2. $r = \cos\theta$, $0 \le \theta \le \pi/6$.

$$A = \int_0^{\pi/6} \tfrac{1}{2} r^2\, d\theta = \int_0^{\pi/6} \tfrac{1}{2} \cos^2\theta\, d\theta = \tfrac{1}{2} \int_0^{\pi/6} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta = \tfrac{1}{4} \left[\theta + \tfrac{1}{2} \sin 2\theta \right]_0^{\pi/6}$$
$$= \tfrac{1}{4} \left(\tfrac{\pi}{6} + \tfrac{1}{2} \cdot \tfrac{1}{2}\sqrt{3} \right) = \tfrac{\pi}{24} + \tfrac{1}{16}\sqrt{3}$$

3. $r^2 = 9 \sin 2\theta$, $r \ge 0$, $0 \le \theta \le \pi/2$.

$$A = \int_0^{\pi/2} \tfrac{1}{2} r^2\, d\theta = \int_0^{\pi/2} \tfrac{1}{2}(9 \sin 2\theta)\, d\theta = \tfrac{9}{2} \left[-\tfrac{1}{2} \cos 2\theta \right]_0^{\pi/2} = -\tfrac{9}{4}(-1 - 1) = \tfrac{9}{2}$$

4. $r = \tan\theta,\ \pi/6 \le \theta \le \pi/3$.

$$A = \int_{\pi/6}^{\pi/3} \tfrac{1}{2} r^2\, d\theta = \int_{\pi/6}^{\pi/3} \tfrac{1}{2} \tan^2\theta\, d\theta = \int_{\pi/6}^{\pi/3} \tfrac{1}{2}(\sec^2\theta - 1)\, d\theta = \tfrac{1}{2}\big[\tan\theta - \theta\big]_{\pi/6}^{\pi/3}$$

$$= \tfrac{1}{2}\big[\big(\sqrt{3} - \tfrac{\pi}{3}\big) - \big(\tfrac{1}{3}\sqrt{3} - \tfrac{\pi}{6}\big)\big] = \tfrac{1}{2}\big[\tfrac{2}{3}\sqrt{3} - \tfrac{\pi}{6}\big] = \tfrac{1}{3}\sqrt{3} - \tfrac{\pi}{12}$$

5. $r = \sqrt{\theta},\ 0 \le \theta \le 2\pi$. $\quad A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}\left(\sqrt{\theta}\right)^2 d\theta = \int_0^{2\pi} \tfrac{1}{2}\theta\, d\theta = \big[\tfrac{1}{4}\theta^2\big]_0^{2\pi} = \pi^2$

6. $r = 1 + \cos\theta,\ 0 \le \theta \le \pi$.

$$A = \int_0^{\pi} \tfrac{1}{2}(1+\cos\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta = \tfrac{1}{2}\int_0^{\pi} \big[1 + 2\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\big]\, d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi} \big(\tfrac{3}{2} + 2\cos\theta + \tfrac{1}{2}\cos 2\theta\big)\, d\theta = \tfrac{1}{2}\big[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\big]_0^{\pi} = \tfrac{1}{2}\big(\tfrac{3}{2}\pi + 0 + 0\big) - \tfrac{1}{2}(0) = \tfrac{3\pi}{4}$$

7. $r = 4 + 3\sin\theta,\ -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}$.

$$A = \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}((4 + 3\sin\theta)^2\, d\theta = \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 24\sin\theta + 9\sin^2\theta)\, d\theta$$

$$= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 9\sin^2\theta)\, d\theta \qquad \text{[by Theorem 4.5.6(b) [ET 5.5.7(b)]]}$$

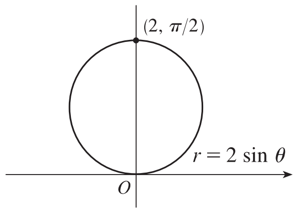
$$= \tfrac{1}{2}\cdot 2\int_0^{\pi/2} \big[16 + 9\cdot\tfrac{1}{2}(1 - \cos 2\theta)\big]\, d\theta \qquad \text{[by Theorem 4.5.6(a) [ET 5.5.7(a)]]}$$

$$= \int_0^{\pi/2} \big(\tfrac{41}{2} - \tfrac{9}{2}\cos 2\theta\big)\, d\theta = \big[\tfrac{41}{2}\theta - \tfrac{9}{4}\sin 2\theta\big]_0^{\pi/2} = \big(\tfrac{41\pi}{4} - 0\big) - (0 - 0) = \tfrac{41\pi}{4}$$

8. $r = \sin 2\theta,\ 0 \le \theta \le \tfrac{\pi}{2}$.

$$A = \int_0^{\pi/2} \tfrac{1}{2}\sin^2 2\theta\, d\theta = \tfrac{1}{2}\int_0^{\pi/2} \tfrac{1}{2}(1 - \cos 4\theta)\, d\theta = \tfrac{1}{4}\big[\theta - \tfrac{1}{4}\sin 4\theta\big]_0^{\pi/2} = \tfrac{1}{4}\big(\tfrac{\pi}{2}\big) = \tfrac{\pi}{8}$$

9. The area is bounded by $r = 2\sin\theta$ for $\theta = 0$ to $\theta = \pi$.

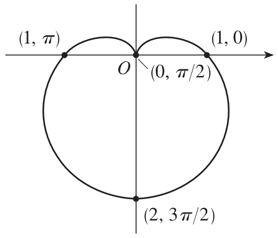


$$A = \int_0^{\pi} \tfrac{1}{2} r^2\, d\theta = \tfrac{1}{2}\int_0^{\pi} (2\sin\theta)^2\, d\theta = \tfrac{1}{2}\int_0^{\pi} 4\sin^2\theta\, d\theta$$

$$= 2\int_0^{\pi} \tfrac{1}{2}(1 - \cos 2\theta) d\theta = \Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi} = \pi$$

Also, note that this is a circle with radius 1, so its area is $\pi(1)^2 = \pi$.

10. $A = \int_0^{2\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{2\pi} \tfrac{1}{2}(1 - \sin\theta)^2\, d\theta$



$$= \tfrac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta + \sin^2\theta)\, d\theta = \tfrac{1}{2}\int_0^{2\pi} \big[1 - 2\sin\theta + \tfrac{1}{2}(1 - \cos 2\theta)\big]\, d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \big(\tfrac{3}{2} - 2\sin\theta - \tfrac{1}{2}\cos 2\theta\big)\, d\theta = \tfrac{1}{2}\big[\tfrac{3}{2}\theta + 2\cos\theta - \tfrac{1}{4}\sin 2\theta\big]_0^{2\pi}$$

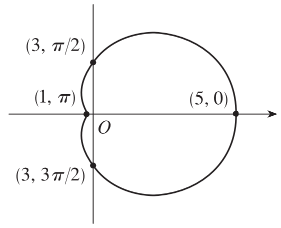
$$= \tfrac{1}{2}[(3\pi + 2) - (2)] = \tfrac{3\pi}{2}$$

11. $A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \int_0^{2\pi} \frac{1}{2}(3+2\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(9+12\cos\theta+4\cos^2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[9+12\cos\theta+4\cdot\frac{1}{2}(1+\cos 2\theta)\right]d\theta$

$= \frac{1}{2}\int_0^{2\pi}(11+12\cos\theta+2\cos 2\theta)\,d\theta = \frac{1}{2}\left[11\theta+12\sin\theta+\sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{2}(22\pi) = 11\pi$

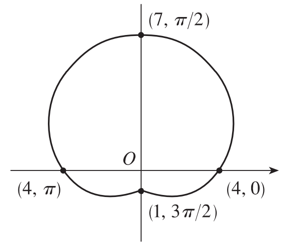


12. $A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \int_0^{2\pi} \frac{1}{2}(4+3\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(16+24\sin\theta+9\sin^2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[16+24\sin\theta+9\cdot\frac{1}{2}(1-\cos 2\theta)\right]d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left(\frac{41}{2}+24\sin\theta-\frac{9}{2}\cos 2\theta\right)d\theta = \frac{1}{2}\left[\frac{41}{2}\theta-24\cos\theta-\frac{9}{4}\sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{2}[(41\pi-24)-(-24)] = \frac{41}{2}\pi$

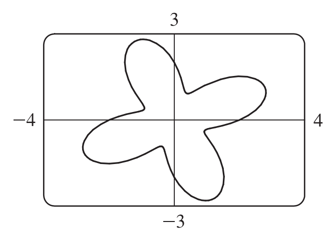


13. $A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \int_0^{2\pi} \frac{1}{2}(2+\sin 4\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(4+4\sin 4\theta+\sin^2 4\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[4+4\sin 4\theta+\frac{1}{2}(1-\cos 8\theta)\right]d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left(\frac{9}{2}+4\sin 4\theta-\frac{1}{2}\cos 8\theta\right)d\theta = \frac{1}{2}\left[\frac{9}{2}\theta-\cos 4\theta-\frac{1}{16}\sin 8\theta\right]_0^{2\pi}$

$= \frac{1}{2}[(9\pi-1)-(-1)] = \frac{9}{2}\pi$

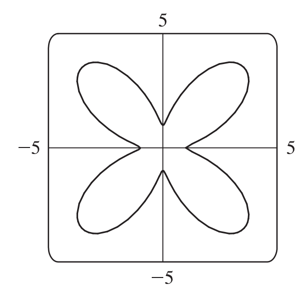


14. $A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \int_0^{2\pi} \frac{1}{2}(3-2\cos 4\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(9-12\cos 4\theta+4\cos^2 4\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[9-12\cos 4\theta+4\cdot\frac{1}{2}(1+\cos 8\theta)\right]d\theta$

$= \frac{1}{2}\int_0^{2\pi}(11-12\cos 4\theta+2\cos 8\theta)\,d\theta = \frac{1}{2}\left[11\theta-3\sin 4\theta+\frac{1}{4}\sin 8\theta\right]_0^{2\pi}$

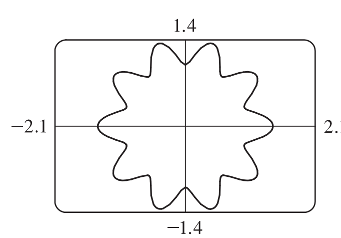
$= \frac{1}{2}(22\pi) = 11\pi$



15. $A = \int_0^{2\pi} \frac{1}{2}r^2\,d\theta = \int_0^{2\pi} \frac{1}{2}\left(\sqrt{1+\cos^2 5\theta}\right)^2 d\theta$

$= \frac{1}{2}\int_0^{2\pi}(1+\cos^2 5\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}\left[1+\frac{1}{2}(1+\cos 10\theta)\right]d\theta$

$= \frac{1}{2}\left[\frac{3}{2}\theta+\frac{1}{20}\sin 10\theta\right]_0^{2\pi} = \frac{1}{2}(3\pi) = \frac{3}{2}\pi$



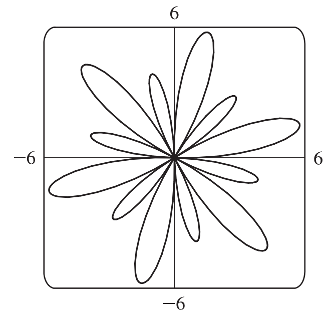
16. $A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2}(1 + 5\sin 6\theta)^2 \, d\theta$

$= \frac{1}{2} \int_0^{2\pi} (1 + 10\sin 6\theta + 25\sin^2 6\theta) \, d\theta$

$= \frac{1}{2} \int_0^{2\pi} \left[1 + 10\sin 6\theta + 25 \cdot \frac{1}{2}(1 - \cos 12\theta)\right] d\theta$

$= \frac{1}{2} \int_0^{2\pi} \left[\frac{27}{2} + 10\sin 6\theta - \frac{25}{2}\cos 12\theta\right] d\theta = \frac{1}{2}\left[\frac{27}{2}\theta - \frac{5}{3}\cos 6\theta - \frac{25}{24}\sin 12\theta\right]_0^{2\pi}$

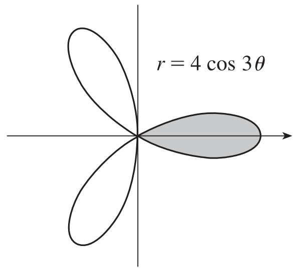
$= \frac{1}{2}\left[\left(27\pi - \frac{5}{3}\right) - \left(-\frac{5}{3}\right)\right] = \frac{27}{2}\pi$



17. The curve passes through the pole when $r = 0 \;\Rightarrow\; 4\cos 3\theta = 0 \;\Rightarrow\; \cos 3\theta = 0 \;\Rightarrow\; 3\theta = \frac{\pi}{2} + \pi n \;\Rightarrow$ $\theta = \frac{\pi}{6} + \frac{\pi}{3}n$. The part of the shaded loop above the polar axis is traced out for $\theta = 0$ to $\theta = \pi/6$, so we'll use $-\pi/6$ and $\pi/6$ as our limits of integration.

$A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}(4\cos 3\theta)^2 \, d\theta = 2\int_0^{\pi/6} \frac{1}{2}(16\cos^2 3\theta) \, d\theta$

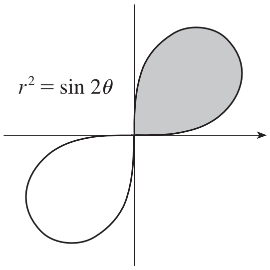
$= 16\int_0^{\pi/6} \frac{1}{2}(1 + \cos 6\theta) \, d\theta = 8\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = 8\left(\frac{\pi}{6}\right) = \frac{4}{3}\pi$



18. For $\theta = 0$ to $\theta = \pi/2$, the shaded loop is traced out by $r = \sqrt{\sin 2\theta}$ and the unshaded loop is traced out by $r = -\sqrt{\sin 2\theta}$.

$A = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/2} \frac{1}{2}\sin 2\theta \, d\theta$

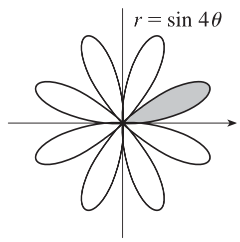
$= \left[-\frac{1}{4}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$



19. $r = 0 \;\Rightarrow\; \sin 4\theta = 0 \;\Rightarrow\; 4\theta = \pi n \;\Rightarrow\; \theta = \frac{\pi}{4}n$.

$A = \int_0^{\pi/4} \frac{1}{2}(\sin 4\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/4} \sin^2 4\theta \, d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{1}{2}(1 - \cos 8\theta) \, d\theta$

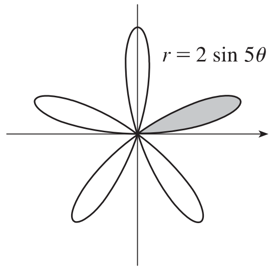
$= \frac{1}{4}\left[\theta - \frac{1}{8}\sin 8\theta\right]_0^{\pi/4} = \frac{1}{4}\left(\frac{\pi}{4}\right) = \frac{1}{16}\pi$

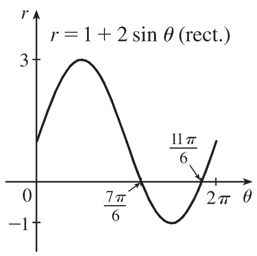


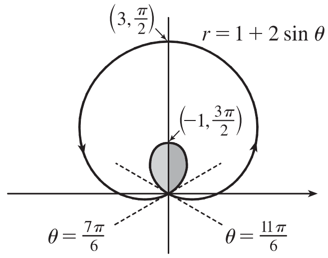
20. $r = 0 \;\Rightarrow\; 2\sin 5\theta = 0 \;\Rightarrow\; \sin 5\theta = 0 \;\Rightarrow\; 5\theta = \pi n \;\Rightarrow\; \theta = \frac{\pi}{5}n$.

$A = \int_0^{\pi/5} \frac{1}{2}(2\sin 5\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/5} 4\sin^2 5\theta \, d\theta$

$= 2\int_0^{\pi/5} \frac{1}{2}(1 - \cos 10\theta) \, d\theta = \left[\theta - \frac{1}{10}\sin 10\theta\right]_0^{\pi/5} = \frac{\pi}{5}$



21. 



This is a limaçon, with inner loop traced out between $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$ [found by solving $r = 0$].

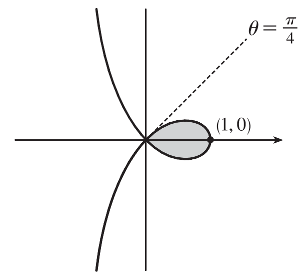
$$A = 2\int_{7\pi/6}^{3\pi/2} \tfrac{1}{2}(1+2\sin\theta)^2\,d\theta = \int_{7\pi/6}^{3\pi/2}\left(1+4\sin\theta+4\sin^2\theta\right)d\theta = \int_{7\pi/6}^{3\pi/2}\left[1+4\sin\theta+4\cdot\tfrac{1}{2}(1-\cos 2\theta)\right]d\theta$$

$$= \left[\theta - 4\cos\theta + 2\theta - \sin 2\theta\right]_{7\pi/6}^{3\pi/2} = \left(\tfrac{9\pi}{2}\right) - \left(\tfrac{7\pi}{2} + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\right) = \pi - \tfrac{3\sqrt{3}}{2}$$

22. To determine when the strophoid $r = 2\cos\theta - \sec\theta$ passes through the pole, we solve

$$r = 0 \;\Rightarrow\; 2\cos\theta - \frac{1}{\cos\theta} = 0 \;\Rightarrow\; 2\cos^2\theta - 1 = 0 \;\Rightarrow\; \cos^2\theta = \frac{1}{2} \;\Rightarrow$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}} \;\Rightarrow\; \theta = \tfrac{\pi}{4} \text{ or } \theta = \tfrac{3\pi}{4} \text{ for } 0 \le \theta \le \pi \text{ with } \theta \ne \tfrac{\pi}{2}.$$



$$A = 2\int_0^{\pi/4} \tfrac{1}{2}(2\cos\theta - \sec\theta)^2\,d\theta = \int_0^{\pi/4}(4\cos^2\theta - 4 + \sec^2\theta)\,d\theta$$

$$= \int_0^{\pi/4}\left[4\cdot\tfrac{1}{2}(1+\cos 2\theta) - 4 + \sec^2\theta\right]d\theta = \int_0^{\pi/4}(-2 + 2\cos 2\theta + \sec^2\theta)\,d\theta$$

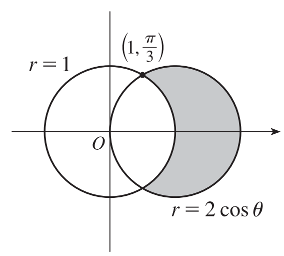
$$= \left[-2\theta + \sin 2\theta + \tan\theta\right]_0^{\pi/4} = \left(-\tfrac{\pi}{2} + 1 + 1\right) - 0 = 2 - \tfrac{\pi}{2}$$

23. $2\cos\theta = 1 \;\Rightarrow\; \cos\theta = \frac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

$$A = 2\int_0^{\pi/3} \tfrac{1}{2}\left[(2\cos\theta)^2 - 1^2\right]d\theta = \int_0^{\pi/3}(4\cos^2\theta - 1)\,d\theta$$

$$= \int_0^{\pi/3}\left\{4\left[\tfrac{1}{2}(1+\cos 2\theta)\right] - 1\right\}d\theta = \int_0^{\pi/3}(1 + 2\cos 2\theta)\,d\theta$$

$$= \left[\theta + \sin 2\theta\right]_0^{\pi/3} = \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2}$$

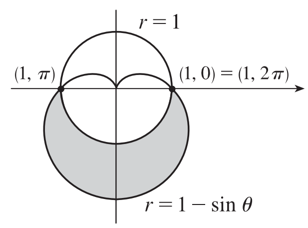


24. $1 - \sin\theta = 1 \;\Rightarrow\; \sin\theta = 0 \;\Rightarrow\; \theta = 0 \text{ or } \pi \;\Rightarrow$

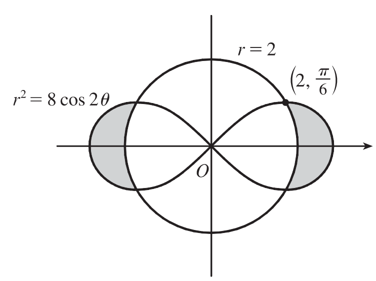
$$A = \int_\pi^{2\pi} \tfrac{1}{2}\left[(1-\sin\theta)^2 - 1\right]d\theta = \tfrac{1}{2}\int_\pi^{2\pi}(\sin^2\theta - 2\sin\theta)\,d\theta$$

$$= \tfrac{1}{4}\int_\pi^{2\pi}(1 - \cos 2\theta - 4\sin\theta)\,d\theta = \tfrac{1}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta + 4\cos\theta\right]_\pi^{2\pi}$$

$$= \tfrac{1}{4}\pi + 2$$



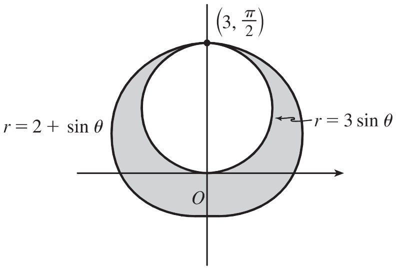
25. To find the area inside the leminiscate $r^2 = 8\cos 2\theta$ and outside the circle $r = 2$, we first note that the two curves intersect when $r^2 = 8\cos 2\theta$ and $r = 2$, that is, when $\cos 2\theta = \frac{1}{2}$. For $-\pi < \theta \le \pi$, $\cos 2\theta = \frac{1}{2} \;\Leftrightarrow\; 2\theta = \pm\pi/3$ or $\pm 5\pi/3 \;\Leftrightarrow\; \theta = \pm\pi/6$ or $\pm 5\pi/6$. The figure shows that the desired area is 4 times the area between the curves from 0 to $\pi/6$. Thus,



$$A = 4\int_0^{\pi/6}\left[\tfrac{1}{2}(8\cos 2\theta) - \tfrac{1}{2}(2)^2\right]d\theta = 8\int_0^{\pi/6}(2\cos 2\theta - 1)\,d\theta$$

$$= 8\left[\sin 2\theta - \theta\right]_0^{\pi/6} = 8\left(\sqrt{3}/2 - \pi/6\right) = 4\sqrt{3} - 4\pi/3$$

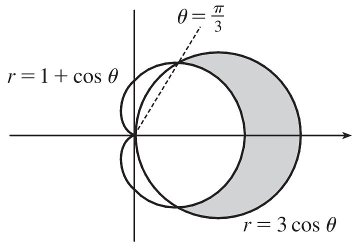
26. To find the shaded area A, we'll find the area A_1 inside the curve $r = 2 + \sin\theta$ and subtract $\pi\left(\frac{3}{2}\right)^2$ since $r = 3\sin\theta$ is a circle with radius $\frac{3}{2}$.



$$\begin{aligned} A_1 &= \int_0^{2\pi} \tfrac{1}{2}(2+\sin\theta)^2\,d\theta = \tfrac{1}{2}\int_0^{2\pi}(4+4\sin\theta+\sin^2\theta)\,d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}\left[4+4\sin\theta+\tfrac{1}{2}\cdot(1-\cos 2\theta)\right]d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi}\left(\tfrac{9}{2}+4\sin\theta-\tfrac{1}{2}\cos 2\theta\right)d\theta \\ &= \tfrac{1}{2}\left[\tfrac{9}{2}\theta-4\cos\theta-\tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{1}{2}[(9\pi-4)-(-4)] = \tfrac{9\pi}{2} \end{aligned}$$

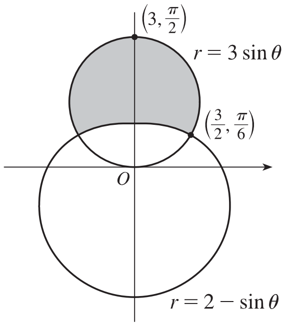
So $A = A_1 - \frac{9\pi}{4} = \frac{9\pi}{2} - \frac{9\pi}{4} = \frac{9\pi}{4}$.

27. $3\cos\theta = 1+\cos\theta \quad\Leftrightarrow\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{3}$ or $-\frac{\pi}{3}$.



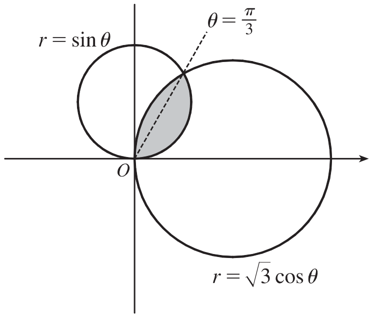
$$\begin{aligned} A &= 2\int_0^{\pi/3}\tfrac{1}{2}[(3\cos\theta)^2-(1+\cos\theta)^2]\,d\theta \\ &= \int_0^{\pi/3}(8\cos^2\theta-2\cos\theta-1)\,d\theta = \int_0^{\pi/3}[4(1+\cos 2\theta)-2\cos\theta-1]\,d\theta \\ &= \int_0^{\pi/3}(3+4\cos 2\theta-2\cos\theta)\,d\theta = \left[3\theta+2\sin 2\theta-2\sin\theta\right]_0^{\pi/3} \\ &= \pi+\sqrt{3}-\sqrt{3} = \pi \end{aligned}$$

28. $3\sin\theta = 2-\sin\theta \quad\Rightarrow\quad 4\sin\theta = 2 \quad\Rightarrow\quad \sin\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.



$$\begin{aligned} A &= 2\int_{\pi/6}^{\pi/2}\tfrac{1}{2}[(3\sin\theta)^2-(2-\sin\theta)^2]\,d\theta \\ &= \int_{\pi/6}^{\pi/2}(9\sin^2\theta-4+4\sin\theta-\sin^2\theta]\,d\theta \\ &= \int_{\pi/6}^{\pi/2}(8\sin^2\theta+4\sin\theta-4)\,d\theta \\ &= 4\int_{\pi/6}^{\pi/2}\left[2\cdot\tfrac{1}{2}(1-\cos 2\theta)+\sin\theta-1\right]d\theta \\ &= 4\int_{\pi/6}^{\pi/2}(\sin\theta-\cos 2\theta)\,d\theta = 4\left[-\cos\theta-\tfrac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2} \\ &= 4\left[(0-0)-\left(-\tfrac{\sqrt{3}}{2}-\tfrac{\sqrt{3}}{4}\right)\right] = 4\left(\tfrac{3\sqrt{3}}{4}\right) = 3\sqrt{3} \end{aligned}$$

29. $\sqrt{3}\cos\theta = \sin\theta \quad\Rightarrow\quad \sqrt{3} = \dfrac{\sin\theta}{\cos\theta} \quad\Rightarrow\quad \tan\theta = \sqrt{3} \quad\Rightarrow\quad \theta = \frac{\pi}{3}$.



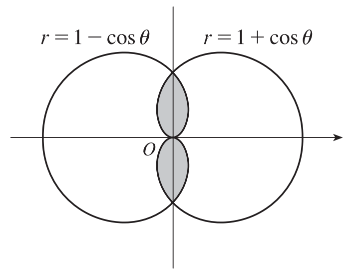
$$\begin{aligned} A &= \int_0^{\pi/3}\tfrac{1}{2}(\sin\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2}\tfrac{1}{2}\left(\sqrt{3}\cos\theta\right)^2\,d\theta \\ &= \int_0^{\pi/3}\tfrac{1}{2}\cdot\tfrac{1}{2}(1-\cos 2\theta)\,d\theta + \int_{\pi/3}^{\pi/2}\tfrac{1}{2}\cdot 3\cdot\tfrac{1}{2}(1+\cos 2\theta)\,d\theta \\ &= \tfrac{1}{4}\left[\theta-\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/3} + \tfrac{3}{4}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} \\ &= \tfrac{1}{4}\left[\left(\tfrac{\pi}{3}-\tfrac{\sqrt{3}}{4}\right)-0\right] + \tfrac{3}{4}\left[\left(\tfrac{\pi}{2}+0\right)-\left(\tfrac{\pi}{3}+\tfrac{\sqrt{3}}{4}\right)\right] \\ &= \tfrac{\pi}{12}-\tfrac{\sqrt{3}}{16}+\tfrac{\pi}{8}-\tfrac{3\sqrt{3}}{16} = \tfrac{5\pi}{24}-\tfrac{\sqrt{3}}{4} \end{aligned}$$

30. $A = 4\int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2\,d\theta = 2\int_0^{\pi/2}(1-2\cos\theta+\cos^2\theta)\,d\theta$

$= 2\int_0^{\pi/2}\left[1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$

$= 2\int_0^{\pi/2}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta = \int_0^{\pi/2}(3-4\cos\theta+\cos 2\theta)\,d\theta$

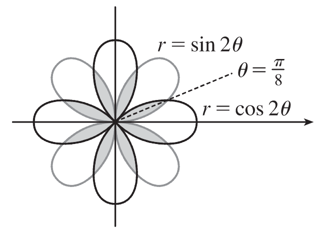
$= \left[3\theta-4\sin\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi}{2}-4$



31. $\sin 2\theta = \cos 2\theta \quad\Rightarrow\quad \dfrac{\sin 2\theta}{\cos 2\theta} = 1 \quad\Rightarrow\quad \tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} \quad\Rightarrow$
$\theta = \frac{\pi}{8} \quad\Rightarrow$

$A = 8\cdot 2\int_0^{\pi/8}\frac{1}{2}\sin^2 2\theta\,d\theta = 8\int_0^{\pi/8}\frac{1}{2}(1-\cos 4\theta)\,d\theta$

$= 4\left[\theta-\frac{1}{4}\sin 4\theta\right]_0^{\pi/8} = 4\left(\frac{\pi}{8}-\frac{1}{4}\cdot 1\right) = \frac{\pi}{2}-1$



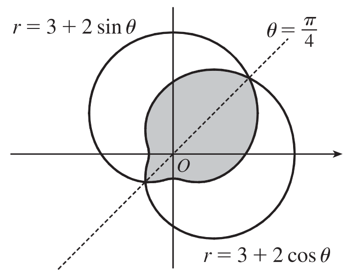
32. $3+2\cos\theta = 3+2\sin\theta \quad\Rightarrow\quad \cos\theta = \sin\theta \quad\Rightarrow\quad \theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

$A = 2\int_{\pi/4}^{5\pi/4}\frac{1}{2}(3+2\cos\theta)^2\,d\theta = \int_{\pi/4}^{5\pi/4}(9+12\cos\theta+4\cos^2\theta)\,d\theta$

$= \int_{\pi/4}^{5\pi/4}\left[9+12\cos\theta+4\cdot\frac{1}{2}(1+\cos 2\theta)\right]d\theta$

$= \int_{\pi/4}^{5\pi/4}(11+12\cos\theta+2\cos 2\theta)\,d\theta = \left[11\theta+12\sin\theta+\sin 2\theta\right]_{\pi/4}^{5\pi/4}$

$= \left(\frac{55\pi}{4}-6\sqrt{2}+1\right)-\left(\frac{11\pi}{4}+6\sqrt{2}+1\right) = 11\pi-12\sqrt{2}$

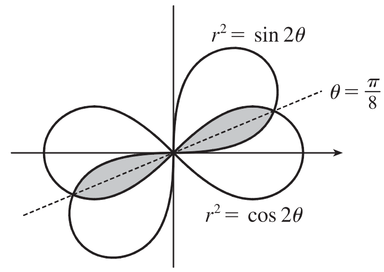


33. $\sin 2\theta = \cos 2\theta \quad\Rightarrow\quad \tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} \quad\Rightarrow\quad \theta = \frac{\pi}{8}$

$A = 4\int_0^{\pi/8}\frac{1}{2}\sin 2\theta\,d\theta \quad [\text{since } r^2 = \sin 2\theta]$

$= \int_0^{\pi/8} 2\sin 2\theta\,d\theta = \left[-\cos 2\theta\right]_0^{\pi/8}$

$= -\frac{1}{2}\sqrt{2}-(-1) = 1-\frac{1}{2}\sqrt{2}$



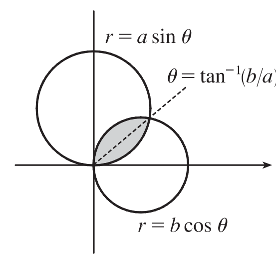
34. Let $\alpha = \tan^{-1}(b/a)$. Then

$A = \int_0^{\alpha}\frac{1}{2}(a\sin\theta)^2\,d\theta + \int_{\alpha}^{\pi/2}\frac{1}{2}(b\cos\theta)^2\,d\theta$

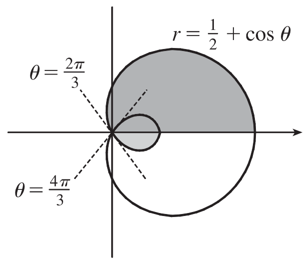
$= \frac{1}{4}a^2\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\alpha} + \frac{1}{4}b^2\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\alpha}^{\pi/2}$

$= \frac{1}{4}\alpha(a^2-b^2)+\frac{1}{8}\pi b^2-\frac{1}{4}(a^2+b^2)(\sin\alpha\,\cos\alpha)$

$= \frac{1}{4}(a^2-b^2)\,\tan^{-1}(b/a)+\frac{1}{8}\pi b^2-\frac{1}{4}ab$

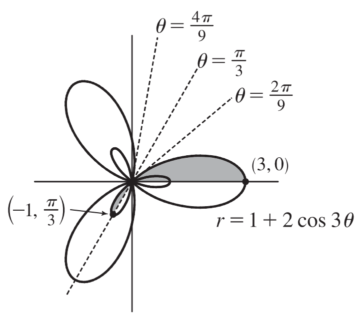


35. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/3$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/3$ to $\theta = \pi$), and then double that difference to obtain the desired area.



$$\begin{aligned}
A &= 2\left[\int_0^{2\pi/3} \tfrac{1}{2}\left(\tfrac{1}{2} + \cos\theta\right)^2\, d\theta - \int_{2\pi/3}^{\pi} \tfrac{1}{2}\left(\tfrac{1}{2} + \cos\theta\right)^2\, d\theta\right] \\
&= \int_0^{2\pi/3} \left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta - \int_{2\pi/3}^{\pi} \left(\tfrac{1}{4} + \cos\theta + \cos^2\theta\right) d\theta \\
&= \int_0^{2\pi/3} \left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\
&\qquad\qquad - \int_{2\pi/3}^{\pi} \left[\tfrac{1}{4} + \cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\
&= \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3} - \left[\frac{\theta}{4} + \sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi} \\
&= \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) - \left(\tfrac{\pi}{4} + \tfrac{\pi}{2}\right) + \left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{3} - \tfrac{\sqrt{3}}{8}\right) \\
&= \tfrac{\pi}{4} + \tfrac{3}{4}\sqrt{3} = \tfrac{1}{4}\left(\pi + 3\sqrt{3}\right)
\end{aligned}$$

36. $r = 0 \;\Rightarrow\; 1 + 2\cos 3\theta = 0 \;\Rightarrow\; \cos 3\theta = -\frac{1}{2} \;\Rightarrow\; 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ [for $0 \le 3\theta \le 2\pi$] $\Rightarrow\; \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$. The darker shaded region (from $\theta = 0$ to $\theta = 2\pi/9$) represents $\frac{1}{2}$ of the desired area plus $\frac{1}{2}$ of the area of the inner loop. From this area, we'll subtract $\frac{1}{2}$ of the area of the inner loop (the lighter shaded region from $\theta = 2\pi/9$ to $\theta = \pi/3$), and then double that difference to obtain the desired area.



$$A = 2\left[\int_0^{2\pi/9} \tfrac{1}{2}(1 + 2\cos 3\theta)^2\, d\theta - \int_{2\pi/9}^{\pi/3} \tfrac{1}{2}(1 + 2\cos 3\theta)^2\, d\theta\right]$$

Now
$$\begin{aligned}
r^2 &= (1 + 2\cos 3\theta)^2 = 1 + 4\cos 3\theta + 4\cos^2 3\theta = 1 + 4\cos 3\theta + 4\cdot\tfrac{1}{2}(1 + \cos 6\theta) \\
&= 1 + 4\cos 3\theta + 2 + 2\cos 6\theta = 3 + 4\cos 3\theta + 2\cos 6\theta
\end{aligned}$$

and $\int r^2 d\theta = 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta + C$, so

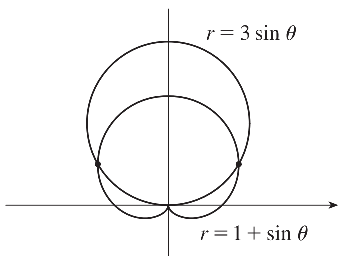
$$\begin{aligned}
A &= \left[3\theta + \tfrac{4}{3}\sin 3\theta + \tfrac{1}{3}\sin 6\theta\right]_0^{2\pi/9} - \left[3\theta + \tfrac{4}{3}\sin 3\theta + \tfrac{1}{3}\sin 6\theta\right]_{2\pi/9}^{\pi/3} \\
&= \left[\left(\tfrac{2\pi}{3} + \tfrac{4}{3}\cdot\tfrac{\sqrt{3}}{2} + \tfrac{1}{3}\cdot\tfrac{-\sqrt{3}}{2}\right) - 0\right] - \left[(\pi + 0 + 0) - \left(\tfrac{2\pi}{3} + \tfrac{4}{3}\cdot\tfrac{\sqrt{3}}{2} + \tfrac{1}{3}\cdot\tfrac{-\sqrt{3}}{2}\right)\right] \\
&= \tfrac{4\pi}{3} + \tfrac{4}{3}\sqrt{3} - \tfrac{1}{3}\sqrt{3} - \pi = \tfrac{\pi}{3} + \sqrt{3}
\end{aligned}$$

37. The pole is a point of intersection.

$1 + \sin\theta = 3\sin\theta \;\Rightarrow\; 1 = 2\sin\theta \;\Rightarrow\; \sin\theta = \frac{1}{2} \;\Rightarrow$

$\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

The other two points of intersection are $\left(\frac{3}{2}, \frac{\pi}{6}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$.

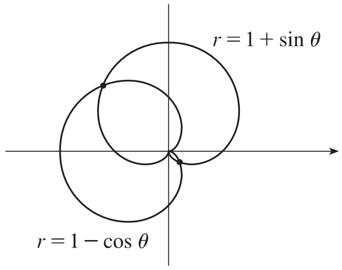


38. The pole is a point of intersection.

$1 - \cos\theta = 1 + \sin\theta \quad\Rightarrow\quad -\cos\theta = \sin\theta \quad\Rightarrow\quad -1 = \tan\theta \quad\Rightarrow$

$\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

The other two points of intersection are $\left(1 + \frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$ and $\left(1 - \frac{\sqrt{2}}{2}, \frac{7\pi}{4}\right)$.



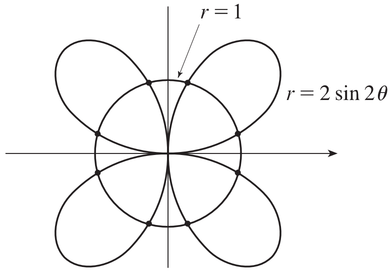
39. $2\sin 2\theta = 1 \quad\Rightarrow\quad \sin 2\theta = \frac{1}{2} \quad\Rightarrow\quad 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$.

By symmetry, the eight points of intersection are given by

$(1, \theta)$, where $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$, and $\frac{17\pi}{12}$, and

$(-1, \theta)$, where $\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$, and $\frac{23\pi}{12}$.

[There are many ways to describe these points.]

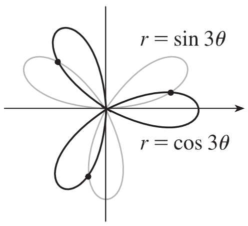


40. Clearly the pole lies on both curves. $\sin 3\theta = \cos 3\theta \quad\Rightarrow\quad \tan 3\theta = 1 \quad\Rightarrow$

$3\theta = \frac{\pi}{4} + n\pi$ [n any integer] $\quad\Rightarrow\quad \theta = \frac{\pi}{12} + \frac{\pi}{3}n \quad\Rightarrow$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$, or $\frac{3\pi}{4}$, so the three remaining intersection points are

$\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right)$, and $\left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)$.

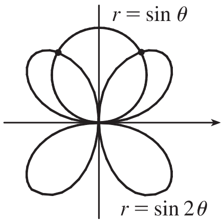


41. The pole is a point of intersection. $\sin\theta = \sin 2\theta = 2\sin\theta\,\cos\theta \quad\Leftrightarrow$

$\sin\theta\,(1 - 2\cos\theta) = 0 \quad\Leftrightarrow\quad \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \quad\Rightarrow$

$\theta = 0, \pi, \frac{\pi}{3}$, or $-\frac{\pi}{3} \quad\Rightarrow$ the other intersection points are $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$

and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$ [by symmetry].

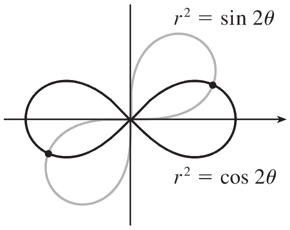


42. Clearly the pole is a point of intersection. $\sin 2\theta = \cos 2\theta \quad\Rightarrow$

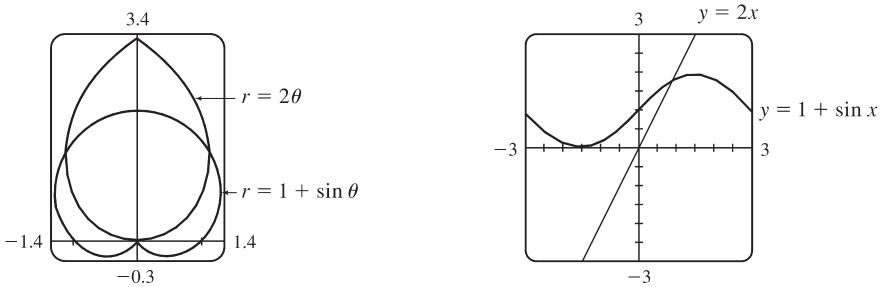
$\tan 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{4} + 2n\pi$ [since $\sin 2\theta$ and $\cos 2\theta$ must be

positive in the equations] $\quad\Rightarrow\quad \theta = \frac{\pi}{8} + n\pi \quad\Rightarrow\quad \theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$.

So the curves also intersect at $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right)$ and $\left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right)$.



43.

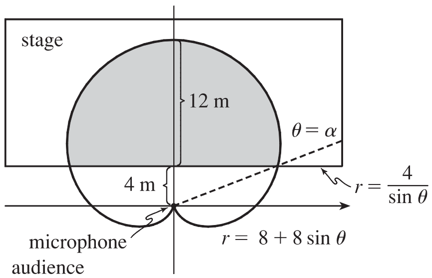


From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the θ-values of the intersection points to be $\alpha \approx 0.88786 \approx 0.89$ and $\pi - \alpha \approx 2.25$. (The first of these values may be more easily estimated by plotting $y = 1 + \sin x$ and $y = 2x$ in rectangular coordinates; see the second graph.) By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$A = 2\int_0^{\alpha} \tfrac{1}{2}(2\theta)^2\, d\theta + 2\int_{\alpha}^{\pi/2} \tfrac{1}{2}(1+\sin\theta)^2\, d\theta = \int_0^{\alpha} 4\theta^2\, d\theta + \int_{\alpha}^{\pi/2} \left[1 + 2\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)\right] d\theta$$

$$= \left[\tfrac{4}{3}\theta^3\right]_0^{\alpha} + \left[\theta - 2\cos\theta + \left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right)\right]_{\alpha}^{\pi/2} = \tfrac{4}{3}\alpha^3 + \left[\left(\tfrac{\pi}{2} + \tfrac{\pi}{4}\right) - \left(\alpha - 2\cos\alpha + \tfrac{1}{2}\alpha - \tfrac{1}{4}\sin 2\alpha\right)\right] \approx 3.4645$$

44.



We need to find the shaded area A in the figure. The horizontal line representing the front of the stage has equation $y = 4 \quad\Leftrightarrow\quad r\sin\theta = 4 \quad\Rightarrow\quad r = 4/\sin\theta$. This line intersects the curve $r = 8 + 8\sin\theta$ when $8 + 8\sin\theta = \dfrac{4}{\sin\theta} \quad\Rightarrow$
$8\sin\theta + 8\sin^2\theta = 4 \quad\Rightarrow\quad 2\sin^2\theta + 2\sin\theta - 1 = 0 \quad\Rightarrow$

$$\sin\theta = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1+\sqrt{3}}{2} \quad \text{[the other value is less than } -1\text{]} \quad\Rightarrow\quad \theta = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right).$$

This angle is about 21.5° and is denoted by α in the figure.

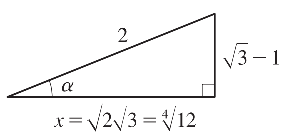
$$A = 2\int_{\alpha}^{\pi/2} \tfrac{1}{2}(8+8\sin\theta)^2\, d\theta - 2\int_{\alpha}^{\pi/2} \tfrac{1}{2}(4\csc\theta)^2\, d\theta = 64\int_{\alpha}^{\pi/2}(1 + 2\sin\theta + \sin^2\theta)\, d\theta - 16\int_{\alpha}^{\pi/2}\csc^2\theta\, d\theta$$

$$= 64\int_{\alpha}^{\pi/2}\left(1 + 2\sin\theta + \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta + 16\int_{\alpha}^{\pi/2}(-\csc^2\theta)\, d\theta = 64\left[\tfrac{3}{2}\theta - 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_{\alpha}^{\pi/2} + 16\left[\cot\theta\right]_{\alpha}^{\pi/2}$$

$$= 16\left[6\theta - 8\cos\theta - \sin 2\theta + \cot\theta\right]_{\alpha}^{\pi/a} = 16[(3\pi - 0 - 0 + 0) - (6\alpha - 8\cos\alpha - \sin 2\alpha + \cot\alpha)]$$

$$= 48\pi - 96\alpha + 128\cos\alpha + 16\sin 2\alpha - 16\cot\alpha$$

From the figure, $x^2 + \left(\sqrt{3} - 1\right)^2 = 2^2 \quad\Rightarrow\quad x^2 = 4 - \left(3 - 2\sqrt{3} + 1\right) \quad\Rightarrow$



$x^2 = 2\sqrt{3} = \sqrt{12}$, so $x = \sqrt{2\sqrt{3}} = \sqrt[4]{12}$. Using the trigonometric relationships for a right triangle and the identity $\sin 2\alpha = 2\sin\alpha\,\cos\alpha$, we continue:

$$A = 48\pi - 96\alpha + 128\cdot\frac{\sqrt[4]{12}}{2} + 16\cdot 2\cdot\frac{\sqrt{3}-1}{2}\cdot\frac{\sqrt[4]{12}}{2} - 16\cdot\frac{\sqrt[4]{12}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= 48\pi - 96\alpha + 64\sqrt[4]{12} + 8\sqrt[4]{12}\left(\sqrt{3}-1\right) - 8\sqrt[4]{12}\left(\sqrt{3}+1\right) = 48\pi + 48\sqrt[4]{12} - 96\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

$$\approx 204.16\ \text{m}^2$$

45. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{\pi} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$

$= \int_0^{\pi} \sqrt{4(\cos^2\theta + \sin^2\theta)}\, d\theta = \int_0^{\pi} \sqrt{4}\, d\theta = \big[2\theta\big]_0^{\pi} = 2\pi$

As a check, note that the curve is a circle of radius 1, so its circumference is $2\pi(1) = 2\pi$.

46. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(5^\theta)^2 + (5^\theta \ln 5)^2}\, d\theta = \int_0^{2\pi} \sqrt{5^{2\theta}[1 + (\ln 5)^2]}\, d\theta$

$= \sqrt{1 + (\ln 5)^2} \int_0^{2\pi} \sqrt{5^{2\theta}}\, d\theta = \sqrt{1 + (\ln 5)^2} \int_0^{2\pi} 5^\theta\, d\theta = \sqrt{1 + (\ln 5)^2} \left[\dfrac{5^\theta}{\ln 5}\right]_0^{2\pi}$

$= \sqrt{1 + (\ln 5)^2} \left(\dfrac{5^{2\pi}}{\ln 5} - \dfrac{1}{\ln 5}\right) = \dfrac{\sqrt{1 + (\ln 5)^2}}{\ln 5}(5^{2\pi} - 1)$

47. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$

$= \int_0^{2\pi} \sqrt{\theta^2(\theta^2 + 4)}\, d\theta = \int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta$

Now let $u = \theta^2 + 4$, so that $du = 2\theta\, d\theta$ $\left[\theta\, d\theta = \frac{1}{2}\, du\right]$ and

$$\int_0^{2\pi} \theta\sqrt{\theta^2 + 4}\, d\theta = \int_4^{4\pi^2 + 4} \tfrac{1}{2}\sqrt{u}\, du = \tfrac{1}{2} \cdot \tfrac{2}{3}\left[u^{3/2}\right]_4^{4(\pi^2+1)} = \tfrac{1}{3}[4^{3/2}(\pi^2 + 1)^{3/2} - 4^{3/2}] = \tfrac{8}{3}[(\pi^2 + 1)^{3/2} - 1]$$

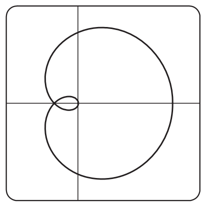
48. $L = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{[2(1 + \cos\theta)]^2 + (-2\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta$

$= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta = \sqrt{8} \int_0^{2\pi} \sqrt{1 + \cos\theta}\, d\theta = \sqrt{8} \int_0^{2\pi} \sqrt{2 \cdot \tfrac{1}{2}(1 + \cos\theta)}\, d\theta$

$= \sqrt{8} \int_0^{2\pi} \sqrt{2\cos^2 \dfrac{\theta}{2}}\, d\theta = \sqrt{8}\sqrt{2} \int_0^{2\pi} \left|\cos \dfrac{\theta}{2}\right| d\theta = 4 \cdot 2 \int_0^{\pi} \cos \dfrac{\theta}{2}\, d\theta$ [by symmetry]

$= 8\left[2\sin \dfrac{\theta}{2}\right]_0^{\pi} = 8(2) = 16$

49. The curve $r = \cos^4(\theta/4)$ is completely traced with $0 \le \theta \le 4\pi$.



$r^2 + (dr/d\theta)^2 = [\cos^4(\theta/4)]^2 + \left[4\cos^3(\theta/4) \cdot (-\sin(\theta/4)) \cdot \tfrac{1}{4}\right]^2$

$= \cos^8(\theta/4) + \cos^6(\theta/4)\sin^2(\theta/4)$

$= \cos^6(\theta/4)[\cos^2(\theta/4) + \sin^2(\theta/4)] = \cos^6(\theta/4)$

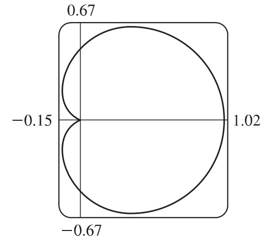
$L = \int_0^{4\pi} \sqrt{\cos^6(\theta/4)}\, d\theta = \int_0^{4\pi} \left|\cos^3(\theta/4)\right|\, d\theta$

$= 2\int_0^{2\pi} \cos^3(\theta/4)\, d\theta$ [since $\cos^3(\theta/4) \ge 0$ for $0 \le \theta \le 2\pi$] $= 8\int_0^{\pi/2} \cos^3 u\, du$ $\left[u = \tfrac{1}{4}\theta\right]$

$= 8\int_0^{\pi/2} (1 - \sin^2 u)\cos u\, du = 8\int_0^1 (1 - x^2)\, dx$ $\left[\begin{matrix} x = \sin u, \\ dx = \cos u\, du \end{matrix}\right]$

$= 8\left[x - \tfrac{1}{3}x^3\right]_0^1 = 8\left(1 - \tfrac{1}{3}\right) = \tfrac{16}{3}$

50. The curve $r = \cos^2(\theta/2)$ is completely traced with $0 \le \theta \le 2\pi$.



$$\begin{aligned} r^2 + (dr/d\theta)^2 &= [\cos^2(\theta/2)]^2 + \left[2\cos(\theta/2)\cdot(-\sin(\theta/2))\cdot\tfrac{1}{2}\right]^2 \\ &= \cos^4(\theta/2) + \cos^2(\theta/2)\sin^2(\theta/2) \\ &= \cos^2(\theta/2)[\cos^2(\theta/2) + \sin^2(\theta/2)] \\ &= \cos^2(\theta/2) \end{aligned}$$

$$L = \int_0^{2\pi} \sqrt{\cos^2(\theta/2)}\, d\theta = \int_0^{2\pi} |\cos(\theta/2)|\, d\theta = 2\int_0^{\pi} \cos(\theta/2)\, d\theta \qquad [\text{since } \cos(\theta/2) \ge 0 \text{ for } 0 \le \theta \le \pi]$$

$$= 4\int_0^{\pi/2} \cos u\, du \quad \left[u = \tfrac{1}{2}\theta\right] \quad = 4\left[\sin u\right]_0^{\pi/2} = 4(1-0) = 4$$

51. One loop of the curve $r = \cos 2\theta$ is traced with $-\pi/4 \le \theta \le \pi/4$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos^2 2\theta + (-2\sin 2\theta)^2 = \cos^2 2\theta + 4\sin^2 2\theta = 1 + 3\sin^2 2\theta \quad\Rightarrow\quad L\int_{-\pi/4}^{\pi/4} \sqrt{1+3\sin^2 2\theta}\, d\theta \approx 2.4221.$$

52. $$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \tan^2\theta + (\sec^2\theta)^2 \quad\Rightarrow\quad L\int_{\pi/6}^{\pi/3} \sqrt{\tan^2\theta + \sec^4\theta}\, d\theta \approx 1.2789$$

53. The curve $r = \sin(6\sin\theta)$ is completely traced with $0 \le \theta \le \pi$. $r = \sin(6\sin\theta) \;\Rightarrow\; \dfrac{dr}{d\theta} = \cos(6\sin\theta)\cdot 6\cos\theta$, so

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2(6\sin\theta) + 36\cos^2\theta\cos^2(6\sin\theta) \quad\Rightarrow\quad L\int_0^{\pi} \sqrt{\sin^2(6\sin\theta) + 36\cos^2\theta\, \cos^2(6\sin\theta)}\, d\theta \approx 8.0091.$$

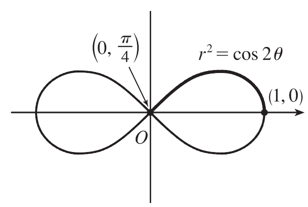
54. The curve $r = \sin(\theta/4)$ is completely traced with $0 \le \theta \le 8\pi$. $r = \sin(\theta/4) \;\Rightarrow\; \dfrac{dr}{d\theta} = \tfrac{1}{4}\cos(\theta/4)$, so

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2(\theta/4) + \tfrac{1}{16}\cos^2(\theta/4) \quad\Rightarrow\quad L\int_0^{8\pi} \sqrt{\sin^2(\theta/4) + \tfrac{1}{16}\cos^2(\theta/4)}\, d\theta \approx 17.1568.$$

55. (a) From (10.2.6),

$$\begin{aligned} S &= \int_a^b 2\pi y\sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta \\ &= \int_a^b 2\pi y\sqrt{r^2 + (dr/d\theta)^2}\, d\theta \qquad [\text{from the derivation of Equation 10.4.5}] \\ &= \int_a^b 2\pi r\sin\theta\sqrt{r^2 + (dr/d\theta)^2}\, d\theta \end{aligned}$$

(b) The curve $r^2 = \cos 2\theta$ goes through the pole when $\cos 2\theta = 0 \;\Rightarrow\; 2\theta = \frac{\pi}{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$. We'll rotate the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ and double this value to obtain the total surface area generated.



$$r^2 = \cos 2\theta \quad\Rightarrow\quad 2r\,\frac{dr}{d\theta} = -2\sin 2\theta \quad\Rightarrow\quad \left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{r^2} = \frac{\sin^2 2\theta}{\cos 2\theta}.$$

$$S = 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta + \left(\sin^2 2\theta\right)/\cos 2\theta}\, d\theta = 4\pi\int_0^{\pi/4} \sqrt{\cos 2\theta}\sin\theta\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

$$= 4\pi\int_0^{\pi/4} \sqrt{\cos 2\theta}\sin\theta\,\frac{1}{\sqrt{\cos 2\theta}}\, d\theta = 4\pi\int_0^{\pi/4} \sin\theta\, d\theta = 4\pi\left[-\cos\theta\right]_0^{\pi/4} = -4\pi\left(\tfrac{\sqrt{2}}{2} - 1\right) = 2\pi\left(2 - \sqrt{2}\right)$$

56. (a) Rotation around $\theta = \frac{\pi}{2}$ is the same as rotation around the y-axis, that is, $S = \int_a^b 2\pi x\, ds$ where $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ for a parametric equation, and for the special case of a polar equation, $x = r\cos\theta$ and $ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$ [see the derivation of Equation 10.4.5]. Therefore, for a polar equation rotated around $\theta = \frac{\pi}{2}$, $S = \int_a^b 2\pi r\cos\theta \sqrt{r^2 + (dr/d\theta)^2}\, d\theta$.

(b) As in the solution for Exercise 55(b), we can double the surface area generated by rotating the curve from $\theta = 0$ to $\theta = \frac{\pi}{4}$ to obtain the total surface area.

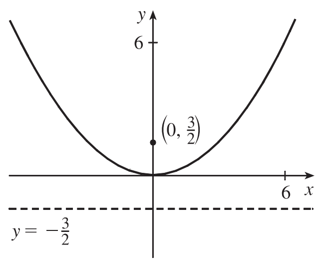
$$S = 2\int_0^{\pi/4} 2\pi\sqrt{\cos 2\theta}\cos\theta\sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta}\, d\theta = 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\cos\theta\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\, d\theta$$

$$= 4\pi\int_0^{\pi/4}\sqrt{\cos 2\theta}\cos\theta\,\frac{1}{\sqrt{\cos 2\theta}}\, d\theta = 4\pi\int_0^{\pi/4}\cos\theta\, d\theta = 4\pi\big[\sin\theta\big]_0^{\pi/4} = 4\pi\left(\frac{\sqrt{2}}{2} - 0\right) = 2\sqrt{2}\,\pi$$

10.5 Conic Sections

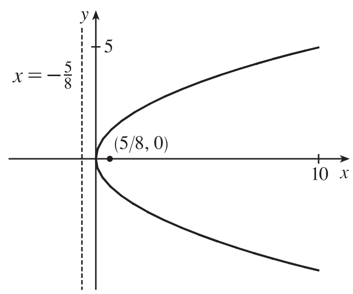
1. $x^2 = 6y$ and $x^2 = 4py \;\Rightarrow\; 4p = 6 \;\Rightarrow\; p = \frac{3}{2}$.

The vertex is $(0, 0)$, the focus is $\left(0, \frac{3}{2}\right)$, and the directrix is $y = -\frac{3}{2}$.



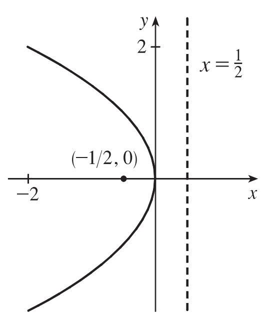
2. $2y^2 = 5x \;\Rightarrow\; y^2 = \frac{5}{2}x$. $4p = \frac{5}{2} \;\Rightarrow\; p = \frac{5}{8}$.

The vertex is $(0, 0)$, the focus is $\left(\frac{5}{8}, 0\right)$, and the directrix is $x = -\frac{5}{8}$.



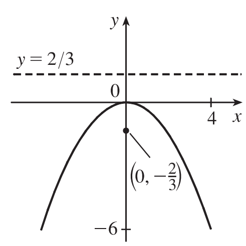
3. $2x = -y^2 \;\Rightarrow\; y^2 = -2x$. $4p = -2 \;\Rightarrow\; p = -\frac{1}{2}$.

The vertex is $(0, 0)$, the focus is $\left(-\frac{1}{2}, 0\right)$, and the directrix is $x = \frac{1}{2}$.

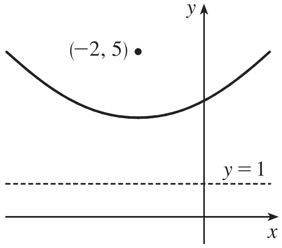


4. $3x^2 + 8y = 0 \;\Rightarrow\; 3x^2 = -8y \;\Rightarrow\; x^2 = -\frac{8}{3}y$.

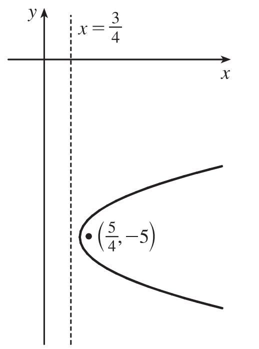
$4p = -\frac{8}{3} \;\Rightarrow\; p = -\frac{2}{3}$. The vertex is $(0, 0)$, the focus is $\left(0, -\frac{2}{3}\right)$, and the directrix is $y = \frac{2}{3}$.



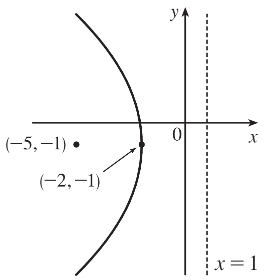
5. $(x+2)^2 = 8(y-3)$. $4p = 8$, so $p = 2$. The vertex is $(-2,3)$, the focus is $(-2,5)$, and the directrix is $y = 1$.



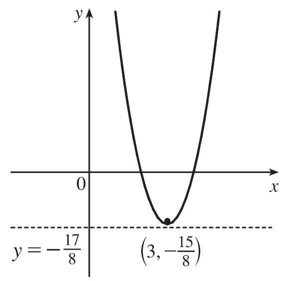
6. $x - 1 = (y+5)^2$. $4p = 1$, so $p = \frac{1}{4}$. The vertex is $(1,-5)$, the focus is $\left(\frac{5}{4},-5\right)$, and the directrix is $x = \frac{3}{4}$.



7. $y^2 + 2y + 12x + 25 = 0 \quad \Rightarrow$
$y^2 + 2y + 1 = -12x - 24 \quad \Rightarrow$
$(y+1)^2 = -12(x+2)$. $4p = -12$, so $p = -3$.
The vertex is $(-2,-1)$, the focus is $(-5,-1)$, and the directrix is $x = 1$.



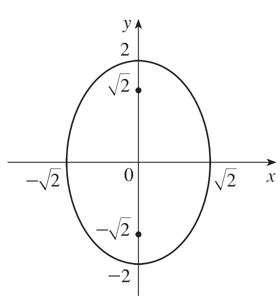
8. $y + 12x - 2x^2 = 16 \quad \Rightarrow \quad 2x^2 - 12x = y - 16 \quad \Rightarrow$
$2(x^2 - 6x + 9) = y - 16 + 18 \quad \Rightarrow$
$2(x-3)^2 = y + 2 \quad \Rightarrow \quad (x-3)^2 = \frac{1}{2}(y+2)$.
$4p = \frac{1}{2}$, so $p = \frac{1}{8}$. The vertex is $(3,-2)$, the focus is $\left(3,-\frac{15}{8}\right)$, and the directrix is $y = -\frac{17}{8}$.



9. The equation has the form $y^2 = 4px$, where $p < 0$. Since the parabola passes through $(-1,1)$, we have $1^2 = 4p(-1)$, so $4p = -1$ and an equation is $y^2 = -x$ or $x = -y^2$. $4p = -1$, so $p = -\frac{1}{4}$ and the focus is $\left(-\frac{1}{4},0\right)$ while the directrix is $x = \frac{1}{4}$.

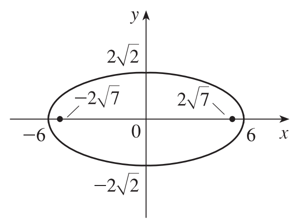
10. The vertex is $(2,-2)$, so the equation is of the form $(x-2)^2 = 4p(y+2)$, where $p > 0$. The point $(0,0)$ is on the parabola, so $4 = 4p(2)$ and $4p = 2$. Thus, an equation is $(x-2)^2 = 2(y+2)$. $4p = 2$, so $p = \frac{1}{2}$ and the focus is $\left(2,-\frac{3}{2}\right)$ while the directrix is $y = -\frac{5}{2}$.

11. $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1 \quad \Rightarrow \quad a = \sqrt{4} = 2$, $b = \sqrt{2}$, $c = \sqrt{a^2 - b^2} = \sqrt{4-2} = \sqrt{2}$. The ellipse is centered at $(0,0)$, with vertices at $(0,\pm 2)$. The foci are $\left(0,\pm\sqrt{2}\right)$.



12. $\dfrac{x^2}{36} + \dfrac{y^2}{8} = 1 \quad \Rightarrow \quad a = \sqrt{36} = 6, b = \sqrt{8},$

$c = \sqrt{a^2 - b^2} = \sqrt{36 - 8} = \sqrt{28} = 2\sqrt{7}$. The ellipse is centered at $(0, 0)$, with vertices at $(\pm 6, 0)$. The foci are $(\pm 2\sqrt{7}, 0)$.

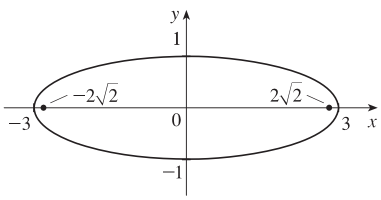


13. $x^2 + 9y^2 = 9 \quad \Leftrightarrow \quad \dfrac{x^2}{9} + \dfrac{y^2}{1} = 1 \quad \Rightarrow \quad a = \sqrt{9} = 3,$

$b = \sqrt{1} = 1$, $c = \sqrt{a^2 - b^2} = \sqrt{9 - 1} = \sqrt{8} = 2\sqrt{2}$.

The ellipse is centered at $(0, 0)$, with vertices $(\pm 3, 0)$.

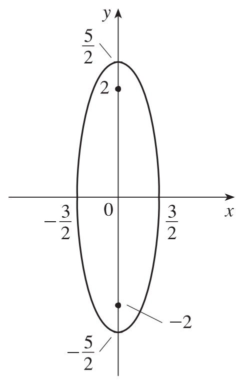
The foci are $(\pm 2\sqrt{2}, 0)$.



14. $100x^2 + 36y^2 = 225 \quad \Leftrightarrow \quad \dfrac{x^2}{\frac{225}{100}} + \dfrac{y^2}{\frac{225}{36}} = 1 \quad \Leftrightarrow$

$\dfrac{x^2}{\frac{9}{4}} + \dfrac{y^2}{\frac{25}{4}} = 1 \quad \Rightarrow \quad a = \sqrt{\frac{25}{4}} = \frac{5}{2}, b = \sqrt{\frac{9}{4}} = \frac{3}{2},$

$c = \sqrt{a^2 - b^2} = \sqrt{\frac{25}{4} - \frac{9}{4}} = 2$. The ellipse is centered at $(0, 0)$, with vertices $\left(0, \pm\frac{5}{2}\right)$. The foci are $(0, \pm 2)$.



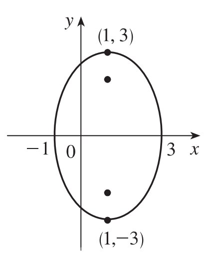
15. $9x^2 - 18x + 4y^2 = 27 \quad \Leftrightarrow$

$9(x^2 - 2x + 1) + 4y^2 = 27 + 9 \quad \Leftrightarrow$

$9(x - 1)^2 + 4y^2 = 36 \quad \Leftrightarrow \quad \dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1 \quad \Rightarrow$

$a = 3$, $b = 2$, $c = \sqrt{5} \quad \Rightarrow \quad$ center $(1, 0)$,

vertices $(1, \pm 3)$, foci $\left(1, \pm\sqrt{5}\right)$



16. $x^2 + 3y^2 + 2x - 12y + 10 = 0 \quad \Leftrightarrow$

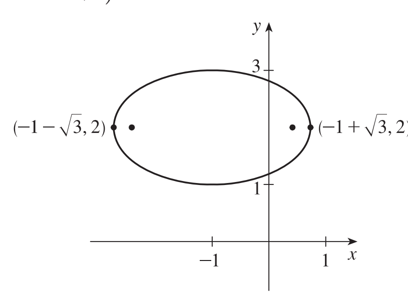
$x^2 + 2x + 1 + 3(y^2 - 4y + 4) = -10 + 1 + 12 \quad \Leftrightarrow$

$(x + 1)^2 + 3(y - 2)^2 = 3 \quad \Leftrightarrow$

$\dfrac{(x+1)^2}{3} + \dfrac{(y-2)^2}{1} = 1 \quad \Rightarrow \quad a = \sqrt{3}, b = 1,$

$c = \sqrt{2} \quad \Rightarrow \quad$ center $(-1, 2)$, vertices $\left(-1 \pm \sqrt{3}, 2\right)$,

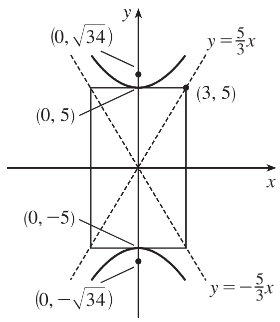
foci $\left(-1 \pm \sqrt{2}, 2\right)$



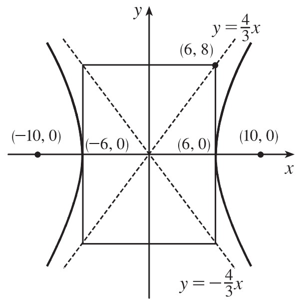
17. The center is $(0, 0)$, $a = 3$, and $b = 2$, so an equation is $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(0, \pm\sqrt{5}\right)$.

18. The ellipse is centered at $(2,1)$, with $a=3$ and $b=2$. An equation is $\dfrac{(x-2)^2}{9}+\dfrac{(y-1)^2}{4}=1$. $c=\sqrt{a^2-b^2}=\sqrt{5}$, so the foci are $\left(2\pm\sqrt{5},1\right)$.

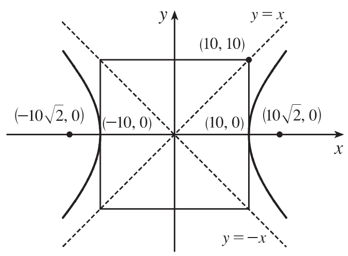
19. $\dfrac{y^2}{25}-\dfrac{x^2}{9}=1 \quad\Rightarrow\quad a=5,\, b=3,\, c=\sqrt{25+9}=\sqrt{34} \quad\Rightarrow$
center $(0,0)$, vertices $(0,\pm 5)$, foci $\left(0,\pm\sqrt{34}\right)$, asymptotes $y=\pm\frac{5}{3}x$.
Note: It is helpful to draw a $2a$-by-$2b$ rectangle whose center is the center of the hyperbola. The asymptotes are the extended diagonals of the rectangle.



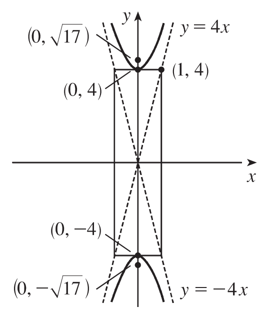
20. $\dfrac{x^2}{36}-\dfrac{y^2}{64}=1 \quad\Rightarrow\quad a=6,\, b=8,\, c=\sqrt{36+64}=10 \quad\Rightarrow$
center $(0,0)$, vertices $(\pm 6,0)$, foci $(\pm 10,0)$, asymptotes $y=\pm\frac{8}{6}x=\pm\frac{4}{3}x$



21. $x^2-y^2=100 \quad\Leftrightarrow\quad \dfrac{x^2}{100}-\dfrac{y^2}{100}=1 \quad\Rightarrow\quad a=b=10,$
$c=\sqrt{100+100}=10\sqrt{2} \quad\Rightarrow\quad$ center $(0,0)$, vertices $(\pm 10,0)$,
foci $\left(\pm 10\sqrt{2},0\right)$, asymptotes $y=\pm\frac{10}{10}x=\pm x$



22. $y^2-16x^2=16 \quad\Leftrightarrow\quad \dfrac{y^2}{16}-\dfrac{x^2}{1}=1 \quad\Rightarrow\quad a=4,\, b=1,$
$c=\sqrt{16+1}=\sqrt{17} \quad\Rightarrow\quad$ center $(0,0)$, vertices $(0,\pm 4)$,
foci $\left(0,\pm\sqrt{17}\right)$, asymptotes $y=\pm\frac{4}{1}x=\pm 4x$



23. $4x^2 - y^2 - 24x - 4y + 28 = 0 \quad \Leftrightarrow$

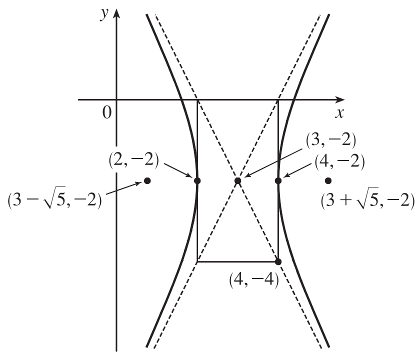
$4(x^2 - 6x + 9) - (y^2 + 4y + 4) = -28 + 36 - 4 \quad \Leftrightarrow$

$4(x-3)^2 - (y+2)^2 = 4 \quad \Leftrightarrow \quad \dfrac{(x-3)^2}{1} - \dfrac{(y+2)^2}{4} = 1 \quad \Rightarrow$

$a = \sqrt{1} = 1$, $b = \sqrt{4} = 2$, $c = \sqrt{1+4} = \sqrt{5} \quad \Rightarrow$

center $(3, -2)$, vertices $(4, -2)$ and $(2, -2)$, foci $\left(3 \pm \sqrt{5}, -2\right)$,

asymptotes $y + 2 = \pm 2(x - 3)$.



24. $y^2 - 4x^2 - 2y + 16x = 31 \quad \Leftrightarrow$

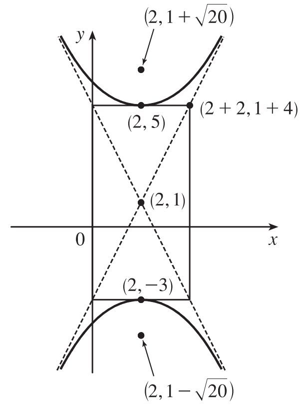
$(y^2 - 2y + 1) - 4(x^2 - 4x + 4) = 31 + 1 - 16 \quad \Leftrightarrow$

$(y-1)^2 - 4(x-2)^2 = 16 \quad \Leftrightarrow$

$\dfrac{(y-1)^2}{16} - \dfrac{(x-2)^2}{4} = 1 \quad \Rightarrow \quad a = \sqrt{16} = 4$, $b = \sqrt{4} = 2$,

$c = \sqrt{16+4} = \sqrt{20} \quad \Rightarrow \quad$ center $(2, 1)$, vertices $(2, 1 \pm 4)$,

foci $\left(2, 1 \pm \sqrt{20}\right)$, asymptotes $y - 1 = \pm 2(x - 2)$.



25. $x^2 = y + 1 \quad \Leftrightarrow \quad x^2 = 1(y+1)$. This is an equation of a *parabola* with $4p = 1$, so $p = \frac{1}{4}$. The vertex is $(0, -1)$ and the focus is $\left(0, -\frac{3}{4}\right)$.

26. $x^2 = y^2 + 1 \quad \Leftrightarrow \quad x^2 - y^2 = 1$. This is an equation of a *hyperbola* with vertices $(\pm 1, 0)$. The foci are at $\left(\pm\sqrt{1+1}, 0\right) = \left(\pm\sqrt{2}, 0\right)$.

27. $x^2 = 4y - 2y^2 \quad \Leftrightarrow \quad x^2 + 2y^2 - 4y = 0 \quad \Leftrightarrow \quad x^2 + 2(y^2 - 2y + 1) = 2 \quad \Leftrightarrow \quad x^2 + 2(y-1)^2 = 2 \quad \Leftrightarrow$ $\dfrac{x^2}{2} + \dfrac{(y-1)^2}{1} = 1$. This is an equation of an *ellipse* with vertices at $\left(\pm\sqrt{2}, 1\right)$. The foci are at $\left(\pm\sqrt{2-1}, 1\right) = (\pm 1, 1)$.

28. $y^2 - 8y = 6x - 16 \quad \Leftrightarrow \quad y^2 - 8y + 16 = 6x \quad \Leftrightarrow \quad (y-4)^2 = 6x$. This is an equation of a *parabola* with $4p = 6$, so $p = \frac{3}{2}$. The vertex is $(0, 4)$ and the focus is $\left(\frac{3}{2}, 4\right)$.

29. $y^2 + 2y = 4x^2 + 3 \quad \Leftrightarrow \quad y^2 + 2y + 1 = 4x^2 + 4 \quad \Leftrightarrow \quad (y+1)^2 - 4x^2 = 4 \quad \Leftrightarrow \quad \dfrac{(y+1)^2}{4} - x^2 = 1$. This is an equation of a *hyperbola* with vertices $(0, -1 \pm 2) = (0, 1)$ and $(0, -3)$. The foci are at $\left(0, -1 \pm \sqrt{4+1}\right) = \left(0, -1 \pm \sqrt{5}\right)$.

30. $4x^2 + 4x + y^2 = 0 \quad \Leftrightarrow \quad 4\left(x^2 + x + \frac{1}{4}\right) + y^2 = 1 \quad \Leftrightarrow \quad 4\left(x + \frac{1}{2}\right)^2 + y^2 = 1 \quad \Leftrightarrow \quad \dfrac{\left(x + \frac{1}{2}\right)^2}{1/4} + y^2 = 1$. This is an equation of an *ellipse* with vertices $\left(-\frac{1}{2}, 0 \pm 1\right) = \left(-\frac{1}{2}, \pm 1\right)$. The foci are at $\left(-\frac{1}{2}, 0 \pm \sqrt{1 - \frac{1}{4}}\right) = \left(-\frac{1}{2}, \pm\sqrt{3}/2\right)$.

31. The parabola with vertex $(0, 0)$ and focus $(1, 0)$ opens to the right and has $p = 1$, so its equation is $y^2 = 4px$, or $y^2 = 4x$.

32. The parabola with focus $(0, 0)$ and directrix $y = 6$ has vertex $(0, 3)$ and opens downward, so $p = -3$ and its equation is $(x - 0)^2 = 4p(y - 3)$, or $x^2 = -12(y - 3)$.

33. The distance from the focus $(-4, 0)$ to the directrix $x = 2$ is $2 - (-4) = 6$, so the distance from the focus to the vertex is $\frac{1}{2}(6) = 3$ and the vertex is $(-1, 0)$. Since the focus is to the left of the vertex, $p = -3$. An equation is $y^2 = 4p(x + 1) \quad \Rightarrow$ $y^2 = -12(x + 1)$.

34. The distance from the focus $(3, 6)$ to the vertex $(3, 2)$ is $6 - 2 = 4$. Since the focus is above the vertex, $p = 4$. An equation is $(x - 3)^2 = 4p(y - 2) \quad \Rightarrow \quad (x - 3)^2 = 16(y - 2)$.

35. A parabola with vertical axis and vertex $(2, 3)$ has equation $y - 3 = a(x - 2)^2$. Since it passes through $(1, 5)$, we have $5 - 3 = a(1 - 2)^2 \quad \Rightarrow \quad a = 2$, so an equation is $y - 3 = 2(x - 2)^2$.

36. A parabola with horizontal axis has equation $x = ay^2 + by + c$. Since the parabola passes through the point $(-1, 0)$, substitute -1 for x and 0 for y: $-1 = 0 + 0 + c$. Now with $c = -1$, substitute 1 for x and -1 for y: $1 = a - b - 1$ **(1)**; and then 3 for x and 1 for y: $3 = a + b - 1$ **(2)**. Add **(1)** and **(2)** to get $4 = 2a - 2 \quad \Rightarrow \quad a = 3$ and then $b = 1$. Thus, the equation is $x = 3y^2 + y - 1$.

37. The ellipse with foci $(\pm 2, 0)$ and vertices $(\pm 5, 0)$ has center $(0, 0)$ and a horizontal major axis, with $a = 5$ and $c = 2$, so $b^2 = a^2 - c^2 = 25 - 4 = 21$. An equation is $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$.

38. The ellipse with foci $(0, \pm 5)$ and vertices $(0, \pm 13)$ has center $(0, 0)$ and a vertical major axis, with $c = 5$ and $a = 13$, so $b = \sqrt{a^2 - c^2} = 12$. An equation is $\dfrac{x^2}{144} + \dfrac{y^2}{169} = 1$.

39. Since the vertices are $(0, 0)$ and $(0, 8)$, the ellipse has center $(0, 4)$ with a vertical axis and $a = 4$. The foci at $(0, 2)$ and $(0, 6)$ are 2 units from the center, so $c = 2$ and $b = \sqrt{a^2 - c^2} = \sqrt{4^2 - 2^2} = \sqrt{12}$. An equation is $\dfrac{(x - 0)^2}{b^2} + \dfrac{(y - 4)^2}{a^2} = 1 \quad \Rightarrow$ $\dfrac{x^2}{12} + \dfrac{(y - 4)^2}{16} = 1$.

40. Since the foci are $(0, -1)$ and $(8, -1)$, the ellipse has center $(4, -1)$ with a horizontal axis and $c = 4$. The vertex $(9, -1)$ is 5 units from the center, so $a = 5$ and $b = \sqrt{a^2 - c^2} = \sqrt{5^2 - 4^2} = \sqrt{9}$. An equation is $\dfrac{(x - 4)^2}{a^2} + \dfrac{(y + 1)^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{(x - 4)^2}{25} + \dfrac{(y + 1)^2}{9} = 1$.

41. An equation of an ellipse with center $(-1, 4)$ and vertex $(-1, 0)$ is $\dfrac{(x + 1)^2}{b^2} + \dfrac{(y - 4)^2}{4^2} = 1$. The focus $(-1, 6)$ is 2 units from the center, so $c = 2$. Thus, $b^2 + 2^2 = 4^2 \quad \Rightarrow \quad b^2 = 12$, and the equation is $\dfrac{(x + 1)^2}{12} + \dfrac{(y - 4)^2}{16} = 1$.

42. Foci $F_1(-4,0)$ and $F_2(4,0)$ $\Rightarrow$ $c=4$ and an equation is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The ellipse passes through $P(-4,1.8)$, so

$2a=|PF_1|+|PF_2|$ $\Rightarrow$ $2a=1.8+\sqrt{8^2+(1.8)^2}$ $\Rightarrow$ $2a=1.8+8.2$ $\Rightarrow$ $a=5$.

$b^2=a^2-c^2=25-16=9$ and the equation is $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$.

43. An equation of a hyperbola with vertices $(\pm 3,0)$ is $\dfrac{x^2}{3^2}-\dfrac{y^2}{b^2}=1$. Foci $(\pm 5,0)$ $\Rightarrow$ $c=5$ and $3^2+b^2=5^2$ $\Rightarrow$

$b^2=25-9=16$, so the equation is $\dfrac{x^2}{9}-\dfrac{y^2}{16}=1$.

44. An equation of a hyperbola with vertices $(0,\pm 2)$ is $\dfrac{y^2}{2^2}-\dfrac{x^2}{b^2}=1$. Foci $(0,\pm 5)$ $\Rightarrow$ $c=5$ and $2^2+b^2=5^2$ $\Rightarrow$

$b^2=25-4=21$, so the equation is $\dfrac{y^2}{4}-\dfrac{x^2}{21}=1$.

45. The center of a hyperbola with vertices $(-3,-4)$ and $(-3,6)$ is $(-3,1)$, so $a=5$ and an equation is

$\dfrac{(y-1)^2}{5^2}-\dfrac{(x+3)^2}{b^2}=1$. Foci $(-3,-7)$ and $(-3,9)$ $\Rightarrow$ $c=8$, so $5^2+b^2=8^2$ $\Rightarrow$ $b^2=64-25=39$ and the

equation is $\dfrac{(y-1)^2}{25}-\dfrac{(x+3)^2}{39}=1$.

46. The center of a hyperbola with vertices $(-1,2)$ and $(7,2)$ is $(3,2)$, so $a=4$ and an equation is $\dfrac{(x-3)^2}{4^2}-\dfrac{(y-2)^2}{b^2}=1$.

Foci $(-2,2)$ and $(8,2)$ $\Rightarrow$ $c=5$, so $4^2+b^2=5^2$ $\Rightarrow$ $b^2=25-16=9$ and the equation is

$\dfrac{(x-3)^2}{16}-\dfrac{(y-2)^2}{9}=1$.

47. The center of a hyperbola with vertices $(\pm 3,0)$ is $(0,0)$, so $a=3$ and an equation is $\dfrac{x^2}{3^2}-\dfrac{y^2}{b^2}=1$.

Asymptotes $y=\pm 2x$ $\Rightarrow$ $\dfrac{b}{a}=2$ $\Rightarrow$ $b=2(3)=6$ and the equation is $\dfrac{x^2}{9}-\dfrac{y^2}{36}=1$.

48. The center of a hyperbola with foci $(2,0)$ and $(2,8)$ is $(2,4)$, so $c=4$ and an equation is $\dfrac{(y-4)^2}{a^2}-\dfrac{(x-2)^2}{b^2}=1$.

The asymptote $y=3+\frac{1}{2}x$ has slope $\frac{1}{2}$, so $\dfrac{a}{b}=\dfrac{1}{2}$ $\Rightarrow$ $b=2a$ and $a^2+b^2=c^2$ $\Rightarrow$ $a^2+(2a)^2=4^2$ $\Rightarrow$

$5a^2=16$ $\Rightarrow$ $a^2=\frac{16}{5}$ and so $b^2=16-\frac{16}{5}=\frac{64}{5}$. Thus, an equation is $\dfrac{(y-4)^2}{16/5}-\dfrac{(x-2)^2}{64/5}=1$.

49. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance $a-c$ from it) while the farthest point is the other vertex (at a distance of $a+c$). So for this lunar orbit, $(a-c)+(a+c)=2a=(1728+110)+(1728+314)$, or $a=1940$; and $(a+c)-(a-c)=2c=314-110$, or $c=102$. Thus, $b^2=a^2-c^2=3{,}753{,}196$, and the equation is $\dfrac{x^2}{3{,}763{,}600}+\dfrac{y^2}{3{,}753{,}196}=1$.

50. (a) Choose V to be the origin, with x-axis through V and F. Then F is $(p, 0)$, A is $(p, 5)$, so substituting A into the equation $y^2 = 4px$ gives $25 = 4p^2$ so $p = \frac{5}{2}$ and $y^2 = 10x$.

(b) $x = 11 \quad \Rightarrow \quad y = \sqrt{110} \quad \Rightarrow \quad |CD| = 2\sqrt{110}$

51. (a) Set up the coordinate system so that A is $(-200, 0)$ and B is $(200, 0)$.

$|PA| - |PB| = (1200)(980) = 1{,}176{,}000 \text{ ft} = \frac{2450}{11} \text{ mi} = 2a \quad \Rightarrow \quad a = \frac{1225}{11}$, and $c = 200$ so

$b^2 = c^2 - a^2 = \dfrac{3{,}339{,}375}{121} \quad \Rightarrow \quad \dfrac{121x^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1.$

(b) Due north of $B \quad \Rightarrow \quad x = 200 \quad \Rightarrow \quad \dfrac{(121)(200)^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1 \quad \Rightarrow \quad y = \dfrac{133{,}575}{539} \approx 248$ mi

52. $|PF_1| - |PF_2| = \pm 2a \quad \Leftrightarrow \quad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a \quad \Leftrightarrow$

$\sqrt{(x+c)^2 + y^2} = \sqrt{(x-c)^2 + y^2} \pm 2a \quad \Leftrightarrow \quad (x+c)^2 + y^2 = (x-c)^2 + y^2 + 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2} \quad \Leftrightarrow$

$4cx - 4a^2 = \pm 4a\sqrt{(x-c)^2 + y^2} \quad \Leftrightarrow \quad c^2x^2 - 2a^2cx + a^4 = a^2(x^2 - 2cx + c^2 + y^2) \quad \Leftrightarrow$

$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \quad \Leftrightarrow \quad b^2x^2 - a^2y^2 = a^2b^2$ [where $b^2 = c^2 - a^2$] $\quad \Leftrightarrow \quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

53. The function whose graph is the upper branch of this hyperbola is concave upward. The function is

$y = f(x) = a\sqrt{1 + \dfrac{x^2}{b^2}} = \dfrac{a}{b}\sqrt{b^2 + x^2}$, so $y' = \dfrac{a}{b}x(b^2 + x^2)^{-1/2}$ and

$y'' = \dfrac{a}{b}\left[(b^2 + x^2)^{-1/2} - x^2(b^2 + x^2)^{-3/2}\right] = ab(b^2 + x^2)^{-3/2} > 0$ for all x, and so f is concave upward.

54. We can follow exactly the same sequence of steps as in the derivation of Formula 4, except we use the points $(1, 1)$ and $(-1, -1)$ in the distance formula (first equation of that derivation) so $\sqrt{(x-1)^2 + (y-1)^2} + \sqrt{(x+1)^2 + (y+1)^2} = 4$ will lead (after moving the second term to the right, squaring, and simplifying) to $2\sqrt{(x+1)^2 + (y+1)^2} = x + y + 4$, which, after squaring and simplifying again, leads to $3x^2 - 2xy + 3y^2 = 8$.

55. (a) If $k > 16$, then $k - 16 > 0$, and $\dfrac{x^2}{k} + \dfrac{y^2}{k-16} = 1$ is an *ellipse* since it is the sum of two squares on the left side.

(b) If $0 < k < 16$, then $k - 16 < 0$, and $\dfrac{x^2}{k} + \dfrac{y^2}{k-16} = 1$ is a *hyperbola* since it is the difference of two squares on the left side.

(c) If $k < 0$, then $k - 16 < 0$, and there is *no curve* since the left side is the sum of two negative terms, which cannot equal 1.

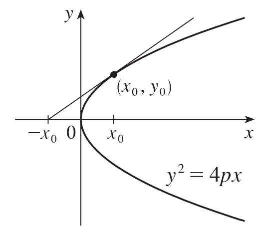
(d) In case (a), $a^2 = k$, $b^2 = k - 16$, and $c^2 = a^2 - b^2 = 16$, so the foci are at $(\pm 4, 0)$. In case (b), $k - 16 < 0$, so $a^2 = k$, $b^2 = 16 - k$, and $c^2 = a^2 + b^2 = 16$, and so again the foci are at $(\pm 4, 0)$.

56. (a) $y^2 = 4px \quad \Rightarrow \quad 2yy' = 4p \quad \Rightarrow \quad y' = \dfrac{2p}{y}$, so the tangent line is

$$y - y_0 = \frac{2p}{y_0}(x - x_0) \quad \Rightarrow \quad yy_0 - y_0^2 = 2p(x - x_0) \quad \Leftrightarrow$$

$$yy_0 - 4px_0 = 2px - 2px_0 \quad \Rightarrow \quad yy_0 = 2p(x + x_0).$$

(b) The x-intercept is $-x_0$.

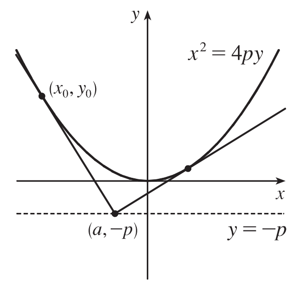


57. $x^2 = 4py \quad \Rightarrow \quad 2x = 4py' \quad \Rightarrow \quad y' = \dfrac{x}{2p}$, so the tangent line at (x_0, y_0) is

$y - \dfrac{x_0^2}{4p} = \dfrac{x_0}{2p}(x - x_0)$. This line passes through the point $(a, -p)$ on the

directrix, so $-p - \dfrac{x_0^2}{4p} = \dfrac{x_0}{2p}(a - x_0) \quad \Rightarrow \quad -4p^2 - x_0^2 = 2ax_0 - 2x_0^2 \quad \Leftrightarrow$

$x_0^2 - 2ax_0 - 4p^2 = 0 \quad \Leftrightarrow \quad x_0^2 - 2ax_0 + a^2 = a^2 + 4p^2 \quad \Leftrightarrow$



$(x_0 - a)^2 = a^2 + 4p^2 \quad \Leftrightarrow \quad x_0 = a \pm \sqrt{a^2 + 4p^2}$. The slopes of the tangent lines at $x = a \pm \sqrt{a^2 + 4p^2}$

are $\dfrac{a \pm \sqrt{a^2 + 4p^2}}{2p}$, so the product of the two slopes is

$$\frac{a + \sqrt{a^2 + 4p^2}}{2p} \cdot \frac{a - \sqrt{a^2 + 4p^2}}{2p} = \frac{a^2 - (a^2 + 4p^2)}{4p^2} = \frac{-4p^2}{4p^2} = -1,$$

showing that the tangent lines are perpendicular.

58. Without a loss of generality, let the ellipse, hyperbola, and foci be as shown in the figure.

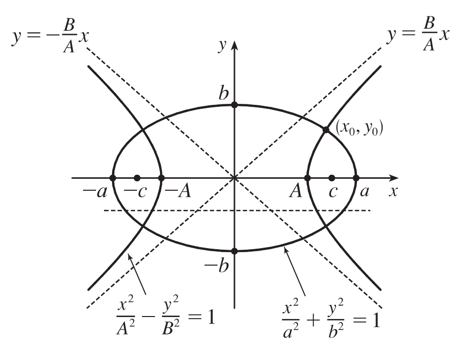
The curves intersect (eliminate y^2) $\quad \Rightarrow$

$$B^2\left(\frac{x^2}{A^2} - \frac{y^2}{B^2}\right) + b^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = B^2 + b^2 \quad \Rightarrow$$

$$\frac{B^2x^2}{A^2} + \frac{b^2x^2}{a^2} = B^2 + b^2 \quad \Rightarrow \quad x^2\left(\frac{B^2}{A^2} + \frac{b^2}{a^2}\right) = B^2 + b^2 \quad \Rightarrow$$

$$x^2 = \frac{B^2 + b^2}{\dfrac{a^2B^2 + b^2A^2}{A^2a^2}} = \frac{A^2a^2(B^2 + b^2)}{a^2B^2 + b^2A^2}.$$

Similarly, $y^2 = \dfrac{B^2b^2(a^2 - A^2)}{b^2A^2 + a^2B^2}$.



Next we find the slopes of the tangent lines of the curves: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \quad \Rightarrow \quad \dfrac{yy'}{b^2} = -\dfrac{x}{a^2} \quad \Rightarrow$

$y_E' = -\dfrac{b^2}{a^2}\dfrac{x}{y}$ and $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1 \quad \Rightarrow \quad \dfrac{2x}{A^2} - \dfrac{2yy'}{B^2} = 0 \quad \Rightarrow \quad \dfrac{yy'}{B^2} = \dfrac{x}{A^2} \quad \Rightarrow \quad y_H' = \dfrac{B^2}{A^2}\dfrac{x}{y}$. The product of the slopes

at (x_0, y_0) is $y_E'\,y_H' = -\dfrac{b^2B^2x_0^2}{a^2A^2y_0^2} = -\dfrac{b^2B^2\left[\dfrac{A^2a^2(B^2 + b^2)}{a^2B^2 + b^2A^2}\right]}{a^2A^2\left[\dfrac{B^2b^2(a^2 - A^2)}{b^2A^2 + a^2B^2}\right]} = -\dfrac{B^2 + b^2}{a^2 - A^2}$. Since $a^2 - b^2 = c^2$ and $A^2 + B^2 = c^2$,

we have $a^2 - b^2 = A^2 + B^2 \quad \Rightarrow \quad a^2 - A^2 = b^2 + B^2$, so the product of the slopes is -1, and hence, the tangent lines at each point of intersection are perpendicular.

59. $9x^2 + 4y^2 = 36 \quad \Leftrightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. We use the parametrization $x = 2\cos t$, $y = 3\sin t$, $0 \le t \le 2\pi$. The circumference is given by

$$L = \int_0^{2\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (3\cos t)^2}\,dt = \int_0^{2\pi} \sqrt{4\sin^2 t + 9\cos^2 t}\,dt$$
$$= \int_0^{2\pi} \sqrt{4 + 5\cos^2 t}\,dt$$

Now use Simpson's Rule with $n = 8$, $\Delta t = \dfrac{2\pi - 0}{8} = \dfrac{\pi}{4}$, and $f(t) = \sqrt{4 + 5\cos^2 t}$ to get

$$L \approx S_8 = \tfrac{\pi/4}{3}\left[f(0) + 4f\left(\tfrac{\pi}{4}\right) + 2f\left(\tfrac{\pi}{2}\right) + 4f\left(\tfrac{3\pi}{4}\right) + 2f(\pi) + 4f\left(\tfrac{5\pi}{4}\right) + 2f\left(\tfrac{3\pi}{2}\right) + 4f\left(\tfrac{7\pi}{4}\right) + f(2\pi)\right] \approx 15.9.$$

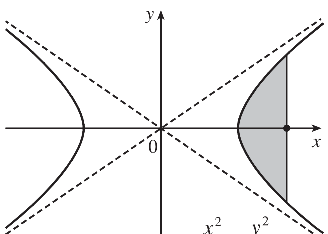
60. The length of the major axis is $2a$, so $a = \frac{1}{2}(1.18 \times 10^{10}) = 5.9 \times 10^9$. The length of the minor axis is $2b$, so $b = \frac{1}{2}(1.14 \times 10^{10}) = 5.7 \times 10^9$. An equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, or converting into parametric equations, $x = a\cos\theta$ and $y = b\sin\theta$. So

$$L = 4\int_0^{\pi/2} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta$$

Using Simpson's Rule with $n = 10$, $\Delta\theta = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$, and $f(\theta) = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$, we get

$$L \approx 4 \cdot S_{10} = 4 \cdot \tfrac{\pi}{20 \cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 2f\left(\tfrac{8\pi}{20}\right) + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 3.64 \times 10^{10}\text{ km}$$

61. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{y^2}{b^2} = \dfrac{x^2 - a^2}{a^2} \quad \Rightarrow \quad y = \pm\dfrac{b}{a}\sqrt{x^2 - a^2}$.

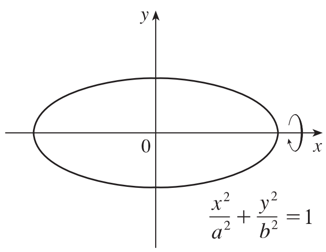


$$A = 2\int_a^c \frac{b}{a}\sqrt{x^2 - a^2}\,dx \overset{39}{=} \frac{2b}{a}\left[\frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right|\right]_a^c$$
$$= \frac{b}{a}\left[c\sqrt{c^2 - a^2} - a^2\ln\left|c + \sqrt{c^2 - a^2}\right| + a^2\ln|a|\right]$$

Since $a^2 + b^2 = c^2$, $c^2 - a^2 = b^2$, and $\sqrt{c^2 - a^2} = b$.

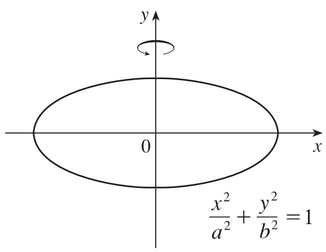
$$= \frac{b}{a}\left[cb - a^2\ln(c + b) + a^2\ln a\right] = \frac{b}{a}\left[cb + a^2(\ln a - \ln(b + c))\right]$$
$$= b^2c/a + ab\ln[a/(b + c)],\ \text{where } c^2 = a^2 + b^2.$$

62. (a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{y^2}{b^2} = \dfrac{a^2 - x^2}{a^2} \quad \Rightarrow \quad y = \pm\dfrac{b}{a}\sqrt{a^2 - x^2}$.



$$V = \int_{-a}^{a} \pi\left(\frac{b}{a}\sqrt{a^2 - x^2}\right)^2 dx = 2\pi\,\frac{b^2}{a^2}\int_0^a (a^2 - x^2)\,dx$$
$$= \frac{2\pi b^2}{a^2}\left[a^2x - \tfrac{1}{3}x^3\right]_0^a = \frac{2\pi b^2}{a^2}\left(\frac{2a^3}{3}\right) = \frac{4}{3}\pi b^2 a$$

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{x^2}{a^2} = \dfrac{b^2 - y^2}{b^2} \quad \Rightarrow \quad x = \pm\dfrac{a}{b}\sqrt{b^2 - y^2}$.



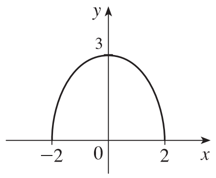
$$V = \int_{-b}^{b} \pi\left(\frac{a}{b}\sqrt{b^2 - y^2}\right)^2 dy = 2\pi\,\frac{a^2}{b^2}\int_0^b (b^2 - y^2)\,dy$$
$$= \frac{2\pi a^2}{b^2}\left[b^2y - \tfrac{1}{3}y^3\right]_0^b = \frac{2\pi a^2}{b^2}\left(\frac{2b^3}{3}\right) = \frac{4}{3}\pi a^2 b$$

63. $9x^2 + 4y^2 = 36 \quad \Leftrightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{9} = 1 \quad \Rightarrow \quad a = 3, b = 2$. By symmetry, $\overline{x} = 0$. By Example 2 in Section 7.3, the area of the top half of the ellipse is $\frac{1}{2}(\pi ab) = 3\pi$. Solve $9x^2 + 4y^2 = 36$ for y to get an equation for the top half of the ellipse:

$9x^2 + 4y^2 = 36 \quad \Leftrightarrow \quad 4y^2 = 36 - 9x^2 \quad \Leftrightarrow \quad y^2 = \frac{9}{4}(4 - x^2) \quad \Rightarrow \quad y = \frac{3}{2}\sqrt{4 - x^2}$. Now

$$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx = \frac{1}{3\pi}\int_{-2}^{2} \frac{1}{2}\left(\frac{3}{2}\sqrt{4 - x^2}\right)^2 dx = \frac{3}{8\pi}\int_{-2}^{2}(4 - x^2)\,dx$$

$$= \frac{3}{8\pi}\cdot 2\int_0^2 (4 - x^2)\,dx = \frac{3}{4\pi}\left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{3}{4\pi}\left(\frac{16}{3}\right) = \frac{4}{\pi}$$



so the centroid is $(0, 4/\pi)$.

64. (a) Consider the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a > b$, so that the major axis is the x-axis. Let the ellipse be parametrized by $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$. Then

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2\sin^2 t + b^2\cos^2 t = a^2(1 - \cos^2 t) + b^2\cos^2 t = a^2 + (b^2 - a^2)\cos^2 t = a^2 - c^2\cos^2 t,$$

where $c^2 = a^2 - b^2$. Using symmetry and rotating the ellipse about the major axis gives us surface area

$$S = \int 2\pi y\,ds = 2\int_0^{\pi/2} 2\pi(b\sin t)\sqrt{a^2 - c^2\cos^2 t}\,dt = 4\pi b\int_c^0 \sqrt{a^2 - u^2}\left(-\frac{1}{c}\,du\right) \qquad \begin{bmatrix} u = c\cos t \\ du = -c\sin t\,dt \end{bmatrix}$$

$$= \frac{4\pi b}{c}\int_0^c \sqrt{a^2 - u^2}\,du \overset{30}{=} \frac{4\pi b}{c}\left[\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)\right]_0^c = \frac{2\pi b}{c}\left[c\sqrt{a^2 - c^2} + a^2\sin^{-1}\left(\frac{c}{a}\right)\right]$$

$$= \frac{2\pi b}{c}\left[bc + a^2\sin^{-1}\left(\frac{c}{a}\right)\right]$$

(b) As in part (a),

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2\sin^2 t + b^2\cos^2 t = a^2\sin^2 t + b^2(1 - \sin^2 t) = b^2 + (a^2 - b^2)\sin^2 t = b^2 + c^2\sin^2 t.$$

Rotating about the minor axis gives us

$$S = \int 2\pi x\,ds = 2\int_0^{\pi/2} 2\pi(a\cos t)\sqrt{b^2 + c^2\sin^2 t}\,dt = 4\pi a\int_0^c \sqrt{b^2 + u^2}\left(\frac{1}{c}\,du\right) \qquad \begin{bmatrix} u = c\sin t \\ du = c\cos t\,dt \end{bmatrix}$$

$$\overset{21}{=} \frac{4\pi a}{c}\left[\frac{u}{2}\sqrt{b^2 + u^2} + \frac{b^2}{2}\ln\left(u + \sqrt{b^2 + u^2}\right)\right]_0^c = \frac{2\pi a}{c}\left[c\sqrt{b^2 + c^2} + b^2\ln\left(c + \sqrt{b^2 + c^2}\right) - b^2\ln b\right]$$

$$= \frac{2\pi a}{c}\left[ac + b^2\ln\left(\frac{a + c}{b}\right)\right]$$

65. Differentiating implicitly, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \quad \Rightarrow \quad y' = -\dfrac{b^2x}{a^2y} \quad [y \ne 0]$. Thus, the slope of the tangent line at P is $-\dfrac{b^2x_1}{a^2y_1}$. The slope of F_1P is $\dfrac{y_1}{x_1 + c}$ and of F_2P is $\dfrac{y_1}{x_1 - c}$. By the formula from Problems Plus, we have

$$\tan\alpha = \frac{\dfrac{y_1}{x_1 + c} + \dfrac{b^2x_1}{a^2y_1}}{1 - \dfrac{b^2x_1y_1}{a^2y_1(x_1 + c)}} = \frac{a^2y_1^2 + b^2x_1(x_1 + c)}{a^2y_1(x_1 + c) - b^2x_1y_1} = \frac{a^2b^2 + b^2cx_1}{c^2x_1y_1 + a^2cy_1} \qquad \begin{bmatrix} \text{using } b^2x_1^2 + a^2y_1^2 = a^2b^2, \\ \text{and } a^2 - b^2 = c^2 \end{bmatrix}$$

$$= \frac{b^2(cx_1 + a^2)}{cy_1(cx_1 + a^2)} = \frac{b^2}{cy_1}$$

[continued]

and
$$\tan\beta = \frac{-\dfrac{b^2x_1}{a^2y_1} - \dfrac{y_1}{x_1-c}}{1 - \dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}} = \frac{-a^2y_1^2 - b^2x_1(x_1-c)}{a^2y_1\,(x_1-c) - b^2x_1y_1} = \frac{-a^2b^2 + b^2cx_1}{c^2x_1y_1 - a^2cy_1} = \frac{b^2\left(cx_1 - a^2\right)}{cy_1(cx_1-a^2)} = \frac{b^2}{cy_1}$$

Thus, $\alpha = \beta$.

66. The slopes of the line segments F_1P and F_2P are $\dfrac{y_1}{x_1+c}$ and $\dfrac{y_1}{x_1-c}$, where P is (x_1, y_1). Differentiating implicitly, $\dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; y' = \dfrac{b^2x}{a^2y} \;\Rightarrow\;$ the slope of the tangent at P is $\dfrac{b^2x_1}{a^2y_1}$, so by the formula in Problem 19 on text page 271,

$$\tan\alpha = \frac{\dfrac{b^2x_1}{a^2y_1} - \dfrac{y_1}{x_1+c}}{1 + \dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}} = \frac{b^2x_1(x_1+c) - a^2y_1^2}{a^2y_1(x_1+c) + b^2x_1y_1} = \frac{b^2(cx_1+a^2)}{cy_1(cx_1+a^2)} \quad \left[\begin{array}{c}\text{using } x_1^2/a^2 - y_1^2/b^2 = 1, \\ \text{and } a^2+b^2 = c^2\end{array}\right] = \frac{b^2}{cy_1}$$

and
$$\tan\beta = \frac{-\dfrac{b^2x_1}{a^2y_1} + \dfrac{y_1}{x_1-c}}{1 + \dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}} = \frac{-b^2x_1(x_1-c) + a^2y_1^2}{a^2y_1(x_1-c) + b^2x_1y_1} = \frac{b^2(cx_1-a^2)}{cy_1(cx_1-a^2)} = \frac{b^2}{cy_1}$$

So $\alpha = \beta$.

10.6 Conic Sections in Polar Coordinates

1. The directrix $x = 4$ is to the right of the focus at the origin, so we use the form with "$+\ e\cos\theta$" in the denominator. (See Theorem 6 and Figure 2.) An equation is $r = \dfrac{ed}{1+e\cos\theta} = \dfrac{\frac{1}{2}\cdot 4}{1+\frac{1}{2}\cos\theta} = \dfrac{4}{2+\cos\theta}$.

2. The directrix $x = -3$ is to the left of the focus at the origin, so we use the form with "$-\ e\cos\theta$" in the denominator. $e = 1$ for a parabola, so an equation is $r = \dfrac{ed}{1-e\cos\theta} = \dfrac{1\cdot 3}{1-1\cos\theta} = \dfrac{3}{1-\cos\theta}$.

3. The directrix $y = 2$ is above the focus at the origin, so we use the form with "$+\ e\sin\theta$" in the denominator. An equation is $r = \dfrac{ed}{1+e\sin\theta} = \dfrac{1.5(2)}{1+1.5\sin\theta} = \dfrac{6}{2+3\sin\theta}$.

4. The directrix $x = 3$ is to the right of the focus at the origin, so we use the form with "$+\ e\cos\theta$" in the denominator. An equation is $r = \dfrac{ed}{1+e\cos\theta} = \dfrac{3\cdot 3}{1+3\cos\theta} = \dfrac{9}{1+3\cos\theta}$.

5. The vertex $(4, 3\pi/2)$ is 4 units below the focus at the origin, so the directrix is 8 units below the focus $(d = 8)$, and we use the form with "$-\ e\sin\theta$" in the denominator. $e = 1$ for a parabola, so an equation is $r = \dfrac{ed}{1-e\sin\theta} = \dfrac{1(8)}{1-1\sin\theta} = \dfrac{8}{1-\sin\theta}$.

6. The vertex $P(1, \pi/2)$ is 1 unit above the focus F at the origin, so $|PF| = 1$ and we use the form with "$+\ e\sin\theta$" in the denominator. The distance from the focus to the directrix l is d, so

$e = \dfrac{|PF|}{|Pl|} \quad \Rightarrow \quad 0.8 = \dfrac{1}{d-1} \quad \Rightarrow \quad 0.8d - 0.8 = 1 \quad \Rightarrow \quad 0.8d = 1.8 \quad \Rightarrow \quad d = 2.25.$

An equation is $r = \dfrac{ed}{1 + e\sin\theta} = \dfrac{0.8(2.25)}{1 + 0.8\sin\theta} \cdot \dfrac{5}{5} = \dfrac{9}{5 + 4\sin\theta}.$

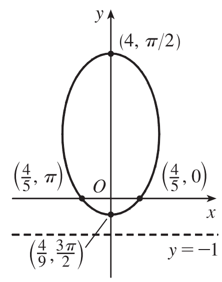
7. The directrix $r = 4\sec\theta$ (equivalent to $r\cos\theta = 4$ or $x = 4$) is to the right of the focus at the origin, so we will use the form with "$+\ e\cos\theta$" in the denominator. The distance from the focus to the directrix is $d = 4$, so an equation is

$$r = \frac{ed}{1 + e\cos\theta} = \frac{\frac{1}{2}(4)}{1 + \frac{1}{2}\cos\theta} \cdot \frac{2}{2} = \frac{4}{2 + \cos\theta}.$$

8. The directrix $r = -6\csc\theta$ (equivalent to $r\sin\theta = -6$ or $y = -6$) is below the focus at the origin, so we will use the form with "$-\ e\sin\theta$" in the denominator. The distance from the focus to the directrix is $d = 6$, so an equation is

$$r = \frac{ed}{1 - e\sin\theta} = \frac{3(6)}{1 - 3\sin\theta} = \frac{18}{1 - 3\sin\theta}.$$

9. $r = \dfrac{4}{5 - 4\sin\theta} \cdot \dfrac{1/5}{1/5} = \dfrac{4/5}{1 - \frac{4}{5}\sin\theta}$, where $e = \frac{4}{5}$ and $ed = \frac{4}{5} \quad \Rightarrow \quad d = 1.$



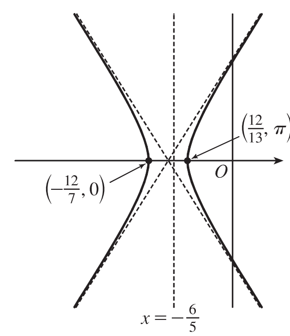
(a) Eccentricity $= e = \frac{4}{5}$

(b) Since $e = \frac{4}{5} < 1$, the conic is an ellipse.

(c) Since "$-\ e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin, $d = |Fl| = 1$, so an equation of the directrix is $y = -1$.

(d) The vertices are $\left(4, \frac{\pi}{2}\right)$ and $\left(\frac{4}{9}, \frac{3\pi}{2}\right)$.

10. $r = \dfrac{12}{3 - 10\cos\theta} \cdot \dfrac{1/3}{1/3} = \dfrac{4}{1 - \frac{10}{3}\cos\theta}$, where $e = \frac{10}{3}$ and $ed = 4 \quad \Rightarrow \quad d = 4\left(\frac{3}{10}\right) = \frac{6}{5}.$



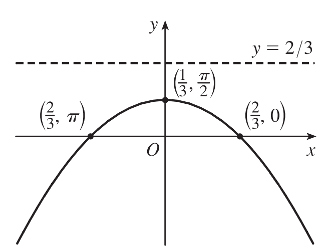
(a) Eccentricity $= e = \frac{10}{3}$

(b) Since $e = \frac{10}{3} > 1$, the conic is a hyperbola.

(c) Since "$-\ e\cos\theta$" appears in the denominator, the directrix is to the left of the focus at the origin. $d = |Fl| = \frac{6}{5}$, so an equation of the directrix is $x = -\frac{6}{5}$.

(d) The vertices are $\left(-\frac{12}{7}, 0\right)$ and $\left(\frac{12}{13}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{120}{91}, \pi\right)$.

11. $r = \dfrac{2}{3 + 3\sin\theta} \cdot \dfrac{1/3}{1/3} = \dfrac{2/3}{1 + 1\sin\theta}$, where $e = 1$ and $ed = \frac{2}{3} \quad \Rightarrow \quad d = \frac{2}{3}.$



(a) Eccentricity $= e = 1$

(b) Since $e = 1$, the conic is a parabola.

(c) Since "$+\ e\sin\theta$" appears in the denominator, the directrix is above the focus at the origin. $d = |Fl| = \frac{2}{3}$, so an equation of the directrix is $y = \frac{2}{3}$.

(d) The vertex is at $\left(\frac{1}{3}, \frac{\pi}{2}\right)$, midway between the focus and directrix.

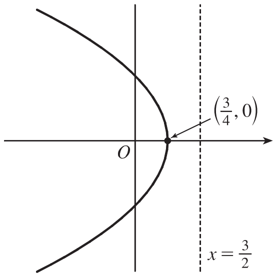
12. $r = \dfrac{3}{2 + 2\cos\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{3/2}{1 + 1\cos\theta}$, where $e = 1$ and $ed = \frac{3}{2} \quad\Rightarrow\quad d = \frac{3}{2}$.

(a) Eccentricity $= e = 1$

(b) Since $e = 1$, the conic is a parabola.

(c) Since "$+e\cos\theta$" appears in the denominator, the directrix is to the right of the focus at the origin. $d = |Fl| = \frac{3}{2}$, so an equation of the directrix is $x = \frac{3}{2}$.

(d) The vertex is at $\left(\frac{3}{4}, 0\right)$, midway between the focus and directrix.



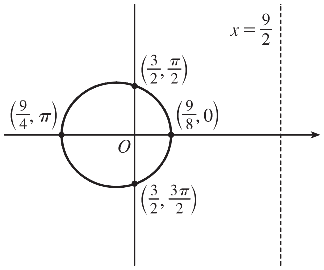
13. $r = \dfrac{9}{6 + 2\cos\theta} \cdot \dfrac{1/6}{1/6} = \dfrac{3/2}{1 + \frac{1}{3}\cos\theta}$, where $e = \frac{1}{3}$ and $ed = \frac{3}{2} \quad\Rightarrow\quad d = \frac{9}{2}$.

(a) Eccentricity $= e = \frac{1}{3}$

(b) Since $e = \frac{1}{3} < 1$, the conic is an ellipse.

(c) Since "$+e\cos\theta$" appears in the denominator, the directrix is to the right of the focus at the origin. $d = |Fl| = \frac{9}{2}$, so an equation of the directrix is $x = \frac{9}{2}$.

(d) The vertices are $\left(\frac{9}{8}, 0\right)$ and $\left(\frac{9}{4}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{9}{16}, \pi\right)$.



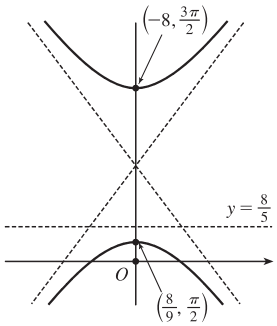
14. $r = \dfrac{8}{4 + 5\sin\theta} \cdot \dfrac{1/4}{1/4} = \dfrac{2}{1 + \frac{5}{4}\sin\theta}$, where $e = \frac{5}{4}$ and $ed = 2 \quad\Rightarrow\quad d = 2\left(\frac{4}{5}\right) = \frac{8}{5}$.

(a) Eccentricity $= e = \frac{5}{4}$

(b) Since $e = \frac{5}{4} > 1$, the conic is a hyperbola.

(c) Since "$+e\sin\theta$" appears in the denominator, the directrix is above the focus at the origin. $d = |Fl| = \frac{8}{5}$, so an equation of the directrix is $y = \frac{8}{5}$.

(d) The vertices are $\left(\frac{8}{9}, \frac{\pi}{2}\right)$ and $\left(-8, \frac{3\pi}{2}\right)$, so the center is midway between them, that is, $\left(\frac{40}{9}, \frac{\pi}{2}\right)$.



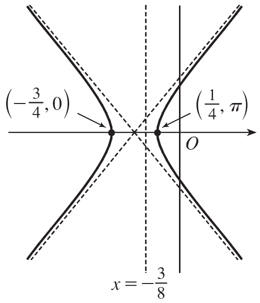
15. $r = \dfrac{3}{4 - 8\cos\theta} \cdot \dfrac{1/4}{1/4} = \dfrac{3/4}{1 - 2\cos\theta}$, where $e = 2$ and $ed = \frac{3}{4} \quad\Rightarrow\quad d = \frac{3}{8}$.

(a) Eccentricity $= e = 2$

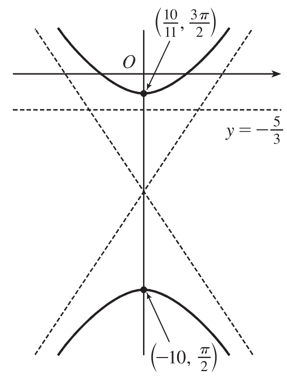
(b) Since $e = 2 > 1$, the conic is a hyperbola.

(c) Since "$-e\cos\theta$" appears in the denominator, the directrix is to the left of the focus at the origin. $d = |Fl| = \frac{3}{8}$, so an equation of the directrix is $x = -\frac{3}{8}$.

(d) The vertices are $\left(-\frac{3}{4}, 0\right)$ and $\left(\frac{1}{4}, \pi\right)$, so the center is midway between them, that is, $\left(\frac{1}{2}, \pi\right)$.



16. $r = \dfrac{10}{5 - 6\sin\theta} \cdot \dfrac{1/5}{1/5} = \dfrac{2}{1 - \frac{6}{5}\sin\theta}$, where $e = \frac{6}{5}$ and $ed = 2 \quad\Rightarrow\quad d = 2\left(\frac{5}{6}\right) = \frac{5}{3}$.



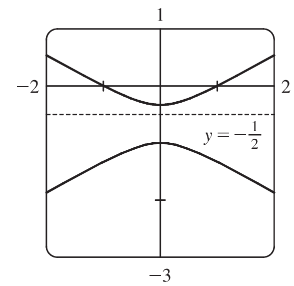
(a) Eccentricity $= e = \frac{6}{5}$

(b) Since $e = \frac{6}{5} > 1$, the conic is a hyperbola.

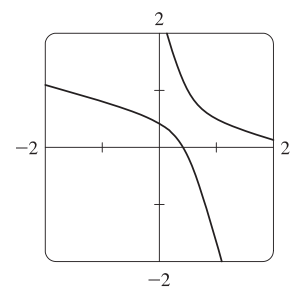
(c) Since "$-e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin. $d = |Fl| = \frac{5}{3}$, so an equation of the directrix is $y = -\frac{5}{3}$.

(d) The vertices are $\left(-10, \frac{\pi}{2}\right)$ and $\left(\frac{10}{11}, \frac{3\pi}{2}\right)$, so the center is midway between them, that is, $\left(\frac{60}{11}, \frac{3\pi}{2}\right)$.

17. (a) $r = \dfrac{1}{1 - 2\sin\theta}$, where $e = 2$ and $ed = 1 \quad\Rightarrow\quad d = \frac{1}{2}$. The eccentricity $e = 2 > 1$, so the conic is a hyperbola. Since "$-e\sin\theta$" appears in the denominator, the directrix is below the focus at the origin. $d = |Fl| = \frac{1}{2}$, so an equation of the directrix is $y = -\frac{1}{2}$. The vertices are $\left(-1, \frac{\pi}{2}\right)$ and $\left(\frac{1}{3}, \frac{3\pi}{2}\right)$, so the center is midway between them, that is, $\left(\frac{2}{3}, \frac{3\pi}{2}\right)$.



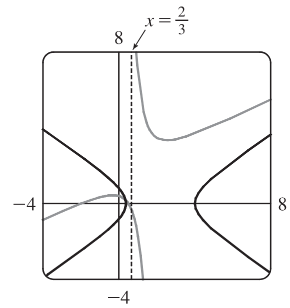
(b) By the discussion that precedes Example 4, the equation is $r = \dfrac{1}{1 - 2\sin\left(\theta - \frac{3\pi}{4}\right)}$.



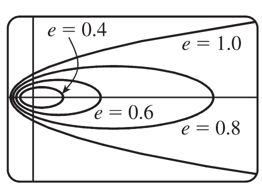
18. $r = \dfrac{4}{5 + 6\cos\theta} = \dfrac{4/5}{1 + \frac{6}{5}\cos\theta}$, so $e = \frac{6}{5}$ and $ed = \frac{4}{5} \quad\Rightarrow\quad d = \frac{2}{3}$.

An equation of the directrix is $x = \frac{2}{3} \quad\Rightarrow\quad r\cos\theta = \frac{2}{3} \quad\Rightarrow\quad r = \dfrac{2}{3\cos\theta}$.

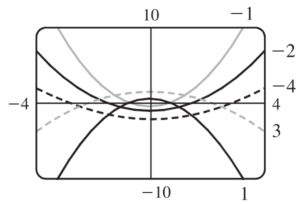
If the hyperbola is rotated about its focus (the origin) through an angle $\pi/3$, its equation is the same as that of the original, with θ replaced by $\theta - \frac{\pi}{3}$ (see Example 4), so $r = \dfrac{4}{5 + 6\cos\left(\theta - \frac{\pi}{3}\right)}$.



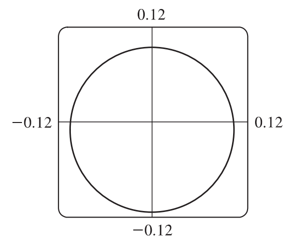
19. For $e < 1$ the curve is an ellipse. It is nearly circular when e is close to 0. As e increases, the graph is stretched out to the right, and grows larger (that is, its right-hand focus moves to the right while its left-hand focus remains at the origin.) At $e = 1$, the curve becomes a parabola with focus at the origin.



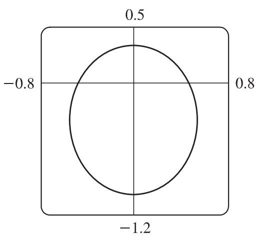
20. (a) The value of d does not seem to affect the shape of the conic (a parabola) at all, just its size, position, and orientation (for $d < 0$ it opens upward, for $d > 0$ it opens downward).



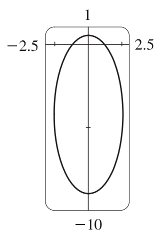
(b) We consider only positive values of e. When $0 < e < 1$, the conic is an ellipse. As $e \to 0^+$, the graph approaches perfect roundness and zero size. As e increases, the ellipse becomes more elongated, until at $e = 1$ it turns into a parabola. For $e > 1$, the conic is a hyperbola, which moves downward and gets broader as e continues to increase.



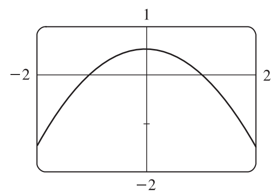
$e = 0.1$



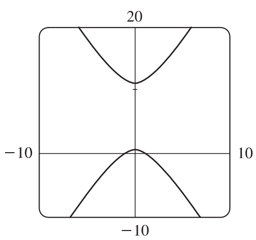
$e = 0.5$



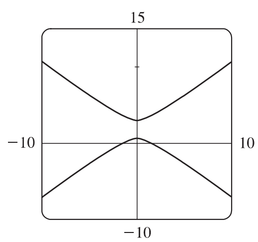
$e = 0.9$



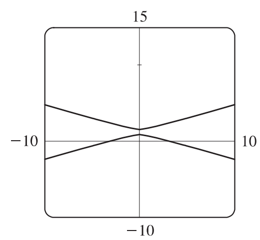
$e = 1$



$e = 1.1$



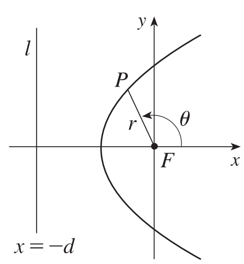
$e = 1.5$



$e = 10$

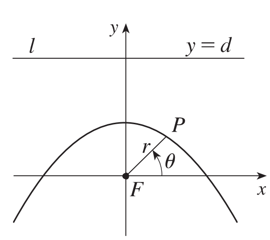
21. $|PF| = e\,|Pl| \quad\Rightarrow\quad r = e[d - r\cos(\pi - \theta)] = e(d + r\cos\theta) \quad\Rightarrow$

$r(1 - e\cos\theta) = ed \quad\Rightarrow\quad r = \dfrac{ed}{1 - e\cos\theta}$



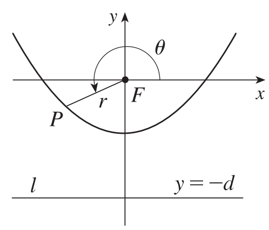
22. $|PF| = e\,|Pl| \quad\Rightarrow\quad r = e[d - r\sin\theta] \quad\Rightarrow\quad r(1 + e\sin\theta) = ed \quad\Rightarrow$

$r = \dfrac{ed}{1 + e\sin\theta}$



23. $|PF| = e\,|Pl| \quad\Rightarrow\quad r = e[d - r\sin(\theta - \pi)] = e(d + r\sin\theta) \quad\Rightarrow$

$r(1 - e\sin\theta) = ed \quad\Rightarrow\quad r = \dfrac{ed}{1 - e\sin\theta}$



24. The parabolas intersect at the two points where $\dfrac{c}{1+\cos\theta} = \dfrac{d}{1-\cos\theta} \quad\Rightarrow\quad \cos\theta = \dfrac{c-d}{c+d} \quad\Rightarrow\quad r = \dfrac{c+d}{2}$.

For the first parabola, $\dfrac{dr}{d\theta} = \dfrac{c\sin\theta}{(1+\cos\theta)^2}$, so

$$\frac{dy}{dx} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{c\sin^2\theta + c\cos\theta(1+\cos\theta)}{c\sin\theta\cos\theta - c\sin\theta(1+\cos\theta)} = \frac{1+\cos\theta}{-\sin\theta}$$

and similarly for the second, $\dfrac{dy}{dx} = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$. Since the product of these slopes is -1, the parabolas intersect at right angles.

25. We are given $e = 0.093$ and $a = 2.28 \times 10^8$. By (7), we have

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{2.28 \times 10^8[1 - (0.093)^2]}{1 + 0.093\cos\theta} \approx \frac{2.26 \times 10^8}{1 + 0.093\cos\theta}$$

26. We are given $e = 0.048$ and $2a = 1.56 \times 10^9 \quad\Rightarrow\quad a = 7.8 \times 10^8$. By (7), we have

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{7.8 \times 10^8[1 - (0.048)^2]}{1 + 0.048\cos\theta} \approx \frac{7.78 \times 10^8}{1 + 0.048\cos\theta}$$

27. Here $2a$ = length of major axis = 36.18 AU $\quad\Rightarrow\quad a = 18.09$ AU and $e = 0.97$. By (7), the equation of the orbit is

$r = \dfrac{18.09[1 - (0.97)^2]}{1 + 0.97\cos\theta} \approx \dfrac{1.07}{1 + 0.97\cos\theta}$. By (8), the maximum distance from the comet to the sun is

$18.09(1 + 0.97) \approx 35.64$ AU or about 3.314 billion miles.

28. Here $2a$ = length of major axis = 356.5 AU $\quad\Rightarrow\quad a = 178.25$ AU and $e = 0.9951$. By (7), the equation of the orbit

is $r = \dfrac{178.25[1 - (0.9951)^2]}{1 + 0.9951\cos\theta} \approx \dfrac{1.7426}{1 + 0.9951\cos\theta}$. By (8), the minimum distance from the comet to the sun is

$178.25(1 - 0.9951) \approx 0.8734$ AU or about 81 million miles.

29. The minimum distance is at perihelion, where $4.6 \times 10^7 = r = a(1-e) = a(1 - 0.206) = a(0.794) \quad\Rightarrow$

$a = 4.6 \times 10^7/0.794$. So the maximum distance, which is at aphelion, is

$r = a(1+e) = \left(4.6 \times 10^7/0.794\right)(1.206) \approx 7.0 \times 10^7$ km.

30. At perihelion, $r = a(1-e) = 4.43 \times 10^9$, and at aphelion, $r = a(1+e) = 7.37 \times 10^9$. Adding, we get $2a = 11.80 \times 10^9$,

so $a = 5.90 \times 10^9$ km. Therefore $1 + e = a(1+e)/a = \frac{7.37}{5.90} \approx 1.249$ and $e \approx 0.249$.

31. From Exercise 29, we have $e = 0.206$ and $a(1-e) = 4.6 \times 10^7$ km. Thus, $a = 4.6 \times 10^7/0.794$. From (7), we can write the equation of Mercury's orbit as $r = a\dfrac{1-e^2}{1+e\cos\theta}$. So since

$$\frac{dr}{d\theta} = \frac{a(1-e^2)e\sin\theta}{(1+e\cos\theta)^2} \quad \Rightarrow$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2(1-e^2)^2}{(1+e\cos\theta)^2} + \frac{a^2(1-e^2)^2\,e^2\sin^2\theta}{(1+e\cos\theta)^4} = \frac{a^2(1-e^2)^2}{(1+e\cos\theta)^4}\,(1+2e\cos\theta+e^2)$$

the length of the orbit is

$$L = \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = a(1-e^2)\int_0^{2\pi} \frac{\sqrt{1+e^2+2e\cos\theta}}{(1+e\cos\theta)^2}\,d\theta \approx 3.6 \times 10^8 \text{ km}$$

This seems reasonable, since Mercury's orbit is nearly circular, and the circumference of a circle of radius a is $2\pi a \approx 3.6 \times 10^8$ km.

10 Review

CONCEPT CHECK

1. (a) A parametric curve is a set of points of the form $(x, y) = (f(t), g(t))$, where f and g are continuous functions of a variable t.

   (b) Sketching a parametric curve, like sketching the graph of a function, is difficult to do in general. We can plot points on the curve by finding $f(t)$ and $g(t)$ for various values of t, either by hand or with a calculator or computer. Sometimes, when f and g are given by formulas, we can eliminate t from the equations $x = f(t)$ and $y = g(t)$ to get a Cartesian equation relating x and y. It may be easier to graph that equation than to work with the original formulas for x and y in terms of t.

2. (a) You can find $\dfrac{dy}{dx}$ as a function of t by calculating $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ [if $dx/dt \neq 0$].

   (b) Calculate the area as $\int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)dt$ [or $\int_\beta^\alpha g(t)\,f'(t)dt$ if the leftmost point is $(f(\beta), g(\beta))$ rather than $(f(\alpha), g(\alpha))$].

3. (a) $L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$

   (b) $S = \int_\alpha^\beta 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$

4. (a) See Figure 5 in Section 10.3.

   (b) $x = r\cos\theta,\ y = r\sin\theta$

   (c) To find a polar representation (r, θ) with $r \geq 0$ and $0 \leq \theta < 2\pi$, first calculate $r = \sqrt{x^2+y^2}$. Then θ is specified by $\cos\theta = x/r$ and $\sin\theta = y/r$.

5. (a) Calculate $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{\frac{d}{d\theta}(y)}{\frac{d}{d\theta}(x)} = \dfrac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \dfrac{\left(\frac{dr}{d\theta}\right)\sin\theta + r\cos\theta}{\left(\frac{dr}{d\theta}\right)\cos\theta - r\sin\theta}$, where $r = f(\theta)$.

   (b) Calculate $A = \int_a^b \frac{1}{2}r^2\,d\theta = \int_a^b \frac{1}{2}[f(\theta)]^2\,d\theta$

   (c) $L = \int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\,d\theta = \int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$

6. (a) A parabola is a set of points in a plane whose distances from a fixed point F (the focus) and a fixed line l (the directrix) are equal.

   (b) $x^2 = 4py$; $y^2 = 4px$

7. (a) An ellipse is a set of points in a plane the sum of whose distances from two fixed points (the foci) is a constant.

   (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$.

8. (a) A hyperbola is a set of points in a plane the difference of whose distances from two fixed points (the foci) is a constant. This difference should be interpreted as the larger distance minus the smaller distance.

   (b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$

   (c) $y = \pm\dfrac{\sqrt{c^2 - a^2}}{a}x$

9. (a) If a conic section has focus F and corresponding directrix l, then the eccentricity e is the fixed ratio $|PF| \,/\, |Pl|$ for points P of the conic section.

   (b) $e < 1$ for an ellipse; $e > 1$ for a hyperbola; $e = 1$ for a parabola.

   (c) $x = d$: $r = \dfrac{ed}{1 + e\cos\theta}$. $x = -d$: $r = \dfrac{ed}{1 - e\cos\theta}$. $y = d$: $r = \dfrac{ed}{1 + e\sin\theta}$. $y = -d$: $r = \dfrac{ed}{1 - e\sin\theta}$.

TRUE-FALSE QUIZ

1. False. Consider the curve defined by $x = f(t) = (t-1)^3$ and $y = g(t) = (t-1)^2$. Then $g'(t) = 2(t-1)$, so $g'(1) = 0$, but its graph has a *vertical* tangent when $t = 1$. *Note:* The statement is true if $f'(1) \neq 0$ when $g'(1) = 0$.

2. False. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$.

3. False. For example, if $f(t) = \cos t$ and $g(t) = \sin t$ for $0 \le t \le 4\pi$, then the curve is a circle of radius 1, hence its length is 2π, but $\int_0^{4\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_0^{4\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^{4\pi} 1\,dt = 4\pi$, since as t increases from 0 to 4π, the circle is traversed twice.

4. False. If $(r, \theta) = (1, \pi)$, then $(x, y) = (-1, 0)$, so $\tan^{-1}(y/x) = \tan^{-1} 0 = 0 \neq \theta$. The statement is true for points in quadrants I and IV.

5. True. The curve $r = 1 - \sin 2\theta$ is unchanged if we rotate it through 180° about O because $1 - \sin 2(\theta + \pi) = 1 - \sin(2\theta + 2\pi) = 1 - \sin 2\theta$. So it's unchanged if we replace r by $-r$. (See the discussion after Example 8 in Section 10.3.) In other words, it's the same curve as $r = -(1 - \sin 2\theta) = \sin 2\theta - 1$.

6. True. The polar equation $r = 2$, the Cartesian equation $x^2 + y^2 = 4$, and the parametric equations $x = 2\sin 3t$, $y = 2\cos 3t$ $[0 \le t \le 2\pi]$ all describe the circle of radius 2 centered at the origin.

7. False. The first pair of equations gives the portion of the parabola $y = x^2$ with $x \geq 0$, whereas the second pair of equations traces out the whole parabola $y = x^2$.

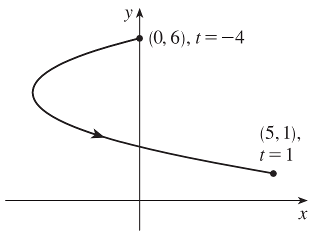
8. True. $y^2 = 2y + 3x \quad \Leftrightarrow \quad (y-1)^2 = 3x + 1 = 3\left(x + \frac{1}{3}\right) = 4\left(\frac{3}{4}\right)\left(x + \frac{1}{3}\right)$, which is the equation of a parabola with vertex $\left(-\frac{1}{3}, 1\right)$ and focus $\left(-\frac{1}{3} + \frac{3}{4}, 1\right)$, opening to the right.

9. True. By rotating and translating the parabola, we can assume it has an equation of the form $y = cx^2$, where $c > 0$. The tangent at the point $\left(a, ca^2\right)$ is the line $y - ca^2 = 2ca(x - a)$; i.e., $y = 2cax - ca^2$. This tangent meets the parabola at the points $\left(x, cx^2\right)$ where $cx^2 = 2cax - ca^2$. This equation is equivalent to $x^2 = 2ax - a^2$ [since $c > 0$]. But $x^2 = 2ax - a^2 \quad \Leftrightarrow \quad x^2 - 2ax + a^2 = 0 \quad \Leftrightarrow \quad (x-a)^2 = 0 \quad \Leftrightarrow \quad x = a \quad \Leftrightarrow$ $\left(x, cx^2\right) = \left(a, ca^2\right)$. This shows that each tangent meets the parabola at exactly one point.

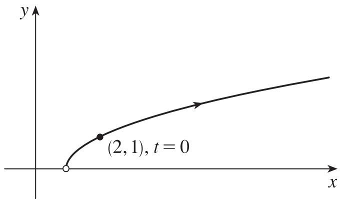
10. True. Consider a hyperbola with focus at the origin, oriented so that its polar equation is $r = \dfrac{ed}{1 + e\cos\theta}$, where $e > 1$. The directrix is $x = d$, but along the hyperbola we have $x = r\cos\theta = \dfrac{ed\cos\theta}{1 + e\cos\theta} = d\left(\dfrac{e\cos\theta}{1 + e\cos\theta}\right) \neq d$.

EXERCISES

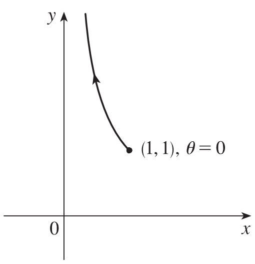
1. $x = t^2 + 4t$, $y = 2 - t$, $-4 \leq t \leq 1$. $t = 2 - y$, so
$x = (2-y)^2 + 4(2-y) = 4 - 4y + y^2 + 8 - 4y = y^2 - 8y + 12 \quad \Leftrightarrow$
$x + 4 = y^2 - 8y + 16 = (y-4)^2$. This is part of a parabola with vertex $(-4, 4)$, opening to the right.



2. $x = 1 + e^{2t}$, $y = e^t$.
$x = 1 + e^{2t} = 1 + (e^t)^2 = 1 + y^2$, $y > 0$.

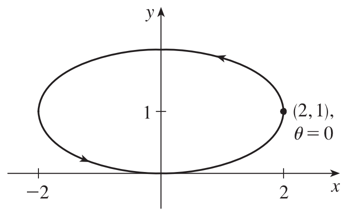


3. $y = \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{x}$. Since $0 \leq \theta \leq \pi/2$, $0 < x \leq 1$ and $y \geq 1$.
This is part of the hyperbola $y = 1/x$.



4. $x = 2\cos\theta$, $y = 1 + \sin\theta$, $\cos^2\theta + \sin^2\theta = 1 \quad \Rightarrow$

$\left(\frac{x}{2}\right)^2 + (y-1)^2 = 1 \quad \Rightarrow \quad \frac{x^2}{4} + (y-1)^2 = 1$. This is an ellipse, centered at $(0, 1)$, with semimajor axis of length 2 and semiminor axis of length 1.



5. Three different sets of parametric equations for the curve $y = \sqrt{x}$ are

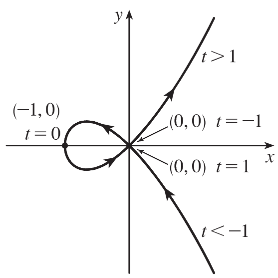
(i) $x = t, \quad y = \sqrt{t}$

(ii) $x = t^4, \quad y = t^2$

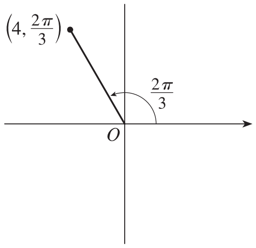
(iii) $x = \tan^2 t, \quad y = \tan t, \quad 0 \le t < \pi/2$

There are many other sets of equations that also give this curve.

6. For $t < -1$, $x > 0$ and $y < 0$ with x decreasing and y increasing. When $t = -1$, $(x, y) = (0, 0)$. When $-1 < t < 0$, we have $-1 < x < 0$ and $0 < y < 1/2$. When $t = 0$, $(x, y) = (-1, 0)$. When $0 < t < 1$, $-1 < x < 0$ and $-\frac{1}{2} < y < 0$. When $t = 1$, $(x, y) = (0, 0)$ again. When $t > 1$, both x and y are positive and increasing.



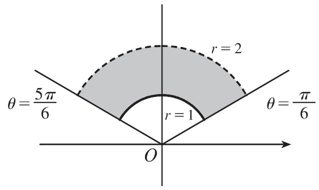
7. (a)



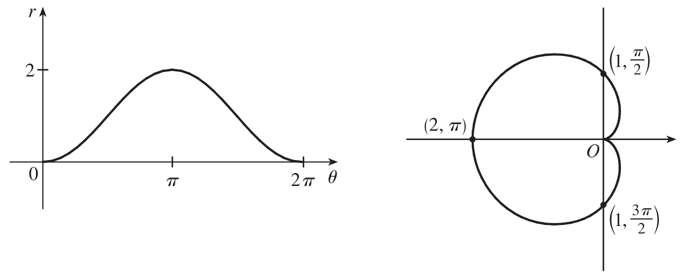
The Cartesian coordinates are $x = 4\cos\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$ and $y = 4\sin\frac{2\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$, that is, the point $\left(-2, 2\sqrt{3}\right)$.

(b) Given $x = -3$ and $y = 3$, we have $r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$. Also, $\tan\theta = \dfrac{y}{x} \quad \Rightarrow \quad \tan\theta = \dfrac{3}{-3}$, and since $(-3, 3)$ is in the second quadrant, $\theta = \frac{3\pi}{4}$. Thus, one set of polar coordinates for $(-3, 3)$ is $\left(3\sqrt{2}, \frac{3\pi}{4}\right)$, and two others are $\left(3\sqrt{2}, \frac{11\pi}{4}\right)$ and $\left(-3\sqrt{2}, \frac{7\pi}{4}\right)$.

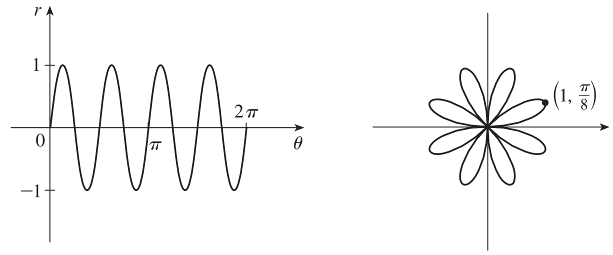
8. $1 \le r < 2, \ \frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$



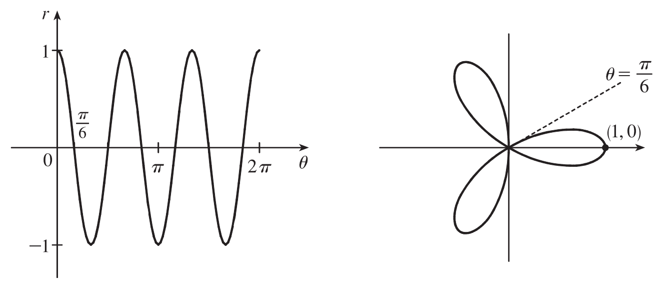
9. $r = 1 - \cos\theta$. This cardioid is symmetric about the polar axis.



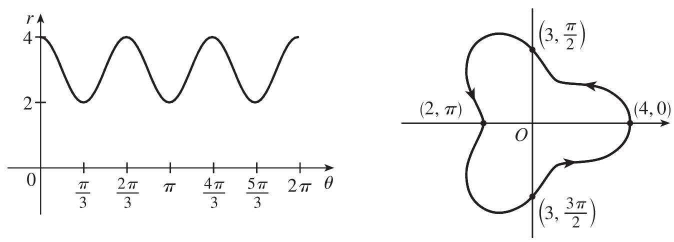
10. $r = \sin 4\theta$. This is an eight-leaved rose.



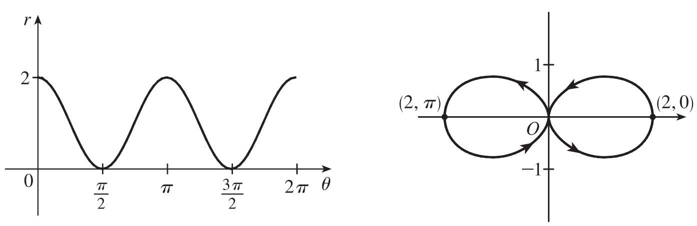
11. $r = \cos 3\theta$. This is a three-leaved rose. The curve is traced twice.



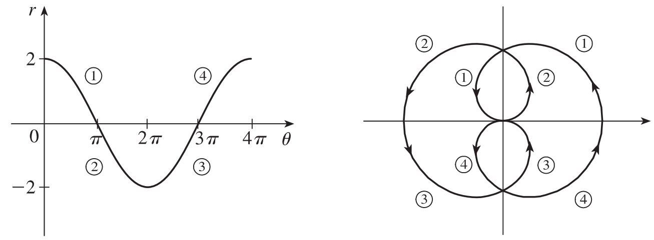
12. $r = 3 + \cos 3\theta$. The curve is symmetric about the horizontal axis.



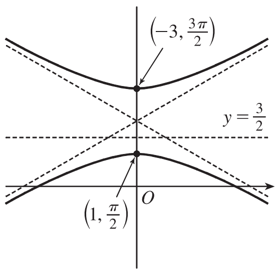
13. $r = 1 + \cos 2\theta$. The curve is symmetric about the pole and both the horizontal and vertical axes.



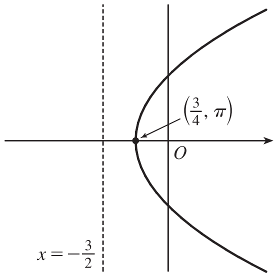
14. $r = 2\cos(\theta/2)$. The curve is symmetric about the pole and both the horizontal and vertical axes.



15. $r = \dfrac{3}{1 + 2\sin\theta} \Rightarrow e = 2 > 1$, so the conic is a hyperbola. $de = 3 \Rightarrow d = \frac{3}{2}$ and the form "$+2\sin\theta$" imply that the directrix is above the focus at the origin and has equation $y = \frac{3}{2}$. The vertices are $\left(1, \frac{\pi}{2}\right)$ and $\left(-3, \frac{3\pi}{2}\right)$.



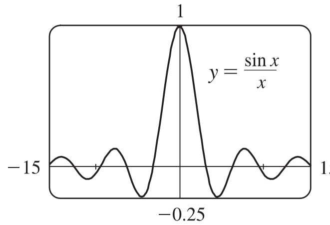
16. $r = \dfrac{3}{2 - 2\cos\theta} \cdot \dfrac{1/2}{1/2} = \dfrac{3/2}{1 - 1\cos\theta} \Rightarrow e = 1$, so the conic is a parabola. $de = \frac{3}{2} \Rightarrow d = \frac{3}{2}$ and the form "$-2\cos\theta$" imply that the directrix is to the left of the focus at the origin and has equation $x = -\frac{3}{2}$. The vertex is $\left(\frac{3}{4}, \pi\right)$.

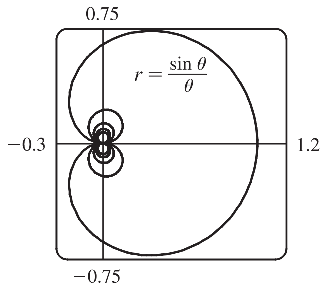


17. $x + y = 2 \Leftrightarrow r\cos\theta + r\sin\theta = 2 \Leftrightarrow r(\cos\theta + \sin\theta) = 2 \Leftrightarrow r = \dfrac{2}{\cos\theta + \sin\theta}$

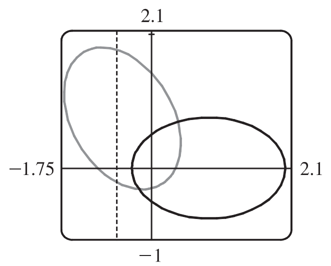
18. $x^2 + y^2 = 2 \Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$. [$r = -\sqrt{2}$ gives the same curve.]

19. $r = (\sin\theta)/\theta$. As $\theta \to \pm\infty$, $r \to 0$. As $\theta \to 0$, $r \to 1$. In the first figure, there are an infinite number of x-intercepts at $x = \pi n$, n a nonzero integer. These correspond to pole points in the second figure.





20. $r = \dfrac{2}{4 - 3\cos\theta} = \dfrac{1/2}{1 - \frac{3}{4}\cos\theta} \Rightarrow e = \frac{3}{4}$ and $d = \frac{2}{3}$. The equation of the directrix is $x = -\frac{2}{3} \Rightarrow r = -2/(3\cos\theta)$. To obtain the equation of the rotated ellipse, we replace θ in the original equation with $\theta - \frac{2\pi}{3}$, and get $r = \dfrac{2}{4 - 3\cos\left(\theta - \frac{2\pi}{3}\right)}$.



21. $x = \ln t$, $y = 1 + t^2$; $t = 1$. $\dfrac{dy}{dt} = 2t$ and $\dfrac{dx}{dt} = \dfrac{1}{t}$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1/t} = 2t^2$.

When $t = 1$, $(x, y) = (0, 2)$ and $dy/dx = 2$.

22. $x = t^3 + 6t + 1$, $y = 2t - t^2$; $t = -1$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2 - 2t}{3t^2 + 6}$. When $t = -1$, $(x, y) = (-6, -3)$ and $\dfrac{dy}{dx} = \dfrac{4}{9}$.

23. $r = e^{-\theta} \quad \Rightarrow \quad y = r\sin\theta = e^{-\theta}\sin\theta$ and $x = r\cos\theta = e^{-\theta}\cos\theta \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{-e^{-\theta}\sin\theta + e^{-\theta}\cos\theta}{-e^{-\theta}\cos\theta - e^{-\theta}\sin\theta}\cdot\frac{-e^{\theta}}{-e^{\theta}} = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}.$$

When $\theta = \pi$, $\dfrac{dy}{dx} = \dfrac{0-(-1)}{-1+0} = \dfrac{1}{-1} = -1$.

24. $r = 3 + \cos 3\theta \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \dfrac{-3\sin 3\theta\sin\theta + (3+\cos 3\theta)\cos\theta}{-3\sin 3\theta\cos\theta - (3+\cos 3\theta)\sin\theta}$.

When $\theta = \pi/2$, $\dfrac{dy}{dx} = \dfrac{(-3)(-1)(1) + (3+0)\cdot 0}{(-3)(-1)(0) - (3+0)\cdot 1} = \dfrac{3}{-3} = -1$.

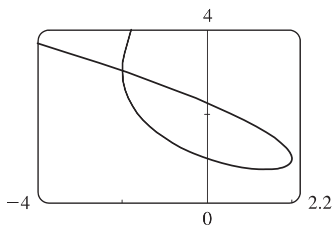
25. $x = t + \sin t,\ y = t - \cos t \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+\sin t}{1+\cos t} \quad \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{(1+\cos t)\cos t - (1+\sin t)(-\sin t)}{(1+\cos t)^2}}{1+\cos t} = \frac{\cos t + \cos^2 t + \sin t + \sin^2 t}{(1+\cos t)^3} = \frac{1+\cos t+\sin t}{(1+\cos t)^3}$$

26. $x = 1 + t^2,\ y = t - t^3$. $\dfrac{dy}{dt} = 1 - 3t^2$ and $\dfrac{dx}{dt} = 2t$, so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1-3t^2}{2t} = \frac{1}{2}t^{-1} - \frac{3}{2}t$.

$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt} = \frac{-\frac{1}{2}t^{-2} - \frac{3}{2}}{2t} = -\tfrac{1}{4}t^{-3} - \tfrac{3}{4}t^{-1} = -\frac{1}{4t^3}\left(1+3t^2\right) = -\frac{3t^2+1}{4t^3}.$$

27. We graph the curve $x = t^3 - 3t$, $y = t^2 + t + 1$ for $-2.2 \le t \le 1.2$. By zooming in or using a cursor, we find that the lowest point is about $(1.4, 0.75)$. To find the exact values, we find the t-value at which $dy/dt = 2t + 1 = 0 \quad \Leftrightarrow \quad t = -\frac{1}{2} \quad \Leftrightarrow \quad (x,y) = \left(\frac{11}{8}, \frac{3}{4}\right)$.



28. We estimate the coordinates of the point of intersection to be $(-2, 3)$. In fact this is exact, since both $t = -2$ and $t = 1$ give the point $(-2, 3)$. So the area enclosed by the loop is

$$\begin{aligned}\int_{t=-2}^{t=1} y\,dx &= \int_{-2}^{1}(t^2+t+1)(3t^2-3)\,dt = \int_{-2}^{1}(3t^4+3t^3-3t-3)\,dt \\ &= \left[\tfrac{3}{5}t^5 + \tfrac{3}{4}t^4 - \tfrac{3}{2}t^2 - 3t\right]_{-2}^{1} = \left(\tfrac{3}{5}+\tfrac{3}{4}-\tfrac{3}{2}-3\right) - \left[-\tfrac{96}{5}+12-6-(-6)\right] = \tfrac{81}{20}\end{aligned}$$

29. $x = 2a\cos t - a\cos 2t \quad \Rightarrow \quad \dfrac{dx}{dt} = -2a\sin t + 2a\sin 2t = 2a\sin t(2\cos t - 1) = 0 \quad \Leftrightarrow$

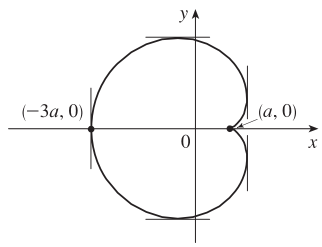
$\sin t = 0$ or $\cos t = \frac{1}{2} \quad \Rightarrow \quad t = 0, \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$.

$y = 2a\sin t - a\sin 2t \quad \Rightarrow \quad \dfrac{dy}{dt} = 2a\cos t - 2a\cos 2t = 2a\left(1 + \cos t - 2\cos^2 t\right) = 2a(1-\cos t)(1+2\cos t) = 0 \quad \Rightarrow$

$t = 0, \frac{2\pi}{3}$, or $\frac{4\pi}{3}$.

Thus the graph has vertical tangents where $t = \frac{\pi}{3}, \pi$ and $\frac{5\pi}{3}$, and horizontal tangents where $t = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$. To determine what the slope is where $t = 0$, we use l'Hospital's Rule to evaluate $\lim\limits_{t \to 0} \dfrac{dy/dt}{dx/dt} = 0$, so there is a horizontal tangent there.

t	x	y
0	a	0
$\frac{\pi}{3}$	$\frac{3}{2}a$	$\frac{\sqrt{3}}{2}a$
$\frac{2\pi}{3}$	$-\frac{1}{2}a$	$\frac{3\sqrt{3}}{2}a$
π	$-3a$	0
$\frac{4\pi}{3}$	$-\frac{1}{2}a$	$-\frac{3\sqrt{3}}{2}a$
$\frac{5\pi}{3}$	$\frac{3}{2}a$	$-\frac{\sqrt{3}}{2}a$



30. From Exercise 29, $x = 2a\cos t - a\cos 2t$, $y = 2a\sin t - a\sin 2t \quad\Rightarrow$

$$A = 2\int_{\pi}^{0}(2a\sin t - a\sin 2t)(-2a\sin t + 2a\sin 2t)\,dt = 4a^2\int_0^{\pi}(2\sin^2 t + \sin^2 2t - 3\sin t\sin 2t)\,dt$$
$$= 4a^2\int_0^{\pi}\left[(1-\cos 2t) + \tfrac{1}{2}(1-\cos 4t) - 6\sin^2 t\,\cos t\right]dt = 4a^2\left[t - \tfrac{1}{2}\sin 2t + \tfrac{1}{2}t - \tfrac{1}{8}\sin 4t - 2\sin^3 t\right]_0^{\pi}$$
$$= 4a^2\left(\tfrac{3}{2}\right)\pi = 6\pi a^2$$

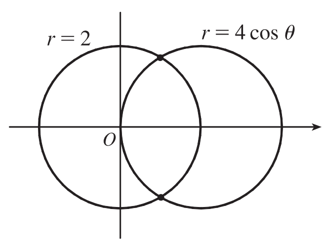
31. The curve $r^2 = 9\cos 5\theta$ has 10 "petals." For instance, for $-\frac{\pi}{10} \le \theta \le \frac{\pi}{10}$, there are two petals, one with $r > 0$ and one with $r < 0$.

$$A = 10\int_{-\pi/10}^{\pi/10}\tfrac{1}{2}r^2\,d\theta = 5\int_{-\pi/10}^{\pi/10}9\cos 5\theta\,d\theta = 5\cdot 9\cdot 2\int_0^{\pi/10}\cos 5\theta\,d\theta = 18\big[\sin 5\theta\big]_0^{\pi/10} = 18$$

32. $r = 1 - 3\sin\theta$. The inner loop is traced out as θ goes from $\alpha = \sin^{-1}\left(\frac{1}{3}\right)$ to $\pi - \alpha$, so

$$A = \int_{\alpha}^{\pi-\alpha}\tfrac{1}{2}r^2\,d\theta = \int_{\alpha}^{\pi/2}(1-3\sin\theta)^2\,d\theta = \int_{\alpha}^{\pi/2}\left[1 - 6\sin\theta + \tfrac{9}{2}(1-\cos 2\theta)\right]d\theta$$
$$= \left[\tfrac{11}{2}\theta + 6\cos\theta - \tfrac{9}{4}\sin 2\theta\right]_{\alpha}^{\pi/2} = \tfrac{11}{4}\pi - \tfrac{11}{2}\sin^{-1}\left(\tfrac{1}{3}\right) - 3\sqrt{2}$$

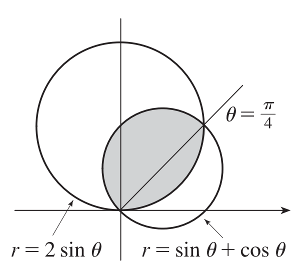
33. The curves intersect when $4\cos\theta = 2 \quad\Rightarrow\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \pm\frac{\pi}{3}$ for $-\pi \le \theta \le \pi$. The points of intersection are $\left(2, \frac{\pi}{3}\right)$ and $\left(2, -\frac{\pi}{3}\right)$.



34. The two curves clearly both contain the pole. For other points of intersection, $\cot\theta = 2\cos(\theta + 2n\pi)$ or $-2\cos(\theta + \pi + 2n\pi)$, both of which reduce to $\cot\theta = 2\cos\theta \quad\Leftrightarrow\quad \cos\theta = 2\sin\theta\cos\theta \quad\Leftrightarrow\quad \cos\theta(1 - 2\sin\theta) = 0 \quad\Rightarrow$ $\cos\theta = 0$ or $\sin\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ or $\frac{3\pi}{2} \quad\Rightarrow\quad$ intersection points are $\left(0, \frac{\pi}{2}\right)$, $\left(\sqrt{3}, \frac{\pi}{6}\right)$, and $\left(\sqrt{3}, \frac{11\pi}{6}\right)$.

35. The curves intersect where $2\sin\theta = \sin\theta + \cos\theta \quad\Rightarrow$ $\sin\theta = \cos\theta \quad\Rightarrow \theta = \frac{\pi}{4}$, and also at the origin (at which $\theta = \frac{3\pi}{4}$ on the second curve).

$$A = \int_0^{\pi/4}\tfrac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\pi/4}^{3\pi/4}\tfrac{1}{2}(\sin\theta + \cos\theta)^2\,d\theta$$
$$= \int_0^{\pi/4}(1 - \cos 2\theta)\,d\theta + \tfrac{1}{2}\int_{\pi/4}^{3\pi/4}(1 + \sin 2\theta)\,d\theta$$
$$= \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} + \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\cos 2\theta\right]_{\pi/4}^{3\pi/4} = \tfrac{1}{2}(\pi - 1)$$

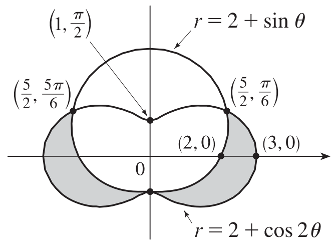


36. $A = 2\int_{-\pi/2}^{\pi/6} \frac{1}{2}\left[(2+\cos 2\theta)^2 - (2+\sin\theta)^2\right] d\theta$

$= \int_{-\pi/2}^{\pi/6} \left[4\cos 2\theta + \cos^2 2\theta - 4\sin\theta - \sin^2\theta\right] d\theta$

$= \left[2\sin 2\theta + \frac{1}{2}\theta + \frac{1}{8}\sin 4\theta + 4\cos\theta - \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/6}$

$= \frac{51}{16}\sqrt{3}$



37. $x = 3t^2,\ y = 2t^3.$

$L = \int_0^2 \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2}\, dt = \int_0^2 \sqrt{36t^2 + 36t^4}\, dt = \int_0^2 \sqrt{36t^2}\sqrt{1+t^2}\, dt$

$= \int_0^2 6\,|t|\,\sqrt{1+t^2}\, dt = 6\int_0^2 t\sqrt{1+t^2}\, dt = 6\int_1^5 u^{1/2}\left(\frac{1}{2}du\right)$ $\quad [u = 1+t^2,\ du = 2t\,dt]$

$= 6\cdot\frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^5 = 2(5^{3/2} - 1) = 2\left(5\sqrt{5} - 1\right)$

38. $x = 2 + 3t,\quad y = \cosh 3t \quad\Rightarrow\quad (dx/dt)^2 + (dy/dt)^2 = 3^2 + (3\sinh 3t)^2 = 9(1 + \sinh^2 3t) = 9\cosh^2 3t$, so

$L = \int_0^1 \sqrt{9\cosh^2 3t}\, dt = \int_0^1 |3\cosh 3t|\ dt = \int_0^1 3\cosh 3t\, dt = \left[\sinh 3t\right]_0^1 = \sinh 3 - \sinh 0 = \sinh 3.$

39. $L = \int_\pi^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_\pi^{2\pi} \sqrt{(1/\theta)^2 + (-1/\theta^2)^2}\, d\theta = \displaystyle\int_\pi^{2\pi} \frac{\sqrt{\theta^2+1}}{\theta^2}\, d\theta$

$\overset{24}{=} \left[-\dfrac{\sqrt{\theta^2+1}}{\theta} + \ln\left(\theta + \sqrt{\theta^2+1}\right)\right]_\pi^{2\pi} = \dfrac{\sqrt{\pi^2+1}}{\pi} - \dfrac{\sqrt{4\pi^2+1}}{2\pi} + \ln\left(\dfrac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right)$

$= \dfrac{2\sqrt{\pi^2+1} - \sqrt{4\pi^2+1}}{2\pi} + \ln\left(\dfrac{2\pi + \sqrt{4\pi^2+1}}{\pi + \sqrt{\pi^2+1}}\right)$

40. $L = \int_0^\pi \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^\pi \sqrt{\sin^6\left(\frac{1}{3}\theta\right) + \sin^4\left(\frac{1}{3}\theta\right)\cos^2\left(\frac{1}{3}\theta\right)}\, d\theta$

$= \int_0^\pi \sin^2\left(\frac{1}{3}\theta\right) d\theta = \left[\frac{1}{2}\left(\theta - \frac{3}{2}\sin\left(\frac{2}{3}\theta\right)\right)\right]_0^\pi = \frac{1}{2}\pi - \frac{3}{8}\sqrt{3}$

41. $x = 4\sqrt{t},\ y = \dfrac{t^3}{3} + \dfrac{1}{2t^2},\ 1 \le t \le 4 \quad\Rightarrow$

$S = \int_1^4 2\pi y\sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \int_1^4 2\pi\left(\frac{1}{3}t^3 + \frac{1}{2}t^{-2}\right)\sqrt{\left(2/\sqrt{t}\right)^2 + (t^2 - t^{-3})^2}\, dt$

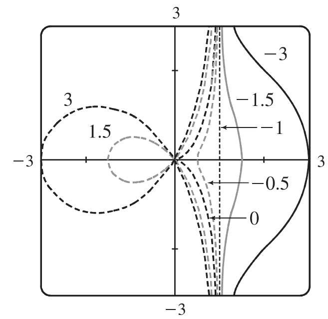
$= 2\pi\int_1^4 \left(\frac{1}{3}t^3 + \frac{1}{2}t^{-2}\right)\sqrt{(t^2 + t^{-3})^2}\, dt = 2\pi\int_1^4 \left(\frac{1}{3}t^5 + \frac{5}{6} + \frac{1}{2}t^{-5}\right) dt = 2\pi\left[\frac{1}{18}t^6 + \frac{5}{6}t - \frac{1}{8}t^{-4}\right]_1^4 = \frac{471{,}295}{1024}\pi$

42. $x = 2 + 3t,\quad y = \cosh 3t \quad\Rightarrow\quad (dx/dt)^2 + (dy/dt)^2 = 3^2 + (3\sinh 3t)^2 = 9(1 + \sinh^2 3t) = 9\cosh^2 3t$, so

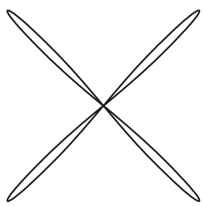
$S = \int_0^1 2\pi y\, ds = \int_0^1 2\pi\cosh 3t\sqrt{9\cosh^2 3t}\, dt = \int_0^1 2\pi\cosh 3t\,|3\cosh 3t|\ dt = \int_0^1 2\pi\cosh 3t\cdot 3\cosh 3t\, dt$

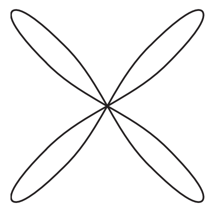
$= 6\pi\int_0^1 \cosh^2 3t\, dt = 6\pi\int_0^1 \frac{1}{2}(1 + \cosh 6t)\, dt = 3\pi\left[t + \frac{1}{6}\sinh 6t\right]_0^1 = 3\pi\left(1 + \frac{1}{6}\sinh 6\right) = 3\pi + \frac{\pi}{2}\sinh 6$

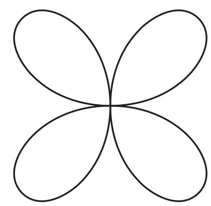
43. For all c except -1, the curve is asymptotic to the line $x = 1$. For $c < -1$, the curve bulges to the right near $y = 0$. As c increases, the bulge becomes smaller, until at $c = -1$ the curve is the straight line $x = 1$. As c continues to increase, the curve bulges to the left, until at $c = 0$ there is a cusp at the origin. For $c > 0$, there is a loop to the left of the origin, whose size and roundness increase as c increases. Note that the x-intercept of the curve is always $-c$.

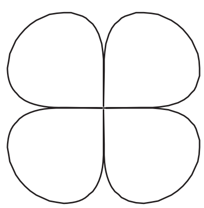


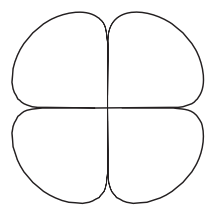
44. For a close to 0, the graph consists of four thin petals. As a increases, the petals get wider, until as $a \to \infty$, each petal occupies almost its entire quarter-circle.

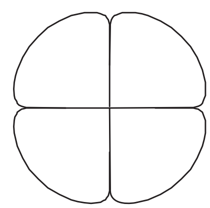

$a = 0.01$


$a = 0.1$

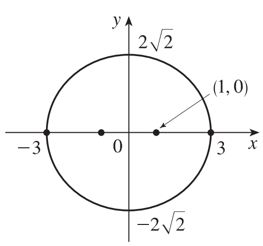

$a = 1$


$a = 5$

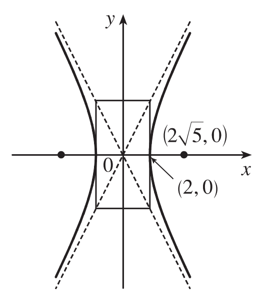

$a = 10$


$a = 25$

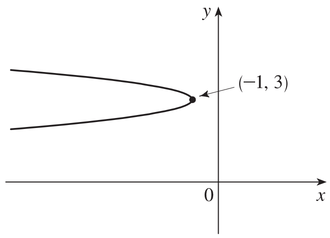
45. $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$ is an ellipse with center $(0, 0)$.
$a = 3$, $b = 2\sqrt{2}$, $c = 1 \quad \Rightarrow$
foci $(\pm 1, 0)$, vertices $(\pm 3, 0)$.



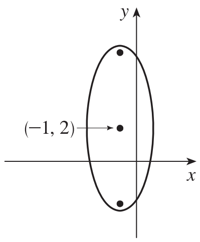
46. $4x^2 - y^2 = 16 \quad \Leftrightarrow \quad \dfrac{x^2}{4} - \dfrac{y^2}{16} = 1$ is a hyperbola with center $(0, 0)$, vertices $(\pm 2, 0)$, $a = 2$, $b = 4$, $c = \sqrt{16 + 4} = 2\sqrt{5}$, foci $\left(\pm 2\sqrt{5}, 0\right)$ and asymptotes $y = \pm 2x$.



47. $6y^2 + x - 36y + 55 = 0 \quad \Leftrightarrow$
$6(y^2 - 6y + 9) = -(x+1) \quad \Leftrightarrow$
$(y-3)^2 = -\frac{1}{6}(x+1)$, a parabola with vertex $(-1,3)$, opening to the left, $p = -\frac{1}{24} \quad \Rightarrow \quad$ focus $\left(-\frac{25}{24}, 3\right)$ and directrix $x = -\frac{23}{24}$.



48. $25x^2 + 4y^2 + 50x - 16y = 59 \quad \Leftrightarrow$
$25(x+1)^2 + 4(y-2)^2 = 100 \quad \Leftrightarrow$
$\frac{1}{4}(x+1)^2 + \frac{1}{25}(y-2)^2 = 1$ is an ellipse centered at $(-1,2)$ with foci on the line $x=-1$, vertices $(-1,7)$ and $(-1,-3)$; $a=5$, $b=2 \quad \Rightarrow \quad c = \sqrt{21} \quad \Rightarrow$ foci $\left(-1, 2 \pm \sqrt{21}\right)$.



49. The ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$ has center $(0,0)$ and a horizontal major axis, with $a=5$ and $c=4$, so $b^2 = a^2 - c^2 = 5^2 - 4^2 = 9$. An equation is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$.

50. The distance from the focus $(2,1)$ to the directrix $x=-4$ is $2-(-4)=6$, so the distance from the focus to the vertex is $\frac{1}{2}(6) = 3$ and the vertex is $(-1,1)$. Since the focus is to the right of the vertex, $p=3$. An equation is $(y-1)^2 = 4 \cdot 3[x-(-1)]$, or $(y-1)^2 = 12(x+1)$.

51. The center of a hyperbola with foci $(0, \pm 4)$ is $(0,0)$, so $c=4$ and an equation is $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$.
The asymptote $y = 3x$ has slope 3, so $\dfrac{a}{b} = \dfrac{3}{1} \quad \Rightarrow \quad a = 3b$ and $a^2 + b^2 = c^2 \quad \Rightarrow \quad (3b)^2 + b^2 = 4^2 \quad \Rightarrow$
$10b^2 = 16 \quad \Rightarrow \quad b^2 = \frac{8}{5}$ and so $a^2 = 16 - \frac{8}{5} = \frac{72}{5}$. Thus, an equation is $\dfrac{y^2}{72/5} - \dfrac{x^2}{8/5} = 1$, or $\dfrac{5y^2}{72} - \dfrac{5x^2}{8} = 1$.

52. Center is $(3,0)$, and $a = \frac{8}{2} = 4$, $c = 2 \quad \Leftrightarrow \quad b = \sqrt{4^2 - 2^2} = \sqrt{12} \quad \Rightarrow$
an equation of the ellipse is $\dfrac{(x-3)^2}{12} + \dfrac{y^2}{16} = 1$.

53. $x^2 = -(y-100)$ has its vertex at $(0,100)$, so one of the vertices of the ellipse is $(0,100)$. Another form of the equation of a parabola is $x^2 = 4p(y-100)$ so $4p(y-100) = -(y-100) \quad \Rightarrow \quad 4p = -1 \quad \Rightarrow \quad p = -\frac{1}{4}$. Therefore the shared focus is found at $\left(0, \frac{399}{4}\right)$ so $2c = \frac{399}{4} - 0 \quad \Rightarrow \quad c = \frac{399}{8}$ and the center of the ellipse is $\left(0, \frac{399}{8}\right)$. So $a = 100 - \frac{399}{8} = \frac{401}{8}$ and $b^2 = a^2 - c^2 = \dfrac{401^2 - 399^2}{8^2} = 25$. So the equation of the ellipse is $\dfrac{x^2}{b^2} + \dfrac{\left(y - \frac{399}{8}\right)^2}{a^2} = 1 \quad \Rightarrow \quad \dfrac{x^2}{25} + \dfrac{\left(y - \frac{399}{8}\right)^2}{\left(\frac{401}{8}\right)^2} = 1$,
or $\dfrac{x^2}{25} + \dfrac{(8y-399)^2}{160{,}801} = 1$.

54. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\dfrac{x}{y}$. Therefore $\dfrac{dy}{dx} = m \quad \Leftrightarrow \quad y = -\dfrac{b^2}{a^2}\dfrac{x}{m}$. Combining this condition with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we find that $x = \pm \dfrac{a^2 m}{\sqrt{a^2 m^2 + b^2}}$. In other words, the two points on the ellipse where the

tangent has slope m are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$. The tangent lines at these points have the equations $y \pm \frac{b^2}{\sqrt{a^2m^2+b^2}} = m\left(x \mp \frac{a^2m}{\sqrt{a^2m^2+b^2}}\right)$ or $y = mx \mp \frac{a^2m^2}{\sqrt{a^2m^2+b^2}} \mp \frac{b^2}{\sqrt{a^2m^2+b^2}} = mx \mp \sqrt{a^2m^2+b^2}$.

55. Directrix $x = 4 \Rightarrow d = 4$, so $e = \frac{1}{3} \Rightarrow r = \frac{ed}{1+e\cos\theta} = \frac{4}{3+\cos\theta}$.

56. See the end of the proof of Theorem 10.6.1. If $e > 1$, then $1 - e^2 < 0$ and Equations 10.6.4 become $a^2 = \frac{e^2d^2}{(e^2-1)^2}$ and $b^2 = \frac{e^2d^2}{e^2-1}$, so $\frac{b^2}{a^2} = e^2 - 1$. The asymptotes $y = \pm\frac{b}{a}x$ have slopes $\pm\frac{b}{a} = \pm\sqrt{e^2-1}$, so the angles they make with the polar axis are $\pm\tan^{-1}\left[\sqrt{e^2-1}\right] = \cos^{-1}(\pm 1/e)$.

57. (a) If (a, b) lies on the curve, then there is some parameter value t_1 such that $\frac{3t_1}{1+t_1^3} = a$ and $\frac{3t_1^2}{1+t_1^3} = b$. If $t_1 = 0$, the point is $(0, 0)$, which lies on the line $y = x$. If $t_1 \neq 0$, then the point corresponding to $t = \frac{1}{t_1}$ is given by $x = \frac{3(1/t_1)}{1+(1/t_1)^3} = \frac{3t_1^2}{t_1^3+1} = b$, $y = \frac{3(1/t_1)^2}{1+(1/t_1)^3} = \frac{3t_1}{t_1^3+1} = a$. So (b, a) also lies on the curve. [Another way to see this is to do part (e) first; the result is immediate.] The curve intersects the line $y = x$ when $\frac{3t}{1+t^3} = \frac{3t^2}{1+t^3} \Rightarrow t = t^2 \Rightarrow t = 0$ or 1, so the points are $(0, 0)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$.

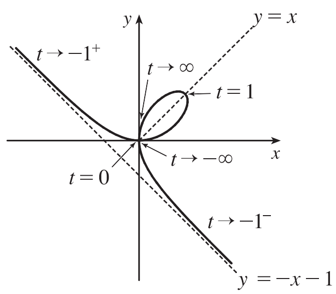
(b) $\frac{dy}{dt} = \frac{(1+t^3)(6t) - 3t^2(3t^2)}{(1+t^3)^2} = \frac{6t - 3t^4}{(1+t^3)^2} = 0$ when $6t - 3t^4 = 3t(2 - t^3) = 0 \Rightarrow t = 0$ or $t = \sqrt[3]{2}$, so there are horizontal tangents at $(0, 0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using the symmetry from part (a), we see that there are vertical tangents at $(0, 0)$ and $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$.

(c) Notice that as $t \to -1^+$, we have $x \to -\infty$ and $y \to \infty$. As $t \to -1^-$, we have $x \to \infty$ and $y \to -\infty$. Also $y - (-x - 1) = y + x + 1 = \frac{3t + 3t^2 + (1+t^3)}{1+t^3} = \frac{(t+1)^3}{1+t^3} = \frac{(t+1)^2}{t^2 - t + 1} \to 0$ as $t \to -1$. So $y = -x - 1$ is a slant asymptote.

(d) $\frac{dx}{dt} = \frac{(1+t^3)(3) - 3t(3t^2)}{(1+t^3)^2} = \frac{3 - 6t^3}{(1+t^3)^2}$ and from part (b) we have $\frac{dy}{dt} = \frac{6t - 3t^4}{(1+t^3)^2}$. So $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t(2-t^3)}{1-2t^3}$.

Also $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{2(1+t^3)^4}{3(1-2t^3)^3} > 0 \Leftrightarrow t < \frac{1}{\sqrt[3]{2}}$.

So the curve is concave upward there and has a minimum point at $(0, 0)$ and a maximum point at $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using this together with the information from parts (a), (b), and (c), we sketch the curve.



(e) $x^3 + y^3 = \left(\dfrac{3t}{1+t^3}\right)^3 + \left(\dfrac{3t^2}{1+t^3}\right)^3 = \dfrac{27t^3 + 27t^6}{(1+t^3)^3} = \dfrac{27t^3(1+t^3)}{(1+t^3)^3} = \dfrac{27t^3}{(1+t^3)^2}$ and

$3xy = 3\left(\dfrac{3t}{1+t^3}\right)\left(\dfrac{3t^2}{1+t^3}\right) = \dfrac{27t^3}{(1+t^3)^2}$, so $x^3 + y^3 = 3xy$.

(f) We start with the equation from part (e) and substitute $x = r\cos\theta$, $y = r\sin\theta$. Then $x^3 + y^3 = 3xy \Rightarrow$ $r^3\cos^3\theta + r^3\sin^3\theta = 3r^2\cos\theta\,\sin\theta$. For $r \neq 0$, this gives $r = \dfrac{3\cos\theta\,\sin\theta}{\cos^3\theta + \sin^3\theta}$. Dividing numerator and denominator by $\cos^3\theta$, we obtain $r = \dfrac{3\left(\dfrac{1}{\cos\theta}\right)\dfrac{\sin\theta}{\cos\theta}}{1 + \dfrac{\sin^3\theta}{\cos^3\theta}} = \dfrac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}$.

(g) The loop corresponds to $\theta \in \left(0, \frac{\pi}{2}\right)$, so its area is

$$A = \int_0^{\pi/2} \frac{r^2}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}\right)^2 d\theta = \frac{9}{2}\int_0^{\pi/2}\frac{\sec^2\theta\,\tan^2\theta}{(1+\tan^3\theta)^2}\,d\theta = \frac{9}{2}\int_0^{\infty}\frac{u^2\,du}{(1+u^3)^2} \quad [\text{let } u = \tan\theta]$$

$$= \lim_{b\to\infty} \tfrac{9}{2}\left[-\tfrac{1}{3}(1+u^3)^{-1}\right]_0^b = \tfrac{3}{2}$$

(h) By symmetry, the area between the folium and the line $y = -x - 1$ is equal to the enclosed area in the third quadrant, plus twice the enclosed area in the fourth quadrant. The area in the third quadrant is $\frac{1}{2}$, and since $y = -x - 1 \Rightarrow$ $r\sin\theta = -r\cos\theta - 1 \Rightarrow r = -\dfrac{1}{\sin\theta + \cos\theta}$, the area in the fourth quadrant is

$\dfrac{1}{2}\displaystyle\int_{-\pi/2}^{-\pi/4}\left[\left(-\frac{1}{\sin\theta+\cos\theta}\right)^2 - \left(\frac{3\sec\theta\,\tan\theta}{1+\tan^3\theta}\right)^2\right] d\theta \overset{\text{CAS}}{=} \dfrac{1}{2}$. Therefore, the total area is $\frac{1}{2} + 2\left(\frac{1}{2}\right) = \frac{3}{2}$.

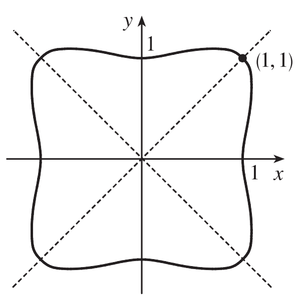
PROBLEMS PLUS

1. $x = \int_1^t \frac{\cos u}{u}\,du$, $y = \int_1^t \frac{\sin u}{u}\,du$, so by FTC1, we have $\frac{dx}{dt} = \frac{\cos t}{t}$ and $\frac{dy}{dt} = \frac{\sin t}{t}$. Vertical tangent lines occur when $\frac{dx}{dt} = 0 \quad \Leftrightarrow \quad \cos t = 0$. The parameter value corresponding to $(x, y) = (0, 0)$ is $t = 1$, so the nearest vertical tangent occurs when $t = \frac{\pi}{2}$. Therefore, the arc length between these points is

$$L = \int_1^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^{\pi/2} \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}}\,dt = \int_1^{\pi/2} \frac{dt}{t} = \big[\ln t\big]_1^{\pi/2} = \ln \tfrac{\pi}{2}$$

2. (a) The curve $x^4 + y^4 = x^2 + y^2$ is symmetric about both axes and about the line $y = x$ (since interchanging x and y does not change the equation) so we need only consider $y \geq x \geq 0$ to begin with. Implicit differentiation gives $4x^3 + 4y^3 y' = 2x + 2yy' \quad \Rightarrow \quad y' = \frac{x(1 - 2x^2)}{y(2y^2 - 1)} \quad \Rightarrow \quad y' = 0$ when $x = 0$ and when $x = \pm\frac{1}{\sqrt{2}}$. If $x = 0$, then $y^4 = y^2 \quad \Rightarrow \quad y^2(y^2 - 1) = 0 \quad \Rightarrow \quad y = 0$ or ± 1. The point $(0, 0)$ can't be a highest or lowest point because it is isolated. [If $-1 < x < 1$ and $-1 < y < 1$, then $x^4 < x^2$ and $y^4 < y^2 \quad \Rightarrow \quad x^4 + y^4 < x^2 + y^2$, except for $(0, 0)$.] If $x = \frac{1}{\sqrt{2}}$, then $x^2 = \frac{1}{2}$, $x^4 = \frac{1}{4}$, so $\frac{1}{4} + y^4 = \frac{1}{2} + y^2 \quad \Rightarrow \quad 4y^4 - 4y^2 - 1 = 0 \quad \Rightarrow \quad y^2 = \frac{4 \pm \sqrt{16+16}}{8} = \frac{1 \pm \sqrt{2}}{2}$. But $y^2 > 0$, so $y^2 = \frac{1+\sqrt{2}}{2} \quad \Rightarrow \quad y = \pm\sqrt{\frac{1}{2}\left(1 + \sqrt{2}\right)}$. Near the point $(0, 1)$, the denominator of y' is positive and the numerator changes from negative to positive as x increases through 0, so $(0, 1)$ is a local minimum point. At $\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$, y' changes from positive to negative, so that point gives a maximum. By symmetry, the highest points on the curve are $\left(\pm\frac{1}{\sqrt{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right)$ and the lowest points are $\left(\pm\frac{1}{\sqrt{2}}, -\sqrt{\frac{1+\sqrt{2}}{2}}\right)$.

(b) We use the information from part (a), together with symmetry with respect to the axes and the lines $y = \pm x$, to sketch the curve.

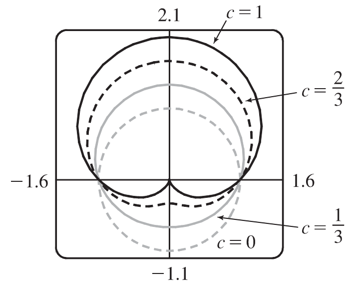


(c) In polar coordinates, $x^4 + y^4 = x^2 + y^2$ becomes $r^4 \cos^4\theta + r^4 \sin^4\theta = r^2$ or $r^2 = \frac{1}{\cos^4\theta + \sin^4\theta}$. By the symmetry shown in part (b), the area enclosed by the curve is $A = 8\int_0^{\pi/4} \frac{1}{2}\,r^2\,d\theta = 4\int_0^{\pi/4} \frac{d\theta}{\cos^4\theta + \sin^4\theta} \overset{\text{CAS}}{=} \sqrt{2}\,\pi$.

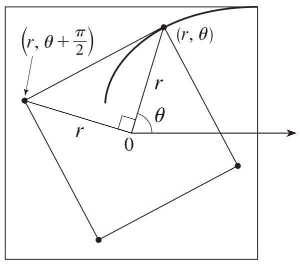
3. In terms of x and y, we have $x = r\cos\theta = (1 + c\sin\theta)\cos\theta = \cos\theta + c\sin\theta\cos\theta = \cos\theta + \frac{1}{2}c\sin 2\theta$ and $y = r\sin\theta = (1 + c\sin\theta)\sin\theta = \sin\theta + c\sin^2\theta$. Now $-1 \leq \sin\theta \leq 1 \quad \Rightarrow \quad -1 \leq \sin\theta + c\sin^2\theta \leq 1 + c \leq 2$, so $-1 \leq y \leq 2$. Furthermore, $y = 2$ when $c = 1$ and $\theta = \frac{\pi}{2}$, while $y = -1$ for $c = 0$ and $\theta = \frac{3\pi}{2}$. Therefore, we need a viewing rectangle with $-1 \leq y \leq 2$.

To find the x-values, look at the equation $x = \cos\theta + \frac{1}{2}c\sin 2\theta$ and use the fact that $\sin 2\theta \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$ and $\sin 2\theta \leq 0$ for $-\frac{\pi}{2} \leq \theta \leq 0$. [Because $r = 1 + c\sin\theta$ is symmetric about the y-axis, we only need to consider

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.] So for $-\frac{\pi}{2} \le \theta \le 0$, x has a maximum value when $c = 0$ and then $x = \cos\theta$ has a maximum value of 1 at $\theta = 0$. Thus, the maximum value of x must occur on $\left[0, \frac{\pi}{2}\right]$ with $c = 1$. Then $x = \cos\theta + \frac{1}{2}\sin 2\theta \quad\Rightarrow$ $\frac{dx}{d\theta} = -\sin\theta + \cos 2\theta = -\sin\theta + 1 - 2\sin^2\theta \quad\Rightarrow\quad \frac{dx}{d\theta} = -(2\sin\theta - 1)(\sin\theta + 1) = 0$ when $\sin\theta = -1$ or $\frac{1}{2}$ [but $\sin\theta \ne -1$ for $0 \le \theta \le \frac{\pi}{2}$]. If $\sin\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$ and $x = \cos\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3} = \frac{3}{4}\sqrt{3}$. Thus, the maximum value of x is $\frac{3}{4}\sqrt{3}$, and, by symmetry, the minimum value is $-\frac{3}{4}\sqrt{3}$. Therefore, the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c\sin\theta$, where $0 \le c \le 1$, is $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$.



4. (a) Let us find the polar equation of the path of the bug that starts in the upper right corner of the square. If the polar coordinates of this bug, at a particular moment, are (r, θ), then the polar coordinates of the bug that it is crawling toward must be $\left(r, \theta + \frac{\pi}{2}\right)$. (The next bug must be the same distance from the origin and the angle between the lines joining the bugs to the pole must be $\frac{\pi}{2}$.) The Cartesian coordinates of the first bug are $(r\cos\theta, r\sin\theta)$ and for the second bug we have $x = r\cos\left(\theta + \frac{\pi}{2}\right) = -r\sin\theta$, $y = r\sin\left(\theta + \frac{\pi}{2}\right) = r\cos\theta$. So the slope of the line joining the bugs is $\dfrac{r\cos\theta - r\sin\theta}{-r\sin\theta - r\cos\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$. This must be equal to the slope of the tangent line at (r, θ), so by Equation 10.3.3 we have $\dfrac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$. Solving for $\dfrac{dr}{d\theta}$, we get



$\dfrac{dr}{d\theta}\sin^2\theta + \dfrac{dr}{d\theta}\sin\theta\cos\theta + r\sin\theta\cos\theta + r\cos^2\theta = \dfrac{dr}{d\theta}\sin\theta\cos\theta - \dfrac{dr}{d\theta}\cos^2\theta - r\sin^2\theta + r\sin\theta\cos\theta \Rightarrow$ $\dfrac{dr}{d\theta}\left(\sin^2\theta + \cos^2\theta\right) + r\left(\cos^2\theta + \sin^2\theta\right) = 0 \quad\Rightarrow\quad \dfrac{dr}{d\theta} = -r$. Solving this differential equation as a separable equation (as in Section 9.3), or using Theorem 9.4.2 with $k = -1$, we get $r = Ce^{-\theta}$. To determine C we use the fact that, at its starting position, $\theta = \frac{\pi}{4}$ and $r = \frac{1}{\sqrt{2}}a$, so $\frac{1}{\sqrt{2}}a = Ce^{-\pi/4} \quad\Rightarrow\quad C = \frac{1}{\sqrt{2}}ae^{\pi/4}$. Therefore, a polar equation of the bug's path is $r = \frac{1}{\sqrt{2}}ae^{\pi/4}e^{-\theta}$ or $r = \frac{1}{\sqrt{2}}ae^{(\pi/4)-\theta}$.

(b) The distance traveled by this bug is $L = \int_{\pi/4}^{\infty}\sqrt{r^2 + (dr/d\theta)^2}\,d\theta$, where $\dfrac{dr}{d\theta} = \dfrac{a}{\sqrt{2}}e^{\pi/4}(-e^{-\theta})$ and so $r^2 + (dr/d\theta)^2 = \frac{1}{2}a^2e^{\pi/2}e^{-2\theta} + \frac{1}{2}a^2e^{\pi/2}e^{-2\theta} = a^2e^{\pi/2}e^{-2\theta}$. Thus

$$L = \int_{\pi/4}^{\infty} ae^{\pi/4}e^{-\theta}\,d\theta = ae^{\pi/4}\lim_{t\to\infty}\int_{\pi/4}^{t} e^{-\theta}\,d\theta = ae^{\pi/4}\lim_{t\to\infty}\left[-e^{-\theta}\right]_{\pi/4}^{t}$$
$$= ae^{\pi/4}\lim_{t\to\infty}\left[e^{-\pi/4} - e^{-t}\right] = ae^{\pi/4}e^{-\pi/4} = a$$

5. Without loss of generality, assume the hyperbola has equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Use implicit differentiation to get $\dfrac{2x}{a^2} - \dfrac{2y\,y'}{b^2} = 0$, so $y' = \dfrac{b^2x}{a^2y}$. The tangent line at the point (c, d) on the hyperbola has equation $y - d = \dfrac{b^2c}{a^2d}(x - c)$.

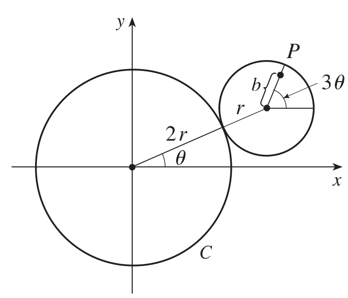
The tangent line intersects the asymptote $y = \dfrac{b}{a}x$ when $\dfrac{b}{a}x - d = \dfrac{b^2c}{a^2d}(x - c) \;\Rightarrow\; abdx - a^2d^2 = b^2cx - b^2c^2 \;\Rightarrow$ $abdx - b^2cx = a^2d^2 - b^2c^2 \;\Rightarrow\; x = \dfrac{a^2d^2 - b^2c^2}{b(ad - bc)} = \dfrac{ad + bc}{b}$ and the y-value is $\dfrac{b}{a}\,\dfrac{ad + bc}{b} = \dfrac{ad + bc}{a}$.

Similarly, the tangent line intersects $y = -\dfrac{b}{a}x$ at $\left(\dfrac{bc - ad}{b}, \dfrac{ad - bc}{a}\right)$. The midpoint of these intersection points is

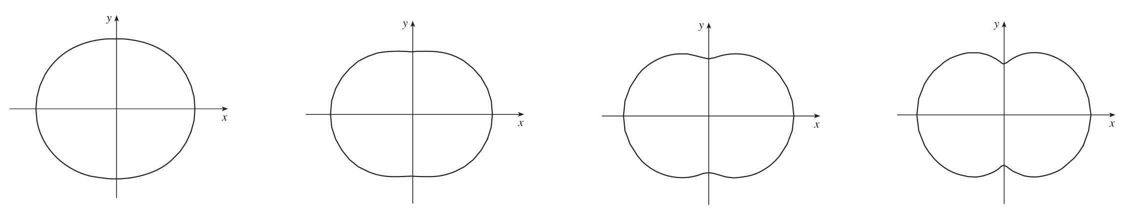
$\left(\dfrac{1}{2}\left(\dfrac{ad + bc}{b} + \dfrac{bc - ad}{b}\right), \dfrac{1}{2}\left(\dfrac{ad + bc}{a} + \dfrac{ad - bc}{a}\right)\right) = \left(\dfrac{1}{2}\dfrac{2bc}{b}, \dfrac{1}{2}\dfrac{2ad}{a}\right) = (c, d)$, the point of tangency.

Note: If $y = 0$, then at $(\pm a, 0)$, the tangent line is $x = \pm a$, and the points of intersection are clearly equidistant from the point of tangency.

6. (a) Since the smaller circle rolls without slipping around C, the amount of arc traversed on C ($2r\theta$ in the figure) must equal the amount of arc of the smaller circle that has been in contact with C. Since the smaller circle has radius r, it must have turned through an angle of $2r\theta/r = 2\theta$. In addition to turning through an angle 2θ, the little circle has rolled through an angle θ against C. Thus, P has turned through an angle of 3θ as shown in the figure. (If the little circle had turned through an angle of 2θ with its center pinned to the x-axis, then P would have turned only 2θ instead of 3θ. The movement of the little circle around C adds θ to the angle.) From the figure, we see that the center of the small circle has coordinates $(3r\cos\theta, 3r\sin\theta)$. Thus, P has coordinates (x, y), where $x = b\cos 3\theta + 3r\cos\theta$ and $y = b\sin 3\theta + 3r\sin\theta$.

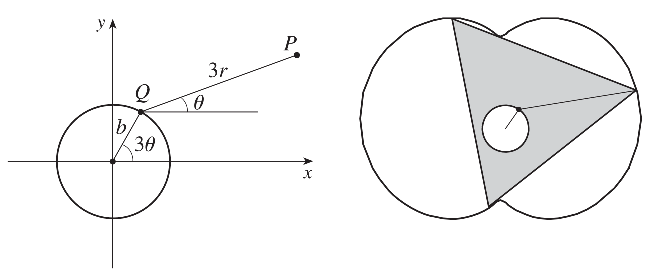


(b)



$b = \frac{1}{5}r$ $b = \frac{2}{5}r$ $b = \frac{3}{5}r$ $b = \frac{4}{5}r$

(c) The diagram gives an alternate description of point P on the epitrochoid. Q moves around a circle of radius b, and P rotates one-third as fast with respect to Q at a distance of $3r$. Place an equilateral triangle with sides of length $3\sqrt{3}r$ so that its centroid is at Q and one vertex is at P. (The distance from the centroid to a vertex is $\frac{1}{\sqrt{3}}$ times the length of a side of the equilateral triangle.)



As θ increases by $\frac{2\pi}{3}$, the point Q travels once around the circle of radius b, returning to its original position. At the same time, P (and the rest of the triangle) rotate through an angle of $\frac{2\pi}{3}$ about Q, so P's position is occupied by another vertex. In this way, we see that the epitrochoid traced out by P is simultaneously traced out by the other two vertices as well. The whole equilateral triangle sits inside the epitrochoid (touching it only with its vertices) and each vertex traces out the curve once while the centroid moves around the circle three times.

(d) We view the epitrochoid as being traced out in the same way as in part (c), by a rotor for which the distance from its center to each vertex is $3r$, so it has radius $6r$. To show that the rotor fits inside the epitrochoid, it suffices to show that for any position of the tracing point P, there are no points on the opposite side of the rotor which are outside the epitrochoid. But the most likely case of intersection is when P is on the y-axis, so as long as the diameter of the rotor $\left(\text{which is } 3\sqrt{3}\,r\right)$ is less than the distance between the y-intercepts, the rotor will fit. The y-intercepts occur when $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ $\Rightarrow$ $y = -b + 3r$ or $y = b - 3r$, so the distance between the intercepts is $(-b + 3r) - (b - 3r) = 6r - 2b$, and the rotor will fit if $3\sqrt{3}\,r \leq 6r - 2b \Leftrightarrow 2b \leq 6r - 3\sqrt{3}\,r \Leftrightarrow b \leq \frac{3}{2}\left(2 - \sqrt{3}\right)r$.