Chapter 1. Heat Equation

Section 1.2

1.2.9 (d) Circular cross section means that $P = 2\pi r, A = \pi r^2$, and thus $P/A = 2/r$, where r is the radius. Also $\gamma = 0$.

1.2.9 (e) $u(x,t) = u(t)$ implies that

$$c\rho\,\frac{du}{dt} = -\frac{2h}{r}u\,.$$

The solution of this first-order linear differential equation with constant coefficients, which satisfies the initial condition $u(0) = u_0$, is

$$u(t) = u_0 \exp\left[-\frac{2h}{c\rho r}t\right].$$

Section 1.3

1.3.2 $\partial u/\partial x$ is continuous if $K_0(x_0-) = K_0(x_0+)$, that is, if the conductivity is continuous.

Section 1.4

1.4.1 (a) Equilibrium satisfies (1.4.14), $d^2u/dx^2 = 0$, whose general solution is (1.4.17), $u = c_1 + c_2 x$. The boundary condition $u(0) = 0$ implies $c_1 = 0$ and $u(L) = T$ implies $c_2 = T/L$ so that $u = Tx/L$.

1.4.1 (d) Equilibrium satisfies (1.4.14), $d^2u/dx^2 = 0$, whose general solution (1.4.17), $u = c_1 + c_2 x$. From the boundary conditions, $u(0) = T$ yields $T = c_1$ and $du/dx(L) = \alpha$ yields $\alpha = c_2$. Thus $u = T + \alpha x$.

1.4.1 (f) In equilibrium, (1.2.9) becomes $d^2u/dx^2 = -Q/K_0 = -x^2$, whose general solution (by integrating twice) is $u = -x^4/12 + c_1 + c_2 x$. The boundary condition $u(0) = T$ yields $c_1 = T$, while $du/dx(L) = 0$ yields $c_2 = L^3/3$. Thus $u = -x^4/12 + L^3 x/3 + T$.

1.4.1 (h) Equilibrium satisfies $d^2u/dx^2 = 0$. One integration yields $du/dx = c_2$, the second integration yields the general solution $u = c_1 + c_2 x$.

$$\begin{aligned} x = 0:&\quad c_2 - (c_1 - T) = 0 \\ x = L:&\quad c_2 = \alpha \text{ and thus } c_1 = T + \alpha. \end{aligned}$$

Therefore, $u = (T + \alpha) + \alpha x = T + \alpha(x + 1)$.

1.4.7 (a) For equilibrium:

$$\frac{d^2u}{dx^2} = -1 \text{ implies } u = -\frac{x^2}{2} + c_1 x + c_2 \text{ and } \frac{du}{dx} = -x + c_1.$$

From the boundary conditions $\frac{du}{dx}(0) = 1$ and $\frac{du}{dx}(L) = \beta, c_1 = 1$ and $-L + c_1 = \beta$ which is consistent only if $\beta + L = 1$. If $\beta = 1 - L$, there is an equilibrium solution ($u = -\frac{x^2}{2} + x + c_2$). If $\beta \neq 1 - L$, there isn't an equilibrium solution. The difficulty is caused by the heat flow being specified at both ends and a source specified inside. An equilibrium will exist only if these three are in balance. This balance can be mathematically verified from conservation of energy:

$$\frac{d}{dt}\int_0^L c\rho u\;dx = -\frac{du}{dx}(0) + \frac{du}{dx}(L) + \int_0^L Q_0\;dx = -1 + \beta + L.$$

If $\beta + L = 1$, then the total thermal energy is constant and the initial energy = the final energy:

$$\int_0^L f(x)\;dx = \int_0^L \left(-\frac{x^2}{2} + x + c_2\right)\;dx, \quad \text{which determines} \quad c_2.$$

If $\beta + L \neq 1$, then the total thermal energy is always changing in time and an equilibrium is never reached.

Section 1.5

1.5.9 (a) In equilibrium, (1.5.14) using (1.5.19) becomes $\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$. Integrating once yields $rdu/dr = c_1$ and integrating a second time (after dividing by r) yields $u = c_1 \ln r + c_2$. An alternate general solution is $u = c_1 \ln(r/r_1) + c_3$. The boundary condition $u(r_1) = T_1$ yields $c_3 = T_1$, while $u(r_2) = T_2$ yields $c_1 = (T_2 - T_1)/\ln(r_2/r_1)$. Thus, $u = \frac{1}{\ln(r_2/r_1)}\left[(T_2 - T_1)\ln r/r_1 + T_1 \ln(r_2/r_1)\right]$.

1.5.11 For equilibrium, the radial flow at $r = a$, $2\pi a\beta$, must equal the radial flow at $r = b$, $2\pi b$. Thus $\beta = b/a$.

1.5.13 From exercise 1.5.12, in equilibrium $\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$. Integrating once yields $r^2 du/dr = c_1$ and integrating a second time (after dividing by r^2) yields $u = -c_1/r + c_2$. The boundary conditions $u(4) = 80$ and $u(1) = 0$ yields $80 = -c_1/4 + c_2$ and $0 = -c_1 + c_2$. Thus $c_1 = c_2 = 320/3$ or $u = \frac{320}{3}\left(1 - \frac{1}{r}\right)$.

Chapter 2. Method of Separation of Variables

Section 2.3

2.3.1 (a) $u(r,t) = \phi(r)h(t)$ yields $\phi \frac{dh}{dt} = \frac{kh}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)$. Dividing by $k\phi h$ yields $\frac{1}{kh}\frac{dh}{dt} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda$ or $\frac{dh}{dt} = -\lambda kh$ and $\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda\phi$.

2.3.1 (c) $u(x,y) = \phi(x)h(y)$ yields $h\frac{d^2\phi}{dx^2} + \phi\frac{d^2h}{dy^2} = 0$. Dividing by ϕh yields $\frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\frac{1}{h}\frac{d^2h}{dy^2} = -\lambda$ or $\frac{d^2\phi}{dx^2} = -\lambda\phi$ and $\frac{d^2h}{dy^2} = \lambda h$.

2.3.1 (e) $u(x,t) = \phi(x)h(t)$ yields $\phi(x)\frac{dh}{dt} = kh(t)\frac{d^4\phi}{dx^4}$. Dividing by $k\phi h$, yields $\frac{1}{kh}\frac{dh}{dt} = \frac{1}{\phi}\frac{d^4\phi}{dx^4} = \lambda$.

2.3.1 (f) $u(x,t) = \phi(x)h(t)$ yields $\phi(x)\frac{d^2h}{dt^2} = c^2h(t)\frac{d^2\phi}{dx^2}$. Dividing by $c^2\phi h$, yields $\frac{1}{c^2h}\frac{d^2h}{dt^2} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$.

2.3.2 (b) $\lambda = (n\pi/L)^2$ with $L = 1$ so that $\lambda = n^2\pi^2,\ n = 1, 2, \ldots$

2.3.2 (d)

(i) If $\lambda > 0, \phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. $\phi(0) = 0$ implies $c_1 = 0$, while $\frac{d\phi}{dx}(L) = 0$ implies $c_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$. Thus $\sqrt{\lambda}L = -\pi/2 + n\pi (n = 1, 2, \ldots)$.

(ii) If $\lambda = 0, \phi = c_1 + c_2x$. $\phi(0) = 0$ implies $c_1 = 0$ and $d\phi/dx(L) = 0$ implies $c_2 = 0$. Therefore $\lambda = 0$ is not an eigenvalue.

(iii) If $\lambda < 0$, let $\lambda = -s$ and $\phi = c_1 \cosh\sqrt{s}x + c_2 \sinh\sqrt{s}x$. $\phi(0) = 0$ implies $c_1 = 0$ and $d\phi/dx(L) = 0$ implies $c_2\sqrt{s}\cosh\sqrt{s}L = 0$. Thus $c_2 = 0$ and hence there are no eigenvalues with $\lambda < 0$.

2.3.2 (f) The simpliest method is to let $x' = x - a$. Then $d^2\phi/dx'^2 + \lambda\phi = 0$ with $\phi(0) = 0$ and $\phi(b-a) = 0$. Thus (from p. 46) $L = b - a$ and $\lambda = [n\pi/(b-a)]^2,\ n = 1, 2, \ldots$.

2.3.3 From (2.3.30), $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2t}$. The initial condition yields $2\cos\frac{3\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$. From (2.3.35), $B_n = \frac{2}{L}\int_0^L 2\cos\frac{3\pi x}{L}\sin\frac{n\pi x}{L}\ dx$.

2.3.4 (a) Total heat energy $= \int_0^L c\rho uA\ dx = c\rho A\sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{L}\right)^2t}\frac{1-\cos n\pi}{\frac{n\pi}{L}}$, using (2.3.30) where B_n satisfies (2.3.35).

2.3.4 (b)

heat flux to right $= -K_0\partial u/\partial x$
total heat flow to right $= -K_0A\partial u/\partial x$
heat flow out at $x = 0 = K_0A\frac{\partial u}{\partial x}\Big|_{x=0}$
heat flow out $(x = L) = -K_0A\frac{\partial u}{\partial x}\Big|_{x=L}$

2.3.4 (c) From conservation of thermal energy, $\frac{d}{dt}\int_0^L u\ dx = k\frac{\partial u}{\partial x}\Big|_0^L = k\frac{\partial u}{\partial x}(L) - k\frac{\partial u}{\partial x}(0)$. Integrating from $t = 0$ yields

$$\underbrace{\int_0^L u(x,t)\ dx}_{\text{heat energy at } t} - \underbrace{\int_0^L u(x,0)\ dx}_{\text{initial heat energy}} = k\underbrace{\int_0^t \left[\frac{\partial u}{\partial x}(L)\right.}_{\text{integral of flow in at } x=L} - \underbrace{\left.\frac{\partial u}{\partial x}(0)\right] dx}_{\text{integral of flow out at } x=L}\ .$$

2.3.8 (a) The general solution of $k\frac{d^2u}{dx^2} = \alpha u\ (\alpha > 0)$ is $u(x) = a\cosh\sqrt{\frac{\alpha}{k}}x + b\sinh\sqrt{\frac{\alpha}{k}}x$. The boundary condition $u(0) = 0$ yields $a = 0$, while $u(L) = 0$ yields $b = 0$. Thus $u = 0$.

2.3.8 (b) Separation of variables, $u = \phi(x)h(t)$ or $\phi\frac{dh}{dt} + \alpha\phi h = kh\frac{d^2\phi}{dx^2}$, yields two ordinary differential equations (divide by $k\phi h$): $\frac{1}{kh}\frac{dh}{dt} + \frac{\alpha}{k} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$. Applying the boundary conditions, yields the eigenvalues $\lambda = (n\pi/L)^2$ and corresponding eigenfunctions $\phi = \sin\frac{n\pi x}{L}$. The time-dependent part are exponentials, $h = e^{-\lambda kt}e^{-\alpha t}$. Thus by superposition, $u(x,t) = e^{-\alpha t}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}$, where the initial conditions $u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ yields $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$. As $t \to \infty$, $u \to 0$, the only equilibrium solution.

2.3.9 (a) If $\alpha < 0$, the general equilibrium solution is $u(x) = a\cos\sqrt{\frac{-\alpha}{k}}x + b\sin\sqrt{\frac{-\alpha}{k}}x$. The boundary condition $u(0) = 0$ yields $a = 0$, while $u(L) = 0$ yields $b\sin\sqrt{\frac{-\alpha}{k}}L = 0$. Thus if $\sqrt{\frac{-\alpha}{k}}L \neq n\pi, u = 0$ is the only equilibrium solution. However, if $\sqrt{\frac{-\alpha}{k}}L = n\pi$, then $u = A\sin\frac{n\pi x}{L}$ is an equilibrium solution.

2.3.9 (b) Solution obtained in 2.3.8 is correct. If $-\frac{\alpha}{k} = \left(\frac{\pi}{L}\right)^2, u(x,t) \to b_1\sin\frac{\pi x}{L}$, the equilibrium solution. If $-\frac{\alpha}{k} < \left(\frac{\pi}{L}\right)^2$, then $u \to 0$ as $t \to \infty$. However, if $-\frac{\alpha}{k} > \left(\frac{\pi}{L}\right)^2, u \to \infty$ (if $b_1 \neq 0$). Note that $b_1 > 0$ if $f(x) \geq 0$. Other more unusual events can occur if $b_1 = 0$. [Essentially, the other possible equilibrium solutions are unstable.]

Section 2.4

2.4.1 The solution is given by (2.4.19), where the coefficients satisfy (2.4.21) and hence (2.4.23-24).

(a) $A_0 = \frac{1}{L}\int_{L/2}^{L} 1dx = \frac{1}{2}, A_n = \frac{2}{L}\int_{L/2}^{L}\cos\frac{n\pi x}{L}dx = \frac{2}{L}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{L/2}^{L} = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$

(b) by inspection $A_0 = 6, A_3 = 4$, others $= 0$.

(c) $A_0 = \frac{-2}{L}\int_0^L \sin\frac{\pi x}{L}\,dx = \frac{2}{\pi}\cos\frac{\pi x}{L}\Big|_0^L = \frac{2}{\pi}(1-\cos\pi) = 4/\pi, A_n = \frac{-4}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx$

(d) by inspection $A_8 = -3$, others $= 0$.

2.4.3 Let $x' = x - \pi$. Then the boundary value problem becomes $d^2\phi/dx'^2 = -\lambda\phi$ subject to $\phi(-\pi) = \phi(\pi)$ and $d\phi/dx'(-\pi) = d\phi/dx'(\pi)$. Thus, the eigenvalues are $\lambda = (n\pi/L)^2 = n^2\pi^2$, since $L = \pi, n = 0, 1, 2, ...$ with the corresponding eigenfunctions being both $\sin n\pi x'/L = \sin n(x-\pi) = (-1)^n \sin nx =>$ $\sin nx$ and $\cos n\pi x'/L = \cos n(x-\pi) = (-1)^n\cos nx => \cos nx$.

Section 2.5

2.5.1 (a) Separation of variables, $u(x,y) = h(x)\phi(y)$, implies that $\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = -\lambda$. Thus $d^2h/dx^2 = -\lambda h$ subject to $h'(0) = 0$ and $h'(L) = 0$. Thus as before, $\lambda = (n\pi/L)^2, n = 0, l, 2, \ldots$ with $h(x) = \cos n\pi x/L$. Furthermore, $\frac{d^2\phi}{dy^2} = \lambda\phi = \left(\frac{n\pi}{L}\right)^2\phi$ so that

$n = 0 : \phi = c_1 + c_2 y$, where $\phi(0) = 0$ yields $c_1 = 0$

$n \neq 0 : \phi = c_1\cosh\frac{n\pi y}{L} + c_2\sinh\frac{n\pi y}{L}$, where $\phi(0) = 0$ yields $c_1 = 0$.

The result of superposition is

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}.$$

The nonhomogeneous boundary condition yields

$$f(x) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi H}{L}\cos\frac{n\pi x}{L},$$

so that

$$A_0 H = \frac{1}{L}\int_0^L f(x)\,dx \text{ and } A_n\sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

2.5.1 (c) Separation of variables, $u = h(x)\phi(y)$, yields $\frac{1}{h}\frac{d^2h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = \lambda$. The boundary conditions $\phi(0) = 0$ and $\phi(H) = 0$ yield an eigenvalue problem in y, whose solution is $\lambda = (n\pi/H)^2$ with $\phi = \sin n\pi y/H, n = 1, 2, 3, \ldots$ The solution of the x-dependent equation is $h(x) = \cosh\ n\pi x/H$ using $dh/dx(0) = 0$. By superposition:

$$u(x, y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi x}{H} \sin \frac{n\pi y}{H}.$$

The nonhomogeneous boundary condition at $x = L$ yields $g(y) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi L}{H} \sin \frac{n\pi y}{H}$, so that A_n is determined by $A_n \cosh \frac{n\pi L}{H} = \frac{2}{H}\int_0^H g(y) \sin \frac{n\pi y}{H}\ dy$.

2.5.1 (e) Separation of variables, $u = \phi(x)h(y)$, yields the eigenvalues $\lambda = (n\pi/L)^2$ and corresponding eigenfunctions $\phi = \sin n\pi x/L, n = 1, 2, 3, \ldots$ The y-dependent differential equation, $\frac{d^2h}{dy^2} = \left(\frac{n\pi}{L}\right)^2 h$, satisfies $h(0) - \frac{dh}{dy}(0) = 0$. The general solution $h = c_1 \cosh \frac{n\pi y}{L} + c_2 \sinh \frac{n\pi y}{L}$ obeys $h(0) = c_1$, while $\frac{dh}{dy} = \frac{n\pi}{L}\left(c_1 \sinh \frac{n\pi y}{L} + c_2 \cosh \frac{n\pi y}{L}\right)$ obeys $\frac{dh}{dy}(0) = c_2 \frac{n\pi}{L}$. Thus, $c_1 = c_2 \frac{n\pi}{L}$ and hence $h_n(y) = \cosh \frac{n\pi y}{L} + \frac{L}{n\pi} \sinh \frac{n\pi y}{L}$. Superposition yields

$$u(x, y) = \sum_{n=1}^{\infty} A_n h_n(y) \sin n\pi x/L,$$

where A_n is determined from the boundary condition, $f(x) = \sum_{n=1}^{\infty} A_n h_n(H) \sin n\pi x/L$, and hence

$$A_n h_n(H) = \frac{2}{L}\int_0^L f(x) \sin n\pi x/L\ dx\ .$$

2.5.2 (a) From physical reasoning (or exercise 1.5.8), the total heat flow across the boundary must equal zero in equilibrium (without sources, i.e. Laplace's equation). Thus $\int_0^L f(x)\ dx = 0$ for a solution.

2.5.3 In order for u to be bounded as $r \to \infty, c_1 = 0$ in (2.5.43) and $\bar{c}_2 = 0$ in (2.5.44). Thus,

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^{-n} \cos n\theta + \sum_{n=1}^{\infty} B_n r^{-n} \sin n\theta.$$

(a) The boundary condition yields $A_0 = \ln 2, A_3 a^{-3} = 4$, other $A_n = 0, B_n = 0$.

(b) The boundary conditions yield (2.5.46) with a^{-n} replacing a^n. Thus, the coefficients are determined by (2.5.47) with a^n replaced by a^{-n}

2.5.4 By substituting (2.5.47) into (2.5.45) and interchanging the orders of summation and integration

$$u(r, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta}) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left(\cos n\theta \cos n\bar{\theta} + \sin n\theta \sin n\bar{\theta}\right)\right]\ d\bar{\theta}.$$

Noting the trigonometric addition formula and $\cos z = R_e[e^{iz}]$, we obtain

$$u(r, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\bar{\theta}) \left[-\frac{1}{2} + Re \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta - \bar{\theta})}\right]\ d\bar{\theta}.$$

Summing the geometric series enables the bracketed term to be replaced by

$$-\frac{1}{2} + Re \frac{1}{1 - \frac{r}{a} e^{i(\theta - \bar{\theta})}} = -\frac{1}{2} + \frac{1 - \frac{r}{a}\cos(\theta - \bar{\theta})}{1 + \frac{r^2}{a^2} - \frac{2r}{a}\cos(\theta - \bar{\theta})} = \frac{\frac{1}{2} - \frac{1}{2}\frac{r^2}{a^2}}{1 + \frac{r^2}{a^2} - \frac{2r}{a}\cos(\theta - \bar{\theta})}.$$

2.5.5 (a) The eigenvalue problem is $d^2\phi/d\theta^2 = -\lambda\phi$ subject to $d\phi/d\theta(0) = 0$ and $\phi(\pi/2) = 0$. It can be shown that $\lambda > 0$ so that $\phi = \cos\sqrt{\lambda}\theta$ where $\phi(\pi/2) = 0$ implies that $\cos\sqrt{\lambda}\pi/2 = 0$ or $\sqrt{\lambda}\pi/2 = -\pi/2 + n\pi, n = 1, 2, 3, \ldots$ The eigenvalues are $\lambda = (2n-1)^2$. The radially dependent term satisfies (2.5.40), and hence the boundedness condition at $r = 0$ yields $G(r) = r^{2n-1}$. Superposition yields

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n-1} \cos(2n-1)\theta.$$

The nonhomogeneous boundary condition becomes

$$f(\theta) = \sum_{n=1}^{\infty} A_n \cos(2n-1)\theta \quad \text{or} \quad A_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\cos(2n-1)\theta \, d\theta.$$

2.5.5 (c) The boundary conditions of (2.5.37) must be replaced by $\phi(0) = 0$ and $\phi(\pi/2) = 0$. Thus, $L = \pi/2$, so that $\lambda = (n\pi/L)^2 = (2n)^2$ and $\phi = \sin\frac{n\pi\theta}{L} = \sin 2n\theta$. The radial part that remains bounded at $r = 0$ is $G = r^{\sqrt{\lambda}} = r^{2n}$. By superposition,

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin 2n\theta \ .$$

To apply the nonhomogeneous boundary condition, we differentiate with respect to r:

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} A_n (2n) r^{2n-1} \sin 2n\theta \ .$$

The bc at $r = 1$, $f(\theta) = \sum_{n=1}^{\infty} 2nA_n \sin 2n\theta$, determines A_n, $2nA_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin 2n\theta \ d\theta$.

2.5.6 (a) The boundary conditions of (2.5.37) must be replaced by $\phi(0) = 0$ and $\phi(\pi) = 0$. Thus $L = \pi$, so that the eigenvalues are $\lambda = (n\pi/L)^2 = n^2$ and corresponding eigenfunctions $\phi = \sin n\pi\theta/L = \sin n\theta, n = 1, 2, 3, ...$ The radial part which is bounded at $r = 0$ is $G = r^{\sqrt{\lambda}} = r^n$. Thus by superposition

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta \ .$$

The bc at $r = a$, $g(\theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta$, determines A_n, $A_n a^n = \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin n\theta \ d\theta$.

2.5.7 (b) The boundary conditions of (2.5.37) must be replaced by $\phi'(0) = 0$ and $\phi'(\pi/3) = 0$. This will yield a cosine series with $L = \pi/3, \lambda = (n\pi/L)^2 = (3n)^2$ and $\phi = \cos n\pi\theta/L = \cos 3n\theta, n = 0, 1, 2, \ldots$. The radial part which is bounded at $r = 0$ is $G = r^{\sqrt{\lambda}} = r^{3n}$. Thus by superposition

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^{3n} \cos 3n\theta \ .$$

The boundary condition at $r = a$, $g(\theta) = \sum_{n=0}^{\infty} A_n a^{3n} \cos 3n\theta$, determines A_n: $A_0 = \frac{3}{\pi}\int_0^{\pi/3} g(\theta) \ d\theta$ and $(n \neq 0) A_n a^{3n} = \frac{6}{\pi}\int_0^{\pi/3} g(\theta)\cos 3n\theta \ d\theta$.

2.5.8 (a) There is a full Fourier series in θ. It is easier (but equivalent) to choose radial solutions that satisfy the corresponding homogeneous boundary condition. Instead of r^n and r^{-n} (1 and $\ln r$ for $n = 0$), we choose $\phi_1(r)$ such that $\phi_1(a) = 0$ and $\phi_2(r)$ such that $\phi_2(b) = 0$:

$$\phi_1(r) = \begin{cases} \ln(r/a) & n = 0 \\ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n & n \neq 0 \end{cases} \quad \phi_2(r) = \begin{cases} \ln(r/b) & n = 0 \\ \left(\frac{r}{b}\right)^n - \left(\frac{b}{r}\right)^n & n \neq 0 \end{cases} .$$

Then by superposition

$$u(r,\theta)=\sum_{n=0}^{\infty}\cos n\theta\left[A_n\phi_1(r)+B_n\phi_2(r)\right]+\sum_{n=1}^{\infty}\sin n\theta\left[C_n\phi_1(r)+D_n\phi_2(r)\right].$$

The boundary conditions at $r=a$ and $r=b$,

$$f(\theta)=\sum_{n=0}^{\infty}\cos n\theta\left[A_n\phi_1(a)+B_n\phi_2(a)\right]+\sum_{n=1}^{\infty}\sin n\theta\left[C_n\phi_1(a)+D_n\phi_2(a)\right]$$

$$g(\theta)=\sum_{n=0}^{\infty}\cos n\theta\left[A_n\phi_1(b)+B_n\phi_2(b)\right]+\sum_{n=1}^{\infty}\sin n\theta\left[C_n\phi_1(b)+D_n\phi_2(b)\right]$$

easily determine A_n, B_n, C_n, D_n since $\phi_1(a)=0$ and $\phi_2(b)=0: D_n\phi_2(a)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\sin n\theta\,d\theta,$ etc.

2.5.9 (a) The boundary conditions of (2.5.37) must be replaced by $\phi(0)=0$ and $\phi(\pi/2)=0$. This is a sine series with $L=\pi/2$ so that $\lambda=(n\pi/L)^2=(2n)^2$ and the eigenfunctions are $\phi=\sin n\pi\theta/L=\sin 2n\theta, n=1,2,3,\ldots$. The radial part which is zero at $r=a$ is $G=(r/a)^{2n}-(a/r)^{2n}$. Thus by superposition,

$$u(r,\theta)=\sum_{n=1}^{\infty}A_n\left[\left(\frac{r}{a}\right)^{2n}-\left(\frac{a}{r}\right)^{2n}\right]\sin 2n\theta.$$

The nonhomogeneous boundary condition, $f(\theta)=\sum_{n=1}^{\infty}A_n\left[\left(\frac{b}{a}\right)^{2n}-\left(\frac{a}{b}\right)^{2n}\right]\sin 2n\theta$, determines A_n : $A_n\left[\left(\frac{b}{a}\right)^{2n}-\left(\frac{a}{b}\right)^{2n}\right]=\frac{4}{\pi}\int_0^{\pi/2}f(\theta)\sin 2n\theta\,d\theta.$

2.5.9 (b) The two homogeneous boundary conditions are in r, and hence $\phi(r)$ must be an eigenvalue problem. By separation of variables, $u=\phi(r)G(\theta), d^2G/d\theta^2=\lambda G$ and $r^2\frac{d^2\phi}{dr^2}+r\frac{d\phi}{dr}+\lambda\phi=0$. The radial equation is equidimensional (see p.78) and solutions are in the form $\phi=r^p$. Thus $p^2=-\lambda$ (with $\lambda>0$) so that $p=\pm i\sqrt{\lambda}$. $r^{\pm i\sqrt{\lambda}}=e^{\pm i\sqrt{\lambda}\ln r}$. Thus real solutions are $\cos(\sqrt{\lambda}\ln r)$ and $\sin(\sqrt{\lambda}\ln r)$. It is more convenient to use independent solutions which simplify at $r=a$, $\cos[\sqrt{\lambda}\ln(r/a)]$ and $\sin[\sqrt{\lambda}\ln(r/a)]$. Thus the general solution is

$$\phi=c_1\cos[\sqrt{\lambda}\ln(r/a)]+c_2\sin[\sqrt{\lambda}\ln(r/a)].$$

The homogeneous condition $\phi(a)=0$ yields $0=c_1$, while $\phi(b)=0$ implies $\sin[\sqrt{\lambda}\ln(r/a)]=0$. Thus $\sqrt{\lambda}\ln(b/a)=n\pi,\ n=1,2,3,...$ and the corresponding eigenfunctions are $\phi=\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]$. The solution of the θ -equation satisfying $G(0)=0$ is $G=\sinh\sqrt{\lambda}\theta=\sinh\frac{n\pi\theta}{\ln(b/a)}$. Thus by superposition

$$u=\sum_{n=1}^{\infty}A_n\sinh\frac{n\pi\theta}{\ln(b/a)}\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right].$$

The nonhomogeneous boundary condition,

$$f(r)=\sum_{n=1}^{\infty}A_n\sinh\frac{n\pi^2}{2\ln(b/a)}\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right],$$

will determine A_n. One method (for another, see exercise 5.3.9) is to let $z=\ln(r/a)/\ln(b/a)$. Then $a<r<b$, lets $0<z<1$. This is a sine series in z (with $L=1$) and hence

$$A_n\sinh\frac{n\pi^2}{2\ln(b/a)}=2\int_0^1 f(r)\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]dz.$$

But $dz=dr/r\ln(b/a)$. Thus

$$A_n\sinh\frac{n\pi^2}{2\ln(b/a)}=\frac{2}{\ln(b/a)}\int_0^1 f(r)\sin\left[n\pi\frac{\ln(r/a)}{\ln(b/a)}\right]dr/r.$$