Chapter 2

Problem 2.1

a)
Let

$$w_k = x + \mathrm{j}\, y$$

$$p(-k) = a + \mathrm{j}\, b$$

We may then write

$$\begin{aligned} f =& w_k p^*(-k) \\ =& (x + \mathrm{j}\, y)(a - \mathrm{j}\, b) \\ =& (ax + by) + \mathrm{j}(ay - bx) \end{aligned}$$

Letting

$$f = u + \mathrm{j}\, v$$

where

$$u = ax + by$$

$$v = ay - bx$$

Hence,

$$\frac{\partial u}{\partial x} = a \qquad \frac{\partial u}{\partial y} = b$$

$$\frac{\partial v}{\partial y} = a \qquad \frac{\partial v}{\partial x} = -b$$

From these results we can immediately see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

In other words, the product term $w_k p^*(-k)$ satisfies the Cauchy-Riemann equations, and so this term is analytic.

b)

Let

$$\begin{aligned} f =& w_k p^*(-k) \\ =& (x - \mathrm{j}\, y)(a + \mathrm{j}\, b) \\ =& (ax + by) + \mathrm{j}(bx - ay) \end{aligned}$$

Let

$$f = u + jv$$

with

$$u = ax + by$$

$$v = bx - ay$$

Hence,

$$\begin{aligned} \frac{\partial u}{\partial x} =& a \qquad \frac{\partial u}{\partial y} = b \\ \frac{\partial v}{\partial x} =& b \qquad \frac{\partial v}{\partial y} = -a \end{aligned}$$

From these results we immediately see that

$$\begin{aligned} \frac{\partial u}{\partial x} \neq& \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} =& -\frac{\partial u}{\partial y} \end{aligned}$$

In other words, the product term $w_k^* p(-k)$ does not satisfy the Cauchy-Riemann equations, and so this term is *not* analytic.

Problem 2.2

a)
From the Wiener-Hopf equation, we have

$$\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p} \tag{1}$$

We are given that

$$\mathbf{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

Hence the inverse of $\mathbf{R}$ is

$$\begin{aligned} \mathbf{R}^{-1} &= \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{0.75}\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}^{-1} \end{aligned}$$

Using Equation (1), we therefore get

$$\begin{aligned} \mathbf{w}_0 &= \frac{1}{0.75}\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} \\ &= \frac{1}{0.75}\begin{bmatrix} 0.375 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \end{aligned}$$

b)
The minimum mean-square error is

$$\begin{aligned} J_{\text{min}} &= \sigma_d^2 - \mathbf{p}^H\mathbf{w}_0 \\ &= \sigma_d^2 - \begin{bmatrix} 0.5 & 0.25 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \\ &= \sigma_d^2 - 0.25 \end{aligned}$$

c)
The eigenvalues of the matrix $\mathbf{R}$ are roots of the characteristic equation:

$$(1-\lambda)^2 - (0.5)^2 = 0$$

That is, the two roots are

$$\lambda_1 = 0.5 \quad \text{and } \lambda_2 = 1.5$$

The associated eigenvectors are defined by

$$\mathbf{Rq} = \lambda\mathbf{q}$$

For $\lambda_1 = 0.5$, we have

$$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = 0.5 \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix}$$

Expanded this becomes

$$q_{11} + 0.5q_{12} = 0.5q_{11}$$

$$0.5q_{11} + q_{12} = 0.5q_{12}$$

Therefore,

$$q_{11} = -q_{12}$$

Normalizing the eigenvector $\mathbf{q}_1$ to unit length, we therefore have

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly, for the eigenvalue $\lambda_2 = 1.5$, we may show that

$$\mathbf{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Accordingly, we may express the Wiener filter in terms of its eigenvalues and eigenvectors as follows:

$$\begin{aligned}
\mathbf{w}_0 &= \left(\sum_{i=1}^{2} \frac{1}{\lambda_i}\mathbf{q}_i\mathbf{q}_i^H\right)\mathbf{p} \\
&= \left(\frac{1}{\lambda_1}\mathbf{q}_1\mathbf{q}_1^H + \frac{1}{\lambda_2}\mathbf{q}_2\mathbf{q}_2^H\right)\mathbf{p} \\
&= \left(\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}\right)\begin{bmatrix}0.5\\0.25\end{bmatrix} \\
&= \left(\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right)\begin{bmatrix}0.5\\0.25\end{bmatrix} \\
&= \begin{bmatrix}\frac{4}{3} & -\frac{2}{3}\\-\frac{2}{3} & \frac{4}{3}\end{bmatrix}\begin{bmatrix}0.5\\0.25\end{bmatrix} \\
&= \begin{bmatrix}\frac{4}{6} - \frac{1}{6}\\-\frac{1}{3} + \frac{1}{3}\end{bmatrix} \\
&= \begin{bmatrix}0.5\\0\end{bmatrix}
\end{aligned}$$

Problem 2.3

a)

From the Wiener-Hopf equation we have

$$\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p} \tag{1}$$

We are given

$$\mathbf{R} = \begin{bmatrix}1 & 0.5 & 0.25\\0.5 & 1 & 0.5\\0.25 & 0.5 & 1\end{bmatrix}$$

and

$$\mathbf{p} = \begin{bmatrix}0.5 & 0.25 & 0.125\end{bmatrix}^T$$

Hence, the use of these values in Equation (1) yields

$$\begin{aligned}\mathbf{w}_0 =& \mathbf{R}^{-1}\mathbf{p} \\ =& \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ 0.25 \\ 0.125 \end{bmatrix} \\ =& \begin{bmatrix} 1.33 & -0.67 & 0 \\ -0.67 & 1.67 & -0.67 \\ 0 & -0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.25 \\ 0.125 \end{bmatrix} \\ \mathbf{w}_0 =& \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix}^T\end{aligned}$$

b)
The Minimum mean-square error is

$$\begin{aligned}J_{\text{min}} =& \sigma_d^2 - \mathbf{p}^H \mathbf{w}_0 \\ =& \sigma_d^2 - \begin{bmatrix} 0.5 & 0.25 & 0.125 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix} \\ =& \sigma_d^2 - 0.25\end{aligned}$$

c)
The eigenvalues of the matrix $\mathbf{R}$ are

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0.4069 & 0.75 & 1.8431 \end{bmatrix}$$

The corresponding eigenvectors constitute the orthogonal matrix:

$$\mathbf{Q} = \begin{bmatrix} -0.4544 & -0.7071 & 0.5418 \\ 0.7662 & 0 & 0.6426 \\ -0.4544 & 0.7071 & 0.5418 \end{bmatrix}$$

Accordingly, we may express the Wiener filter in terms of its eigenvalues and eigenvectors as follows:

$$\mathbf{w}_0 = \left(\sum_{i=1}^{3} \frac{1}{\lambda_i} \mathbf{q}_i \mathbf{q}_i^H \right) \mathbf{p}$$

$$\mathbf{w}_0 = \left(\frac{1}{0.4069} \begin{bmatrix} -0.4544 \\ 0.7662 \\ -0.4544 \end{bmatrix} \begin{bmatrix} -0.4544 & 0.7662 & -0.4544 \end{bmatrix} + \frac{1}{0.75} \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix} \begin{bmatrix} -0.7071 & 0 & -0.7071 \end{bmatrix} + \frac{1}{1.8431} \begin{bmatrix} 0.5418 \\ 0.6426 \\ 0.5418 \end{bmatrix} \begin{bmatrix} 0.5418 & 0.6426 & 0.5418 \end{bmatrix} \right) \times \begin{bmatrix} 0.5 \\ 0.25 \\ 0.125 \end{bmatrix}$$

$$\mathbf{w}_0 = \left(\frac{1}{0.4069} \begin{bmatrix} 0.2065 & -0.3482 & 0.2065 \\ -0.3482 & 0.5871 & -0.3482 \\ 0.2065 & -0.3482 & 0.2065 \end{bmatrix} + \frac{1}{0.75} \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix} + \frac{1}{1.8431} \begin{bmatrix} 0.2935 & 0.3482 & 0.2935 \\ 0.3482 & 0.4129 & 0.3482 \\ 0.2935 & 0.3482 & 0.2935 \end{bmatrix} \right) \times \begin{bmatrix} 0.5 \\ 0.25 \\ 0.125 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}$$

Problem 2.4

By definition, the correlation matrix

$$\mathbf{R} = \mathbb{E}[\mathbf{u}(n)\mathbf{u}^H(n)]$$

Where

$$\mathbf{u}(n) = \begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$

Invoking the ergodicity theorem,

$$\mathbf{R}(N) = \frac{1}{N+1} \sum_{n=0}^{N} \mathbf{u}(n)\mathbf{u}^H(n)$$

Likewise, we may compute the cross-correlation vector

$$\mathbf{p} = \mathbb{E}[\mathbf{u}(n)d^*(n)]$$

as the time average

$$\mathbf{p}(N) = \frac{1}{N+1}\sum_{n=0}^{N}\mathbf{u}(n)d^*(n)$$

The tap-weight vector of the wiener filter is thus defined by the matrix product

$$\mathbf{w}_0(N) = \left(\sum_{n=0}^{N}\mathbf{u}(n)\mathbf{u}^H(n)\right)^{-1}\left(\sum_{n=0}^{N}\mathbf{u}(n)d^*(n)\right)$$

Problem 2.5

a)

$$\begin{aligned}\mathbf{R} =&\mathbb{E}[\mathbf{u}(n)\mathbf{u}^H(n)]\\ =&\mathbb{E}[(\alpha(n)\mathbf{s}(n)+\mathbf{v}(n))(\alpha^*(n)\mathbf{s}^H(n)+\mathbf{v}^H(n))]\end{aligned}$$

With $\alpha(n)$ uncorrelated with $v(n)$, we have

$$\begin{aligned}\mathbf{R} =&\mathbb{E}[|\alpha(n)|^2]\mathbf{s}(n)\mathbf{s}^H(n)+\mathbb{E}[\mathbf{v}(n)\mathbf{v}^H(n)]\\ =&\sigma_\alpha^2\mathbf{s}(n)\mathbf{s}^H(n)+\mathbf{R}_v \qquad (1)\end{aligned}$$

where $\mathbf{R}_v$ is the correlation matrix of $\mathbf{v}$

b)

The cross-correlation vector between the input vector $\mathbf{u}(n)$ and the desired response $d(n)$ is

$$\mathbf{p} = \mathbb{E}[\mathbf{u}(n)d^*(n)] \qquad (2)$$

If $d(n)$ is uncorrelated with $\mathbf{u}(n)$, we have

$$\mathbf{p} = \mathbf{0}$$

Hence, the tap-weight of the wiener filter is

$$\begin{aligned}\mathbf{w}_0 =&\mathbf{R}^{-1}\mathbf{p}\\ =&\mathbf{0}\end{aligned}$$

c)
With $\sigma_\alpha^2 = 0$, Equation (1) reduces to

$$\mathbf{R} = \mathbf{R}_v$$

with the desired response

$$d(n) = v(n-k)$$

Equation (2) yields

$$\begin{aligned}
\mathbf{p} =& \mathbb{E}[(\alpha(n)\mathbf{s}(n) + \mathbf{v}(n)v^*(n-k))] \\
=& \mathbb{E}[(\mathbf{v}(n)v^*(n-k))] \\
=& \mathbb{E}\left[\begin{bmatrix} v(n) \\ v(n-1) \\ \vdots \\ v(n-M+1) \end{bmatrix} (v^*(n-k))\right] \\
=& \mathbb{E}\begin{bmatrix} r_v(n) \\ r_v(n-1) \\ \vdots \\ r_v(k-M+1) \end{bmatrix}, \quad 0 \le k \le M-1
\end{aligned} \tag{3}$$

where $r_v(k)$ is the autocorrelation of $v(n)$ for lag k. Accordingly, the tap-weight vector of the (optimum) wiener filter is

$$\begin{aligned}
\mathbf{w}_0 =& \mathbf{R}^{-1}\mathbf{p} \\
=& \mathbf{R}_v^{-1}\mathbf{p}
\end{aligned}$$

where $\mathbf{p}$ is defined in Equation (3).

d)
For a desired response

$$d(n) = \alpha(n)\exp(-\mathrm{j}\,\omega\tau)$$

The cross-correlation vector $\mathbf{p}$ is

$$\begin{aligned}
\mathbf{p} =& \mathbb{E}[\mathbf{u}(n)(d^*n)] \\
=& \mathbb{E}[(\alpha(n)\mathbf{s}(n) + \mathbf{v}(n))\,\alpha^*(n)\exp(-\mathrm{j}\,\omega\tau)] \\
=& \mathbf{s}(n)\exp(\mathrm{j}\,\omega\tau)\mathbb{E}[|\alpha(n)|^2] \\
=& \sigma_\alpha^2\mathbf{s}(n)\exp(\mathrm{j}\,\omega\tau) \\
=& \sigma_\alpha^2 \begin{bmatrix} 1 \\ \exp(-\mathrm{j}\,\omega) \\ \vdots \\ \exp((-\mathrm{j}\,\omega)(M-1)) \end{bmatrix} \exp(\mathrm{j}\,\omega\tau) \\
=& \sigma_\alpha^2 \begin{bmatrix} \exp(\mathrm{j}\,\omega\tau) \\ \exp(\mathrm{j}\,\omega(\tau-1)) \\ \vdots \\ \exp((\mathrm{j}\,\omega)(\tau-M+1)) \end{bmatrix}
\end{aligned}$$

The corresponding value of the tap-weight vector of the Wiener filter is

$$\begin{aligned}
\mathbf{w}_0 =& \sigma_\alpha^2(\sigma_\alpha^2\mathbf{s}(n)\mathbf{s}^H(n) + \mathbf{R}_v)^{-1} \begin{bmatrix} \exp(\mathrm{j}\,\omega\tau) \\ \exp(\mathrm{j}\,\omega(\tau-1)) \\ \vdots \\ \exp((\mathrm{j}\,\omega)(\tau-M+1)) \end{bmatrix} \\
=& \left(\mathbf{s}(n)\mathbf{s}^H(n) + \frac{1}{\sigma_\alpha^2}\mathbf{R}_v\right)^{-1} \begin{bmatrix} \exp(\mathrm{j}\,\omega\tau) \\ \exp(\mathrm{j}\,\omega(\tau-1)) \\ \vdots \\ \exp((\mathrm{j}\,\omega)(\tau-M+1)) \end{bmatrix}
\end{aligned}$$

Problem 2.6

The optimum filtering solution is defined by the Wiener-Hopf equation

$$\mathbf{R}\mathbf{w}_0 = \mathbf{p} \tag{1}$$

for which the minimum mean-square error is

$$J_{\text{min}} = \sigma_d^2 - \mathbf{p}^H\mathbf{w}_0 \tag{2}$$

Combine Equations (1) and Equation(2) into a single relation:

$$\begin{bmatrix} \sigma_d^2 & \mathbf{p}^H \\ \mathbf{p} & \mathbf{R} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{w}_0 \end{bmatrix} = \begin{bmatrix} J_{\text{min}} \\ \mathbf{0} \end{bmatrix}$$

Define

$$\mathbf{A} = \begin{bmatrix} \sigma_d^2 & \mathbf{p}^H \\ \mathbf{p} & \mathbf{R} \end{bmatrix} \tag{3}$$

Since

$$\sigma_d^2 = \mathbb{E}[d(n)d^*(n)]$$

$$\mathbf{p} = \mathbb{E}[\mathbf{u}(n)d^*(n)]$$

$$\mathbf{R} = \mathbb{E}[\mathbf{u}(n)\mathbf{u}^H(n)]$$

we may rewrite Equation (3) as

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbb{E}[d(n)d^*(n)] & \mathbb{E}[d(n)\mathbf{u}^H(n)] \\ \mathbb{E}[\mathbf{u}(n)d^*(n)] & \mathbb{E}[\mathbf{u}(n)\mathbf{u}^H(n)] \end{bmatrix} \\ &= \mathbb{E}\left\{ \begin{bmatrix} d(n) \\ \mathbf{u}(n) \end{bmatrix} \begin{bmatrix} d^*(n) & \mathbf{u}^H(n) \end{bmatrix} \right\} \end{aligned}$$

The minimum mean-square error equals

$$J_{\text{min}} = \sigma_d^2 - \mathbf{p}^H \mathbf{w}_0 \tag{4}$$

Eliminating σ_d^2 between Equation (1) and Equation (4):

$$J(\mathbf{w}) = J_{\text{min}} + \mathbf{p}^H \mathbf{w}_0 - \mathbf{p}^H \mathbf{R} \mathbf{w} - \mathbf{w}^H \mathbf{R} \mathbf{w}_0 + \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{5}$$

Eliminating $\mathbf{p}$ between Equation (2) and Equation (5)

$$J(\mathbf{w}) = J_{\text{min}} + \mathbf{w}_0^H \mathbf{R} \mathbf{w}_0 - \mathbf{w}_0^H \mathbf{R} \mathbf{w} - \mathbf{w}^H \mathbf{R} \mathbf{w}_0 + \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{6}$$

where we have used the property $\mathbf{R}^H = \mathbf{R}$. We may rewrite Equation (6) as

$$J(\mathbf{w}) = J_{\mathbf{min}} + (\mathbf{w} - \mathbf{w}_0)^H \mathbf{R} (\mathbf{w} - \mathbf{w}_0)$$

which clearly shows that $J(\mathbf{w}_0) = J_{\mathbf{min}}$

Problem 2.7

The minimum mean-square error is

$$J_{\text{min}} = \sigma_d^2 - \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} \tag{1}$$

Using the spectral theorem, we may express the correlation matrix $\mathbf{R}$ as

$$\mathbf{R} = \mathbf{Q}\Lambda\mathbf{Q}^H$$

$$\mathbf{R} = \sum_{k=1}^{M} \lambda_k \mathbf{q}_k \mathbf{q}_k^H \tag{2}$$

Substituting Equation (2) into Equation (1)

$$\begin{aligned} J_{\text{min}} =& \sigma_d^2 - \sum_{k=1}^{M} \frac{1}{\lambda_k} \mathbf{p}^H \mathbf{q}_k \mathbf{p}^H \mathbf{q}_k \\ =& \sigma_d^2 - \sum_{k=1}^{M} \frac{1}{\lambda_k} |\mathbf{p}^H \mathbf{q}_k|^2 \end{aligned}$$

Problem 2.8

When the length of the Wiener filter is greater than the model order m, the tail end of the tap-weight vector of the Wiener filter is zero; thus,

$$\mathbf{w}_0 = \begin{bmatrix} \mathbf{a}_m \\ \mathbf{0} \end{bmatrix}$$

Therefore, the only possible solution for the case of an over-fitted model is

$$\mathbf{w}_0 = \begin{bmatrix} \mathbf{a}_m \\ \mathbf{0} \end{bmatrix}$$

Problem 2.9

a)

The Wiener solution is defined by

$$\mathbf{R}_M \mathbf{a}_M = \mathbf{p}_M$$

$$\begin{bmatrix} \mathbf{R}_M & \mathbf{r}_{M-m} \\ \mathbf{r}_{M-m}^H & \mathbf{R}_{M-m,M-m} \end{bmatrix} \begin{bmatrix} \mathbf{a}_m \\ \mathbf{0}_{M-m} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_m \\ \mathbf{p}_{M-m} \end{bmatrix}$$

$$\mathbf{R}_M \mathbf{a}_m = \mathbf{p}_m$$

$$\mathbf{r}_{M-m}^H \mathbf{a}_m = \mathbf{p}_{M-m}$$

$$\mathbf{p}_{M-m} = \mathbf{r}_{M-m}^H \mathbf{a}_m = \mathbf{r}_{M-m}^H \mathbf{R}_M^{-1} \mathbf{p}_m \qquad (1)$$

b)

Applying the conditions of Equation (1) to the example in Section 2.7 in the textbook

$$\mathbf{r}_{M-m}^H = \begin{bmatrix} -0.05 & 0.1 & 0.15 \end{bmatrix}$$

$$\mathbf{a}_m = \begin{bmatrix} 0.8719 \\ -0.9129 \\ 0.2444 \end{bmatrix}$$

The last entry in the 4-by-1 vector $\mathbf{p}$ is therefore

$$\begin{aligned} \mathbf{r}_{M-m}^H \mathbf{a}_m &= -0.0436 - 0.0912 + 0.1222 \\ &= -0.0126 \end{aligned}$$

Problem 2.10

$$\begin{aligned} J_{\text{min}} &= \sigma_d^2 - \mathbf{p}^H \mathbf{w}_0 \\ &= \sigma_d^2 - \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} \end{aligned}$$

when $m = 0$,

$$\begin{aligned} J_{\text{min}} &= \sigma_d^2 \\ &= 1.0 \end{aligned}$$

When $m = 1$,

$$\begin{aligned} J_{\text{min}} &= 1 - 0.5 \times \frac{1}{1.1} \times 0.5 \\ &= 0.9773 \end{aligned}$$

when $m = 2$

$$\begin{aligned}
J_{\text{min}} &= 1 - \begin{bmatrix} 0.5 & -0.4 \end{bmatrix} \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ -0.4 \end{bmatrix} \\
&= 1 - 0.6781 \\
&= 0.3219
\end{aligned}$$

when $m = 3$,

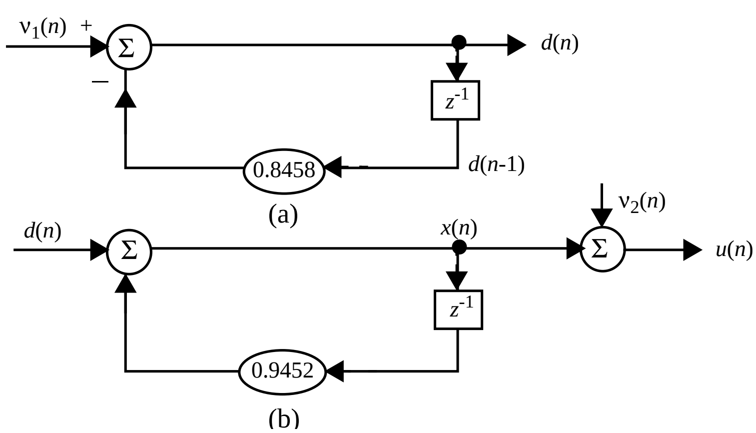
$$\begin{aligned}
J_{\text{min}} &= 1 - \begin{bmatrix} 0.5 & -0.4 & -0.2 \end{bmatrix} \begin{bmatrix} 1.1 & 0.5 & 0.1 \\ 0.5 & 1.1 & 0.5 \\ 0.1 & 0.5 & 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5 \\ -0.4 \\ -0.2 \end{bmatrix} \\
&= 1 - 0.6859 \\
&= 0.3141
\end{aligned}$$

when $m = 4$,

$$\begin{aligned}
J_{\text{min}} &= 1 - 0.6859 \\
&= 0.3141
\end{aligned}$$

Thus any further increase in the filter order beyond $m = 3$ does not produce any meaningful reduction in the minimum mean-square error.

Problem 2.11



a)

$$u(n) = x(n) + v_2(n) \tag{1}$$
$$d(n) = -d(n-1) \times 0.8458 + v_1(n) \tag{2}$$
$$x(n) = d(n) + 0.9458x(n-1) \tag{3}$$

Equation (3) rearranged to solve for $d(n)$ is

$$d(n) = x(n) - 0.9458x(n-1)$$

Using Equation (2) and Equation (3):

$$x(n) - 0.9458x(n-1) = 0.8458[-x(n-1) + 0.9458x(n-2)] + v_1(n)$$

Rearranging the terms this produces:

$$\begin{aligned} x(n) =& (0.9458 - 8.8458)x(n-1) + 0.8x(n-2) + v_1(n) \\ =& (0.1)x(n-1) + 0.8x(n-2) + v_1(n) \end{aligned}$$

b)

$$u(n) = x(n) + v_2(n)$$

where $x(n)$ and $v_2(n)$ are uncorrelated, therefore

$$\mathbf{R} = \mathbf{R}_x + \mathbf{R}_{v_2}$$

$$\mathbf{R}_x = \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}$$

$$\begin{aligned} r_x(0) =& \sigma_x^2 \\ =& \frac{1+a_2}{1-a_2} \frac{\sigma_1^2}{(1+a_2)^2 - a_1^2} = 1 \end{aligned}$$

$$r_x(1) = \frac{-a_1}{1+a_2}$$

$$r_x(1) = 0.5$$

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\mathbf{R}_{v_2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{R}_x + \mathbf{R}_{v_2} = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$$

$$p(k) = \mathbb{E}[u(n-k)d(n)], \quad k = 0, 1$$

$$\begin{aligned} p(0) =& r_x(0) + b_1 r_x(-1) \\ =& 1 - 0.9458 \times 0.5 \\ =& 0.5272 \end{aligned}$$

$$\begin{aligned} p(1) =& r_x(1) + b_1 r_x(0 \\ =& 0.5 - 0.9458 \\ =& -0.4458 \end{aligned}$$

Therefore,

$$\mathbf{p} = \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix}$$

c)

The optimal weight vector is given by the equation $\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p}$; hence,

$$\begin{aligned} \mathbf{w}_0 &= \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}^{-1} \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix} \\ &= \begin{bmatrix} 0.8363 \\ -0.7853 \end{bmatrix} \end{aligned}$$

Problem 2.12

a)

For $M = 3$ taps, the correlation matrix of the tap inputs is

$$\mathbf{R} = \begin{bmatrix} 1.1 & 0.5 & 0.85 \\ 0.5 & 1.1 & 0.5 \\ 0.85 & 0.5 & 1.1 \end{bmatrix}$$

The cross-correlation vector between the tap inputs and the desired response is

$$\mathbf{p} = \begin{bmatrix} 0.527 \\ -0.446 \\ 0.377 \end{bmatrix}$$

b)

The inverse of the correlation matrix is

$$\mathbf{R}^{-1} = \begin{bmatrix} 2.234 & -0.304 & -1.666 \\ -0.304 & 1.186 & -0.304 \\ -1.66 & -0.304 & 2.234 \end{bmatrix}$$

Hence, the optimum weight vector is

$$\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p} = \begin{bmatrix} 0.738 \\ -0.803 \\ 0.138 \end{bmatrix}$$

The minimum mean-square error is

$$J_{\text{min}} = 0.15$$

Problem 2.13

a)

The correlation matrix $\mathbf{R}$ is

$$\begin{aligned}\mathbf{R} =&\mathbb{E}[\mathbf{u}(n)\mathbf{u}^H(n)]\\ =&\mathbb{E}[|A_1|^2]\begin{bmatrix} e^{-\mathrm{j}\,\omega_1 n} \\ e^{-\mathrm{j}\,\omega_1 (n-1)} \\ \vdots \\ e^{-\mathrm{j}\,\omega_1 (n-M+1)} \end{bmatrix}\begin{bmatrix} e^{+\mathrm{j}\,\omega_1 n} & e^{+\mathrm{j}\,\omega_1 (n-1)} & \dots & e^{+\mathrm{j}\,\omega_1 (n-M+1)} \end{bmatrix}\\ =&\mathbb{E}[|A_1|^2]\mathbf{s}(\omega_1)\mathbf{s}^H(\omega_1) + \mathbf{I}\mathbb{E}[|v(n)|^2]\\ =&\sigma_1^2\mathbf{s}(\omega_1)\mathbf{s}^H(\omega_1) + \sigma_v^2\mathbf{I}\end{aligned}$$

where $\mathbf{I}$ is the identity matrix.

b)

The tap-weights vector of the Wiener filter is

$$\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p}$$

From part **a)**,

$$\mathbf{R} = \sigma_1^2\mathbf{s}(\omega_1)\mathbf{s}^H(\omega_1) + \sigma_v^2\mathbf{I}$$

We are given

$$\mathbf{p} = \sigma_0^2\mathbf{s}(\omega_0)$$

To invert the matrix $\mathbf{R}$, we use the matrix inversion lemma (see Chapter 10), as described here:

If:

$$\mathbf{A} = \mathbf{B}^{-1} + \mathbf{C}\mathbf{D}^{-1}\mathbf{C}^H$$

then:

$$\mathbf{A}^{-1} = \mathbf{B} - \mathbf{B}\mathbf{C}(\mathbf{D} + \mathbf{C}^H\mathbf{B}\mathbf{C})^{-1}\mathbf{C}^H\mathbf{B}$$

In our case:

$$\mathbf{A} = \sigma_v^2\mathbf{I}$$

$$\mathbf{B}^{-1} = \sigma_v^2 \mathbf{I}$$
$$\mathbf{D}^{-1} = \sigma_1^2$$
$$\mathbf{C} = \mathbf{s}(\omega_1)$$

Hence,

$$\mathbf{R}^{-1} = \frac{1}{\sigma_v^2}\mathbf{I} - \frac{\frac{1}{\sigma_v^2}\mathbf{s}(\omega_1)\mathbf{s}^H(\omega_1)}{\frac{\sigma_v^2}{\sigma_1^2} + \mathbf{s}^H(\omega_1)\mathbf{s}(\omega_1)}$$

The corresponding value of the Wiener tap-weight vector is

$$\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{p}$$

$$\mathbf{w}_0 = \frac{\sigma_0^2}{\sigma_v^2}\mathbf{s}(\omega_0) - \frac{\frac{\sigma_0^2}{\sigma_v^2}\mathbf{s}(\omega_1)\mathbf{s}^H(\omega_1)}{\frac{\sigma_v^2}{\sigma_1^2} + \mathbf{s}^H(\omega_1)\mathbf{s}(\omega_1)}\mathbf{s}(\omega_0)$$

we note that

$$\mathbf{s}^H(\omega_1)\mathbf{s}(\omega_1) = M$$

which is a scalar hence,

$$\mathbf{w}_0 = \frac{\sigma_0^2}{\sigma_v^2}\mathbf{s}(\omega_0) - \left(\frac{\sigma_0^2}{\sigma_v^2}\frac{\mathbf{s}^H(\omega_1)\mathbf{s}(\omega_1)}{\frac{\sigma_v^2}{\sigma_0^2} + M}\mathbf{s}(\omega_0)\right)$$

Problem 2.14

The output of the array processor equals

$$e(n) = u(1,n) - wu(2,n)$$

The mean-square error equals

$$\begin{aligned} J(w) =& \mathbb{E}[|e(n)|^2] \\ =& \mathbb{E}[(u(1,n) - wu(2,n))(u^*(1,n) - w^*u^*(2,n))] \\ =& \mathbb{E}[|u(1,n)|^2] + |w|^2\mathbb{E}[|u(2,n)|^2] - w\mathbb{E}[u(2,n)u^*(1,n)] - w\mathbb{E}[u(1,n)u^*(2,n)] \end{aligned}$$

Differentiating $J(w)$ with respect to w:

$$\frac{\partial J}{\partial w} = -2\mathbb{E}[u(1,n)u^*(2,n)] + 2w\mathbb{E}[|u(2,n)|^2]$$

Putting $\frac{\partial J}{\partial w} = 0$ and solving for the optimum value of w:

$$w_0 = \frac{\mathbb{E}[u(1,n)u^*(2,n)]}{\mathbb{E}[|u(2,n)|^2]}$$

Problem 2.15

Define the index of the performance (i.e., cost function)

$$J(\mathbf{w}) = \mathbb{E}[|e(n)|^2] + \mathbf{c}^H\mathbf{s}^H\mathbf{w} + \mathbf{w}^H\mathbf{s}\mathbf{c} - 2\mathbf{c}^H\mathbf{D}^{1/2}\mathbf{1}$$

$$J(\mathbf{w}) = \mathbf{w}^H\mathbf{R}\mathbf{w} + \mathbf{c}^H\mathbf{s}^H\mathbf{w} + \mathbf{w}^H\mathbf{s}\mathbf{c} - 2\mathbf{c}^H\mathbf{D}^{1/2}\mathbf{1}$$

Differentiate $J(\mathbf{w})$ with respect to $\mathbf{w}$ and set the result equal to zero:

$$\frac{\partial J}{\partial \mathbf{w}} = 2\mathbf{R}\mathbf{w} + 2\mathbf{s}\mathbf{c} = \mathbf{0}$$

Hence,

$$\mathbf{w}_0 = -\mathbf{R}^{-1}\mathbf{s}\mathbf{c}$$

But, we must constrain $\mathbf{w}_0$ as

$$\mathbf{s}^H\mathbf{w}_0 = \mathbf{D}^{1/2}\mathbf{1}$$

therefore, the vector $\mathbf{c}$ equals

$$\mathbf{c} = -(\mathbf{s}^H\mathbf{R}^{-1}\mathbf{s})^{-1}\mathbf{D}^{1/2}\mathbf{1}$$

Correspondingly, the optimum weight vector equals

$$\mathbf{w}_0 = \mathbf{R}^{-1}\mathbf{s}(\mathbf{s}^H\mathbf{R}^{-1}\mathbf{s})^{-1}\mathbf{D}^{1/2}\mathbf{1}$$

Problem 2.16

The weight vector $\mathbf{w}$ of the beamformer that maximizes the output signal-to-noise ratio:

$$(\text{SNR})_0 = \frac{\mathbf{w}^H \mathbf{R}_S \mathbf{w}}{\mathbf{w}^H \mathbf{R}_v \mathbf{w}}$$

is derived in part **b)** of the problem 2.18; there it is shown that the optimum weight vector $\mathbf{w}_{SN}$ so defined is given by

$$\mathbf{w}_{SN} = \mathbf{R}_v^{-1} \mathbf{s} \tag{1}$$

where $\mathbf{s}$ is the signal component and $\mathbf{R}_v$ is the correlation matrix of the noise $\mathbf{v}(n)$. On the other hand, the optimum weight vector of the LCMV beamformer is defined by

$$\mathbf{w}_0 = g^* \frac{\mathbf{R}^{-1} s(\phi)}{s^H(\phi) \mathbf{R}^{-1} s(\phi)} \tag{2}$$

where $s(\phi)$ is the steering vector. In general, the formulas (1) and (2) yield different values for the weight vector of the beamformer.

Problem 2.17

Let τ_i be the propagation delay, measured from the zero-time reference to the ith element of a nonuniformly spaced array, for a plane wave arriving from a direction defined by angle θ with respect to the perpendicular to the array. For a signal of angular frequency ω, this delay amounts to a phase shift equal to $-\omega\tau_i$. Let the phase shifts for all elements of the array be collected together in a column vector denoted by $\mathbf{d}(\omega, \theta)$. The response of a beamformer with weight vector $\mathbf{w}$ to a signal (with angular frequency ω) originates from angle $\theta = \mathbf{w}^H \mathbf{d}(\omega, \theta)$. Hence, constraining the response of the array at ω and θ to some value g involves the linear constraint

$$\mathbf{w}^H \mathbf{d}(\omega, \theta) = g$$

Thus, the constraint vector $\mathbf{d}(\omega, \theta)$ serves the purpose of generalizing the idea of an LCMV beamformer beyond simply the case of a uniformly spaced array. Everything else is the same as before, except for the fact that the correlation matrix of the received signal is no longer Toeplitz for the case of a nonuniformly spaced array

Problem 2.18

a)

Under hypothesis H_1, we have

$$\mathbf{u} = \mathbf{s} + \mathbf{v}$$

The correlation matrix of $\mathbf{u}$ equals

$$\mathbf{R} = \mathbb{E}[\mathbf{u}\mathbf{u}^T]$$

$$\mathbf{R} = \mathbf{s}\mathbf{s}^T + \mathbf{R}_N, \quad \text{where } \mathbf{R}_N = \mathbb{E}[\mathbf{v}\mathbf{v}^T]$$

The tap-weight vector $\mathbf{w}_k$ is chosen so that $\mathbf{w}_k^T\mathbf{u}$ yields an optimum estimate of the kth element of $\mathbf{s}$. Thus, with $s(k)$ treated as the desired response, the cross-correlation vector between $\mathbf{u}$ and $s(k)$ equals

$$\begin{aligned}\mathbf{p}_k =& \mathbb{E}[\mathbf{u}s(k)] \\ =& \mathbf{s}\mathbf{s}(k), \quad k = 1, 2, \dots, m\end{aligned}$$

Hence, the Wiener-Hopf equation yields the optimum value of $\mathbf{w}_k$ as

$$\mathbf{w}_{k0} = \mathbf{R}^{-1}\mathbf{p}_k$$

$$\mathbf{w}_{k0} = (\mathbf{s}\mathbf{s}^T + \mathbf{R}_N)^{-1}\mathbf{s}s(k), \quad k = 1, 2, \dots, M \tag{1}$$

To apply the matrix inversion lemma (introduced in Problem 2.13), we let

$$\mathbf{A} = \mathbf{R}$$

$$\mathbf{B}^{-1} = \mathbf{R}_N$$

$$\mathbf{C} = \mathbf{s}$$

$$\mathbf{D} = 1$$

Hence,

$$\mathbf{R}^{-1} = \mathbf{R}_N^{-1} - \frac{\mathbf{R}_N^{-1}\mathbf{s}\mathbf{s}^T\mathbf{R}_N^{-1}}{1 + \mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{s}} \tag{2}$$

Substituting Equation (2) into Equation (1) yields:

$$\mathbf{w}_{k0} = \left(\mathbf{R}_N^{-1} - \frac{\mathbf{R}_N^{-1}\mathbf{s}\mathbf{s}^T\mathbf{R}_N^{-1}}{1 + \mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{s}}\right)\mathbf{s}s(k)$$

$$\mathbf{w}_{k0} = \frac{\mathbf{R}_N^{-1}\mathbf{s}(1 + \mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{s}) - \mathbf{R}_N^{-1}\mathbf{s}\mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{s}}{1 + \mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{s}} s(k)$$

$$\mathbf{w}_{k0} = \frac{s(k)}{1 + \mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{s}} \mathbf{R}_N^{-1}\mathbf{s}$$

b)

The output signal-to-noise ratio is

$$\begin{aligned} \text{SNR} &= \frac{\mathbb{E}[(\mathbf{w}^T\mathbf{s})^2]}{\mathbb{E}[(\mathbf{w}^T\mathbf{v})^2]} \\ &= \frac{\mathbf{w}^T\mathbf{s}\mathbf{s}^T\mathbf{w}}{\mathbf{w}^T\mathbb{E}[\mathbf{v}\mathbf{v}^T]\mathbf{w}} \\ &= \frac{\mathbf{w}^T\mathbf{s}\mathbf{s}^T\mathbf{w}}{\mathbf{w}^T\mathbf{R}_N\mathbf{w}} \end{aligned} \tag{3}$$

Since $\mathbf{R}_N$ is positive definite, we may write,

$$\mathbf{R}_N = \mathbf{R}_N^{1/2}\mathbf{R}_N^{1/2}$$

Define the vector

$$\mathbf{a} = \mathbf{R}_N^{1/2}\mathbf{w}$$

or equivalently,

$$\mathbf{w} = \mathbf{R}_N^{-1/2}\mathbf{a} \tag{4}$$

Accordingly, we may rewrite Equation (3) as follows

$$\text{SNR} = \frac{\mathbf{a}^T\mathbf{R}_N^{1/2}\mathbf{s}\mathbf{s}^T\mathbf{R}_N^{1/2}\mathbf{a}}{\mathbf{a}^T\mathbf{a}} \tag{5}$$

where we have used the symmetric property of $\mathbf{R}_N$. Define the normalized vector

$$\bar{\mathbf{a}} = \frac{\mathbf{a}}{||\mathbf{a}||}$$

where $||\mathbf{a}||$ is the norm of $\mathbf{a}$. Equation (5) may be rewritten as:

$$\text{SNR} = \bar{\mathbf{a}}^T\mathbf{R}_N^{1/2}\mathbf{s}\mathbf{s}^T\mathbf{R}_N^{1/2}\bar{\mathbf{a}}$$

$$\text{SNR} = \left|\bar{\mathbf{a}}^T \mathbf{R}_N^{1/2}\mathbf{s}\right|^2$$

Thus the output signal-to-noise ratio SNR equals the squared magnitude of the inner product of the two vectors $\bar{\mathbf{a}}$ and $\mathbf{R}_N^{1/2}\mathbf{s}$. This inner product is maximized when $\mathbf{a}$ equals $\mathbf{R}_N^{-1/2}$. That is,

$$\mathbf{a}_{SN} = \mathbf{R}_N^{-1/2}\mathbf{s} \tag{6}$$

Let $\mathbf{w}_{SN}$ denote the value of the tap-weight vector that corresponds to Equation (6). Hence, the use of Equation (4) in Equation (6) yields

$$\mathbf{w}_{SN} = \mathbf{R}_N^{-1/2}(\mathbf{R}_N^{-1/2}\mathbf{s})$$

$$\mathbf{w}_{SN} = \mathbf{R}_N^{-1}\mathbf{s}$$

c)

Since the noise vector $\mathbf{v}(n)$ is Gaussian, its joint probability density function equals

$$f_{\mathbf{v}}(\mathbf{v}) = \frac{1}{(2\pi)^{M/2}(\text{det}(\mathbf{R}_N))^{1/2}} \exp\left(-\frac{1}{2}\mathbf{v}^T\mathbf{R}_N^{-1}\mathbf{v}\right)$$

Under the hypothesis H_0 we have

$$\mathbf{u} = \mathbf{v}$$

and

$$f_{\mathbf{u}}(\mathbf{u}|H_0) = \frac{1}{(2\pi)^{M/2}(\text{det}\mathbf{R}_N)^{1/2}} \exp\left(-\frac{1}{2}\mathbf{u}^T\mathbf{R}_N^{-1}\mathbf{u}\right)$$

Under hypothesis H_1 we have

$$\mathbf{u} = \mathbf{s} + \mathbf{v}$$

and

$$f_{\mathbf{u}}(\mathbf{u}|H_1) = \frac{1}{(2\pi)^{M/2}(\text{det}\mathbf{R}_N)^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{u}-\mathbf{s})^T\mathbf{R}_N^{-1}(\mathbf{u}-\mathbf{s})\right)$$

Hence, the likelihood ratio is defined by

$$\begin{aligned}\Lambda =& \frac{f_{\mathbf{u}}(\mathbf{u}|H_1)}{f_{\mathbf{u}}(\mathbf{u}|H_0)} \\ =& \exp\left(-\frac{1}{2}\mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{s} + \mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{u}\right)\end{aligned}$$

The natural logarithm of the likelihood ratio equals

$$\ln \Lambda = -\frac{1}{2}\mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{s} + \mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{u} \tag{7}$$

The first term in (7) represents a constant. Hence, testing $\ln \Lambda$ against a threshold is equivalent to the test

$$\mathbf{s}^T\mathbf{R}_N^{-1}\mathbf{u} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

where λ is some threshold. Equivalently, we may write

$$\mathbf{w}_{ML} = \mathbf{R}_N^{-1}\mathbf{s}$$

where $\mathbf{w}_{ML}$ is the maximum likelihood weight vector.

The results of parts **a)**, **b)**, and **c)** show that the three criteria discussed here yield the same optimum value for the weight vector, except for a scaling factor.

Problem 2.19

a)

Assuming the use of a noncausal Wiener filter, we write

$$\sum_{i=-\infty}^{\infty} w_{0i} r(i-k) = p(-k), \quad k = 0, \pm 1, \pm 2, \dots, \pm\infty \tag{1}$$

where the sum now extends from $i = -\infty$ to $i = \infty$. Define the z-transforms:

$$S(z) = \sum_{k=-\infty}^{\infty} r(k) z^{-k}, \qquad H_u(z) = \sum_{k=-\infty}^{\infty} w_{0,k} z^{-k}$$

$$P(z) = \sum_{k=-\infty}^{\infty} p(-k) z^{-k} = P(z^{-1})$$

Hence, applying the z-transform to Equation (1):

$$H_u(z)S(z) = P(z^{-1})$$

$$H_u(z) = \frac{P(1/z)}{S(z)} \tag{2}$$

b)

$$P(z) = \frac{0.36}{\left(1 - \frac{0.2}{z}\right)(1 - 0.2z)}$$

$$P(1/z) = \frac{0.36}{(1 - 0.2z)\left(1 - \frac{0.2}{z}\right)}$$

$$S(z) = 1.37\frac{(1 - 0.146z^{-1})(1 - 0.146z)}{(1 - 0.2z^{-1})(1 - 0.2z)}$$

Thus, applying Equation (2) yields

$$\begin{aligned} H_u(z) &= \frac{0.36}{1.37(1 - 0.146z^{-1})(1 - 0.146z)} \\ &= \frac{0.36z^{-1}}{1.37(1 - 0.146z^{-1})(z^{-1} - 0.146)} \\ &= \frac{0.2685}{1 - 0.146z^{-1}} + \frac{0.0392}{z^{-1} - 0.146} \end{aligned}$$

Clearly, this system is noncausal. Its impulse response is $h(n)$ = inverse z-transform of $H_u(z)$ is given by

$$h(n) = 0.2685(0.146)^n u_{\text{step}}(n) - \frac{0.0392}{0.146}\left(\frac{1}{0.146}\right)^n u_{\text{step}}(-n)$$

where $u_{\text{step}}(n)$ is the unit-step function:

$$u_{\text{step}}(n) = \begin{cases} 1 \text{ for } n = 0, 1, 2, \ldots \\ 0 \text{ for } n = -1, -2, \ldots \end{cases}$$

and $u_{\text{step}}(-n)$ is its mirror image:

$$u_{\text{step}}(-n) = \begin{cases} 1 \text{ for } n = 0, -1, -2, \ldots \\ 0 \text{ for } n = 1, 2, \ldots \end{cases}$$

Simplifying,

$$h_u(n) = 0.2685 \times (0.146)^n u_{\text{step}}(n) - 0.2685 \times (6.849)^{-n} u_{\text{step}}(-n)$$

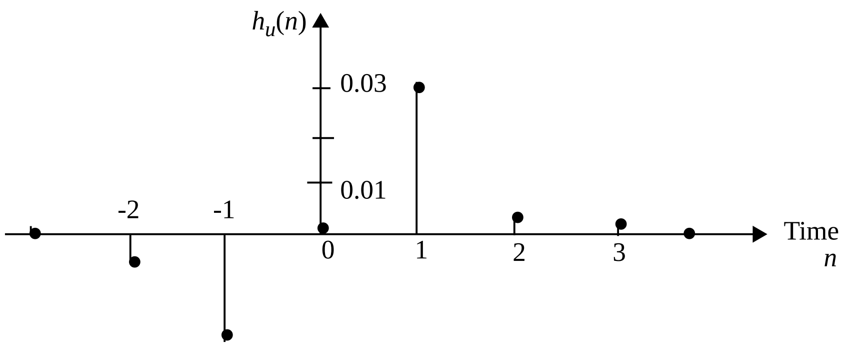
Evaluating $h_u(n)$ for varying n:

$$h_u(0) = 0$$

$$h_u(1) = 0.03, \qquad h_u(2) = 0.005, \qquad h_u(3) = 0.0008$$

$$h_u(-1) = -0.03, \quad h_u(-2) = -0.005, \quad h_u(-3) = -0.0008$$

The preceding values for $h_u(n)$ are plotted in the following figure:



c)
A delay of 3 time units applied to the impulse response will make the system causal and therefore realizable.